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Some fractional Hermite–Hadamard-type inequalities for interval-valued coordinated functions

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Abstract

The primary objective of this paper is establishing new Hermite–Hadamard-type inequalities for interval-valued coordinated functions via Riemann–Liouville-type fractional integrals. Moreover, we obtain some fractional Hermite–Hadamard-type inequalities for the product of two coordinated h-convex interval-valued functions. Our results generalize several well-known inequalities.

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1 Introduction

The classical Hermite–Hadamard inequalities state that

$$f\left(\frac{o+\varsigma}{2}\right)\le\frac{1}{\varsigma-o}\int_o^\varsigma f(\chi)\,d\chi\le\frac{f(o)+f(\varsigma)}{2},\qquad(1.1)$$

where $f:\mathcal{I}\to\mathbb{R}$ is a convex function on the closed bounded interval $\mathcal{I}$ of $\mathbb{R}$, and $o,\varsigma\in\mathcal{I}$ with $o<\varsigma$. Since they play a crucial role in convex analysis and can be a very powerful tool for measuring and computing errors, many authors have devoted their efforts to generalize inequalities (1.1); see [1–6]. It is worth noting that Sarikaya et al. [7] established new Hermite–Hadamard-type inequalities via the Riemann–Liouville fractional integrals. Since then, many papers have generalized different forms of fractional integrals and presented new and interesting refinements of Hermite–Hadamard-type inequalities using these integrals. Fernandez and Mohammed [8] established some Hermite–Hadamard-type inequalities for the Atangana–Baleanu fractional integral. Mohammed and Abdeljawad [9] proved new Hermite–Hadamard-type inequalities in the context of fractional calculus with respect to functions involving nonsingular kernels. For other related results, we refer the readers to [7–19].

On the other hand, to improve the reliability of the calculation results and automatic operation error analysis, Moore [20] introduced the theory of interval analysis. Interval analysis has a strong model for handling interval uncertainty and has been widely applied and stretched in control theory [21], dynamical game theory [22], and many others. Recently, numerous famous inequalities have been extended to interval-valued functions. Chalco-Cano et al. [23] obtained Ostrowski-type inequalities for interval-valued functions by using the Hukuhara derivative. Román-Flores et al. [24] derived the Minkowski and Beckenbach-type inequalities for interval-valued functions. Liu et al. [18] proved Hermite–Hadamard-type inequalities via interval Riemann–Liouville-type fractional integrals for interval-valued functions. Very recently, Zhao et al. [25, 26] established Hermite–Hadamard-type inequalities for interval-valued coordinated functions. Budak et al. [27] gave a definition of Riemann–Liouville-type fractional integrals for interval-valued coordinated functions and presented some new Hermite–Hadamard-type inequalities.

Motivated by Zhao et al. [25, 26] and Budak et al. [27], we present a new class of Hermite–Hadamard-type inequalities for coordinated h-convex interval-valued functions via Riemann–Liouville-type fractional integrals. We also establish Hermite–Hadamard-type inequalities for the products of two interval-valued coordinated functions.

The paper is organized as follows. Section 2 contains some necessary preliminaries. In Sect. 3, we establish some new Hermite–Hadamard-type inequalities for coordinated h-convex interval-valued functions via Riemann–Liouville-type fractional integrals. We end with Sect. 4 of conclusions.

2 Preliminaries

In this section, we recall some basic definitions and results on interval analysis. We denote by $\mathbb{R}_{\mathcal{I}}$ the set of closed bounded intervals of $\mathbb{R}$ and by $\mathbb{R}^+$ and $\mathbb{R}_{\mathcal{I}}^+$ the sets of positive real numbers and positive intervals, respectively. We also denote $\triangle = [o, \varsigma] \times [\rho, q]$. For more notions on interval-valued functions, see [28, 29].

Definition 2.1 ([29]) Let $h : [0,1] \to \mathbb{R}^+$. We say that $f : [o, \varsigma] \to \mathbb{R}_{\mathcal{I}}^+$ is an h-convex interval-valued function if for all $\chi, \gamma \in [o, \varsigma]$ and $\tau \in [0,1]$, we have

$$f\big(\tau\chi + (1-\tau)\gamma\big) \supseteq h(\tau)f(\chi) + h(1-\tau)f(\gamma).$$

We denote the set of all h-convex interval-valued functions by $SX(h, [o, \varsigma], \mathbb{R}_{\mathcal{I}}^+)$.

Definition 2.2 ([26]) A function $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}^+$ is said to be a coordinated convex interval-valued function if

$$\begin{aligned}&\mathcal{F}\big(\tau\chi + (1-\tau)\gamma, \theta\mu + (1-\theta)\nu\big)\\&\quad\supseteq \tau\theta\mathcal{F}(\chi,\mu) + \tau(1-\theta)\mathcal{F}(\chi,\nu) + (1-\tau)\theta\mathcal{F}(\gamma,\mu) + (1-\tau)(1-\theta)\mathcal{F}(\gamma,\nu)\end{aligned}$$

for all $(\chi, \gamma), (\mu, \nu) \in \triangle$ and $\tau, \theta \in [0,1]$.

Definition 2.3 ([25]) Let $h : [0,1] \to \mathbb{R}^+$. Then $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}^+$ is called a coordinated h-convex interval-valued function on $\triangle$ if the partial mappings

$$\begin{aligned}&\mathcal{F}_\gamma : [o, \varsigma] \to \mathbb{R}_{\mathcal{I}}^+, \qquad \mathcal{F}_\gamma(\chi) = \mathcal{F}(\chi, \gamma),\\&\mathcal{F}_\chi : [\rho, q] \to \mathbb{R}_{\mathcal{I}}^+, \qquad \mathcal{F}_\chi(\gamma) = \mathcal{F}(\chi, \gamma)\end{aligned}$$

are h-convex for all $\gamma \in [\rho, q]$ and $\chi \in [o, \varsigma]$. We denote the set of all coordinated h-convex interval-valued functions on $\triangle$ by $SX(ch, \triangle, \mathbb{R}^+_{\mathcal{I}})$.

The families of all Riemann-integrable real-valued functions on $[o, \varsigma]$, interval-valued functions on $[o, \varsigma]$ and on $\triangle$ are denoted by $\mathcal{R}_{([o,\varsigma])}$, $\mathcal{IR}_{([o,\varsigma])}$, and $\mathcal{ID}_{(\triangle)}$. We have the following:

Theorem 2.4 ([30]) *Let $f : [o, \varsigma] \to \mathbb{R}_{\mathcal{I}}$ be such that $f = [\underline{f}, \overline{f}]$. Then $f \in \mathcal{IR}_{([o,\varsigma])}$ iff $\underline{f}, \overline{f} \in \mathcal{R}_{([o,\varsigma])}$ and*

$$(\mathcal{IR}) \int_o^\varsigma f(s)\,ds = \left[(\mathcal{R}) \int_o^\varsigma \underline{f}(s)\,ds, (\mathcal{R}) \int_o^\varsigma \overline{f}(s)\,ds\right].$$

Theorem 2.5 ([31]) *Let $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}$. If $\mathcal{F} \in \mathcal{ID}_{(\triangle)}$, then*

$$(\mathcal{ID}) \iint_\triangle \mathcal{F}(t,s)\,dt\,ds = (\mathcal{IR}) \int_o^\varsigma dt (\mathcal{IR}) \int_\rho^q \mathcal{F}(t,s)\,ds.$$

Definition 2.6 ([16]) Let $f : [o, \varsigma] \to \mathbb{R}_{\mathcal{I}}$ and $f \in \mathcal{IR}_{([o,\varsigma])}$. Then the interval Riemann–Liouville-type integrals of f are defined by

$$\begin{aligned}
\mathfrak{J}^\alpha_{o^+} f(\vartheta) &= \frac{1}{\Gamma(\alpha)} \int_o^\vartheta (\vartheta - \chi)^{\alpha-1} f(\chi)\,d\chi, \quad \vartheta > o,\\
\mathfrak{J}^\alpha_{\varsigma^-} f(\vartheta) &= \frac{1}{\Gamma(\alpha)} \int_\vartheta^\varsigma (\chi - \vartheta)^{\alpha-1} f(\chi)\,d\chi, \quad \vartheta < \varsigma,
\end{aligned}$$

where $\alpha > 0$, and Γ is the gamma function.

Theorem 2.7 ([32]) *Let $f : [o, \varsigma] \to \mathbb{R}^+_{\mathcal{I}}$, $f \in \mathcal{IR}_{([o,\varsigma])}$, and $h : [0, 1] \to \mathbb{R}^+$. If $f \in SX(h, [o, \varsigma], \mathbb{R}^+_{\mathcal{I}})$, then*

$$\begin{aligned}
\frac{1}{\alpha h(\frac{1}{2})} f\left(\frac{o+\varsigma}{2}\right) &\supseteq \frac{\Gamma(\alpha)}{(\varsigma - o)^\alpha} \left[\mathfrak{J}^\alpha_{o^+} f(\varsigma) + \mathfrak{J}^\alpha_{\varsigma^-} f(o)\right]\\
&\supseteq \left[f(o) + f(\varsigma)\right] \int_0^1 \tau^{\alpha-1} \left[h(\tau) + h(1-\tau)\right] d\tau
\end{aligned} \tag{2.1}$$

with $\alpha > 0$.

The Riemann–Liouville-type fractional integrals of interval-valued coordinated functions $\mathcal{F}(t, s)$ are given as follows.

Definition 2.8 ([27]) Let $\mathcal{F} : \triangle \to \mathbb{R}^+_{\mathcal{I}}$ and $\mathcal{F} \in \mathcal{ID}_{(\triangle)}$. The Riemann–Liouville-type integrals $\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}$, $\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}$, $\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}$, $\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}$ of $\mathcal{F}$ of order $\alpha, \beta > 0$ are defined by

$$\begin{aligned}
\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+} \mathcal{F}(\chi, \gamma) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_o^\chi \int_\rho^\gamma (\chi - t)^{\alpha-1} (\gamma - s)^{\beta-1} \mathcal{F}(t,s)\,ds\,dt, \quad \chi > o, \gamma > \rho,\\
\mathfrak{J}^{\alpha,\beta}_{o^+,q^-} \mathcal{F}(\chi, \gamma) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_o^\chi \int_\gamma^q (\chi - t)^{\alpha-1} (s - \gamma)^{\beta-1} \mathcal{F}(t,s)\,ds\,dt, \quad \chi > o, \gamma < q,
\end{aligned}$$

$$\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(\chi,\gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\chi^\varsigma\int_\rho^\gamma (t-\chi)^{\alpha-1}(\gamma-s)^{\beta-1}\mathcal{F}(t,s)\,ds\,dt, \quad \chi<\varsigma, \gamma>\rho,$$

$$\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(\chi,\gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\chi^\varsigma\int_\gamma^q (t-\chi)^{\alpha-1}(s-\gamma)^{\beta-1}\mathcal{F}(t,s)\,ds\,dt, \quad \chi<\varsigma, \gamma<q.$$

3 Main results

In this section, we prove some new Hermite–Hadamard-type inequalities for coordinated h-convex interval-valued functions via the interval Riemann–Liouville-type integrals.

Theorem 3.1 *Let $\mathcal{F}:\triangle\to\mathbb{R}^+_{\mathcal{I}}$ be such that $\mathcal{F}=[\underline{\mathcal{F}},\overline{\mathcal{F}}]$ and $\mathcal{F}\in\mathcal{ID}_{(\triangle)}$, and let $h:[0,1]\to\mathbb{R}^+$. If $\mathcal{F}\in SX(ch,\triangle,\mathbb{R}^+_{\mathcal{I}})$, then*

$$\begin{aligned}
&\frac{1}{\alpha\beta h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\\
&\quad\supseteq \frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^\alpha(q-\rho)^\beta}\\
&\quad\quad\times\left[\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q)+\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\right]\\
&\quad\supseteq\left[\mathcal{F}(o,\rho)+\mathcal{F}(o,q)+\mathcal{F}(\varsigma,\rho)+\mathcal{F}(\varsigma,q)\right]\\
&\quad\quad\times\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h(\tau)h(\theta)+h(1-\tau)h(\theta)\\
&\quad\quad+h(\tau)h(1-\theta)+h(1-\tau)h(1-\theta)\big]\,d\tau\,d\theta
\end{aligned}\tag{3.1}$$

with $\alpha,\beta>0$.

Proof Since $\mathcal{F}\in SX(ch,\triangle,\mathbb{R}^+_{\mathcal{I}})$, we have

$$\frac{1}{h^2(\frac{1}{2})}\mathcal{F}\left(\frac{\chi+\gamma}{2},\frac{\mu+\nu}{2}\right)\supseteq\mathcal{F}(\chi,\mu)+\mathcal{F}(\gamma,\mu)+\mathcal{F}(\chi,\nu)+\mathcal{F}(\gamma,\nu).$$

Let $\chi=\tau o+(1-\tau)\varsigma$, $\gamma=(1-\tau)o+\tau\varsigma$, $\mu=\theta\rho+(1-\theta)q$, $w=(1-\theta)\rho+\theta q$, $\tau,\theta\in[0,1]$. Then

$$\begin{aligned}
&\frac{1}{h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\\
&\quad\supseteq\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)+\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\quad+\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)+\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big).
\end{aligned}\tag{3.2}$$

Consequently,

$$\begin{aligned}
&\frac{1}{\alpha\beta h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\\
&\quad=\frac{1}{h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\,d\theta\,d\tau\\
&\quad\supseteq\bigg[\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\,d\theta\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\,d\theta\,d\tau \\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\,d\theta\,d\tau \\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\,d\theta\,d\tau\Big] \\
={}&\Big[\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\big[\underline{\mathcal{F}}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big), \\
&\overline{\mathcal{F}}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)\,dq\big)\big]\,d\theta\,d\tau \\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\big[\underline{\mathcal{F}}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big), \\
&\overline{\mathcal{F}}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\big]\,d\theta\,d\tau \\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\big[\underline{\mathcal{F}}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big), \\
&\overline{\mathcal{F}}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\big]\,d\theta\,d\tau \\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\big[\underline{\mathcal{F}}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big), \qquad (3.3)\\
&\overline{\mathcal{F}}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\big]\,d\theta\,d\tau\Big] \\
={}&\Big[\int_\varsigma^o\int_q^\rho \big(\eta(\chi)\big)^{\alpha-1}\big(\zeta(\gamma)\big)^{\beta-1}\underline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{\rho-q}\frac{d\chi}{o-\varsigma}, \\
&\int_\varsigma^o\int_q^\rho \big(\eta(\chi)\big)^{\alpha-1}\big(\zeta(\gamma)\big)^{\beta-1}\overline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{\rho-q}\frac{d\chi}{o-\varsigma}\Big] \\
&+\Big[\int_o^\varsigma\int_q^\rho \big(1-\eta(\chi)\big)^{\alpha-1}\big(\zeta(\gamma)\big)^{\beta-1}\underline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{\rho-q}\frac{d\chi}{\varsigma-o}, \\
&\int_o^\varsigma\int_q^\rho \big(1-\eta(\chi)\big)^{\alpha-1}\big(\zeta(\gamma)\big)^{\beta-1}\overline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{\rho-q}\frac{d\chi}{\varsigma-a}\Big] \\
&+\Big[\int_\varsigma^o\int_\rho^q \big(\eta(\chi)\big)^{\alpha-1}\big(1-\zeta(\gamma)\big)^{\beta-1}\underline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{q-\rho}\frac{d\chi}{o-\varsigma}, \\
&\int_\varsigma^o\int_\rho^q \big(\eta(\chi)\big)^{\alpha-1}\big(1-\zeta(\gamma)\big)^{\beta-1}\overline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{q-\rho}\frac{d\chi}{o-\varsigma}\Big] \\
&+\Big[\int_o^\varsigma\int_\rho^q \big(1-\eta(\chi)\big)^{\alpha-1}\big(1-\zeta(\gamma)\big)^{\beta-1}\underline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{q-\rho}\frac{d\chi}{\varsigma-o}, \\
&\int_o^\varsigma\int_\rho^q \big(1-\eta(\chi)\big)^{\alpha-1}\big(1-\zeta(\gamma)\big)^{\beta-1}\overline{\mathcal{F}}(\chi,\gamma)\frac{d\gamma}{q-\rho}\frac{d\chi}{\varsigma-o}\Big] \\
={}&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^\alpha(q-\rho)^\beta}\big[\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\underline{\mathcal{F}}(\varsigma,q)+\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\underline{\mathcal{F}}(\varsigma,\rho)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\underline{\mathcal{F}}(o,q)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\underline{\mathcal{F}}(o,\rho), \\
&\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\overline{\mathcal{F}}(\varsigma,q)+\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\overline{\mathcal{F}}(\varsigma,\rho)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\overline{\mathcal{F}}(o,q)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\overline{F}(o,\rho)\big] \\
={}&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^\alpha(q-\rho)^\beta}\big[\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q)+\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\big],
\end{aligned}
$$

where $\eta(\chi)=\frac{\varsigma-\chi}{\varsigma-o}$, $\zeta(\gamma)=\frac{q-\gamma}{q-\rho}$.

Similarly, since $\mathcal{F} \in SX(ch, \triangle, \mathbb{R}^+_{\mathcal{I}})$,

$$\begin{aligned}
&\mathcal{F}\big(\tau o + (1-\tau)\varsigma, \theta\rho + (1-\theta)q\big) + \mathcal{F}\big((1-\tau)o + \tau\varsigma, \theta\rho + (1-\theta)q\big) \\
&\quad + \mathcal{F}\big(\tau o + (1-\tau)\varsigma, (1-\theta)\rho + \theta q\big) + \mathcal{F}\big((1-\tau)o + \tau\varsigma, (1-\theta)\rho + \theta q\big) \\
&\supseteq \big[h(\tau)h(\theta) + h(1-\tau)h(\theta) + h(\tau)h(1-\theta) + h(1-\tau)h(1-\theta)\big] \\
&\quad \times \big[\mathcal{F}(o,\rho) + \mathcal{F}(o,q) + \mathcal{F}(\varsigma,\rho) + \mathcal{F}(\varsigma,q)\big].
\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $\tau^{\alpha-1}\theta^{\beta-1}$ and integrating on $[0,1] \times [0,1]$, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^\alpha(q-\rho)^\beta}\big[\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q) + \mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho) + \mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q) + \mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\big] \\
&\supseteq \big[\mathcal{F}(o,\rho) + \mathcal{F}(o,q) + \mathcal{F}(\varsigma,\rho) + \mathcal{F}(\varsigma,q)\big] \\
&\quad \times \int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\big[h(\tau)h(\theta) + h(1-\tau)h(\theta) \\
&\quad + h(\tau)h(1-\theta) + h(1-\tau)h(1-\theta)\big]\,d\tau\,d\theta.
\end{aligned} \tag{3.5}$$

Using inequalities (3.3) and (3.5) completes the proof. □

Example 3.2 Let $\triangle = [0,2] \times [0,2]$. Let $h(\theta) = \theta$, $\alpha = \beta = \frac{1}{2}$, and $\mathcal{F}(\chi,\gamma) = [(2-\sqrt{\chi})(2-\sqrt{\gamma}), (2+\sqrt{\chi})(2+\sqrt{\gamma})]$. Then

$$\begin{aligned}
&\frac{1}{\alpha\beta h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2}, \frac{\rho+q}{2}\right) = [16, 144], \\
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^\alpha(q-\rho)^\beta}\big[\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q) + \mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho) + \mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q) + \mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\big] \\
&= \left[66 - 16\sqrt{2} - 8\sqrt{2}\pi + 2\pi + \frac{\pi^2}{2}, 66 + 16\sqrt{2} + 8\sqrt{2}\pi + 2\pi + \frac{\pi^2}{2}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\big[\mathcal{F}(o,\rho) + \mathcal{F}(o,q) + \mathcal{F}(\varsigma,\rho) + \mathcal{F}(\varsigma,q)\big] \\
&\quad \times \int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\big[h(\tau)h(\theta) + h(1-\tau)h(\theta) \\
&\quad + h(\tau)h(1-\theta) + h(1-\tau)h(1-\theta)\big]\,d\tau\,d\theta \\
&= [72 - 32\sqrt{2}, 72 + 32\sqrt{2}].
\end{aligned}$$

Therefore

$$\begin{aligned}
[16, 144] &\supseteq \left[66 - 16\sqrt{2} - 8\sqrt{2}\pi + 2\pi + \frac{\pi^2}{2}, 66 + 16\sqrt{2} + 8\sqrt{2}\pi + 2\pi + \frac{\pi^2}{2}\right] \\
&\supseteq [72 - 32\sqrt{2}, 72 + 32\sqrt{2}].
\end{aligned}$$

Consequently, Theorem 3.1 is verified.

Remark 3.3 If $\underline{\mathcal{F}} = \overline{\mathcal{F}}$ and $h(\theta) = \theta$, then we get Theorem 3 of [33]. If $\underline{\mathcal{F}} = \overline{\mathcal{F}}$, $h(\theta) = \theta$ and $\alpha = \beta = 1$, then we get Theorem 1 of [34].

Theorem 3.4 *Let $\mathcal{F} : \triangle \to \mathbb{R}^+_{\mathcal{I}}$ be such that $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$ and $\mathcal{F} \in \mathcal{ID}_{(\triangle)}$, and let $h : [0,1] \to \mathbb{R}^+$. If $\mathcal{F} \in SX(ch, \triangle, \mathbb{R}^+_{\mathcal{I}})$, then*

$$\begin{aligned}
&\frac{1}{h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2}, \frac{\rho+q}{2}\right) \\
&\supseteq \frac{\Gamma(\alpha+1)}{2h(\frac{1}{2})(\varsigma-o)^\alpha}\left[\mathfrak{I}^\alpha_{o^+}\mathcal{F}\left(\varsigma, \frac{\rho+q}{2}\right) + \mathfrak{I}^\alpha_{\varsigma^-}\mathcal{F}\left(o, \frac{\rho+q}{2}\right)\right] \\
&\quad + \frac{\Gamma(\beta+1)}{2h(\frac{1}{2})(q-\rho)^\beta}\left[\mathfrak{I}^\beta_{\rho^+}\mathcal{F}\left(\frac{o+\varsigma}{2}, q\right) + \mathfrak{I}^\beta_{q^-}\mathcal{F}\left(\frac{o+\varsigma}{2}, \rho\right)\right] \\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\varsigma-o)^\alpha(q-\rho)^\beta} \\
&\quad \times \left[\mathfrak{I}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q) + \mathfrak{I}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho) + \mathfrak{I}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q) + \mathfrak{I}^{\alpha,\beta}_{\varsigma^-,q^-}F(o,\rho)\right] \\
&\supseteq \frac{\beta\Gamma(\alpha+1)}{(\varsigma-o)^\alpha}\left[\mathfrak{I}^\alpha_{o^+}\mathcal{F}(\varsigma,\rho) + \mathfrak{I}^\alpha_{o^+}\mathcal{F}(\varsigma,q) + \mathfrak{I}^\alpha_{\varsigma^-}\mathcal{F}(\varsigma,\rho) + \mathfrak{I}^\alpha_{\varsigma^-}\mathcal{F}(\varsigma,q)\right] \\
&\quad \times \int_0^1 \theta^{\beta-1}\left[h(\theta) + h(1-\theta)\right]d\theta \\
&\quad + \frac{\alpha\Gamma(\beta+1)}{(q-\rho)^\beta}\left[\mathfrak{I}^\beta_{\rho^+}\mathcal{F}(o,q) + \mathfrak{I}^\beta_{\rho^+}\mathcal{F}(\varsigma,q) + \mathfrak{I}^\beta_{q^-}\mathcal{F}(o,\rho) + \mathfrak{I}^\beta_{q^-}\mathcal{F}(\varsigma,\rho)\right] \\
&\quad \times \int_0^1 \tau^{\alpha-1}\left[h(\tau) + h(1-\tau)\right]d\tau \\
&\supseteq \alpha\beta\left[\mathcal{F}(o,\rho) + \mathcal{F}(o,q) + \mathcal{F}(\varsigma,\rho) + \mathcal{F}(\varsigma,q)\right] \\
&\quad \times \int_0^1 \tau^{\alpha-1}\left[h(\tau) + h(1-\tau)\right]d\tau \int_0^1 \theta^{\beta-1}\left[h(\theta) + h(1-\theta)\right]d\theta.
\end{aligned} \tag{3.6}$$

Proof Using Theorem 2.7 and $\mathcal{F} \in SX(ch, \triangle, \mathbb{R}^+_{\mathcal{I}})$, we get

$$\begin{aligned}
\frac{1}{\beta h(\frac{1}{2})}\mathcal{F}_\chi\left(\frac{\rho+q}{2}\right) &\supseteq \frac{\Gamma(\beta)}{(q-\rho)^\beta}\left[\mathfrak{I}^\beta_{\rho^+}\mathcal{F}_\chi(q) + \mathfrak{I}^\beta_{q^-}\mathcal{F}_\chi(\rho)\right] \\
&\supseteq \left[\mathcal{F}_\chi(\rho) + \mathcal{F}_\chi(q)\right]\int_0^1 \theta^{\beta-1}\left[h(\theta) + h(1-\theta)\right]d\theta,
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{1}{\beta h(\frac{1}{2})}\mathcal{F}\left(\chi, \frac{\rho+q}{2}\right) \\
&\supseteq \frac{1}{(q-\rho)^\beta}\left[\int_\rho^q (q-\gamma)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma + \int_\rho^q (\gamma-\rho)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\right] \\
&\supseteq \left[\mathcal{F}(\chi,\rho) + \mathcal{F}(\chi,q)\right]\int_0^1 \theta^{\beta-1}\left[h(\theta) + h(1-\theta)\right]d\theta
\end{aligned}$$

for all $\chi \in [o, \varsigma]$. Moreover, we have

$$\begin{aligned} &\frac{1}{\beta(\varsigma-o)^{\alpha}h(\frac{1}{2})}\int_{o}^{\varsigma}(\varsigma-\chi)^{\alpha-1}\mathcal{F}\left(\chi,\frac{\rho+q}{2}\right)d\chi \\ &\quad\supseteq \frac{1}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\left[\int_{o}^{\varsigma}\int_{\rho}^{q}(\varsigma-\chi)^{\alpha-1}(q-\gamma)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi \right. \\ &\qquad\left. +\int_{o}^{\varsigma}\int_{\rho}^{q}(\varsigma-\chi)^{\alpha-1}(\gamma-\rho)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi\right] \\ &\quad\supseteq \frac{1}{(\varsigma-o)^{\alpha}}\int_{o}^{\varsigma}\int_{0}^{1}(\varsigma-\chi)^{\alpha-1}\big[\mathcal{F}(\chi,\rho)+\mathcal{F}(\chi,q)\big]\theta^{\beta-1}\big[h(\theta)+h(1-\theta)\big]\,d\theta\,d\chi \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\frac{1}{\beta(\varsigma-o)^{\alpha}h(\frac{1}{2})}\int_{o}^{\varsigma}(\chi-o)^{\alpha-1}\mathcal{F}\left(\chi,\frac{\rho+q}{2}\right)d\chi \\ &\quad\supseteq \frac{1}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\left[\int_{o}^{\varsigma}\int_{\rho}^{q}(\chi-o)^{\alpha-1}(q-\gamma)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi \right. \\ &\qquad\left. +\int_{o}^{\varsigma}\int_{\rho}^{q}(\chi-o)^{\alpha-1}(\gamma-\rho)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi\right] \\ &\quad\supseteq \frac{1}{(\varsigma-o)^{\alpha}}\int_{o}^{\varsigma}\int_{0}^{1}(\chi-o)^{\alpha-1}\big[\mathcal{F}(\chi,\rho)+\mathcal{F}(\chi,q)\big]\theta^{\beta-1}\big[h(\theta)+h(1-\theta)\big]\,d\theta\,d\chi. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned} &\frac{1}{\alpha(q-\rho)^{\beta}h(\frac{1}{2})}\int_{\rho}^{q}(q-\gamma)^{\beta-1}\mathcal{F}\left(\frac{o+\varsigma}{2},\gamma\right)d\gamma \\ &\quad\supseteq \frac{1}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\left[\int_{o}^{\varsigma}\int_{\rho}^{q}(\varsigma-\chi)^{\alpha-1}(q-\gamma)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi \right. \\ &\qquad\left. +\int_{o}^{\varsigma}\int_{\rho}^{q}(\chi-o)^{\alpha-1}(q-\gamma)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi\right] \\ &\quad\supseteq \frac{1}{(q-\rho)^{\beta}}\int_{\rho}^{q}\int_{0}^{1}(q-\gamma)^{\beta-1}\big[\mathcal{F}(o,\gamma)+\mathcal{F}(\varsigma,\gamma)\big]\tau^{\alpha-1}\big[h(\tau)+h(1-\tau)\big]\,d\tau\,d\gamma \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\frac{1}{\alpha(q-\rho)^{\beta}h(\frac{1}{2})}\int_{\rho}^{q}(\gamma-\rho)^{\beta-1}\mathcal{F}\left(\frac{o+\varsigma}{2},\gamma\right)d\gamma \\ &\quad\supseteq \frac{1}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\left[\int_{o}^{\varsigma}\int_{\rho}^{q}(\varsigma-\chi)^{\alpha-1}(\gamma-\rho)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi \right. \\ &\qquad\left. +\int_{o}^{\varsigma}\int_{\rho}^{q}(\chi-o)^{\alpha-1}(\gamma-\rho)^{\beta-1}\mathcal{F}(\chi,\gamma)\,d\gamma\,d\chi\right] \\ &\quad\supseteq \frac{1}{(q-\rho)^{\beta}}\int_{\rho}^{q}\int_{0}^{1}(\gamma-\rho)^{\beta-1}\big[\mathcal{F}(o,\gamma)+\mathcal{F}(\varsigma,\gamma)\big]\tau^{\alpha-1}\big[h(\tau)+h(1-\tau)\big]\,d\tau\,d\gamma. \end{aligned} \tag{3.10}$$

Summing inequalities (3.7)–(3.10), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2h(\frac{1}{2})(\varsigma-o)^{\alpha}}\left[\mathfrak{J}_{o^+}^{\alpha}\mathcal{F}\left(\varsigma,\frac{\rho+q}{2}\right)+\mathfrak{J}_{\varsigma^-}^{\alpha}\mathcal{F}\left(o,\frac{\rho+q}{2}\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{2h(\frac{1}{2})(q-\rho)^{\beta}}\left[\mathfrak{J}_{\rho^+}^{\beta}\mathcal{F}\left(\frac{o+\varsigma}{2},q\right)+\mathfrak{J}_{q^-}^{\beta}\mathcal{F}\left(\frac{o+\varsigma}{2},\varsigma\right)\right]\\
&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\left[\mathfrak{J}_{o^+,\rho^+}^{\alpha,\beta}\mathcal{F}(\varsigma,q)+\mathfrak{J}_{o^+,q^-}^{\alpha,\beta}\mathcal{F}(\varsigma,\rho)+\mathfrak{J}_{\varsigma^-,\rho^+}^{\alpha,\beta}\mathcal{F}(o,q)+\mathfrak{J}_{\varsigma^-,q^-}^{\alpha,\beta}\mathcal{F}(o,\rho)\right]\\
&\supseteq\frac{\beta\Gamma(\alpha+1)}{(\varsigma-o)^{\alpha}}\left[\mathfrak{J}_{o^+}^{\alpha}\mathcal{F}(\varsigma,\rho)+\mathfrak{J}_{o^+}^{\alpha}\mathcal{F}(\varsigma,q)+\mathfrak{J}_{\varsigma^-}^{\alpha}\mathcal{F}(\varsigma,\rho)+\mathfrak{J}_{\varsigma^-}^{\alpha}\mathcal{F}(\varsigma,q)\right]\\
&\quad\times\int_0^1\theta^{\beta-1}\left[h(\theta)+h(1-\theta)\right]d\theta\\
&\quad+\frac{\alpha\Gamma(\beta+1)}{(q-\rho)^{\beta}}\left[\mathfrak{J}_{\rho^+}^{\beta}\mathcal{F}(o,q)+\mathfrak{J}_{\rho^+}^{\beta}\mathcal{F}(\varsigma,q)+\mathfrak{J}_{q^-}^{\beta}\mathcal{F}(o,\varsigma)+\mathfrak{J}_{q^-}^{\beta}\mathcal{F}(\varsigma,\rho)\right]\\
&\quad\times\int_0^1\tau^{\alpha-1}\left[h(\tau)+h(1-\tau)\right]d\tau,
\end{aligned}$$

which gives the second and third inequalities in (3.6).

Using the first inequality in (2.1), we get

$$\frac{1}{h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\supseteq\frac{\Gamma(\alpha+1)}{h(\frac{1}{2})(\varsigma-o)^{\alpha}}\left[\mathfrak{J}_{\varsigma^-}^{\alpha}\mathcal{F}\left(o,\frac{\rho+q}{2}\right)+\mathfrak{J}_{o^+}^{\alpha}\mathcal{F}\left(\varsigma,\frac{\rho+q}{2}\right)\right]\tag{3.11}$$

and

$$\frac{1}{h^2(\frac{1}{2})}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\supseteq\frac{\Gamma(\beta+1)}{h(\frac{1}{2})(q-\rho)^{\beta}}\left[\mathfrak{J}_{q^-}^{\beta}\mathcal{F}\left(\frac{o+\varsigma}{2},\rho\right)+\mathfrak{J}_{\rho^+}^{\beta}\mathcal{F}\left(\frac{o+\varsigma}{2},q\right)\right].\tag{3.12}$$

Summing inequalities (3.11) and (3.12), we get the first inequality in (3.6).

Using the second inequality in (2.1), we also state

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\varsigma-o)^{\alpha}}\left[\mathfrak{J}_{\varsigma^-}^{\alpha}\mathcal{F}(o,\rho)+\mathfrak{J}_{o^+}^{\alpha}\mathcal{F}(\varsigma,\rho)\right]\supseteq\left[\mathcal{F}(o,\rho)+\mathcal{F}(\varsigma,\rho)\right]\int_0^1\tau^{\alpha-1}\left[h(\tau)+h(1-\tau)\right]d\tau,\\
&\frac{\Gamma(\alpha)}{(\varsigma-o)^{\alpha}}\left[\mathfrak{J}_{\varsigma^-}^{\alpha}\mathcal{F}(o,q)+\mathfrak{J}_{o^+}^{\alpha}\mathcal{F}(\varsigma,q)\right]\supseteq\left[\mathcal{F}(o,q)+\mathcal{F}(\varsigma,q)\right]\int_0^1\tau^{\alpha-1}\left[h(\tau)+h(1-\tau)\right]d\tau,\\
&\frac{\Gamma(\beta)}{(q-\rho)^{\beta}}\left[\mathfrak{J}_{q^-}^{\beta}\mathcal{F}(o,\rho)+\mathfrak{J}_{\rho^+}^{\beta}\mathcal{F}(o,q)\right]\supseteq\left[\mathcal{F}o,\rho)+\mathcal{F}(o,q)\right]\int_0^1\theta^{\beta-1}\left[h(\theta)+h(1-\theta)\right]d\theta,
\end{aligned}$$

and

$$\frac{\Gamma(\beta)}{(q-\rho)^{\beta}}\left[\mathfrak{J}_{q^-}^{\beta}\mathcal{F}(\varsigma,\rho)+\mathfrak{J}_{\rho^+}^{\beta}F(\varsigma,q)\right]\supseteq\left[F(\varsigma,\rho)+F(\varsigma,q)\right]\int_0^1\theta^{\beta-1}\left[h(\theta)+h(1-\theta)\right]d\theta,$$

which gives the last inequality in (3.6). This completes the proof. □

Remark 3.5 If $\underline{\mathcal{F}}=\overline{\mathcal{F}}$ and $h(\theta)=\theta$, then we get Theorem 4 of [33]. If $\alpha=\beta=1$, then we get Theorem 3.5 of [25]. If $\alpha=\beta=1$ and $h(\theta)=\theta$, then we get Theorem 7 of [26].

Theorem 3.6 *Let $\mathcal{F},\mathcal{G}:\triangle\to\mathbb{R}_{\mathcal{I}}^{+}$ be such that $\mathcal{F}=[\underline{\mathcal{F}},\overline{\mathcal{F}}]$, $\mathcal{G}=[\underline{\mathcal{G}},\overline{\mathcal{G}}]$, and $\mathcal{F}\mathcal{G}\in\mathcal{ID}_{(\triangle)}$, and let $h:[0,1]\to\mathbb{R}^{+}$. If $\mathcal{F}\in SX(ch_1,\triangle,\mathbb{R}_{\mathcal{I}}^{+})$ and $\mathcal{G}\in SX(ch_2,\triangle,\mathbb{R}_{\mathcal{I}}^{+})$, then*

$$\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\big[\mathfrak{J}_{\varsigma^-,\rho^+}^{\alpha,\beta}\mathcal{F}(o,q)\mathcal{G}(o,q)+\mathfrak{J}_{\varsigma^-,q^-}^{\alpha,\beta}\mathcal{F}(o,\rho)\mathcal{G}(o,\rho)\\
&\quad+\mathfrak{J}_{o^+,\rho^+}^{\alpha,\beta}\mathcal{F}(\varsigma,q)\mathcal{G}(\varsigma,q)+\mathfrak{J}_{o^+,q^-}^{\alpha,\beta}\mathcal{F}(\varsigma,\rho)\mathcal{G}(\varsigma,\rho)\big]\\
&\quad\supseteq\mathcal{M}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(\theta)+h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta)\\
&\quad+h_1(\tau)h_1(\theta)h_2(\tau)h_2(\theta)\big]\,d\tau\,d\theta\\
&\quad+\mathcal{N}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta)+h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(\theta)\\
&\quad+h_1(\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)\big]\,d\tau\,d\theta\\
&\quad+\mathcal{P}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(1-\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta)+h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(\theta)\\
&\quad+h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(\theta)\big]\,d\tau\,d\theta\\
&\quad+\mathcal{Q}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta)+h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)\\
&\quad+h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\big]\,d\tau\,d\theta,
\end{aligned}\tag{3.13}$$

where

$$\begin{aligned}
\mathcal{M}(o,\varsigma,\rho,q)&=\mathcal{F}(o,\rho)\mathcal{G}(o,\rho)+\mathcal{F}(\varsigma,\rho)\mathcal{G}(\varsigma,\rho)+\mathcal{F}(o,q)\mathcal{G}(o,q)+\mathcal{F}(\varsigma,q)\mathcal{G}(\varsigma,q),\\
\mathcal{N}(o,\varsigma,\rho,q)&=\mathcal{F}(o,\rho)\mathcal{G}(o,q)+\mathcal{F}(\varsigma,\rho)\mathcal{G}(\varsigma,q)+\mathcal{F}(o,q)\mathcal{G}(o,\rho)+\mathcal{F}(\varsigma,q)\mathcal{G}(\varsigma,\rho),\\
\mathcal{P}(o,\varsigma,\rho,q)&=\mathcal{F}(o,\rho)\mathcal{G}(\varsigma,\rho)+\mathcal{F}(\varsigma,\rho)\mathcal{G}(o,\rho)+\mathcal{F}(o,q)\mathcal{G}(\varsigma,q)+\mathcal{F}(\varsigma,q)\mathcal{G}(o,q),\\
\mathcal{Q}(o,\varsigma,\rho,q)&=\mathcal{F}(o,\rho)\mathcal{G}(\varsigma,q)+\mathcal{F}(\varsigma,\rho)\mathcal{G}(o,q)+\mathcal{F}(o,q)\mathcal{G}(\varsigma,\rho)+\mathcal{F}(\varsigma,q)\mathcal{G}(o,\rho).
\end{aligned}$$

Proof Since $\mathcal{F}\in SX(ch_1,\triangle,\mathbb{R}_{\mathcal{I}}^{+})$ and $\mathcal{G}\in SX(ch_2,\triangle,\mathbb{R}_{\mathcal{I}}^{+})$, we have

$$\begin{aligned}
&\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\supseteq h_1(\tau)h_1(\theta)\mathcal{F}(o,\rho)+h_1(\tau)h_1(1-\theta)\mathcal{F}(o,q)\\
&\qquad+h_1(1-\tau)h_1(\theta)\mathcal{F}(\varsigma,\rho)+h_1(1-\tau)h_1(1-\theta)\mathcal{F}(\varsigma,q),\\
&\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\supseteq h_1(\tau)h_1(1-\theta)\mathcal{F}(o,\rho)+h_1(\tau)h_1(\theta)\mathcal{F}(o,q)\\
&\qquad+h_1(1-\tau)h_1(1-\theta)\mathcal{F}(\varsigma,\rho)+h_1(1-\tau)h_1(\theta)\mathcal{F}(\varsigma,q),
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\supseteq h_1(1-\tau)h_1(\theta)\mathcal{F}(o,\rho)+h_1(1-\tau)h_1(1-\theta)\mathcal{F}(o,q)\\
&\qquad+h_1(\tau)h_1(\theta)\mathcal{F}(\varsigma,\rho)+h_1(\tau)h_1(1-\theta)\mathcal{F}(\varsigma,q),\\
&\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\supseteq h_1(1-\tau)h_1(1-\theta)\mathcal{F}(o,\rho)+h_1(1-\tau)h_1(\theta)\mathcal{F}(o,q)\\
&\qquad+h_1(\tau)h_1(1-\theta)\mathcal{F}(\varsigma,\rho)+h_1(\tau)h_1(\theta)\mathcal{F}(\varsigma,q),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{G}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\supseteq h_2(\tau)h_2(\theta)\mathcal{G}(o,\rho)+h_2(\tau)h_2(1-\theta)\mathcal{G}(o,q)\\
&\qquad+h_2(1-\tau)h_2(\theta)\mathcal{G}(\varsigma,\rho)+h_2(1-\tau)h_2(1-\theta)\mathcal{G}(\varsigma,q),\\
&\mathcal{G}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\supseteq h_2(\tau)h_2(1-\theta)\mathcal{G}(o,\rho)+h_2(\tau)h_2(\theta)\mathcal{G}(o,q)\\
&\qquad+h_2(1-\tau)h_2(1-\theta)\mathcal{G}(\varsigma,\rho)+h_2(1-\tau)h_2(\theta)\mathcal{G}(\varsigma,q),\\
&\mathcal{G}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\supseteq h_2(1-\tau)h_2(\theta)\mathcal{G}(o,\rho)+h_2(1-\tau)h_2(1-\theta)\mathcal{G}(o,q)\\
&\qquad+h_2(\tau)h_2(\theta)\mathcal{G}(\varsigma,\rho)+h_2(\tau)h_2(1-\theta)\mathcal{G}(\varsigma,q),\\
&\mathcal{G}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\supseteq h_2(1-\tau)h_2(1-\theta)\mathcal{G}(o,\rho)+h_2(1-\tau)h_2(\theta)\mathcal{G}(o,q)\\
&\qquad+h_2(\tau)h_2(1-\theta)\mathcal{G}(\varsigma,\rho)+h_2(\tau)h_2(\theta)\mathcal{G}(\varsigma,q).
\end{aligned}
$$

Since $\mathcal{F},\mathcal{G}\in\mathbb{R}^+_{\mathcal{I}}$, we have

$$
\begin{aligned}
&\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\mathcal{G}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\\
&\qquad+\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\mathcal{G}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\\
&\qquad+\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\mathcal{G}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\qquad+\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\mathcal{G}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\supseteq\mathcal{M}(o,\varsigma,\rho,q)\\
&\qquad\times\big[h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta)+h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(\theta)\\
&\qquad+h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta)+h_1(\tau)h_1(\theta)h_2(\tau)h_2(\theta)\big]\\
&\qquad+\mathcal{N}(o,\varsigma,\rho,q)\big[h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)+h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta)\\
&\qquad+h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(\theta)+h_1(\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)\big]\\
&\qquad+\mathcal{P}(o,\varsigma,\rho,q)\big[h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta)+h_1(1-\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta)\\
&\qquad+h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(\theta)+h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(\theta)\big]
\end{aligned}
$$

$$+ \mathcal{Q}(o,\varsigma,\rho,q)\big[h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta) + h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta) + h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(1-\theta) + h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\big].$$

Moreover, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\times\mathcal{G}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\,d\tau\,d\theta\\
&\quad+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\times\mathcal{G}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\,d\tau\,d\theta\\
&\quad+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\times\mathcal{G}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\,d\tau\,d\theta\\
&\quad+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad\times\mathcal{G}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\,d\tau\,d\theta\\
&\supseteq\mathcal{M}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(\theta)+h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta)\\
&\quad+h_1(\tau)h_1(\theta)h_2(\tau)h_2(\theta)\big]\,d\tau\,d\theta\\
&\quad+\mathcal{N}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta)+h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(\theta)\\
&\quad+h_1(\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)\big]\,d\tau\,d\theta\\
&\quad+\mathcal{P}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(1-\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta)+h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(\theta)\\
&\quad+h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(\theta)\big]\,d\tau\,d\theta\\
&\quad+\mathcal{Q}(o,\varsigma,\rho,q)\int_0^1\int_0^1\tau^{\alpha-1}\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta)+h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)\\
&\quad+h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)\big]\,d\tau\,d\theta.
\end{aligned}\tag{3.14}$$

By Definition 2.8 we get

$$\begin{aligned}
&\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad\times\mathcal{G}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\,d\tau\,d\theta
\end{aligned}$$

$$\begin{aligned}
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\\
&\times\mathcal{G}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\,d\tau\,d\theta\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\times\mathcal{G}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\,d\tau\,d\theta\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\theta^{\beta-1}\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\\
&\times\mathcal{G}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\,d\tau\,d\theta\\
={}&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\big[\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q)\mathcal{G}(o,q)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\mathcal{G}(o,\rho)\\
&+\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q)\mathcal{G}(\varsigma,q)+\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho)\mathcal{G}(\varsigma,\rho)\big].
\end{aligned}\tag{3.15}$$

From inequalities (3.14)–(3.15) we obtain inequalities (3.13). □

Remark 3.7 If $\alpha=\beta=1$ and $h(\theta)=\theta$, then we get Theorem 8 of [26]. If $\underline{\mathcal{F}}=\overline{\mathcal{F}}$, $h(\theta)=\theta$, and $\alpha=\beta=1$, then we get Theorem 4 of [35].

Theorem 3.8 *Let $\mathcal{F},\mathcal{G}:[o,\varsigma]\times[\rho,q]\to\mathbb{R}^+_{\mathcal{I}}$ be such that $\mathcal{F}=[\underline{\mathcal{F}},\overline{\mathcal{F}}]$, $\mathcal{G}=[\underline{\mathcal{G}},\overline{\mathcal{G}}]$, and $\mathcal{F}\mathcal{G}\in\mathcal{ID}_{(\triangle)}$, and let $h:[0,1]\to\mathbb{R}^+$. If $\mathcal{F}\in SX(ch_1,\triangle,\mathbb{R}^+_{\mathcal{I}})$ and $\mathcal{G}\in SX(ch_2,\triangle,\mathbb{R}^+_{\mathcal{I}})$, then*

$$\begin{aligned}
&\frac{1}{2\alpha\beta {h_1}^2(\frac{1}{2}){h_2}^2(\frac{1}{2})}\mathcal{F}\Big(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\Big)\mathcal{G}\Big(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\Big)\\
&\supseteq\frac{\Gamma(\alpha)\Gamma(\beta)}{2(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\big[\mathfrak{J}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q)\mathcal{G}(\varsigma,q)+\mathfrak{J}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho)\mathcal{G}(\varsigma,\rho)\\
&\quad+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q)\mathcal{G}(o,q)+\mathfrak{J}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\mathcal{G}(o,\rho)\big]\\
&\quad+\mathcal{M}(o,\varsigma,\rho,q)\int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(\tau)h_2(1-\theta)\\
&\quad+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\big]\,d\theta\\
&\quad+\mathcal{N}(o,\varsigma,\rho,q)\int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(\tau)h_2(\theta)\\
&\quad+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\big]\,d\theta\\
&\quad+\mathcal{P}(o,\varsigma,\rho,q)\int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(1-\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)\big]\big]\,d\theta\\
&\quad+\mathcal{Q}(o,\varsigma,\rho,q)\int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(1-\tau)h_2(\theta)
\end{aligned}\tag{3.16}$$

$$+h_2(\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)\big]$$
$$+h_1(\tau)h_1(1-\theta)\big[h_2(1-\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\big]\,d\theta.$$

Proof Since $\mathcal{F}\in SX(ch_1,\triangle,\mathbb{R}^+_{\mathcal{I}})$ and $\mathcal{G}\in SX(ch_2,\triangle,\mathbb{R}^+_{\mathcal{I}})$, we have

$$\begin{aligned}
&\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\mathcal{G}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\\
&=\mathcal{F}\left(\frac{\tau o+(1-\tau)\varsigma}{2}+\frac{(1-\tau)o+\tau\varsigma}{2},\frac{\theta\rho+(1-\theta)q}{2}+\frac{(1-\theta)\rho+\theta q}{2}\right)\\
&\quad\times\mathcal{G}\left(\frac{\tau o+(1-\tau)\varsigma}{2}+\frac{(1-\tau)o+\tau\varsigma}{2},\frac{\theta\rho+(1-\theta)q}{2}+\frac{(1-\theta)\rho+\theta q}{2}\right)\\
&\supseteq {h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\\
&\quad\times\big[\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)+\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad+\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)+\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\big]\\
&\quad\times\big[\mathcal{G}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)+\mathcal{G}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad+\mathcal{G}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)+\mathcal{G}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\big]\\
&\supseteq {h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\\
&\quad\times\big[\mathcal{F}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\mathcal{G}\big(\tau o+(1-\tau)\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad+\mathcal{F}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\mathcal{G}\big((1-\tau)o+\tau\varsigma,\theta\rho+(1-\theta)q\big)\\
&\quad+\mathcal{F}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\mathcal{G}\big(\tau o+(1-\tau)\varsigma,(1-\theta)\rho+\theta q\big)\\
&\quad+\mathcal{F}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\mathcal{G}\big((1-\tau)o+\tau\varsigma,(1-\theta)\rho+\theta q\big)\big]\\
&\quad+{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\\
&\quad\times\big[h_1(\tau)h_1(\theta)\big[h_2(\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\\
&\quad+h_1(1-\tau)h_1(\theta)\big[h_2(1-\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\\
&\quad+h_1(1-\tau)h_1(1-\theta)\big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\mathcal{M}(o,\varsigma,\rho,q)\big]\\
&\quad+{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\big[h_1(\tau)h_1(\theta)\big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\\
&\quad+h_1(1-\tau)h_1(\theta)\big[h_2(1-\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)\big]\\
&\quad+h_1(1-\tau)h_1(1-\theta)\\
&\quad\times\big[h_2(1-\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\mathcal{N}(o,\varsigma,\rho,q)\big]\\
&\quad+{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \big[h_1(\tau)h_1(\theta)\big[h_2(1-\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\\
&+h_1(\tau)h_1(1-\theta)\big[h_2(1-\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)\big]\\
&+h_1(1-\tau)h_1(\theta)\big[h_2(\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\\
&+h_1(1-\tau)h_1(1-\theta)\\
&\times \big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\mathcal{P}(o,\varsigma,\rho,q)\big]\\
&+{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\big[h_1(\tau)h_1(\theta)\big[h_2(1-\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)\big]\\
&+h_1(\tau)h_1(1-\theta)\big[h_2(1-\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\\
&+h_1(1-\tau)h_1(\theta)\big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\\
&+h_1(1-\tau)h_1(1-\theta)\\
&\times \big[h_2(\tau)h_2(1-\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\mathcal{Q}(o,\varsigma,\rho,q)\big].
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\frac{1}{\alpha\beta}\mathcal{F}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\mathcal{G}\left(\frac{o+\varsigma}{2},\frac{\rho+q}{2}\right)\\
&\quad\supseteq \frac{\Gamma(\alpha)\Gamma(\beta){h_1}^2(\frac{1}{2}){h_2}^2(\frac{1}{2})}{(\varsigma-o)^{\alpha}(q-\rho)^{\beta}}\\
&\quad\times \big[\mathfrak{I}^{\alpha,\beta}_{o^+,\rho^+}\mathcal{F}(\varsigma,q)+\mathfrak{I}^{\alpha,\beta}_{o^+,q^-}\mathcal{F}(\varsigma,\rho)+\mathfrak{I}^{\alpha,\beta}_{\varsigma^-,\rho^+}\mathcal{F}(o,q)+\mathfrak{I}^{\alpha,\beta}_{\varsigma^-,q^-}\mathcal{F}(o,\rho)\big]\\
&\quad+2{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\mathcal{M}(o,\varsigma,\rho,q)\\
&\quad\times \int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(\tau)h_2(1-\theta)\\
&\quad+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\big]\,d\theta\\
&\quad+2{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\mathcal{N}(o,\varsigma,\rho,q)\\
&\quad\times \int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(\tau)h_2(\theta)\\
&\quad+h_2(1-\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(\tau)h_2(1-\theta)+h_2(1-\tau)h_2(\theta)+h_2(1-\tau)h_2(1-\theta)\big]\big]\,d\theta\\
&\quad+2{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\mathcal{P}(o,\varsigma,\rho,q)\\
&\quad\times \int_0^1\tau^{\alpha-1}\,d\tau\int_0^1\theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(1-\tau)h_2(1-\theta)\\
&\quad+h_2(\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)\big]\\
&\quad+h_1(\tau)h_1(1-\theta)\big[h_2(1-\tau)h_2(\theta)+h_2(\tau)h_2(1-\theta)+h_2(\tau)h_2(\theta)\big]\big]\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ 2{h_1}^2\left(\frac{1}{2}\right){h_2}^2\left(\frac{1}{2}\right)\mathcal{Q}(o,\varsigma,\rho,q) \\
&\times \int_0^1 \tau^{\alpha-1}\,d\tau \int_0^1 \theta^{\beta-1}\big[h_1(\tau)h_1(\theta)\big[h_2(1-\tau)h_2(\theta) \\
&+ h_2(\tau)h_2(1-\theta) + h_2(\tau)h_2(\theta)\big] \\
&+ h_1(\tau)h_1(1-\theta)\big[h_2(1-\tau)h_2(1-\theta) + h_2(\tau)h_2(\theta) + h_2(\tau)h_2(1-\theta)\big]\big]\,d\theta,
\end{aligned}
$$

which rearranges to the required result. □

Remark 3.9 If $\alpha = \beta = 1$ and $h(\theta) = \theta$, then we get Theorem 9 of [26]. If $\underline{\mathcal{F}} = \overline{\mathcal{F}}$, $h(\theta) = \theta$, and $\alpha = \beta = 1$, then we get Theorem 5 of [35].

4 Conclusion

In this paper, we proved some new Hermite–Hadamard-type inequalities for coordinated h-convex interval-valued functions via Riemann–Liouville-type fractional integrals. The results generalize the previous results given in [25–27, 33, 35]. Moreover, in the future investigation, these results may be extended for different kinds of convexities and fractional integrals.

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The authors declare that they have no competing interests.

Authors' contributions
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