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Dynamics of a class of host–parasitoid models with external stocking upon parasitoids

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Abstract

This paper is motivated by the series of research papers that consider parasitoids' external input upon the host–parasitoid interactions. We explore a class of host–parasitoid models with variable release and constant release of parasitoids. We assume that the host population has a constant rate of increase, but we do not assume any density dependence regulation other than parasitism acting on the host population. We compare the obtained results for constant stocking with the results for proportional stocking. We observe that under a specific condition, the release of a constant number of parasitoids can eventually drive the host population (pests) to extinction. There is always a boundary equilibrium where the host population extinct occurs, and the parasitoid population is stabilized at the constant stocking level. The constant and variable stocking can decrease the host population level in the unique interior equilibrium point; on the other hand, the parasitoid population level stays constant and does not depend on stocking. We prove the existence of Neimark–Sacker bifurcation and compute the approximation of the closed invariant curve. Then we consider a few host–parasitoid models with proportional and constant stocking, where we choose well-known probability functions of parasitism. By using the software package Mathematica we provide numerical simulations to support our study.

Keywords: Difference equations; Equilibrium; Host–parasitoid; Neimark–Sacker bifurcation; Stability; Stocking

1 Introduction

Mathematical models of host–parasitoid interaction can exhibit an exciting and complex dynamics. Among host–parasitoid models, competitive models can have simple dynamics. There is a competitive exclusion principle in ecology that states that two-species competing for the same limited resource cannot coexist indefinitely. (Similar competition scenarios take part in mathematical models from biomathematics involving chemotaxis and/or reaction–diffusion phenomena for a single species: see [19, 21].) One population will drive away another. If any of the depleted population members remained, it would happen because they adapted to different niches.

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There are a lot of research papers about the host–parasitoid models under different ecological factors and different assumptions; see, for example, [2, 3, 6, 10–13, 15–17, 20] and references therein.

The following system of difference equations gives a general model describing the host–parasitoid behavior in discrete time:

$$\begin{aligned} x_{n+1} &= rx_n f(x_n, y_n), \\ y_{n+1} &= ex_n(1 - f(x_n, y_n)), \end{aligned}$$

where x_n and y_n are the adult host and adult parasitoid densities, respectively, at generation n . The parameter $r > 0$ is the number of eggs laid by a host that survives through the larvae, pupae, and adult stages, and $f(x_n, y_n)$ denotes the portion of host larvae that are safe from parasitism. The parameter $e > 0$ is the number of eggs laid by a parasitoid on a single host that survives through larvae, pupae, and adult stages. In the classical discrete-time Nicholson–Bailey model, $f(x_n, y_n) = e^{-ay_n}$, where a is the searching efficiency of the parasitoid, referred to as the “area of discovery.”

Lauwerier and Metz [20] investigated the following generalization of the Nicholson–Bailey model:

$$\begin{aligned} x_{n+1} &= rx_n f(y_n), \\ y_{n+1} &= rx_n(1 - f(y_n)), \end{aligned} \tag{1}$$

where $f(y_n)$ is the fraction of host larvae that escapes parasitism, $1 - f(y_n)$ is the fraction of host larvae that will be parasitized. The function $f(y)$ is differentiable such that $f'(y) < 0$, $f(0) = r$, and $f(\infty) < 1$. The Neimark–Sacker bifurcation of nontrivial equilibrium of (1) was studied in [20] for various combinations of the parameters. The obtained results were applied to the several well-known special cases of the function f . The authors investigated in particular the so-called simple (S) model where $f(y) = 1/(1 + y^m)$, $m > 0$, the Hassell–Varley (HV) model where $f(y) = e^{-y^m}$, $0 < m \leq 1$ (Hassell and Varley, 1969, [11]), the parasitoid–parasitoid interaction (PP) model where $f(y) = e^{-\frac{\sqrt{1+y}-1}{m}}$, $m > 0$ (Metz, Vaz Nunez 1977, [20]), and the Hassell (H) model where $f(y) = [me^{-y} + (1 - m)e^{-by}]$, $0 < m < 1$, $b > 0$ (Hassell, 1984, [10]). If $m < 1$, then there is mutual interference between parasitoids. If $m > 1$, then there is cooperative hunting. If $m = 1$, then there is no cooperation among parasitoids.

In [3], we examined how the possibility of a proportional refuge for a host affects the dynamics of model (1), that is, we consider the model

$$\begin{aligned} x_{n+1} &= ax_n + bx_n f(y_n), \\ y_{n+1} &= cx_n(1 - f(y_n)), \end{aligned}$$

where $f(y)$ is a differentiable function such that $f'(y) < 0$, $f(0) = 1$, and $f(\infty) = 0$.

In [17], by using the analytical approach we explore the global behavior and bifurcation in a class of host–parasitoid models when a constant number of the hosts are safe from

parasitism:

$$\begin{aligned}x_{n+1} &= a + bx_n f(y_n), \\ y_{n+1} &= cx_n(1 - f(y_n)),\end{aligned}$$

where $a, b, c > 0$, $f(y_n)$ is the proportion of hosts that are safe from parasitism and satisfies the following assumption:

(H_1) $f \in C[0, \infty) \cap C^3(0, \infty)$, $f(y) > 0$, $f'(y) < 0$, $f''(y) \geq 0$ for $y > 0$, and $f(0) = 1$, $f(\infty) = 0$. The parameter b can be interpreted as the intrinsic growth rate of the host population unsafe from attack by parasitoids.

Biological control is a method of reducing pest populations by natural enemies, also known as biological control agents. Many natural host populations are pests. Many wasps and some flies are parasitoids, and most of the parasitoids have a narrow host range that can be used as biological control agents. When natural enemies are not present in adequate quantities to reduce pests, they can sometimes be increased with natural enemies' release. There are two inoculative and inundative releases. Inoculative releases are applied when the pest populations are low, and relatively few natural enemies are released, usually once or twice over the season. Inundative releases involve releasing large numbers of natural enemies, often several times over a growing season.

Many authors studied the effects of inoculative release or external stocking in the discrete-time host–parasitoid models; see, for example, [1, 4, 6, 8, 9, 13, 14, 18, 26] and references therein. Yakubu [30] showed that a stocking with constant or variable rate or harvesting may be used to obtain stable coexistence of two competing species that would otherwise exclude each other with no stocking or harvesting occurs. Selgrade and Roberds [26] introduced a constant stocking or harvesting into a general class of two-dimensional discrete populations models where each fitness function depends on a linear combination of the densities of the interacting populations. They obtained conditions under which the stocking or harvesting will reverse the period-doubling bifurcation and stabilize the equilibrium, which otherwise, with no stocking or harvesting, would lose stability through a period-doubling bifurcation. In a recent paper [31] the authors consider a two-species competitive model described by differential equations with Michaelis–Menten-type harvesting in the first species. They show that nonlinear harvesting can exhibit more complicated dynamics compared to linear harvesting.

Insect populations frequently suffer some density-dependent effect in addition to mortality from insect parasitoid [23]. In [23], authors were examining the question “Does the ordering of density dependence and parasitism in the host life cycle have a significant effect on the dynamics of the interaction?” To discuss this question, they examined three host–parasitoid model properties, each having the host density-dependence acting differently. They concluded that the relative position in the life cycle of parasitism and density dependence in models does indeed affect their dynamical properties. In [14] the authors considered a simple discrete-time host–parasitoid model to explore the impact of external input of parasitoids upon the host–parasitoid interactions, where it was assumed that parasitism occurs first followed by density-dependence. Jang [13] studied two general two-dimensional host–parasitoid systems, one with no stocking and the other with stocking. In both models, it was assumed that the density-dependence of the host acts first followed by parasitism, unlike the models investigated in [14]. The pest population is considered

to host, and the parasitoids are natural enemies of the hosts. She found that a constant stocking can eliminate the pests if the stocking is sufficiently large. Moreover, stocking can simplify the host–parasitoid interaction by stabilizing a coexisting steady state.

Motivated by the research papers mentioned above, the primary goal of this paper is investigating a class of host–parasitoid models with variable and constant parasitoid population stock rates. Assuming that the number of parasitoids released per generation into host–parasitoid interaction is a function of density y_n , we denote the stocking rate of parasitoids by $g(y_n)$, where the function $g : [0, \infty) \rightarrow [0, \infty)$ is continuous and nonnegative. That is, we consider the following two-dimensional host–parasitoid model:

$$\begin{aligned} x_{n+1} &= ax_n f(y_n), \\ y_{n+1} &= bx_n(1 - f(y_n)) + g(y_n), \end{aligned} \tag{2}$$

where $a, b, c,$ and d are positive real constants. The function f satisfies the following assumptions: f is a real nonnegative function that is sufficiently smooth on $(0, \infty)$ with $f(0) = 1$ and $f(\infty) = 0, f'(y)$ exists for all $y > 0,$ and $f'(y) < 0.$ The function f denotes the proportion of host that escapes parasitism. In the absence of stocking $g(y_n) \equiv 0,$ if $a = b,$ then we have a host–parasitoid system explored in [20]. In variable stocking, a fixed portion of parasitoids is released, $g(y_n) = cy_n,$ where $c > 0.$ In constant stocking, a fixed number of parasitoids is added during each generation of parasitoids population, $g(y_n) = d,$ where d is positive constant. Here we do not include density dependence regulation other than parasitism acting on the host population. More precisely, we investigate the following two host–parasitoid systems:

$$\begin{aligned} x_{n+1} &= ax_n f(y_n), \\ y_{n+1} &= bx_n(1 - f(y_n)) + cy_n, \end{aligned} \tag{3}$$

and

$$\begin{aligned} x_{n+1} &= ax_n f(y_n), \\ y_{n+1} &= bx_n(1 - f(y_n)) + d. \end{aligned} \tag{4}$$

In system (3), we assume that the proportional level of parasitoids is added per generation into host–parasitoid interaction to control the host population. In system (4), it is assumed that a constant number of parasitoids is added per generation into host–parasitoid interaction to control the host population. We compare the long-term behavior of these two systems and compare the long-term behavior of each system with the system investigated in [20].

We organize the rest of the paper as follows. In Sect. 2, we present linearized stability results for system (2) with no stocking. Then we analyze system (3) with variable released entirely. In the case of the coexisting steady state, we show that if the host growth rate is greater than 1 and if $0 < c < 1,$ then the level of the host population decreases as a function of c in the unique interior equilibrium. The level of parasitoids population stays constant. Similarly to the system with no stocking, the extinction equilibrium with variable stocking

can be globally asymptotically stable. The effect of constant stocking of parasitoids is studied in Sect. 2.3. We can see that the constant stocking of parasitoids can eliminate the host (pests) population. See Theorem 2 in this subsection. We obtain that the constant stocking eliminates the equilibrium for which the host population survives and the parasitoid extinct. This equilibrium type exists in model (2) with no stocking ($g(y_n) \equiv 0$) and model (3) with variable stocking, and in both systems, this equilibrium is semistable. The constant stocking eliminates the system extinction equilibrium with no stocking and the system with variable stocking. Moreover, the constant stocking of the parasitoids can lower the host level in the unique coexisting equilibrium. In each model, a Neimark–Sacker bifurcation occurs when the interior equilibrium becomes nonhyperbolic. Thus the oscillatory behavior of two species appears about the interior steady state. A few examples of host–parasitoid models with proportional and constant stocking with well-known functions of probability of escaping parasitism are given in Sect. 2.4. Using the software package Mathematica, we provide numerical simulations for these examples to support our findings. Finally, the conclusion is given in Sect. 3.

2 Systems with stocking of parasitoids

First, we state linearized stability results for a simple model of host–parasitoid interaction (2) with no stocking. Like we said, the Neimark–Sacker bifurcation of model (2) with no stocking and $a = b$ was studied in [20]. Then we provide a detailed analysis of system (2) with proportional and constant stocking of parasitoids.

2.1 System with no stocking

The host–parasitoid interaction without external stocking of parasitoids is given by

$$\begin{aligned} x_{n+1} &= ax_n f(y_n), \\ y_{n+1} &= bx_n(1 - f(y_n)). \end{aligned} \tag{5}$$

Equilibria (\bar{x}, \bar{y}) of (5) satisfy the following equations:

$$\begin{aligned} \bar{x} &= a\bar{x}f(\bar{y}), \\ \bar{y} &= b\bar{x}(1 - f(\bar{y})). \end{aligned} \tag{6}$$

Obviously, system (5) has an extinction equilibrium $E_0 = (0, 0)$ where both populations become extinct. The extinction equilibrium always exists. For $a > 1$, system (5) has an interior equilibrium $E_1 = (\frac{af^{-1}(\frac{1}{a})}{b(a-1)}, f^{-1}(\frac{1}{a}))$ in \mathbb{R}_+^2 , where both populations coexist. Moreover, if $a = 1$ and $f(\bar{y}) = 1$, then system (5) has another boundary equilibrium $E_2 = (\bar{x}, 0)$, where the host population survives, and the parasitoid population becomes extinct.

The Jacobian matrix of system (5) has the following form

$$J(x, y) = \begin{pmatrix} af(y) & axf'(y) \\ b(1 - f(y)) & -bxf'(y) \end{pmatrix}.$$

The Jacobian matrix evaluated at $E_0 = (0, 0)$ is

$$J(0, 0) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = a$ and $\lambda_2 = 0$. We further assume that $f'_+(0)$ and $f''_+(0)$ exist. The proof of the following lemma is straightforward and is omitted.

Lemma 1

- (i) If $a < 1$, then E_0 is globally asymptotically stable in \mathbb{R}_+^2 .
- (ii) If $a > 1$, then E_0 is unstable (a saddle point) with stable manifold $M^s = \{(x, y) : x = 0, 0 < y < \infty\}$ and unstable manifold $M^u = \{(x, y) : 0 < x < \infty, y = 0\}$.
- (iii) If $a = 1$, then E_0 is globally attracting.

The Jacobian matrix evaluated at $E_1 = (f^{-1}(\frac{1}{a}), f^{-1}(\frac{1}{a}))$ has the form

$$J(E_1) = \begin{pmatrix} 1 & a\bar{x}f'(\bar{y}) \\ \frac{b(a-1)}{a} & -b\bar{x}f'(\bar{y}) \end{pmatrix}.$$

The trace and determinant of the Jacobian matrix are

$$\text{Tr}(J) = 1 - b\bar{x}f'(\bar{y}), \quad \text{Det}(J) = -ab\bar{x}f'(\bar{y}),$$

respectively. Notice that both expressions are positive.

Using the linearized stability results, we obtain the following lemma, which describes the interior local character of equilibrium.

Lemma 2 *The following statements hold:*

- (i) If $-\bar{x}f'(\bar{y}) < \frac{1}{ab}$, then E_1 is locally asymptotically stable.
- (ii) If $-\bar{x}f'(\bar{y}) > \frac{1}{ab}$, then E_1 is a repeller.
- (iii) If $-\bar{x}f'(\bar{y}) = \frac{1}{ab}$, then E_1 is nonhyperbolic with complex conjugate eigenvalues $\lambda_1 = \frac{1-a+i\sqrt{(a+1)(3a-1)}}{2a}, \lambda_2 = \frac{1-a-i\sqrt{(a+1)(3a-1)}}{2a}$ of modulus 1.

The third statement of the lemma shows the occurrence of the Neimark–Sacker bifurcation described in [20].

At the boundary equilibrium $E_2 = (\bar{x}, 0)$ the Jacobian is

$$J(E_2) = \begin{pmatrix} 1 & \bar{x} \\ 0 & -b\bar{x}f'(y) \end{pmatrix}.$$

The trace and determinant of this Jacobian are $\text{Tr}(J) = 1 - b\bar{x}f'(0)$ and $\text{Det}(J) = -b\bar{x}f'(0)$. Observe that $\text{Tr}(J) = 1 + \text{Det}(J)$, which implies that the boundary equilibrium is nonhyperbolic. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -b\bar{x}f'(0)$. We apply the center manifold theory to answer whether E_2 is locally asymptotically stable or unstable. To shift the equilibrium $(\bar{x}, 0)$ to the origin, we set $u_n = x_n - \bar{x}, v_n = y_n$. System (5) becomes

$$\begin{aligned} u_{n+1} &= (u_n + \bar{x})f(v_n) - \bar{x}, \\ v_{n+1} &= b(u_n + \bar{x})(1 - f(v_n)). \end{aligned}$$

Set $F(u, v) = (u + \bar{x})f(v) - \bar{x}$, $G(u, v) = b(u + \bar{x})(1 - f(v))$. Taylor’s expansions of the functions $F(u, v)$ and $G(u, v)$ at $(0, 0)$ are, respectively,

$$F(u, v) = u + \bar{x}f'(0)v + f'(0)uv + \frac{1}{2}\bar{x}f''(0)v^2 + O_3,$$

$$G(u, v) = -b\bar{x}f'(0)v - bf'(0)uv - \frac{1}{2}f''(0)\bar{x} + O_3.$$

Now we can write the system in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f'(0)uv + \frac{1}{2}\bar{x}f''(0)v^2 + O_3 \\ -bf'(0)uv - \frac{1}{2}f''(0)\bar{x} + O_3 \end{pmatrix}.$$

Set $A = \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0) \end{pmatrix}$. The eigenvalues associated with the matrix A are $\lambda_1 = 1$ and $\lambda_2 = -b\bar{x}f'(0)$, and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -\frac{1+b\bar{x}f'(0)}{\bar{x}f'(0)} \end{pmatrix}$, respectively. Set $T = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{1+b\bar{x}f'(0)}{\bar{x}f'(0)} \end{pmatrix}$. The inverse matrix of T is $T^{-1} = \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -\frac{1+b\bar{x}f'(0)}{\bar{x}f'(0)} \end{pmatrix}$.

Letting $\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, we have $u = \omega_1 + \omega_2$ and $v = -\frac{1+b\bar{x}f'(0)}{\bar{x}f'(0)}\omega_2$.

Now we have

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0) \end{pmatrix} T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} F_1(\omega_1, \omega_2) \\ G_1(\omega_1, \omega_2) \end{pmatrix},$$

where

$$F_1(\omega_1, \omega_2) = -f'(0)\frac{1 + b\bar{x}f'(0)}{\bar{x}f'(0)}\omega_1\omega_2 + \left(\frac{\bar{x}f''(0)}{2}\left(\frac{1 + b\bar{x}f'(0)}{\bar{x}f'(0)}\right)^2 - f'(0)\frac{1 + b\bar{x}f'(0)}{\bar{x}f'(0)}\right)\omega_2^2 + O_3,$$

and

$$G_1(\omega_1, \omega_2) = bf'(0)\frac{1 + b\bar{x}f'(0)}{\bar{x}f'(0)}\omega_1\omega_2 + \left(\frac{bf'(0)(1 + b\bar{x}f'(0))}{\bar{x}f'(0)} - \frac{b\bar{x}f''(0)}{2}\left(\frac{1 + \bar{x}f'(0)}{\bar{x}f'(0)}\right)^2\right)\omega_2^2 + O_3.$$

We have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -b\bar{x}f'(0) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} F_2(\omega_1, \omega_2) \\ G_2(\omega_1, \omega_2) \end{pmatrix}, \quad \text{where}$$

$$\begin{pmatrix} F_2(\omega_1, \omega_2) \\ G_2(\omega_1, \omega_2) \end{pmatrix} = T^{-1} \begin{pmatrix} F_1(\omega_1, \omega_2) \\ G_1(\omega_1, \omega_2) \end{pmatrix}, \quad \text{and}$$

$$F_2(\omega_1, \omega_2) = \frac{-1}{\bar{x}}\omega_1\omega_2 + \left(\frac{-1}{\bar{x}} + \frac{\bar{x}f''(0)(1 + b\bar{x}f'(0))}{2(\bar{x}f'(0))^2}\right)\omega_2^2,$$

$$G_2(\omega_1, \omega_2) = -bf'(0)\omega_1\omega_2 - \left(bf'(0) - \frac{\bar{x}f''(0)(1 + b\bar{x}f'(0))}{2\bar{x}f'(0)}\right)\omega_2^2.$$

Assume that the center manifold has the form $\omega_1 = h(\omega_2) = d_1\omega_2^2 + d_2\omega_2^3$. We compute the coefficients d_1, d_2 from the equation

$$h((-b\bar{x}f'(0))\omega_2 + G_2((h(\omega_2)), \omega_2)) - h(\omega_2) - F_2(h(\omega_2), \omega_2) = 0.$$

Then we equate equal powers of ω_2 , and in that way we can compute $h(\omega_2)$. After tedious calculations we obtain that the dynamics of the system is reduced to the dynamics of the following one-dimensional function:

$$\omega_2 \mapsto (-b\bar{x}f'(0))\omega_2 - \left(bf'(0) - \frac{b\bar{x}f'(0)(1 + b\bar{x}f'(0))}{2\bar{x}f'(0)} \right) \omega_2^2 - bf'(0)d_1\omega_2^3.$$

Now we easily see that if $-b\bar{x}f'(0) < 1$, then the zero equilibrium is locally asymptotically stable, and if $-b\bar{x}f'(0) > 1$, then it is unstable. As result of the above discussion, we have the following lemma.

Lemma 3 *For the boundary equilibrium of system (5), the following statements hold:*

- (i) *If $-b\bar{x}f'(0) < 1$, then $(\bar{x}, 0)$ is locally asymptotically stable.*
- (ii) *If $-b\bar{x}f'(0) > 1$, then $(\bar{x}, 0)$ is unstable.*

2.2 The host–parasitoid system with variable stocking

In this subsection, we investigate the host–parasitoid model with variable stocking

$$\begin{aligned} x_{n+1} &= ax_n f(y_n), \\ y_{n+1} &= bx_n(1 - f(y_n)) + cy_n, \end{aligned} \tag{7}$$

where a, b, c are positive real parameters, and $f : \mathbb{R}^+ \rightarrow [0, 1]$ is such that $f(y) > 0, f'(y) < 0$, and $f(0) = 1, f(\infty) = 0$.

Before we compute equilibrium points and analyze their stability, we note the following. If $c > 1$, then the second equation of system (7) implies $y_n > c^{n-1}y_0$, where y_0 is a given initial state of parasitoids. Hence $y_n \rightarrow \infty$ as $n \rightarrow \infty$. From the first equation of system (7) we have that $x_{n+1} < af(c^{n-1}y_0)x_n$. Since the function f is continuous and decreasing with $f(\infty) = 0$, there exists $n_0 > 1$ such that $af(c^{n-1}y_0) < af(c^{n_0-1}y_0) < 1$ for all $n \geq n_0$, which implies that $x_n \rightarrow 0$ as $n \rightarrow \infty$. So we have the following lemma.

Lemma 4 *Let $c > 1$. Then $(x_n, y_n) \rightarrow (0, \infty)$ as $n \rightarrow \infty$.*

Lemma 4 states that if the coefficient of releasement c is greater than one, then the host population goes to extinction asymptotically.

The equilibrium points (\bar{x}, \bar{y}) of system (7) are solutions of the system

$$\begin{aligned} \bar{x} &= a\bar{x}f(\bar{y}), \\ \bar{y} &= b\bar{x}(1 - f(\bar{y})) + c\bar{y}. \end{aligned}$$

We easily see that if $\bar{x} = 0$ and $c \neq 1$, then $\bar{y} = 0$, and the only solution is $E_0 = (0, 0)$. However, if $c = 1$, then there is an infinite number of solutions of the form $(0, \bar{y})$ with $\bar{y} \in \mathbb{R}^+$. Also,

if $\bar{y} = 0$ and $a \neq 1$, then the only solution is $E_0 = (0, 0)$. However, if $a = 1$, then there is an infinite number of solutions of the form $(\bar{x}, 0)$ with $\bar{x} \in \mathbb{R}^+$. Now we assume that $\bar{x} \neq 0$ and $\bar{y} \neq 0$. Then the first equation of system (7) implies $f(\bar{y}) = \frac{1}{a}$ or, equivalently, $\bar{y} = f^{-1}(\frac{1}{a})$. Now the second equation implies $\bar{x} = \frac{a(1-c)}{b(a-1)}f^{-1}(\frac{1}{a})$. So, we obtain the equilibrium point $E_p = (\frac{a(1-c)}{b(a-1)}f^{-1}(\frac{1}{a}), f^{-1}(\frac{1}{a}))$, which exists only if $a > 1$ and $0 < c < 1$.

To examine the stability of the equilibrium points, we associate the following map with system (7). Set

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} axf(y) \\ bx(1 - f(y)) + cy \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}. \tag{8}$$

The Jacobian associated with map (8) is

$$J(T) = \begin{pmatrix} af'(y) & axf''(y) \\ b(1 - f(y)) & -bx f''(y) + c \end{pmatrix}. \tag{9}$$

The following lemma holds for the extinction equilibrium $E_0 = (0, 0)$. We further assume that $f'_+(0), f''_+(0)$ exist.

Lemma 5 *The following lemma holds for $E_0 = (0, 0)$:*

- (i) *If $0 < a < 1$ and $0 < c < 1$, then E_0 is locally asymptotically stable. If, in addition, $f''(y) > 0$ for all $y \in \mathbb{R}_+$, then E_0 is globally asymptotically stable.*
- (ii) *If $a > 1$ and $c > 1$, then E_0 is a repeller.*
- (iii) *If either $a > 1$ and $0 < c < 1$ or $0 < a < 1$ and $c > 1$, then E_0 is a saddle point with stable manifold M_1^s and unstable manifold M_1^u , where either*

$$M_1^s = \{(x, y) : x = 0, 0 < y < \infty\}, \quad M_1^u = \{(x, y) : 0 < x < \infty, y = 0\}$$

or

$$M_1^s = \{(x, y) : 0 < x < \infty, y = 0\}, \quad M_1^u = \{(x, y) : x = 0, 0 < y < \infty\}.$$

Proof We prove statement (i), since the proofs of statements (ii) and (iii) follow directly from the linearized stability result. The Jacobian of the map T at the point E_0 is

$$J = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Sine $a < 1$, the first equation of system (7) implies $x_{n+1} < ax_n$ and $x_n \rightarrow 0$ as $n \rightarrow \infty$. That is, for all $\varepsilon > 0$, there exists n_0 such that $x_n < \varepsilon$ for all $n \geq n_0$. Now, for $n \geq n_0$, from the second equation we have $y_{n+1} < b\varepsilon(1 - f(y_n)) + cy_n$. Set $z_{n+1} = b\varepsilon(1 - f(z_n)) + cz_n$ and $h(z) = b\varepsilon(1 - f(z)) + cz$. We have $h'(z) = -b\varepsilon f'(z) + c > 0$. Since $f'(z)$ is a decreasing function with $f(\infty) = 0$ and $\varepsilon > 0$ is arbitrarily small, $-b\varepsilon f'(0)$ can be arbitrarily small, which implies that $h'(0) = -b\varepsilon f'(0) + c$ can be less than 1. On the other hand, $h''(z) = -b\varepsilon f''(z) < 0$, since $f''(z) > 0$. Lemma 3 from [2] implies that $z = 0$ is a unique fixed point of the function $h(z)$, which is globally asymptotically stable. Now we conclude $y_n \rightarrow 0$ as $n \rightarrow \infty$. Hence $(0, 0)$ is globally asymptotically stable. □

Lemma 6

- (i) If $a = 1$ and $0 < c < 1$ or $c > 1$, then the equilibrium $(\bar{x}, 0)$ of system (7) is nonhyperbolic, and its dynamics is reduced to the one-dimensional map $\omega_2 \mapsto (-b\bar{x}f'(0) + c)\omega_2 - (bf'(0) - \frac{b\bar{x}f''(0)(1+b\bar{x}f'(0)-c)}{2\bar{x}f'(0)})\omega_2^2 - bf'(0)D_1\omega_2^3$, where $D_1 = \frac{1}{(1-b\bar{x}f'(0)+c)^2}(\frac{f''(0)(c-1)}{\bar{x}f'(0)} + \frac{\bar{x}f''(0)}{\bar{x}f'(0)} - \frac{f''(0)(c-1)}{\bar{x}f'(0)})$. If $0 < c < 1$ and $-b\bar{x}f'(0) < 1 - c$, then $(\bar{x}, 0)$ is locally asymptotically stable. If $c > 1$, then $-b\bar{x}f'(0) > 1 - c$, that is, $(\bar{x}, 0)$ is unstable.
- (ii) If $c = 1$ and $0 < a < 1$ or $a > 1$, then the equilibrium $(0, \bar{y})$ of system (7) is nonhyperbolic, and its dynamics is reduced to the one-dimensional map $w_1 \mapsto af(\bar{y})w_1 + f'(\bar{y})(w_1^2 + c_1w_1^3 + c_2w_1^4)$, where

$$c_1 = \frac{f'(\bar{y})(f(\bar{y})(a + 1) - 2)}{a^2(f(\bar{y}))^2(f(\bar{y}) - 1) - f(\bar{y}) + 1 - f'(\bar{y})(f(\bar{y})(a + 1) - 2)}$$

and

$$c_2 = \frac{-2ac_1f(\bar{y})f'(\bar{y})(f(\bar{y}) - 1)}{a^3(f(\bar{y}))^3(f(\bar{y}) - 1) - f(\bar{y}) + 1 + f'(\bar{y})(f(\bar{y})(a + 1) - 2)}.$$

If $af(\bar{y}) < 1$, then $(0, \bar{y})$ is locally asymptotically stable. If $af(\bar{y}) > 1$, then $(0, \bar{y})$ is unstable.

Proof (i) If $a = 1$ and $0 < c < 1$ or $c > 1$, then the Jacobian matrix at $(\bar{x}, 0)$ has the form $J = \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0)+c \end{pmatrix}$, and we see that $\text{Tr}(J) = 1 + \text{Det}(J)$, which means that $(\bar{x}, 0)$ is nonhyperbolic. Now we compute the corresponding center manifold. First, we shift the equilibrium point $(\bar{x}, 0)$ of system (7) to the origin by letting $u_n = x_n - \bar{x}$, $v_n = y_n$. So, we have the system

$$\begin{aligned} u_{n+1} &= (u_n + \bar{x})f(v_n) - \bar{x}, \\ v_{n+1} &= b(u_n + \bar{x})(1 - f(v_n)) + cv_n. \end{aligned} \tag{10}$$

System (10) has $(0, 0)$ as an equilibrium point. The map associated with system (10) is given by

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u + \bar{x})f(v) - \bar{x} \\ b(u + \bar{x})(1 - f(v)) + cv \end{pmatrix}.$$

Set

$$F(u, v) = (u + \bar{x})f(v) - \bar{x} \quad \text{and} \quad G(u, v) = b(u + \bar{x})(1 - f(v)) + cv.$$

Taylor’s expansions of these two maps at the point $(0, 0)$ are given by

$$\begin{aligned} F(u, v) &= u + \bar{x}f'(0)v + f'(0)uv + \frac{\bar{x}f''(0)}{2}v^2 + O_3, \\ G(u, v) &= (-b\bar{x}f'(0) + c)v - bf'(0)uv - \frac{b\bar{x}f''(0)}{2}v^2 + O_3, \end{aligned}$$

respectively.

Now system (10) can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0) + c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f'(0)uv + \frac{\bar{x}f''(0)}{2}v^2 + O_3 \\ -bf'(0)uv - \frac{b\bar{x}f''(0)}{2}v^2 + O_3 \end{pmatrix}.$$

Set $A = \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0) + c \end{pmatrix}$. The eigenvalues associated with the matrix A are $\lambda_1 = 1$ and $\lambda_2 = -b\bar{x}f'(0) + c$, and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -\frac{1+b\bar{x}f'(0)-c}{\bar{x}f'(0)} \end{pmatrix}$, respectively. Set $T = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{1+b\bar{x}f'(0)-c}{\bar{x}f'(0)} \end{pmatrix}$. The inverse matrix of T is $T^{-1} = \begin{pmatrix} 1 & \frac{\bar{x}f'(0)}{1+b\bar{x}f'(0)-c} \\ 0 & -\frac{\bar{x}f'(0)}{1+b\bar{x}f'(0)-c} \end{pmatrix}$.

Letting $\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, we have $u = \omega_1 + \omega_2$ and $v = -\frac{1+b\bar{x}f'(0)-c}{\bar{x}f'(0)}\omega_2$.

Now we have

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \bar{x}f'(0) \\ 0 & -b\bar{x}f'(0) + c \end{pmatrix} T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} F_1(\omega_1, \omega_2) \\ G_1(\omega_1, \omega_2) \end{pmatrix},$$

where

$$\begin{aligned} F_1(\omega_1, \omega_2) &= -f'(0) \left(\frac{1 + b\bar{x}f'(0) - c}{\bar{x}f'(0)} \omega_1 \omega_2 \right. \\ &\quad \left. + \left(\frac{\bar{x}f''(0)}{2} \left(\frac{1 + b\bar{x}f'(0) - c}{\bar{x}f'(0)} \right)^2 - f'(0) \frac{1 + b\bar{x}f'(0) - c}{\bar{x}f'(0)} \right) \omega_2^2 + O_3, \right. \end{aligned}$$

and

$$\begin{aligned} G_1(\omega_1, \omega_2) &= bf'(0) \frac{1 + b\bar{x}f'(0) - c}{\bar{x}f'(0)} \omega_1 \omega_2 \\ &\quad + \left(\frac{bf'(0)(1 + b\bar{x}f'(0) - c)}{\bar{x}f'(0)} - \frac{b\bar{x}f''(0)}{2} \left(\frac{1 + \bar{x}f'(0) - c}{\bar{x}f'(0)} \right)^2 \right) \omega_2^2 + O_3. \end{aligned}$$

We have

$$\begin{aligned} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -b\bar{x}f'(0) + c \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} F_2(\omega_1, \omega_2) \\ G_2(\omega_1, \omega_2) \end{pmatrix}, \quad \text{where} \\ \begin{pmatrix} F_2(\omega_1, \omega_2) \\ G_2(\omega_1, \omega_2) \end{pmatrix} &= T^{-1} \begin{pmatrix} F_1(\omega_1, \omega_2) \\ G_1(\omega_1, \omega_2) \end{pmatrix}, \quad \text{and} \\ F_2(\omega_1, \omega_2) &= \frac{f'(0)(c-1)}{\bar{x}f'(0)} \omega_1 \omega_2 + \left(\frac{f'(0)(c-1)}{\bar{x}f'(0)} + \frac{\bar{x}f''(0)(1-c)(1+b\bar{x}f'(0)-c)}{2(\bar{x}f'(0))^2} \right) \omega_2^2, \\ G_2(\omega_1, \omega_2) &= -bf'(0)\omega_1\omega_2 - \left(bf'(0) - \frac{\bar{x}f''(0)(1+b\bar{x}f'(0)-c)}{2\bar{x}f'(0)} \right) \omega_2^2. \end{aligned}$$

Assume that the center manifold has the form $\omega_1 = h(\omega_2) = D_1\omega_2^2 + D_2\omega_2^3$. We compute the coefficients D_1, D_2 from the equation

$$h\left(-b\bar{x}f'(0) + c\right)\omega_2 + G_1\left(h(\omega_2), \omega_2\right) - h(\omega_2) - F_1\left(h(\omega_2), \omega_2\right) = 0.$$

We then equate equal powers of ω_2 , and then in that way we can compute $h(\omega_2)$. After tedious calculations we obtain that the dynamics of the system is reduced to the dynamics of the following one-dimensional map:

$$\omega_2 \mapsto (-b\bar{x}f'(0) + c)\omega_2 - \left(bf'(0) - \frac{b\bar{x}f'(0)(1 + b\bar{x}f'(0) - c)}{2\bar{x}f'(0)} \right) \omega_2^2 - bf'(0)D_1\omega_2^3.$$

Now we easily see that for $0 < c < 1$, if $-b\bar{x}f'(0) < 1 - c$, then $(\bar{x}, 0)$ is locally asymptotically stable. On the other hand, for $c > 1$, we have $-b\bar{x}f'(0) > 1 - c$, that is, $(\bar{x}, 0)$ is unstable.

(ii) If $c = 1$ and $0 < a < 1$ or $a > 1$, then the Jacobian matrix at $(0, \bar{y})$ has the form $J = \begin{pmatrix} f(\bar{y}) & 0 \\ b(1-f(\bar{y})) & 1 \end{pmatrix}$, and we see that $\text{Tr}(J) = 1 + \text{Det}(J)$, which means that $(\bar{x}, 0)$ is nonhyperbolic. In both cases, we easily see that the condition for nonhyperbolic equilibrium $\text{Tr}(J) = 1 + \text{Det}(J)$ is satisfied. Now we compute the corresponding center manifold. By substitution $u_n = x_n$, $v_n = y_n - \bar{y}$ system (7) becomes

$$\begin{aligned} u_{n+1} &= au_n f(v_n + \bar{y}), \\ v_{n+1} &= bu_n(1 - f(v_n + \bar{y})) + v_n, \end{aligned} \tag{11}$$

and $(0, 0)$ is an equilibrium point of this system.

Now we proceed as in the previous case. The linearized system at $(0, 0)$ is

$$\begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix} = \begin{pmatrix} af(\bar{y}) & 0 \\ b(1-f(\bar{y})) & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f'(\bar{y})uv \\ -bf'(\bar{y})uv \end{pmatrix}.$$

The matrix of the eigenvectors is $T = \begin{pmatrix} 1 & 0 \\ \frac{b(1-f(\bar{y}))}{af(\bar{y})-1} & 1 \end{pmatrix}$. Its inverse is $T^{-1} = \begin{pmatrix} \frac{b(f(\bar{y})-1)}{1-af(\bar{y})} & 0 \\ \frac{b(1-f(\bar{y}))}{1-af(\bar{y})} & 1 \end{pmatrix}$. Now take $\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, that is, $u = w_1$ and $v = \frac{b(1-f(\bar{y}))}{af(\bar{y})-1} w_1 + w_2$. The system becomes

$$T \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} af(\bar{y}) & 0 \\ b(1-f(\bar{y})) & 1 \end{pmatrix} T \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} f'(\bar{y})w_1(\frac{b(1-f(\bar{y}))}{af(\bar{y})-1} w_1 + w_2) \\ -bf'(\bar{y})w_1(\frac{b(1-f(\bar{y}))}{af(\bar{y})-1} w_1 + w_2) \end{pmatrix}.$$

By applying T^{-1} we obtain

$$\begin{pmatrix} w_1 & w_2 \end{pmatrix} \mapsto \begin{pmatrix} af(\bar{y}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} H_1(w_1, w_2) \\ H_2(w_1, w_2) \end{pmatrix},$$

where

$$\begin{aligned} H_1(w_1, w_2) &= f'(\bar{y})(w_1^2 + w_1 w_2), \\ H_2(w_1, w_2) &= \frac{f'(\bar{y})(f(\bar{y})(a + 1) - 2)}{f(\bar{y}) - 1}(w_1^2 + w_1 w_2). \end{aligned}$$

Assume that the center manifold has the form $w_2 = h(w_1) = c_1 w_1^2 + c_2 w_1^3$. We compute the coefficients c_1 and c_2 from the equation

$$h(af(\bar{y})w_1 + f'(\bar{y})(w_1^2 + w_1 h(w_1))) - h(w_1) - \frac{f'(\bar{y})(f(\bar{y})(a + 1) - 2)}{f(\bar{y}) - 1}(w_1^2 + w_1 h(w_1)) = 0.$$

We then equate equal powers of w_1 and obtain that

$$c_1 = \frac{f'(\bar{y})(f(\bar{y})(a + 1) - 2)}{a^2(f(\bar{y}))^2(f(\bar{y}) - 1) - f(\bar{y}) + 1 - f'(\bar{y})(f(\bar{y})(a + 1) - 2)}$$

and

$$c_2 = \frac{-2ac_1f'(\bar{y})f'(\bar{y})(f(\bar{y}) - 1)}{a^3(f(\bar{y}))^3(f(\bar{y}) - 1) - f(\bar{y}) + 1 + f'(\bar{y})(f(\bar{y})(a + 1) - 2)}.$$

So, we compute $h(w_1)$. Now the dynamics of the system is reduced to the one-dimensional map $w_1 \mapsto af(\bar{y})w_1 + f'(\bar{y})(w_1^2 + c_1w_1^3 + c_2w_1^4)$. Now we easily see that the zero is an equilibrium point of this map and if $af(\bar{y}) < 1$, then the zero equilibrium is locally asymptotically stable. If $af(\bar{y}) > 1$, then the zero equilibrium is unstable. \square

Now we investigate the stability of the coexistence equilibrium $E_p(\bar{x}, \bar{y}) = (\frac{a(1-c)}{b(a-1)}f^{-1}(\frac{1}{a}), f^{-1}(\frac{1}{a})) = (\frac{a(1-c)}{b(a-1)}\bar{y}, \bar{y})$. The Jacobian evaluated at E_p is given by

$$J = \begin{pmatrix} 1 & a\bar{x}f'(\bar{y}) \\ \frac{b(a-1)}{a} & -b\bar{x}f'(\bar{y}) + c \end{pmatrix}.$$

Lemma 7 *Let $0 < c < 1$ and $a > 1$. Then the following statements hold:*

- (i) *If $-\bar{x}f'(\bar{y}) < \frac{1-c}{ab}$ or, equivalently, $-\bar{y}f'(\bar{y}) < \frac{a-1}{a^2}$, then E_p is locally asymptotically stable.*
- (ii) *If $-\bar{x}f'(\bar{y}) > \frac{1-c}{ab}$ or, equivalently, $-\bar{y}f'(\bar{y}) > \frac{a-1}{a^2}$, then E_p is a repeller.*
- (iii) *If $-\bar{x}f'(\bar{y}) = \frac{1-c}{ab}$ or, equivalently, $-\bar{y}f'(\bar{y}) = \frac{a-1}{a^2}$, then there are complex-conjugate eigenvalues μ_1 and $\bar{\mu}_1$, where*

$$\mu_1 = \frac{(a + 1) + c(a - 1) + i\sqrt{(1 - c)(a - 1)(3a + 1 + c(a - 1))}}{2a}$$

and $|\mu_1| = 1$.

Proof The proof is based on the linearized stability theorem. The trace and determinant of the Jacobian at the point E_p are

$$\text{Tr}(J) = 1 - b\bar{x}f'(\bar{y}) + c > 0,$$

$$\text{Det}(J) = -b\bar{x}f'(\bar{y}) + c - b(a - 1)\bar{x}f'(\bar{y}) = c - ab\bar{x}f'(\bar{y}) > 0,$$

respectively. Note that the trace and determinant are always greater than zero. Since $0 < c < 1$ and $a > 1$, we see that the determinant is always greater than zero. Now we check the condition

$$|\text{Tr}(J)| < 1 + \text{Det}(J) < 2.$$

This condition is equivalent to

$$1 - b\bar{x}f'(\bar{y}) + c < 1 + c - ba\bar{x}f'(\bar{y}) < 2.$$

We easily see that the left inequality is always true. On the other hand, the second inequality holds if $-\bar{x}f'(\bar{y}) < \frac{1-c}{ab}$ and statement (i) is true. Now we check the conditions

$$|\text{Tr}(J)| < 1 + \text{Det}(J) \quad \text{and} \quad |\text{Det}(J)| > 1.$$

These conditions are equivalent to

$$1 - b\bar{x}f'(\bar{y}) + c < 1 + c - ab \quad \text{and} \quad c - ab\bar{x}f'(\bar{y}) > 1.$$

The first inequality is always true, and the second holds if

$$-\bar{x}f'(\bar{y}) > \frac{1-c}{ab},$$

that is, the equilibrium is a repeller.

If $-\bar{x}f'(\bar{y}) = \frac{1-c}{ab}$, then the Jacobian at E_p is

$$J(E_p) = \begin{pmatrix} 1 & \frac{c-1}{b} \\ \frac{b(a-1)}{a} & \frac{1-c+ac}{a} \end{pmatrix}.$$

Straightforward calculations show that $\text{Det}(J) = 1$ and the eigenvalues are μ_1 and $\bar{\mu}_1$, where $|\mu_1| = 1$ and $\mu_1 = \frac{(a+1)+c(a-1)+i\sqrt{(1-c)(a-1)(3a+1+c(a-1))}}{2a}$. □

The third statement of the lemma shows that the Neimark–Sacker bifurcation can occur.

To see the kind of dependence between the interior equilibrium and the coefficients of the release c , we consider \bar{x} as a function of c . Observe that \bar{y} does not depend on c . The first derivative is $\bar{x}'(c) = -\frac{af^{-1}(\frac{1}{a})}{b(a-1)}$. For $a > 1$, the host population level at the interior steady state decreases. On the other hand, the level of the parasitoid population is constant.

As we saw in Lemma 7, for $a > 1$, a Neimark–Sacker bifurcation can occur at the interior equilibrium. In the following subsection, we prove the existence and compute the direction of the Neimark–Sacker bifurcation.

2.2.1 Neimark–Sacker bifurcation

When parameter $a > 1$ varies, a Neimark–Sacker bifurcation occurs at E_p . Therefore E_p becomes unstable, and a stable invariant curve appears around E_p . Moreover, we compute the asymptotic approximation of this invariant curve caused by Neimark–Sacker bifurcation. The occurrence of Neimark–Sacker bifurcations is very significant in biological systems since it suggests the existence of oscillatory behavior of the populations.

By using procedure given in [25] we find an explicit form of the limit curve near the positive equilibrium point of (7). According to the procedure, by change of variable $u_n = x_n - \frac{a(1-c)\bar{y}}{(a-1)b}$ and $v_n = y_n - \bar{y}$ we shift the equilibrium point $(\bar{x}, \bar{y}) = (\frac{a(1-c)\bar{y}}{(a-1)b}, \bar{y})$ to the origin. We have

$$\begin{cases} u_{n+1} = a(\frac{a(1-c)\bar{y}}{(a-1)b} + u_n)f(v_n + \bar{y}) - \frac{a(1-c)\bar{y}}{(a-1)b}, \\ v_{n+1} = b(\frac{a(1-c)\bar{y}}{(a-1)b} + u_n)(1 - f(v_n + \bar{y})) + c(v_n + \bar{y}) - \bar{y}. \end{cases} \tag{12}$$

Let F_1 be the map associated with system (12), that is,

$$F_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a\left(\frac{a(1-c)\bar{y}}{(a-1)b} + u\right)f(\bar{y} + v) - \frac{a(1-c)\bar{y}}{(a-1)b} \\ b\left(\frac{a(1-c)\bar{y}}{(a-1)b} + u\right)(1 - f(\bar{y} + v)) + c(\bar{y} + v) - \bar{y} \end{pmatrix}.$$

The Jacobian matrix associated with the map F_1 is

$$\text{Jac}_{F_1}(u, v) = \begin{pmatrix} af(v + \bar{y}) & a\left(u - \frac{a(1-c)\bar{y}}{(a-1)b}\right)f'(v + \bar{y}) \\ b(1 - f(v + \bar{y})) & c - b\left(u - \frac{a(1-c)\bar{y}}{(a-1)b}\right)f'(v + \bar{y}) \end{pmatrix}.$$

Now $(0, 0)$ is a fixed point of F_1 , and

$$\text{Jac}_{F_1}(0, 0) = \begin{pmatrix} 1 & \frac{a^2(1-c)\bar{y}f'(\bar{y})}{(a-1)b} \\ \frac{(a-1)b}{a} & c + \frac{a(1-c)\bar{y}f'(\bar{y})}{a-1} \end{pmatrix}. \tag{13}$$

We further prove the existence of the Neimark–Sacker bifurcation. We need the following lemma.

Lemma 8 *Let $a_0 > 1$ be such that*

$$f^{(-1)}\left(\frac{1}{a_0}\right)f'\left(f^{(-1)}\left(\frac{1}{a_0}\right)\right) = \frac{1 - a_0}{a_0^2}.$$

Then $(0, 0)$ is an equilibrium point of the map F_1 . If $\mu_1(a_0)$ and $\overline{\mu_1(a_0)}$ are the corresponding eigenvalues of $A_1 = \text{Jac}_{F_1}(0, 0)|_{a=a_0}$, then

$$\mu_1(a_0) = \frac{(a_0 - 1)c + a_0 + i\Delta_1 + 1}{2a_0},$$

where

$$\Delta_1 = \sqrt{1 - c}\sqrt{(a_0 - 1)(a_0(c + 3) - c + 1)}.$$

Furthermore:

- (a) $|\mu_1(a_0)| = 1$;
- (b) $\mu_1(a_0)^k \neq 1$ for $k = 1, 2, 3, 4$;
- (c) $d_1 = d_1(a_0) = \frac{d}{da}|\mu_1(a)|_{a=a_0} = \frac{(c-1)(a_0^3\bar{y}^2f''(\bar{y}) - 2a_0^2 + 4a_0 - 2)}{2(a_0 - 1)^2a_0}$;
- (d) *The corresponding eigenvectors are*

$$q_1(a_0) = \left(\frac{-a_0c + a_0 + c + i\Delta_1 - 1}{2(a_0 - 1)b}, 1 \right)^T$$

and

$$p_1(a_0) = \left(-\frac{i(a_0 - 1)b}{\Delta_1}, \frac{2a_0}{a_0(c + 3) - c - i\Delta_1 + 1} \right),$$

where $A_1q_1(a_0) = \mu_1q_1(a_0)$, $p_1(a_0)A_1 = \mu_1p_1(a_0)$, and $p_1(a_0)q_1(a_0) = 1$.

Proof Let a_0 be such that

$$f^{(-1)}\left(\frac{1}{a_0}\right)f'\left(f^{(-1)}\left(\frac{1}{a_0}\right)\right) = \frac{1-a_0}{a_0^2}.$$

Now we have

$$A_1 = \text{Jac}_{F_1}(0, 0) = \begin{pmatrix} 1 & -\frac{(c-1)(1-a_0)}{b(a_0-1)} \\ \frac{b(a_0-1)}{a_0} & c + \frac{(c-1)(1-a_0)}{(a_0-1)a_0} \end{pmatrix}.$$

Then $\mu_1(a_0)$ and $\overline{\mu_1(a_0)}$ are the eigenvalues of the A_1 , where

$$\mu_1(a_0) = \frac{(a_0 - 1)c + a_0 + 1 + i\sqrt{1 - c}\sqrt{(a_0 - 1)(a_0(c + 3) - c + 1)}}{2a_0}.$$

The corresponding eigenvectors are $\mathbf{q}_1(a_0)$ and $\overline{\mathbf{q}_1(a_0)}$, respectively, where

$$\mathbf{q}_1(a_0) = \left(\frac{-a_0c + a_0 + c - 1 + i\sqrt{1 - c}\sqrt{(a_0 - 1)(a_0(c + 3) - c + 1)}}{2(a_0 - 1)b}, 1 \right)^T.$$

A straightforward calculation shows that $|\mu_1(a_0)| = 1$ and $\mu_1^k(a_0) \neq 1$ for $k = 1, 2, 3, 4$. The eigenvalues of (13) are

$$\mu_1(a) = \frac{a(c - 1)\bar{y}f'(\bar{y}) + ac + a - c - 1 + i\Delta(a, c)}{2(a - 1)},$$

where

$$\Delta(a, c) = \sqrt{(1 - c)(a^2(c - 1)\bar{y}^2f'(\bar{y})^2 + 2a(a - 1)\bar{y}(-2a + c + 1)f'(\bar{y}) + (a - 1)^2(c - 1))}.$$

Now we get

$$|\mu_1(a)|^2 = \mu_1(a)\overline{\mu_1(a)} = \frac{a^2(c - 1)\bar{y}f'(\bar{y})}{a - 1} + c = \frac{a^2(c - 1)f^{(-1)}\left(\frac{1}{a}\right)f'\left(f^{(-1)}\left(\frac{1}{a}\right)\right)}{a - 1} + c.$$

The first derivative with respect to a is

$$\begin{aligned} \frac{d}{da}|\mu_1(a)| &= \left((c - 1)\left(f'\left(f^{(-1)}\left(\frac{1}{a}\right)\right)\right)\left((a - 2)af^{(-1)}\left(\frac{1}{a}\right)f'\left(f^{(-1)}\left(\frac{1}{a}\right)\right) - a + 1\right) \right. \\ &\quad \left. - (a - 1)f^{(-1)}\left(\frac{1}{a}\right)f''\left(f^{(-1)}\left(\frac{1}{a}\right)\right) \right) \\ &\quad / \left(2(a - 1)^2f'\left(f^{(-1)}\left(\frac{1}{a}\right)\right)\sqrt{\frac{a^2(c - 1)f^{(-1)}\left(\frac{1}{a}\right)f'\left(f^{(-1)}\left(\frac{1}{a}\right)\right)}{a - 1} + c} \right). \end{aligned}$$

By using the fact

$$f^{(-1)}\left(\frac{1}{a_0}\right)f'\left(f^{(-1)}\left(\frac{1}{a_0}\right)\right) = \frac{1-a_0}{a_0^2},$$

for $a = a_0$, we obtain $|\mu_1(a_0)| = 1$ and

$$d_1 = d_1(a_0) = \frac{d}{da} |\mu_1(a)| \Big|_{a=a_0} = \frac{(c-1)(a_0^3 \bar{y}^2 f''(\bar{y}) - 2a_0^2 + 4a_0 - 2)}{2(a_0 - 1)^2 a_0},$$

where $\bar{y} = \bar{y}(a_0) = f^{(-1)}(\frac{1}{a_0})$. The rest of the proof is almost immediate, so we skip it. \square

For sufficiently small δ , let $a = a_0 + \delta$. By Lemma 8 system (12) can be transformed into the Arnold normal form

$$z_{n+1} = \mu(\delta)z_n + \gamma(\delta)z_n^2 \bar{z}_n + O(|z_n|^4).$$

By polar coordinates we have

$$\begin{pmatrix} r_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} |\mu(\delta)|r_n + \alpha(\delta)r_n^3 + O(r_n^4) \\ \theta_n + \arg \mu(\delta) + b(\delta)r_n^2 + O(r_n^3) \end{pmatrix}, \tag{14}$$

where $\alpha(\delta) = \text{Re}(\gamma(\delta)/\mu(\delta))$ and $\beta(\delta) = \text{Im}(\gamma(\delta)/\mu(\delta))$. By Taylor’s expansion of the coefficients about $\delta = 0$, the first equation of (14) becomes

$$r_{n+1} = (1 + d_1(a_0)\delta)r_n + \alpha_1(0)r_n^3 + O(r_n^4).$$

Further, by the procedure described in [25] we compute \mathbf{K}_{20} and \mathbf{K}_{11} . For $a = a_0$, we have

$$\mathbf{F}_{01} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{A}_1 \begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}_{01} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} \mathbf{G}_{01} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \frac{a_0 b u f(\bar{y}+v) - \frac{a_0(c-1)\bar{y}(a_0 f(\bar{y}+v)-1) - b u - c v + v}{a_0 - 1}}{b} \\ \frac{-a_0 b u f(\bar{y}+v) + b u + (c-1)v}{a_0} + \frac{(c-1)\bar{y}(a_0 f(\bar{y}+v)-1)}{a_0 - 1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a_0^2(1-c)\bar{y}v^3 f^{(3)}(\bar{y})}{6(a_0-1)b} - \frac{a_0^2(c-1)\bar{y}v^2 f''(\bar{y})}{2(a_0-1)b} + \frac{1}{2}a_0 u v^2 f''(\bar{y}) + \frac{(1-a_0)u v}{a_0 \bar{y}} + O((|u| + |v|)^4) \\ -\frac{(1-a_0)b u v}{a_0^2 \bar{y}} + \frac{a_0(c-1)\bar{y}v^3 f^{(3)}(\bar{y})}{6(a_0-1)} + \frac{a_0(c-1)\bar{y}v^2 f''(\bar{y})}{2(a_0-1)} - \frac{1}{2}b u v^2 f''(\bar{y}) + O((|u| + |v|)^4) \end{pmatrix}. \end{aligned}$$

If we take the basis of \mathbb{R}^2 as $\Phi_1 = (\mathbf{q}_1, \bar{\mathbf{q}}_1)$, where $\mathbf{q}_1 = \mathbf{q}_1(a_0)$, then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi_1 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}_1 z + \bar{\mathbf{q}}_1 \bar{z}).$$

Let

$$\mathbf{G}_{01} \left(\Phi_1 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3). \tag{16}$$

By using package Mathematica we obtain

$$\begin{aligned} \mathbf{g}_{20} &= \frac{\partial^2}{\partial z^2} \mathbf{G}_{01} \left(\Phi_1 \left(\begin{matrix} z \\ \bar{z} \end{matrix} \right) \right) \Big|_{z=0} = \begin{pmatrix} \frac{a_0^3(1-c)\bar{y}^2 f''(\bar{y}) + (a_0-1)(a_0(c-1)-c-i\Delta_1+1)}{(a_0-1)a_0 b \bar{y}} \\ \frac{a_0^3(c-1)\bar{y}^2 f''(\bar{y}) + (a_0-1)(-a_0c+a_0+c+i\Delta_1-1)}{(a_0-1)a_0^2 \bar{y}} \end{pmatrix}, \\ \mathbf{g}_{11} &= \frac{\partial^2}{\partial z \partial \bar{z}} \mathbf{G}_{01} \left(\Phi_1 \left(\begin{matrix} z \\ \bar{z} \end{matrix} \right) \right) \Big|_{z=0} = \begin{pmatrix} \frac{(c-1)((a_0-1)^2 - a_0^3 \bar{y}^2 f''(\bar{y}))}{(a_0-1)a_0 b \bar{y}} \\ \frac{a_0(c-1)\bar{y} f''(\bar{y})}{a_0-1} - \frac{(a_0-1)(c-1)}{a_0^2 \bar{y}} \end{pmatrix}. \end{aligned} \tag{17}$$

Set

$$\Delta_2 = a_0^3(c-1)\bar{y}^2 f''(\bar{y}) + (a_0-1)(-a_0c+a_0+c+i\Delta_1-1)$$

and

$$\Delta_3 = -a_0(c+1)(2c-i\Delta_1-2) + a_0^2(c(c+2)-1) + (c-1)(c-i\Delta_1-1).$$

We obtain

$$\begin{aligned} \mathbf{K}_{20} &= (\mu_1^2 I - A_1)^{-1} \mathbf{g}_{20} = \begin{pmatrix} -\frac{a_0 \Delta_2 ((a_0-2)a_0(c^2-1) + i\Delta_1((a_0-1)c+a_0+1) + (c-1)^2)}{(a_0-1)^2 b(c-1)\Delta_3 \bar{y}(a_0(c+2)-c+1)} \\ -\frac{\Delta_2((a_0-1)(-a_0(c+3)(c-1) + 2a_0^2 + (c-1)^2) - i\Delta_1((a_0-1)c+a_0+1))}{(a_0-1)^2(c-1)\Delta_2 \bar{y}(a_0(c+2)-c+1)} \end{pmatrix}, \\ \mathbf{K}_{11} &= (I - A_1)^{-1} \mathbf{g}_{11} = \begin{pmatrix} \frac{(c-1)((a_0-1)^2 - a_0^3 \bar{y}^2 f''(\bar{y}))}{(a_0-1)^2 b \bar{y}} \\ \frac{a_0^2 \bar{y}^2 f''(\bar{y})}{a_0-1} + \frac{1}{a_0} - 1 \end{pmatrix}. \end{aligned} \tag{18}$$

By using \mathbf{K}_{20} and \mathbf{K}_{11} we have

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G}_{01} \left(\Phi_1 \left(\begin{matrix} z \\ \bar{z} \end{matrix} \right) + \frac{1}{2} \mathbf{K}_{20} z^2 + \mathbf{K}_{11} z \bar{z} \right) \Big|_{z=0}. \tag{19}$$

The package Mathematica yields

$$\alpha_1(0) = \alpha_1(a_0) = \frac{1}{2} \operatorname{Re}(\mathbf{p}_1 \mathbf{g}_{21} \bar{\mu}_1) = \frac{(1-c)(2(a_0-1)^3(a_0(c-1)-c) + \Gamma_1(a_0)a_0^3 \bar{y}^2)}{4(a_0-1)^2 a_0^2 \bar{y}^2}, \tag{20}$$

where

$$\Gamma_1(a_0) = a_0^3(c-2)\bar{y}^2 f''(\bar{y})^2 + (1-a_0)(3(a_0-1)c-2a_0+1)f''(\bar{y}) - (a_0-1)a_0 P^* f^{(3)}(\bar{y}).$$

Thus we have the following theorem; see [29].

Theorem 1 *Let $b > 0$ and $0 < c < 1$. Let $a_0 > 1$ be a positive real number such that*

$$f^{(-1)}\left(\frac{1}{a_0}\right) f' \left(f^{(-1)}\left(\frac{1}{a_0}\right) \right) = \frac{1-a_0}{a_0^2}.$$

If $d_1(a_0) \neq 0$ and $\alpha_1(a_0) \neq 0$, then system (7) passes through a Neimark–Sacker bifurcation at the unique positive equilibrium point

$$E_p(\bar{x}, \bar{y}) = \left(\frac{a_0(1-c)}{(a_0-1)b} f^{(-1)}\left(\frac{1}{a_0}\right), f^{(-1)}\left(\frac{1}{a_0}\right) \right)$$

when the parameter a varies in a small neighborhood of a_0 .

If $d_1(a_0) > 0$ and $\alpha_1(a_0) < 0$ (resp., $\alpha_1(a_0) > 0$), then the Neimark–Sacker bifurcation of model (7) at $a = a_0$ is supercritical (resp., subcritical), and there exists an unique closed invariant curve $\Gamma(a)$, which is attracting (resp., repelling), and bifurcates from E_p for $a > a_0$ (resp., $a < a_0$).

If $d_1(a_0) < 0$ and $\alpha_1(a_0) < 0$ (resp., $\alpha_1(a_0) > 0$), then the Neimark–Sacker bifurcation of model (7) at $a = a_0$ is supercritical (resp., subcritical), and there exists an unique closed invariant curve $\Gamma(a)$, which is attracting (resp., repelling), and bifurcates from E_p for $a < a_0$ (resp., $a > a_0$).

Let $a = a_0 + \delta$. Then for small δ , the curve $\Gamma(a)$ can be approximated by

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx (\bar{x}, \bar{y}) + 2\rho_0 \operatorname{Re}(\mathbf{q}_1 e^{i\theta}) + \rho_0^2 (\operatorname{Re}(\mathbf{K}_{20} e^{2i\theta}) + \mathbf{K}_{11}), \quad \theta \in \mathbb{R},$$

where

$$\rho_0 = \sqrt{-\frac{d_1(a_0)}{\alpha_1(a_0)}} \delta.$$

2.3 The host–parasitoid model with constant stocking

We assume that the constant number d of the parasitoid is added to the host–parasitoid interaction to control the host population; that is, we consider the system

$$\begin{aligned} x_{n+1} &= ax_n f(y_n), \\ y_{n+1} &= bx_n(1 - f(y_n)) + d, \end{aligned} \tag{21}$$

where the function f satisfies the same conditions as in model (2), and the constants $a, b, d > 0$. The equilibrium points (\bar{x}, \bar{y}) satisfy

$$\begin{aligned} \bar{x} &= a\bar{x}f(\bar{y}), \\ \bar{y} &= b\bar{x}(1 - f(\bar{y})) + d. \end{aligned}$$

In contrast to the systems with no stocking and variable stocking, observe that system (21) cannot have an extinction equilibrium since d is a positive constant. However, if $\bar{x} = 0$, then $\bar{y} = d$. Hence there is a boundary equilibrium $E_0 = (0, d)$ at which the host (pest) population becomes extinct. We remark that this steady state does not occur in the systems with no stocking and variable stocking. If $a = 1$, $\bar{x} \neq 0$, and $f(\bar{y}) = 1$, then $\bar{y} = d$. There are infinitely many equilibrium points $E_1 = (\bar{x}, d)$ that exist only for $a = 1$. If $a \neq 1$ and $\bar{x} \neq 0$, then there is an interior equilibrium point $E_2 = \left(\frac{a(f^{-1}(\frac{1}{a})-d)}{b(a-1)}, f^{-1}(\frac{1}{a})\right) = \left(\frac{a(\bar{y}-d)}{b(a-1)}, \bar{y}\right)$. Since E_2

must be in \mathbb{R}_+^2 and $\bar{y} = f^{-1}(\frac{1}{a}) = b\bar{x}(1 - f(\bar{y})) + d > d$, we have that E_2 exists if

$$f^{-1}\left(\frac{1}{a}\right) > d, \quad a > 1. \tag{22}$$

The Jacobian matrix associated with system (21) is

$$J(x, y) = \begin{pmatrix} af(y) & axf'(y) \\ b(1 - f(y)) & -bxf'(y) \end{pmatrix}.$$

Now we investigate the stability of the boundary equilibrium $E_0 = (0, d)$. The Jacobian evaluated at the boundary equilibrium $E_0 = (0, d)$ is

$$J(0, d) = \begin{pmatrix} af(d) & 0 \\ b(1 - f(d)) & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = af(d)$ and $\lambda_2 = 0$. Hence, if $af(d) < 1$, then E_0 is locally asymptotically stable, and if $af(d) > 1$, then E_0 is unstable. If $af(d) = 1$, then E_0 is nonhyperbolic.

Note that the second equation of system (21) implies that $y_n > d$ for all n . Since the function f is decreasing, $f(y_n) < f(d)$ for all n . Now the first equation implies $x_{n+1} < af(d)x_n$. If $af(d) < 1$, then $\lim_{n \rightarrow \infty} x_n = 0$, which means that E_0 is globally attracting. Hence, if $af(d) < 1$, then $E_0 = (0, d)$ is globally asymptotically stable.

If $af(d) = 1$, then $x_{n+1} < x_n$. This implies that the sequence $\{x_n\}$ is decreasing and bounded below, so it must converge. Set $\lim_{n \rightarrow \infty} x_n = l$. If $l > 0$, then by taking the limit in $x_{n+1} < af(d)x_n$ we obtain $af(d) > 1$, which is a contradiction. Consequently, $\lim_{n \rightarrow \infty} x_n = 0$ if $af(d) \leq 1$. Now by taking the limit in the second equation of system (21) we obtain $\lim_{n \rightarrow \infty} y_n = d$. We conclude that if $af(d) = 1$, then the boundary equilibrium is globally attracting. To answer whether the boundary equilibrium is locally asymptotically stable or unstable, we use the center manifold theory. By the change of variable $u_n = x_n, v_n = y_n - d$ we transform system (21) to the following system with equilibrium $(0, 0)$:

$$\begin{aligned} u_{n+1} &= au_n f(v_n + d), \\ v_{n+1} &= bu_n(1 - f(v_n + d)). \end{aligned}$$

Set $F(u, v) = au_n f(v + d)$ and $G(u, v) = bu_n(1 - f(v + d))$. Taylors' expansions of these two functions about $(0, 0)$ are, respectively,

$$\begin{aligned} F(u, v) &= u + af'(d)uv + O_3, \\ G(u, v) &= \frac{b(a-1)}{a}u - bf'(d)uv + O_3. \end{aligned}$$

Consider the map

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ \frac{b(a-1)}{a} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} af'(d)uv + O_3 \\ -bf'(d)uv + O_3 \end{pmatrix}.$$

The eigenvalues of the linear part are $\lambda_1 = 1$ and $\lambda_2 = 0$ with the corresponding eigenvectors $\begin{pmatrix} a \\ b(a-1) \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. To find the center manifold, we diagonalize the matrix $\begin{pmatrix} 1 & 0 \\ \frac{b(a-1)}{a} & 0 \end{pmatrix}$ by

using the matrix $T = \begin{pmatrix} a & 0 \\ b(a-1) & 1 \end{pmatrix}$. Letting $\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ yields

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ \frac{b(a-1)}{a} & 0 \end{pmatrix} T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} af'(d)a\omega_1(b(a-1)\omega_1 + \omega_2 + O_3) \\ -bf'(d)a\omega_1(b(a-1)\omega_1 + \omega_2 + O_3) \end{pmatrix}.$$

By applying $T^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b(a-1)}{a} & 1 \end{pmatrix}$ to both sides we obtain

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} a\omega_1 f'(d)(b(a-1)\omega_1 + \omega_2) \\ -ab\omega_1 f'(d)(b(a-1)\omega_1 + \omega_2) \end{pmatrix}.$$

Set $F_1(\omega_1, \omega_2) = a\omega_1 f'(d)(b(a-1)\omega_1 + \omega_2)$, $G_1(\omega_1, \omega_2) = -ab\omega_1 f'(d)(b(a-1)\omega_1 + \omega_2)$. Then

$$M_c = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_2 = h(\omega_1), h(0) = h'(0) = 0\}.$$

The function h has to satisfy the equation

$$h(A\omega_1 + F_1(\omega_1, h(\omega_1)) - Bh(\omega_1) - G_1(\omega_1, h(\omega_1))) = 0.$$

Since $A = 1$ and $B = 0$, the equation becomes

$$h(\omega_1 + F_1(\omega_1, h(\omega_1)) - G_1(\omega_1, h(\omega_1))) = 0. \tag{23}$$

Assume that $h(\omega_1)$ takes the form

$$h(\omega_1) = c_1\omega_1^2 + c_2\omega_1^3 + O(\omega_1^4).$$

Now equation (23) becomes

$$\begin{aligned} &c_1(\omega_1^2 + 2af'(d)\omega_1^2(b(a-1)\omega_1 + c_1\omega_1^2 + c_2\omega_1^3) + a\omega_1^2 f'(d)(b(a-1)\omega_1 + c_1\omega_1^2 + c_2\omega_1^3)^2) \\ &+ c_2(\omega_1^3 + 3\omega_1^3 af'(d)(b(a-1)\omega_1 + c_1\omega_1^2 + c_2\omega_1^3) \\ &+ 3a\omega_1^3 f'(d)(b(a-1)\omega_1 + c_1\omega_1^2 + c_2\omega_1^3)^2 + a^3\omega_1^3 (f'(d))^3 (b(a-1)\omega_1 + c_1\omega_1^2 + c_2\omega_1^3)^3) \\ &+ ab\omega_1^2 f'(d)(b(a-1) + c_1\omega_1 + c_2\omega_1^2) = 0 \end{aligned}$$

or

$$\omega_1^2(c_1 + ab^2 f'(d)(a-1)) + \omega_1^3(c_2 + c_1 ab f'(d)) + O(\omega_1^4) = 0.$$

By equating the coefficients of the same powers with zero we get

$$c_1 = -ab^2 f'(d)(a-1), c_2 = a^2 b^3 (a-1)(f'(d))^2.$$

Thus the central manifold is represented by the graph of the function

$$h(\omega_1) = -abf'(d)(a-1)\omega_1^2 + a^2 b^3 (a-1)(f'(d))^2 \omega_1^3 + O(\omega_1^4),$$

and the map on the center manifold is given by

$$H(\omega_1) = \omega_1 - abf'(d)(a - 1)\omega_1^2 + a^2b^3(a - 1)(f'(d))^2\omega_1^3 + O(\omega_1^4).$$

Note that $\bar{\omega}_1 = 0$ is a fixed point of $H(\omega_1)$ at which $H'(0) = 1$ and $H''(0) = -2abf'(d)(a - 1) \neq 0$. This implies that the origin is unstable (semistable).

As a result of our discussion, we have the following theorem for the boundary equilibrium $E_0 = (0, d)$ of system (21).

Theorem 2 *The boundary equilibrium $E_0 = (0, d)$ of system (21) is globally asymptotically stable in \mathbb{R}_+^2 if $af(d) < 1$, and $E_0 = (0, d)$ is globally attracting semistable in \mathbb{R}_+^2 if $af(d) = 1$.*

In contrast to the system with variable stocking, we see from Theorem 2 that the constant release of the parasitoids into the host–parasitoid interaction drives the host (pest) population to extinction.

Now we investigate the equilibrium $E_1 = (\bar{x}, d)$. The Jacobian evaluated at E_1 is

$$J(E_1) = \begin{pmatrix} 1 & \bar{x}f'(d) \\ 0 & -b\bar{x}f'(d) \end{pmatrix}.$$

The trace and determinant of the Jacobian are $\text{Tr}(J(E_1)) = 1 - b\bar{x}f'(d)$ and $\text{Det}(J(E_1)) = -b\bar{x}f'(d)$. Since $f'(y) > 0, y > 0$, we have $\text{Tr}(J(E_1)) > 0$ and $\text{Det}(J(E_1)) > 0$. We see that $\text{Tr}(E_1) = 1 + \text{Det}(J(E_1))$, which implies that E_1 is nonhyperbolic. The eigenvalues of $J(E_1)$ are $\lambda_1 = 1$ and $\lambda_2 = -b\bar{x}f'(d)$. To answer the question of whether E_1 is locally asymptotically stable or unstable(semistable), we use the center manifold theory.

Letting $u_n = x_n - \bar{x}, v_n = y_n - d$, we obtain the system

$$\begin{aligned} u_{n+1} &= (u_n + \bar{x})f(v_n + d), \\ v_{n+1} &= b(u_n + \bar{x})(1 - f(v_n)). \end{aligned} \tag{24}$$

The origin is a fixed point of system (24). Set

$$F(u, v) = (u + \bar{x})f(v + d) - \bar{x}, G(u, v) = b(u + \bar{x})(1 - f(v + d)).$$

The Taylor expansions of these two maps about the origin are

$$\begin{aligned} F(u, v) &= u + \bar{x}f'(d)v + f'(d)uv + \frac{1}{2}\bar{x}f''(d)v^2 + O_3, \\ G(u, v) &= -b\bar{x}f'(d)v - bf'(d)uv - \frac{1}{2}b\bar{x}f''(d)v^2 + O_3, \end{aligned}$$

respectively. Now we have the map

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & \bar{x}f'(d) \\ 0 & -b\bar{x}f'(d) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f'(d)uv + \frac{1}{2}\bar{x}f''(d)v^2 + O_3 \\ -b\bar{x}f'(d)uv - \frac{1}{2}b\bar{x}f''(d)v^2 + O_3 \end{pmatrix}.$$

The eigenvalues of the map linearized about the origin are

$$\lambda_1 = 1, \quad \lambda_2 = -b\bar{x}f'(d).$$

Therefore there is one-dimensional center manifold, and if $-b\bar{x}f'(d) < 1$, then there is a one-dimensional stable manifold with the orbit structure determined by the orbit structure on the center manifold. The matrix associated with the linear approximation has columns consisting of the eigenvectors associated with the eigenvalues λ_1 and λ_2 , respectively. It is given by

$$T = \begin{pmatrix} 1 & \bar{x}f'(d) \\ 0 & -(1 + b\bar{x}f'(d)) \end{pmatrix} \quad \text{with } T^{-1} = \begin{pmatrix} 1 & \frac{\bar{x}f'(d)}{1+b\bar{x}f'(d)} \\ 0 & -\frac{1}{1+b\bar{x}f'(d)} \end{pmatrix}.$$

Thus letting

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

the map becomes

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \bar{x}f'(d) \\ 0 & -b\bar{x}f'(d) \end{pmatrix} T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} F_1(\omega_1, \omega_2) \\ G_1(\omega_1, \omega_2) \end{pmatrix},$$

where

$$\begin{aligned} F_1(\omega_1, \omega_2) &= -f'(d)(1 + b\bar{x}f'(d))\omega_1\omega_2 \\ &\quad + \bar{x}(1 + b\bar{x}f'(d)) \left((f'(d))^2 + \frac{1}{2}f''(1 + b\bar{x}f'(d)) \right) \omega_2^2 + O_3, \\ G_1(\omega_1, \omega_2) &= bf'(d)(1 + b\bar{x}f'(d))\omega_1\omega_2 \\ &\quad + \bar{x}(1 + b\bar{x}f'(d)) \left(b(f'(d))^2 - \frac{1}{2}f''(d)(1 + b\bar{x}f'(d)) \right) \omega_2^2 + O_3. \end{aligned}$$

By applying T^{-1} we have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -b\bar{x}f'(d) \end{pmatrix} T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} F_2(\omega_1, \omega_2) \\ G_2(\omega_1, \omega_2) \end{pmatrix},$$

where

$$\begin{pmatrix} F_2(\omega_1, \omega_2) \\ G_2(\omega_1, \omega_2) \end{pmatrix} = T^{-1} \begin{pmatrix} F_1(\omega_1, \omega_2) \\ G_1(\omega_1, \omega_2) \end{pmatrix} = \begin{pmatrix} -f'(d)\omega_1\omega_2 - \bar{x}(f'(d))^2\omega_2^2 + O_3 \\ -bf'(d)\omega_1\omega_2 - b\bar{x}(f'(d))^2\omega_2^2 + O_3 \end{pmatrix}.$$

We seek a center manifold

$$M_c(0) = \{(\omega_1, \omega_2) : \omega_2 = h(\omega_1), h(0) = h'(0) = 0\}$$

for ω_1 sufficiently small. We assume that a center manifold has the form

$$h(\omega_1) = c_1\omega_1^2 + c_2\omega_1^3 + O(\omega_1^4)$$

and substitute it into the center manifold equation

$$h(\omega_1 + F_2(\omega_1, h(\omega_1))) + b\bar{x}f'(d)h(\omega_1) - G_2(\omega_1, h(\omega_1)) = 0.$$

Now we have

$$\begin{aligned} &c_1(\omega_1 - f'(d)(c_1\omega_1^3 + c_2\omega_1^4) - \bar{x}(f'(d))^2(c_1\omega_1^2 + c_2\omega_1^3)^2)^2 \\ &+ c_2(\omega_1 - f'(d)(c_1\omega_1^3 + c_2\omega_1^4) - \bar{x}(f'(d))^2(c_1\omega_1^2 + c_2\omega_1^3)^2)^3 + b\bar{x}ff'(d)(c_1\omega_1^2 + c_2\omega_1^3) \\ &+ b\bar{x}f'(d)(c_1\omega_1^3 + c_2\omega_1^4) + b\bar{x}(f'(d))^2(c_1\omega_1^2 + c_2\omega_1^3)^2 = 0 \end{aligned}$$

or

$$\omega_1^2(c_1 + b\bar{x}f'(d)c_1) + \omega_1^3(c_2 + c_2b\bar{x}f'(d) + b\bar{x}f'(d)c_1) + O(\omega_1^4) = 0.$$

By equating the coefficients with the same degrees

$$c_1 = 0, \quad c_2 = 0.$$

Thus the center manifold is given by the graph of

$$h(\omega_1) = 0 + O(\omega_1^4),$$

and the map restricted to the center manifold is given by

$$\omega_1 \mapsto \omega_1 + O(\omega_1^4),$$

whose each point is a fixed point.

When $-b\bar{x}f'(d) > 1$, there is an unstable manifold at $E_1 = (\bar{x}, d)$.

Now we study the stability of the interior equilibrium $E_2(\bar{x}, \bar{y}) = (\frac{a(f^{-1}(\frac{1}{a})-d)}{b(a-1)}, f^{-1}(\frac{1}{a})) = (\frac{a(\bar{y}-d)}{b(a-1)}, \bar{y})$. The Jacobian evaluated at E_2 is given by

$$J(E_2) = \begin{pmatrix} 1 & a\bar{x}f'(\bar{y}) \\ \frac{b(a-1)}{a} & -b\bar{x}f'(\bar{y}) \end{pmatrix}.$$

Observe that $\text{Tr}J(E_2) = 1 - b\bar{x}f'(\bar{y}) > 0$ and $\text{Det}J(E_2) = -ab\bar{x}f'(\bar{y}) > 0$.

By using linearized stability results we have the following lemma.

Lemma 9 *For the interior equilibrium E_2 , the following holds:*

- (i) *If $-b\bar{x}f'(\bar{y}) < \frac{1}{ab}$, then E_2 is locally asymptotically stable.*
- (ii) *If $-b\bar{x}f'(\bar{y}) > \frac{1}{ab}$, then E_2 is a repeller.*
- (iii) *If $-b\bar{x}f'(\bar{y}) = \frac{1}{ab}$, then E_2 is nonhyperbolic with complex conjugate eigenvalues $\mu_2, \bar{\mu}_2$ of modulus 1, where $\mu_2 = \frac{a+1 \pm i\sqrt{(a-1)(3a+1)}}{2a}$.*

To see how does a constant stocking affect the interior equilibrium, we consider \bar{x} as a function of d . We have $\bar{x}(d) = \frac{a(f^{-1}(\frac{1}{a})-d)}{b(a-1)}$ and $\bar{y}(d) = f^{-1}(\frac{1}{a})$. Now $\bar{x}'(d) = -\frac{a}{b(a-1)}$, which

is less than zero since $a > 1$. This implies that the function $\bar{x}(d)$ decreases with respect to d . This means that if a constant input of parasitoids increases, then the level of the host population at interior equilibrium decreases. On the other hand, \bar{y} is constant and does not depend on d .

2.3.1 Neimark–Sacker bifurcation

Now we verify analytically that the system undergoes a Neimark–Sacker bifurcation at the positive equilibrium as parameter a is varied.

By using procedure given in [25] we find an explicit form of the limit curve near the positive equilibrium point of (21). According to the procedure, by change of variable $u_n = x_n - \frac{a(\bar{y}-d)}{(a-1)b}$ and $v_n = y_n - \bar{y}$ we shift the equilibrium point $E_2(\bar{x}, \bar{y}) = (\frac{a(\bar{y}-d)}{(a-1)b}, \bar{y})$ to the origin. We have

$$\begin{cases} u_{n+1} = a(\frac{a(\bar{y}-d)}{(a-1)b} + u_n)f(v_n + \bar{y}) - \frac{a(\bar{y}-d)}{(a-1)b}, \\ v_{n+1} = b(1 - f(\bar{y} + v_n))(\frac{a(\bar{y}-d)}{(a-1)b} + u_n) + d - \bar{y}. \end{cases} \tag{25}$$

Let F_2 be the map associated with system (25), that is,

$$F_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} af(\bar{y} + v)(\frac{a(\bar{y}-d)}{(a-1)b} + u) - \frac{a(\bar{y}-d)}{(a-1)b} \\ b(1 - f(\bar{y} + v))(\frac{a(\bar{y}-d)}{(a-1)b} + u) + d - \bar{y} \end{pmatrix}.$$

Now $(0, 0)$ is a fixed point of the F_2 , and

$$Jac_{F_2}(0, 0) = \begin{pmatrix} 1 & \frac{a^2(\bar{y}-d)f'(\bar{y})}{(a-1)b} \\ \frac{(a-1)b}{a} & \frac{a(d-\bar{y})f'(\bar{y})}{a-1} \end{pmatrix}.$$

The proof of the following lemma is totally analogous to that of Lemma 8.

Lemma 10 *Let $a_0 > 1$ be such that*

$$f' \left(f^{(-1)} \left(\frac{1}{a_0} \right) \right) = \frac{a_0 - 1}{a_0^2(d - f^{(-1)}(\frac{1}{a_0}))}.$$

Then $(0, 0)$ is an equilibrium point of the map F_2 . If $\mu_2(a)$ and $\overline{\mu_2(a)}$ are the corresponding eigenvalues of $A_2 = Jac_{F_2}(0, 0)|_{a=a_0}$, then

$$\mu_2(a_0) = \frac{1 + a_0 + i\sqrt{(a_0 - 1)(3a_0 + 1)}}{2a_0}.$$

Furthermore:

- (a) $|\mu_2(a_0)| = 1$;
- (b) $\mu_2(a_0)^k \neq 1$ for $k = 1, 2, 3, 4$;
- (c) $d_2 = d_2(a_0) = \frac{d}{da} |\mu_2(a)|_{a=a_0} = \frac{1}{a_0} - \frac{a_0^2(d-\bar{y})^2 f''(\bar{y})}{2(a_0-1)^2}$;
- (d) *The corresponding eigenvectors are*

$$q_2(a_0) = \left(\frac{a_0 + i\sqrt{(a_0 - 1)(3a_0 + 1)} - 1}{2(a_0 - 1)b}, 1 \right)^T$$

and

$$p_2(a_0) = \left(\frac{i(1 - a_0)b}{\sqrt{(a_0 - 1)(3a_0 + 1)}}, \frac{2a_0}{1 + 3a_0 - i\sqrt{(a_0 - 1)(3a_0 + 1)}} \right),$$

where $A_2q_2(a_0) = \mu q_2(a_0)$, $p_2(a_0)A_2 = \mu p_2(a_0)$, and $p_2(a_0)q_2(a_0) = 1$.

By the procedure described in [25] we compute K_{20} and K_{11} . For $a = a_0$, we have

$$F_{02} \begin{pmatrix} u \\ v \end{pmatrix} = A_2 \begin{pmatrix} u \\ v \end{pmatrix} + G_{02} \begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$G_{02} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{a_0^3 v^3 f^{(3)}(\bar{y})(d-\bar{y})}{6(a_0-1)b} - \frac{a_0^2 v^2 (d-\bar{y}) f''(\bar{y})}{2(a_0-1)b} + \frac{1}{2} a_0 u v^2 f''(\bar{y}) + \frac{(a_0-1)uv}{a_0(d-\bar{y})} + O(|u| + |v|^4) \\ \frac{(a_0-1)bu v}{a_0^2(\bar{y}-d)} + \frac{v^3 f^{(3)}(\bar{y})(a_0 d - a_0 \bar{y})}{6(a_0-1)} + \frac{v^2(a_0 d - a_0 \bar{y}) f''(\bar{y})}{2(a_0-1)} - \frac{1}{2} b u v^2 f''(\bar{y}) + O(|u| + |v|^4) \end{pmatrix}.$$

If we take, as the basis of \mathbb{R}^2 , $\Phi_2 = (q_2, \bar{q}_2)$, where $q_2 = q_2(a_0)$, then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi_2 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (q_2 z + \bar{q}_2 \bar{z}).$$

Let

$$G_{02} \left(\Phi_2 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) = \frac{1}{2} (g_{20} z^2 + 2g_{11} z \bar{z} + g_{02} \bar{z}^2) + O(|z|^3).$$

By using package Mathematica we obtain

$$g_{20} = \frac{\partial^2}{\partial z^2} G_{02} \left(\Phi_2 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) \Big|_{z=0} = \left(\frac{-a_0^3 (d-\bar{y})^2 f''(\bar{y}) + (a_0-1)(a_0 + i\sqrt{(a_0-1)(3a_0+1)-1})}{(a_0-1)a_0 b (d-\bar{y})}, \frac{a_0^3 (d-\bar{y})^2 f''(\bar{y}) + (a_0-1)(-a_0 - i\sqrt{(a_0-1)(3a_0+1)+1})}{(a_0-1)a_0^2 (d-\bar{y})} \right),$$

$$g_{11} = \frac{\partial^2}{\partial z \partial \bar{z}} G_{02} \left(\Phi_2 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) \Big|_{z=0} = \left(\frac{(a_0-1)^2 - a_0^3 (d-\bar{y})^2 f''(\bar{y})}{(a_0-1)a_0 b (d-\bar{y})}, \frac{a_0 (d-\bar{y}) f''(\bar{y})}{a_0-1} + \frac{a_0-1}{a_0^2 (\bar{y}-d)} \right).$$

Set

$$\Delta_3 = a_0^3 (d - \bar{y})^2 f''(\bar{y}) + (a_0 - 1)(1 - a_0 - i\sqrt{(a_0 - 1)(3a_0 + 1)})$$

and

$$\Delta_4 = a_0(a_0 - i\sqrt{(a_0 - 1)(3a_0 + 1)} - 2) - i\sqrt{(a_0 - 1)(3a_0 + 1)} - 1.$$

We obtain

$$\mathbf{K}_{20} = (\mu^2 I - A_2)^{-1} \mathbf{g}_{20} = \begin{pmatrix} \frac{a_0 \Delta_3}{(a_0-1)^2(2a_0+1)b(d-\bar{y})} \\ -\frac{(a_0+1)(a_0(2a_0-1)-i\sqrt{(a_0-1)(3a_0+1)-1})\Delta_3}{(a_0-1)^2(2a_0+1)\Delta_4(d-\bar{y})} \end{pmatrix},$$

$$\mathbf{K}_{11} = (I - A_2)^{-1} \mathbf{g}_{11} = \begin{pmatrix} \frac{1 - \frac{a_0^3(d-\bar{y})^2 f''(\bar{y})}{(a_0-1)^2}}{b(d-\bar{y})} \\ \frac{a_0^2(\bar{y}-d)f''(\bar{y})}{a_0-1} + \frac{a_0-1}{a_0(d-\bar{y})} \end{pmatrix}.$$

By using \mathbf{K}_{20} , \mathbf{K}_{11} , and

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G}_{02} \left(\Phi_2 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2} \mathbf{K}_{20} z^2 + \mathbf{K}_{11} z \bar{z} \right) \Big|_{z=0},$$

we obtain

$$\alpha_2(0) = \alpha_2(a_0) = \frac{1}{2} \operatorname{Re}(\mathbf{p}_2 \mathbf{g}_{21} \bar{\mu}_2) = \frac{a_0^2 \Gamma_2(a_0, d)(d-\bar{y})^2 - 2(a_0-1)^3}{4(a_0-1)^2 a_0 (d-\bar{y})^2},$$

where

$$\Gamma_2(a_0, d) = (a_0-1)a_0 f^{(3)}(\bar{y})(d-\bar{y}) + f''(\bar{y})(-2a_0^3(d-\bar{y})^2 f''(\bar{y}) + 2a_0^2 - 3a_0 + 1).$$

In summary, by using Lemma 10, we have shown the following theorem; see [29].

Theorem 3 *Let $b > 0$ and $d \geq 0$. Let $a_0 > 1$ be a positive real number such that*

$$f' \left(f^{(-1)} \left(\frac{1}{a_0} \right) \right) = \frac{a_0 - 1}{a_0^2 (d - f^{(-1)}(\frac{1}{a_0}))}.$$

If $d_2(a_0) \neq 0$ and $\alpha_2(a_0) \neq 0$, then system (21) passes through a Neimark–Sacker bifurcation at the unique positive equilibrium point

$$E_2(\bar{x}, \bar{y}) = \left(\frac{a_0(f^{(-1)}(\frac{1}{a_0}) - d)}{(a_0-1)b}, f^{(-1)} \left(\frac{1}{a_0} \right) \right)$$

when the parameter a varies in a small neighborhood of a_0 .

If $d_2(a_0) > 0$ and $\alpha_2(a_0) < 0$ (resp., $\alpha_2(a_0) > 0$), then the Neimark–Sacker bifurcation of model (21) at $a = a_0$ is supercritical (resp., subcritical), and there exists a unique closed invariant curve $\Gamma(a)$, which is attracting (resp., repelling), and bifurcates from E_2 for $a > a_0$ (resp., $a < a_0$).

If $d_2(a_0) < 0$ and $\alpha_2(a_0) < 0$ (resp., $\alpha_2(a_0) > 0$), then the Neimark–Sacker bifurcation of model (21) at $a = a_0$ is supercritical (resp., subcritical), and there exists a unique closed invariant curve $\Gamma(a)$, which is attracting (resp., repelling), and bifurcates from E_2 for $a < a_0$ (resp., $a > a_0$).

Let $a = a_0 + \delta$. Then for small δ , the curve $\Gamma(a)$ can be approximated by

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx (\bar{x}, \bar{y}) + 2\rho_0 \operatorname{Re}(\mathbf{q}_2 e^{i\theta}) + \rho_0^2 (\operatorname{Re}(\mathbf{K}_{20} e^{2i\theta}) + \mathbf{K}_{11}), \quad \theta \in \mathbb{R},$$

where

$$\rho_0 = \sqrt{-\frac{d_2(a_0)}{\alpha_2(a_0)}} \delta.$$

2.4 Examples with numerical simulations

In this subsection, we consider the following examples of a host–parasitoid model function $f(y)$:

- (S): $f(y) = \frac{1}{1+y^m}, m > 0,$
- (HV): $f(y) = e^{-y^m}$ (Hassell and Varley, 1969, [11]),
- (PP): $f(y) = e^{-\frac{\sqrt{1+y}-1}{m}}, m > 0$ (Metz, Vaz Nunez 1977, [20]).

We analyze the interior equilibrium in each of the models with variable and constant stocking.

Note that straightforward calculations show that the functions $f(y) = (1 + \frac{y}{m})^{-m}$ with $m > 0$ (May, 1978, [22]) and $f(y) = \beta_1 e^{-y} + \beta_2 e^{-by}$ with $\beta_1 + \beta_2 = 1, \beta_1, \beta_2, b > 0$ (Hassell, 1984, [10]) also satisfy the conditions $f(0) = 1, f(\infty) = 0,$ and $f'(y) < 0.$ Thus the stability results can also be applied to them.

Here we emphasize that we do not take the numerical values of the parameters from some field studies. We choose them to illustrate the bifurcations and dynamics of the models and our findings.

2.5 Example 1: $f(y) = \frac{1}{1+y^m}$

In this example, we assume that the probability of escaping parasitism $f(y)$ is given by $\frac{1}{1+y^m}$; see [20]. The parameter m can be interpreted as the mutual interference factor between parasitoids if $m < 1$ and as the degree of cooperative hunting if $m > 1.$ There is no cooperation among (parasitoid) predators if $m = 1.$

Now the corresponding (S) model with proportional number of parasitoids released into a parasitoid population is given by

$$\begin{aligned} x_{n+1} &= a \frac{x_n}{1 + y_n^m}, \\ y_{n+1} &= bx_n \left(1 - \frac{1}{1 + y_n^m} \right) + cy_n. \end{aligned} \tag{26}$$

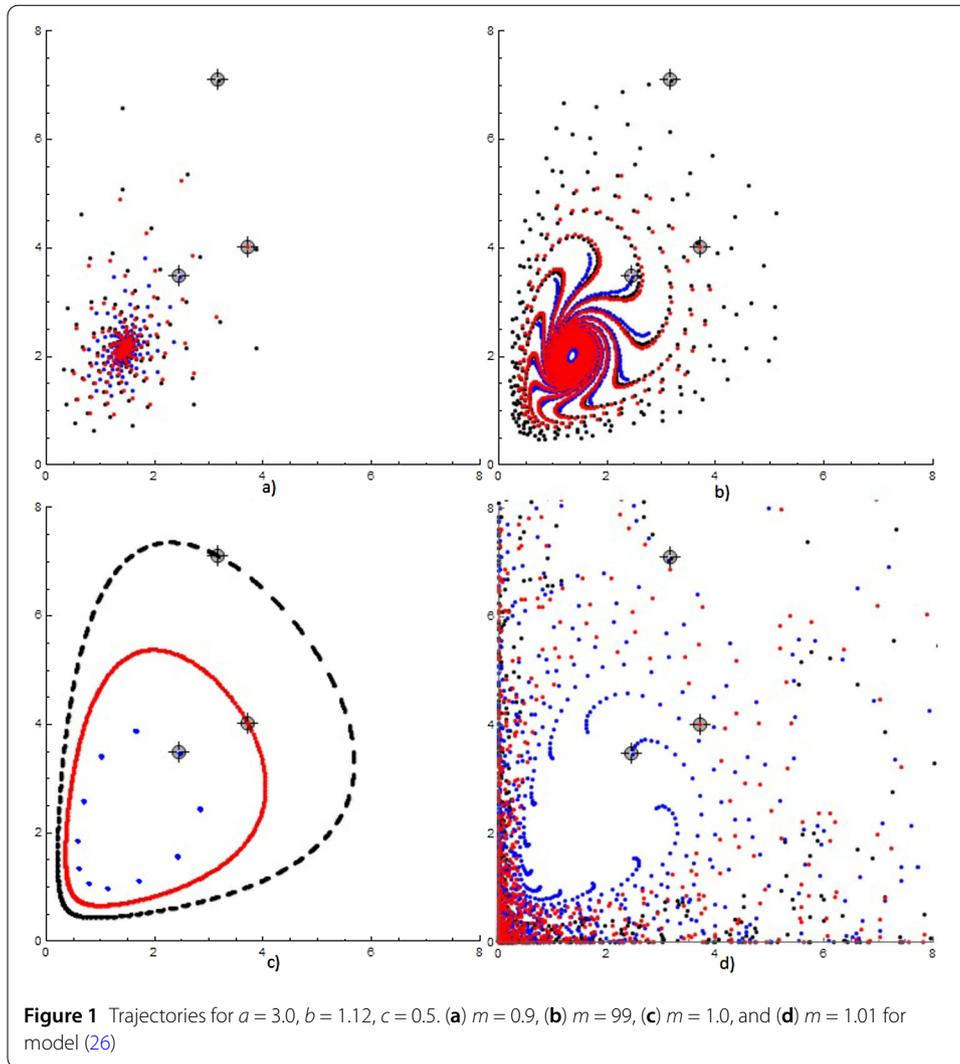
Corollary 1 *Let $m, b > 0.$ If $a > 1$ and $0 < c < 1,$ then (26) has the unique equilibrium point*

$$E_p = \left(\frac{a(1-c)(a-1)^{\frac{1}{m}-1}}{b}, (a-1)^{\frac{1}{m}} \right),$$

and the following statements are true:

- (i) *If $m < 1,$ then E_p is locally asymptotically stable.*
- (ii) *If $m > 1,$ then E_p is a repeller.*
- (iii) *If $m = 1,$ then E_p is nonhyperbolic.*

If $m = 1,$ then for all $a > 1$ and $0 < c < 1,$ we have $\alpha_1 = 0.$ Thus the Neimark–Sacker bifurcation does not occur in this case. We can see that on a logarithmic scale, map (26) is area-preserving. So, it is possible to use KAM theory to investigate the stability property of E_p [15]. Computer simulations suggest that when $m < 1,$ all orbits converge to the interior



equilibrium E_p . On the other hand, when $m > 1$, all orbits tend to infinity. Hence, when the host population intrinsic growth rate a is greater than 1 and the parasitoid attack clamping parameter $m > 1$, computer simulations show that any solution that starts sufficiently far away from the equilibrium goes to infinity. See Fig. 1. This observation suggests that the considered model is not efficient for these parameter values because both populations go to infinity, which is not the usual behavior in nature.

The corresponding (S) model with constant number of parasitoids released into a parasitoid population is

$$\begin{aligned} x_{n+1} &= a \frac{x_n}{1 + y_n^m}, \\ y_{n+1} &= bx_n \left(1 - \frac{1}{1 + y_n^m} \right) + d. \end{aligned} \tag{27}$$

A straightforward calculation gives

$$d_2 = d_2(a_0) = \frac{1}{2} d \left(\frac{m-1}{dm} \right)^{m+1} > 0$$

and

$$\alpha_2 = \alpha_2(a_0) = -\frac{(m-1)^{m+3}((\frac{dm}{m-1})^m + 2)}{4d^2m((dm)^m + (m-1)^m)} < 0,$$

where $a_0 = (\frac{dm}{m-1})^m + 1$. The following corollary follows from the results proved in the previous sections.

Corollary 2 *If $a > 1$, $(a-1)^{\frac{1}{m}} > d$, and $m, b, d > 0$ then (27) has the unique equilibrium point*

$$E_2 = \left(\frac{a((a-1)^{\frac{1}{m}} - d)}{(a-1)b}, (a-1)^{\frac{1}{m}} \right),$$

and the following statements are true:

- (i) *If $m \leq 1$ or $(m > 1 \text{ and } (m-1)(a-1)^{\frac{1}{m}} < dm)$, then E_2 is locally asymptotically stable.*
- (ii) *If $(m-1)(a-1)^{\frac{1}{m}} > dm$, then E_2 is a repeller.*
- (iii) *If $m > 1$ and $(m-1)(a-1)^{\frac{1}{m}} = dm$, then E_2 is nonhyperbolic with complex conjugate eigenvalues of modulus 1. System (27) passes through a supercritical Neimark–Sacker bifurcation at the unique positive equilibrium point E_2 when the parameter a varies in a small neighborhood of $a_0 = (\frac{dm}{m-1})^m + 1$.*

Figure 2 shows the trajectories of System (27) for some numerical values of the parameters.

2.6 Example 2: $f(y) = e^{-y^m}$, $m > 0$

In this example, we assume that the probability of escaping parasitism $f(y)$ is given by e^{-y^m} ; see [20]. Now the corresponding (HV) model with proportional number of parasitoids released into a parasitoid population is

$$\begin{aligned} x_{n+1} &= ax_n e^{-y_n^m}, \\ y_{n+1} &= bx_n(1 - e^{-y_n^m}) + cy_n. \end{aligned} \tag{28}$$

A straightforward calculation gives

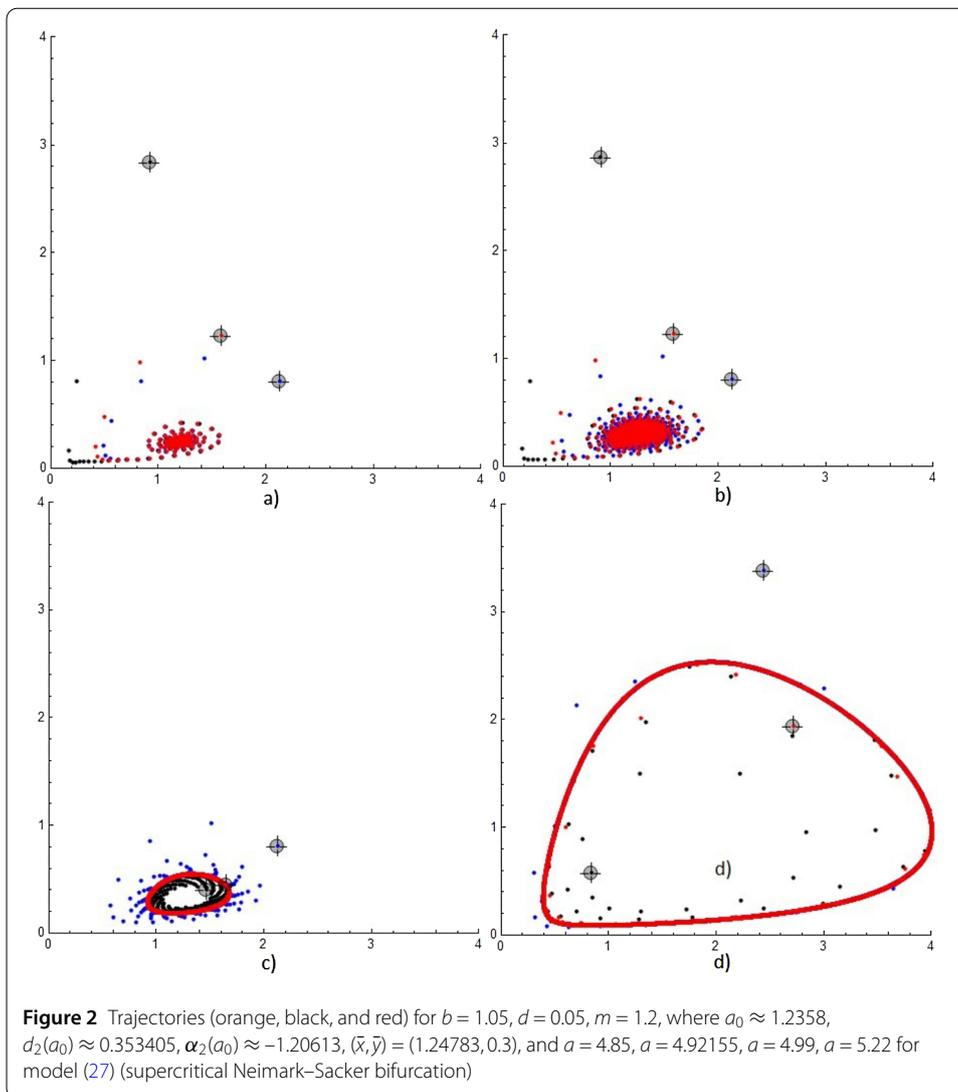
$$d_1 = d_1(a_0) = \frac{(1-c)(a_0 - 1 - \ln(a_0))}{2(a_0 - 1)a_0 \ln(a_0)}$$

and

$$\alpha_1 = \alpha_1(a_0) = -\ln^{-\frac{2}{m}}(a_0) \frac{(1-c)((a_0^2 - 1)c \ln(a_0) + (a_0 - 1)^2(1-c) - a_0(c+1)\ln^2(a_0))}{4a_0^2},$$

where $a_0 m \ln(a_0) = a_0 - 1$.

Let $F(a) = a - 1 - \ln(a)$, $0 < c < 1$, and $a > 1$. Then we have $F'(a) = \frac{a-1}{a} > 0$. Since $F(1) = 0$, we obtain $F(a_0) > F(1) = 0$. Hence $a_0 - 1 - \ln(a_0) > 0$, which implies $d_1(a_0) > 0$. Let $G(c) = (a_0^2 - 1)c \ln(a_0) + (a_0 - 1)^2(1 - c) - a_0(c + 1)\ln^2(a_0)$. We have $G'(c) = (a_0 - \ln(a_0) - 1)(1 - a_0 +$



$a_0 \ln(a_0))$ and $G(0) = (a_0 - 1)^2 - a_0 \ln^2(a_0)$. Let $G_1(a) = 1 - a + a \ln(a)$. From $G'_1(a) = \ln(a) > 0$ and $G_1(1) = 0$ we obtain $G_1(a_0) = 1 - a_0 + a_0 \ln(a_0) > G_1(1) = 0$. Let $G_2(a) = (a - 1)^2 - a \ln^2(a)$. Then $G'_2(a) = 2(a - 1) - \ln(a)(\ln(a) + 2)$ and $G''_2(a) = \frac{2(a - \log(a) - 1)}{a} = \frac{2F(a)}{a} > 0$. This implies $G'_2(a) > G'_1(1) = 0$ for all $a > 1$. Then we obtain $G_2(a_0) = (a_0 - 1)^2 - a_0 \ln^2(a_0) > G_2(1) = 0$. Since $G'(c) = F(a_0)G_1(a_0) > 0$ and $G(0) = G_2(a_0) > 0$, we get $G(c) > G(0) = 0$ for all $0 < c < 1$. This implies $\alpha_1(a_0) < 0$ for $0 < c < 1$ and $a_0 > 1$.

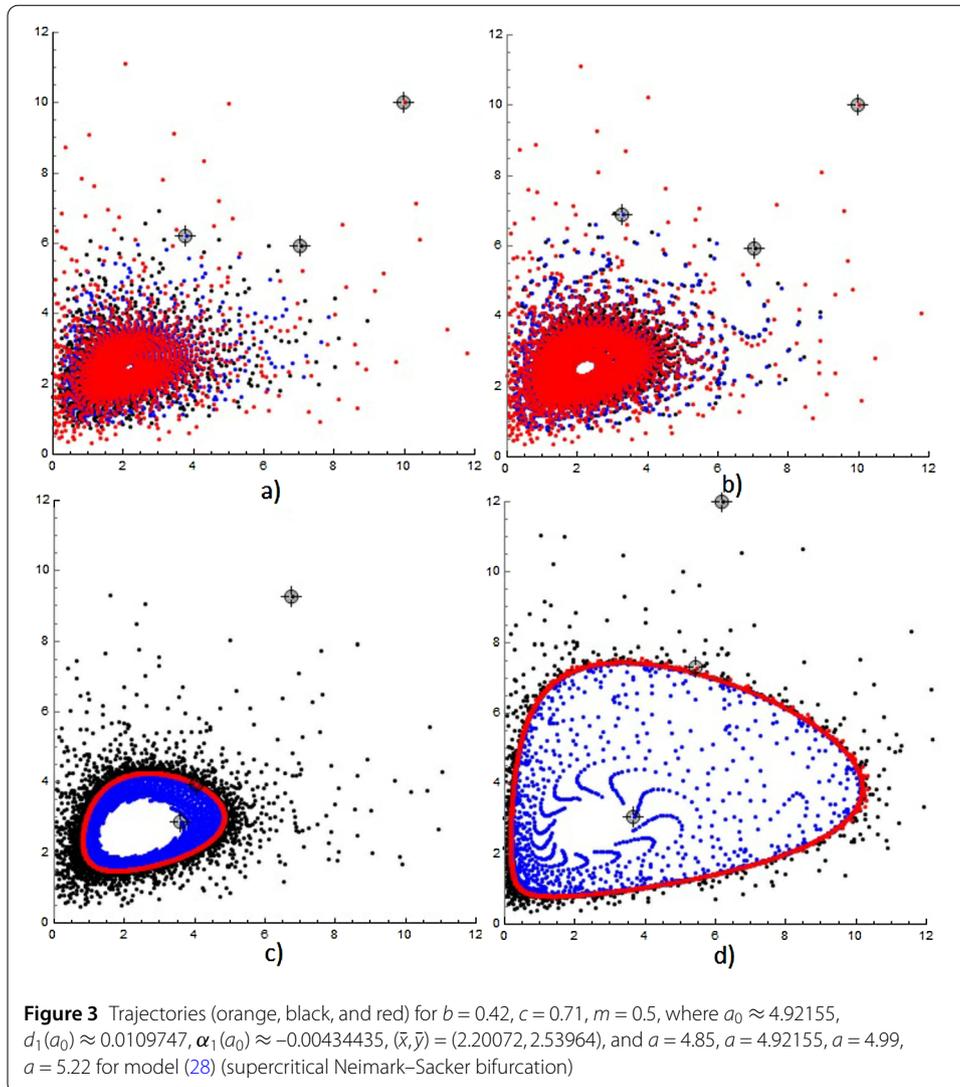
The following corollary follows from the results proved in the previous sections.

Corollary 3 Let $m, b > 0$. If $a > 1$ and $0 < c < 1$, then (28) has the unique equilibrium point

$$E_p = \left(\frac{a(1 - c) \sqrt[m]{\ln(a)}}{(a - 1)b}, \sqrt[m]{\ln(a)} \right),$$

and the following statements are true:

- (i) If $am \ln(a) < a - 1$, then E_p is locally asymptotically stable.
- (ii) If $am \ln(a) > a - 1$, then E_p is a repeller.



(iii) If $am \ln(a) = a - 1$, then E_p is nonhyperbolic with complex conjugate eigenvalues of modulus 1. System (28) passes through a supercritical Neimark–Sacker bifurcation at the unique positive equilibrium point E_p when the parameter a varies in a small neighborhood of a_0 such that $a_0 m \ln(a_0) = a_0 - 1$.

Figure 3 represents the trajectories of System (28) for some parameters' numerical values.

The corresponding (HV) model with constant number of parasitoids released into a parasitoid population is

$$\begin{aligned} x_{n+1} &= ax_n e^{-y_n^m}, \\ y_{n+1} &= bx_n(1 - e^{-y_n^m}) + d. \end{aligned} \tag{29}$$

A straightforward calculation gives

$$d_2 = d_2(a_0) = \frac{m \ln(a_0) + m - 1}{2a_0 m \ln(a_0)}$$

and

$$\alpha_2 = \alpha_2(a_0) = -\frac{m \log^{-\frac{2}{m}}(a_0)(a_0 \ln(a_0)(m \ln(a_0) + m - 1) + (a_0 - 1)(1 - m))}{4(a_0 - 1)},$$

where $a_0 m d = \ln^{\frac{1}{m}-1}(a_0)(a_0 m \ln(a_0) - a_0 + 1)$, $m > 0$, and $a_0 > 1$. Since $d > 0$, we obtain $a_0 m \ln(a_0) - a_0 + 1 > 0$. This implies $m > \frac{a_0 - 1}{a_0 \ln(a_0)}$. From

$$\frac{a_0 - 1}{a_0 \ln(a_0)} - \frac{1}{\ln(a_0) + 1} = \frac{a_0 - \ln(a_0) - 1}{a_0(\ln^2(a_0) + \ln(a_0))} > 0$$

we get $m > \frac{a_0 - 1}{a_0 \ln(a_0)} > \frac{1}{\ln(a_0) + 1}$. We have $m \ln(a_0) + m - 1 > 0$. Hence $d_2(a_0) > 0$. From

$$\frac{a_0 - 1}{a_0 \ln(a_0)} - \frac{a_0 + a_0 \ln(a_0) - 1}{a_0 + a_0 \ln(a_0)(\ln(a_0) + 1) - 1} = \frac{(a_0 - 1)^2 - a_0 \ln^2(a_0)}{a_0 \ln(a_0)(a_0 + a_0 \ln(a_0)(\ln(a_0) + 1) - 1)} > 0,$$

since $G_2(a_0) = (a_0 - 1)^2 - a_0 \ln^2(a_0) > 0$, we obtain

$$m > \frac{a_0 + a_0 \ln(a_0) - 1}{a_0 + a_0 \ln(a_0)(\ln(a_0) + 1) - 1} \Leftrightarrow a_0 \ln(a_0)(m \ln(a_0) + m - 1) + (a_0 - 1)(1 - m) > 0.$$

This implies $\alpha_2(a_0) < 0$. The following corollary follows from the results proved in the previous sections.

Corollary 4 *If $a > 1$, $\sqrt[m]{\ln(a)} > d$, and $m, b, d > 0$ then (29) has the unique equilibrium point*

$$E_2 = \left(\frac{a(\sqrt[m]{\ln(a)} - d)}{(a - 1)b}, \sqrt[m]{\ln(a)} \right),$$

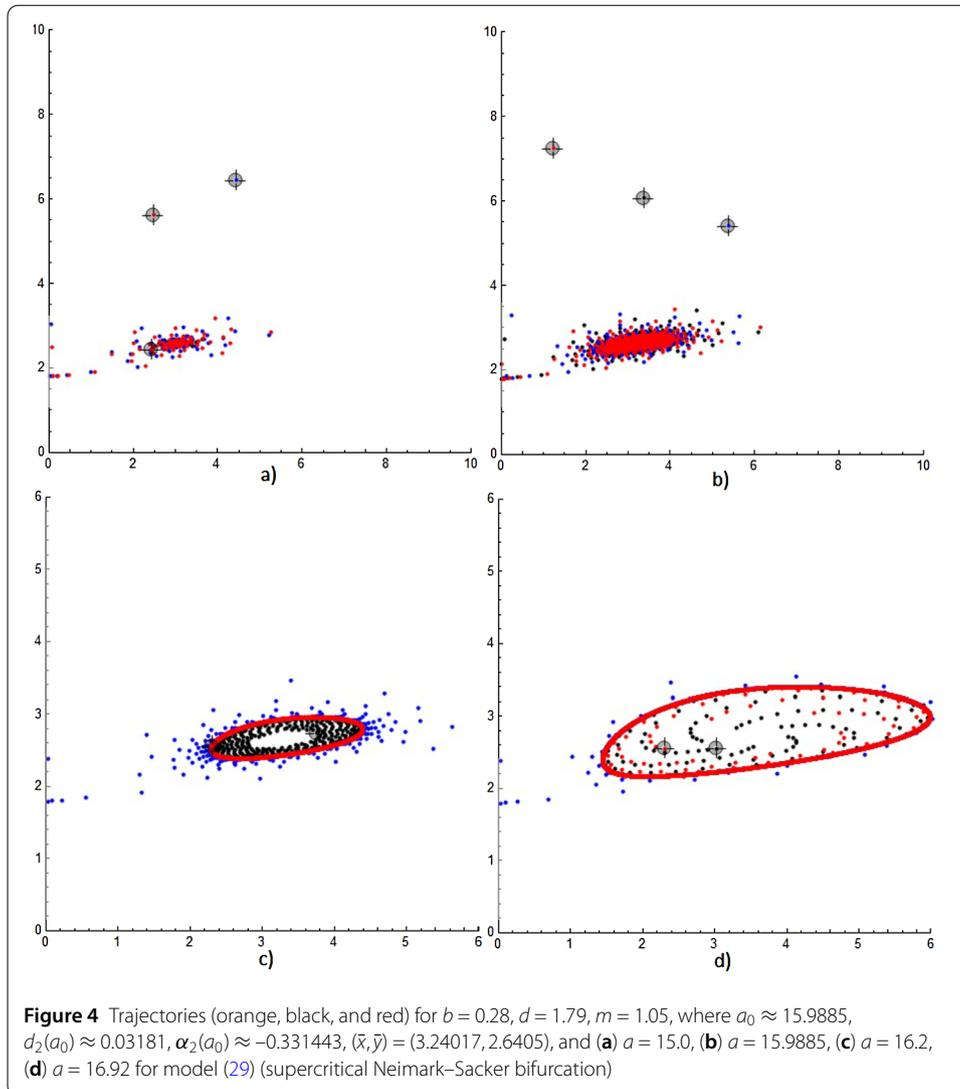
and the following statements are true:

- (i) *If $\sqrt[m]{\ln(a)}(am \ln(a) - a + 1) < adm \ln(a)$, then E_2 is locally asymptotically stable.*
- (ii) *If $\sqrt[m]{\ln(a)}(am \ln(a) - a + 1) > adm \ln(a)$, then E_2 is a repeller.*
- (iii) *If $\sqrt[m]{\ln(a)}(am \ln(a) - a + 1) = adm \ln(a)$, then E_2 is nonhyperbolic with complex conjugate eigenvalues of modulus 1. System (29) passes through a supercritical Neimark–Sacker bifurcation at the unique positive equilibrium point E_2 when the parameter a varies in a small neighborhood of a_0 such that $\sqrt[m]{\ln(a_0)}(a_0 m \ln(a_0) - a_0 + 1) = adm \ln(a_0)$.*

In Fig. 4, we show the trajectories of System (29) with constant releasement of parasitoids.

2.7 Example 3: $f(y) = e^{\frac{1-\sqrt{y+1}}{m}}$, $m > 0$

In this example, we assume that the probability of escaping parasitism $f(y)$ is given by $e^{\frac{1-\sqrt{y+1}}{m}}$; see [20]. Here we emphasize that we will only provide the results for constant stocking. In proportional stocking, using the software package Mathematica, we get very complicated and long expressions, which we skip.



Now the corresponding (PP) model with constant number of parasitoids released into a parasitoid population is

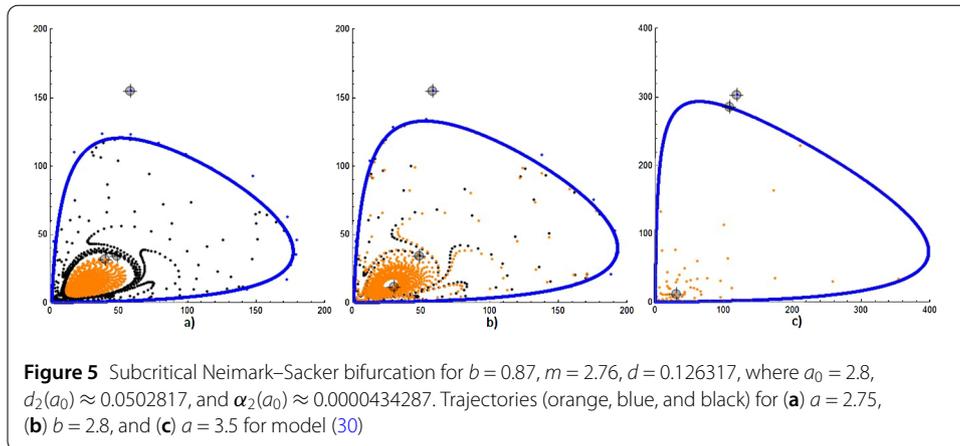
$$\begin{aligned} x_{n+1} &= ax_n e^{\frac{1-\sqrt{y_{n+1}}}{m}}, \\ y_{n+1} &= bx_n \left(1 - e^{\frac{1-\sqrt{y_{n+1}}}{m}}\right) + d. \end{aligned} \tag{30}$$

A straightforward calculation gives

$$d_2 = d_2(a_0) = \frac{m \ln(a_0) - m + 1}{2(a_0 m \ln(a_0) + a_0)}$$

and

$$\alpha_2 = \alpha_2(a_0) = \frac{a_0(m^2 + m - 1) + a_0 m \ln(a_0)(m - m \ln(a_0) - 2) - m^2}{16(a_0 - 1)m^2(m \ln(a_0) + 1)^4},$$



where $m(a_0 m \ln^2(a_0) + 2(a_0 + m - a_0 m) \ln(a_0) + 2) = a_0(d + 2m)$, $m > 0$, and $a_0 > 1$. The following corollary follows from the results proved in the previous sections.

Corollary 5 *If $a > 1$, $m^2 \ln^2(a) + 2m \ln(a) > d$, and $m, b, d > 0$, then (30) has the unique equilibrium point*

$$E_2 = \left(\frac{a(m^2 \ln^2(a) + 2m \ln(a) - d)}{(a - 1)b}, m^2 \ln^2(a) + 2m \ln(a) \right) \in \mathbb{R}_+^2,$$

and the following statements are true:

- (i) *If $m(am \ln^2(a) + 2(a + m - am) \ln(a) + 2) < a(d + 2m)$, then E_2 is locally asymptotically stable.*
- (ii) *If $m(am \ln^2(a) + 2(a + m - am) \ln(a) + 2) > a(d + 2m)$, then E_2 is a repeller.*
- (iii) *If $m(am \ln^2(a) + 2(a + m - am) \ln(a) + 2) = a(d + 2m)$, then E_2 is nonhyperbolic with complex conjugate eigenvalues of modulus 1. System (30) passes through a Neimark–Sacker bifurcation at the unique interior equilibrium point E_2 when the parameter a varies in a small neighborhood of a_0 such that*

$$m(a_0 m \ln^2(a_0) + 2(a_0 + m - a_0 m) \ln(a_0) + 2) = a_0(d + 2m), \quad m > 0, d > 0.$$

Numerical calculations show $d_2(a_0) > 0$. On the other hand, $\alpha_2(a_0)$ changes its sign, which implies the presence of the so-called Chenciner bifurcation. Thus a supercritical and subcritical Neimark–Sacker bifurcation can occur. If $\alpha_2(a_0) > 0$, then by Theorem 1 a repelling closed invariant curve appears surrounding the stable fixed point for $a < a_0$. As a increases, the repelling closed curve decreases in size and merges with the fixed point at $a = a_0$. When $a > a_0$, the repelling closed invariant curve disappears completely, leaving a repelling focus (subcritical Neimark–Sacker bifurcation). In this case the closed repelling curve is generally the boundary of the stable fixed point basin of attraction. Figure 5 shows the typical behavior of the solutions of model (30) if $d_2(a_0) > 0$ and $\alpha_2(a_0) > 0$.

3 Conclusion

Successful biological control means that introduced natural enemies, often parasitoids, can control the pests and suppress them to the level where it can no longer cause economic damage [7, 24, 27, 28]. To address the effect of parasitoid release, we propose the class of

host–parasitoid models with variable and constant release, which is a novelty of this paper. We assume that the host population has a constant rate of increase, but we do not assume any density dependence regulation other than parasitism acting on the host population. The Neimark–Sacker bifurcation at the interior equilibrium of a class two-dimensional discrete host–parasitoid interaction for $a = b$ was analyzed in [20]. We compare the dynamics of these three systems. In contrast to the constant stocking system, systems with no stocking and variable stocking have the extinction equilibrium, which can be globally asymptotically stable (Lemmas 1 and 5). Thus, in these two cases, both populations go to extinction. Neither of these two systems has a boundary equilibrium such that the host population is extinct and the parasitoid population survives. The boundary equilibrium exists in the system with constant stocking, which can be globally asymptotically stable (Theorem 2). Hence adding the constant number of parasitoids into host–parasitoid interaction drives the host population to extinction.

Moreover, the constant stocking eliminates the equilibrium for which the host population survives and the parasitoid is extinct. The constant or variable stocking of the parasitoids can lower the host level in the unique coexisting equilibrium even if the host population intrinsic growth rate is greater than one. The Neimark–Sacker bifurcation appears in each of the investigated systems, and we compute the approximation of the invariant closed curve. The stable closed curve corresponds to the period or quasiperiodic oscillations between the host and parasitoid populations from biology viewpoint. For more detail on this issue, see, for example, [5].

Finally, we apply the results to the several host–parasitoid models with variable and constant stocking, where the function f is one of the well-known functions investigated in [20]. In particular, in a simple (S) model with variable stocking when parasitoid clamping parameter $m = 1$, we show that the Neimark–Sacker bifurcation does not occur. In this case, it is possible to use KAM theory to address the stability of the interior equilibrium point. Using the software package Mathematica, we provide numerical simulations that show that the behavior of examples is similar to that of the considered model. Here we do not take the parameter values from any study field, but we choose them to confirm our theoretical results.

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