Bachar et al. *Advances in Difference Equations* (2021) 2021:22
https://doi.org/10.1186/s13662-020-03176-w

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness of solutions for a class of fractional nonlinear boundary value problems under mild assumptions

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Abstract

We deal with the following Riemann–Liouville fractional nonlinear boundary value problem:

$$\begin{cases} \mathcal{D}^{\alpha} v(x) + f(x, v(x)) = 0, & 2 < \alpha \leq 3, x \in (0,1), \\ v(0) = v'(0) = v(1) = 0. \end{cases}$$

Under mild assumptions, we prove the existence of a unique continuous solution v to this problem satisfying

$$|v(x)| \leq c x^{\alpha-1}(1-x) \quad \text{for all } x \in [0,1] \text{ and some } c > 0.$$

Our results improve those obtained by Zou and He (Appl. Math. Lett. 74:68–73, 2017).

MSC: 34A08; 34B15; 34B27

Keywords: Fractional differential equation; Green's function; Existence and uniqueness of solution; Banach's contraction principle

1 Introduction

Fractional differential equations have attracted great attention due to their ability to model various phenomena in applied sciences. The so-called fractional differential equations are specified by generalizing the standard integer-order derivative to arbitrary order. For more interesting theoretical results and scientific applications of fractional differential equations, we refer to the monographs of Diethelm [2] and Kilbas et al. [3] and references therein.

The existence, uniqueness, and global behavior of solutions for boundary value problems of fractional differential equations have been considered in several recent papers (see, e.g., [1, 4–9] and references therein).

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Zou and He [1] investigated the problem

$$\begin{cases} \mathcal{D}^{\alpha} v(x) + f(x, v(x)) = 0, \quad 2 < \alpha \leq 3, x \in (0,1), \\ v(0) = v'(0) = v(1) = 0, \end{cases} \tag{1.1}$$

where $\mathcal{D}^{\alpha}$ denotes the standard Riemann–Liouville fractional derivative, and f satisfies the following conditions:

(H1) $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 |f(x,0)|\,dx < \infty$;

(H2) There exists $q \in C((0,1), [0,\infty))$ such that

$$\bigl|f(x,v) - f(x,w)\bigr| \leq q(x)|v - w|, \quad \forall x \in (0,1), v, w \in \mathbb{R},$$

and

$$0 < \int_0^1 q(x)\,dx < \infty. \tag{1.2}$$

Let $L > 0$ be the minimum positive constant such that

$$\int_0^1 G_{\alpha}(x,y) q(y) y^{\alpha-1}(1-y)\,dy \leq L x^{\alpha-1}(1-x), \tag{1.3}$$

where $G_{\alpha}(x,y)$ is the Green's function (given later in this paper) associated with problem (1.1). By using Banach's contraction principle on some convenient Banach space they have obtained the following result.

Theorem 1.1 *Under assumptions* (H1)–(H2) *and* $L < 1$, *problem* (1.1) *has a unique solution in* $C([0,1])$.

Motivated by this reault, we prove that the conclusion of Theorem 1.1 remains true under the following weaker assumptions:

(A1) $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 (1-x)^{\alpha-2}|f(x,0)|\,dx < \infty$;

(A2) There exists $q \in C((0,1), [0,\infty))$ such that

$$\bigl|f(x,v) - f(x,w)\bigr| \leq q(x)|v - w|, \quad \forall x \in (0,1), v, w \in \mathbb{R},$$

and

$$0 < M_{q,\alpha} := \frac{1}{\Gamma(\alpha-1)} \int_0^1 x^{\alpha-1}(1-x)^{\alpha-1} q(x)\,dx < \infty. \tag{1.4}$$

Remark 1.2 It is clear that conditions (H1)–(H2) imply (A1)–(A2).

Conversely, for $\beta \in [1, \alpha - 1)$, the function $f(x,v) := (1-x)^{-\beta}(1+v)$ satisfies hypotheses (A1)–(A2) but not conditions (H1)–(H2). So assumptions (A1)–(A2) are weaker.

In this paper, for $\alpha \in [2,3)$, we use the following notations:

• $h(x) := x^{\alpha-1}(1-x)$, $x \in [0,1]$.

• $G_{\alpha}(x,y)$ denotes the Green's function of the operator $v \to -D^{\alpha} v$ with boundary conditions $v(0) = v'(0) = v(1)$.

• $E := \{a > 0 : \int_0^1 G_\alpha(x,y)h(y)q(y)\,dy \leq ah(x), x \in [0,1]\}$ (we will see that $E \neq \emptyset$).

$$\bullet M := \inf E. \tag{1.5}$$

We will prove that M is a positive constant satisfying the following range estimation:

$$M_{q,\alpha+1} \leq M \leq M_{q,\alpha}. \tag{1.6}$$

• For $a \in \mathbb{R}$, $a^+ := \max(a, 0)$.
• $C_h([0,1]) := \{v \in C([0,1]) :$ there is $\sigma > 0$ such that $|v(x)| \leq \sigma h(x), x \in [0,1]\}$.
In the next remark, we list some properties of elements of $C_h([0,1])$.

Remark 1.3
(i) $C_h([0,1])$ is a Banach space equipped with the following h-norm:

$$\|v\|_h := \inf\{\sigma > 0 : |v(x)| \leq \sigma h(x), x \in [0,1]\} = \sup_{x \in (0,1)} \frac{|v(x)|}{h(x)}. \tag{1.7}$$

(ii) $v \in C_h([0,1])$ if and only if $v = h\varphi$, where φ is a bounded continuous function in $(0,1)$.

Our main result is the following:

Theorem 1.4 *Assume that* (A1) *and* (A2) *hold. If* $M < 1$, *then problem* (1.1) *has a unique solution* v *in* $C_h([0,1])$. *In addition, for any* $v_0 \in C_h([0,1])$, *the iterative sequence* $v_k(x) := \int_0^1 G_\alpha(x,y)f(y, v_{k-1}(y))\,dy$ *converges to* v *with respect to the h-norm, and we have*

$$\|v_k - v\|_h \leq \frac{M^k}{1-M}\|v_1 - v_0\|_h. \tag{1.8}$$

Our paper is organized as follows. In Sect. 2, we improve the estimates on Green's function G_α obtained in [1, Lemma 2.2]. This allows us to obtain the range estimation (1.6). Our main result is proved in Sect. 3. Some examples and approximations are given at the end.

2 Preliminaries

Definition 2.1 ([3]) Let $f : (0,\infty) \to \mathbb{R}$ be a measurable function.
(i) The Riemann–Liouville fractional integral of order $\gamma > 0$ for f is defined as

$$I^\gamma f(x) := \frac{1}{\Gamma(\gamma)} \int_0^x (x-y)^{\gamma-1} f(y)\,dy,$$

where Γ is the Euler gamma function.
(ii) The Riemann–Liouville fractional derivative of order $\gamma > 0$ for f is defined as

$$\mathcal{D}^\gamma f(x) := \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n \int_0^x (x-y)^{n-\gamma-1} f(y)\,dy,$$

where $n = [\gamma] + 1$, and $[\gamma]$ is the integer part of γ.

By [10, Lemma 2.2] the Green's function associated with problem (1.1) is given by

$$G_\alpha(x,y)=\frac{1}{\Gamma(\alpha)}\begin{cases}x^{\alpha-1}(1-y)^{\alpha-1}-(x-y)^{\alpha-1} & \text{for } 0\le y\le x\le 1,\\ x^{\alpha-1}(1-y)^{\alpha-1} & \text{for } 0\le x\le y\le 1.\end{cases}\tag{2.1}$$

Lemma 2.2 *The Green's function $G_\alpha(x,y)$ has the following properties*:
(i) *$G_\alpha(x,y)$ is a nonnegative continuous function on* $[0,1]\times[0,1]$.
(ii) *For all $x,y\in[0,1]$, we have*

$$H_\alpha(x,y)\le G_\alpha(x,y)\le(\alpha-1)H_\alpha(x,y),\tag{2.2}$$

where $H_\alpha(x,y):=\frac{1}{\Gamma(\alpha)}x^{\alpha-2}(1-y)^{\alpha-2}\min(x,y)(1-\max(x,y))$.

Proof It is obvious that (i) holds. Now we prove (ii). From (2.1), for all $x,y\in(0,1)$, we have

$$\begin{aligned}\Gamma(\alpha)G_\alpha(x,y)&=x^{\alpha-1}(1-y)^{\alpha-1}-\bigl((x-y)^+\bigr)^{\alpha-1}&(2.3)\\&=x^{\alpha-1}(1-y)^{\alpha-1}\left(1-\left(\frac{(x-y)^+}{x(1-y)}\right)^{\alpha-1}\right).&(2.4)\end{aligned}$$

Since for $\lambda>0$ and $t\in[0,1]$,

$$\min(1,\lambda)(1-t)\le 1-t^\lambda\le\max(1,\lambda)(1-t),$$

we deduce that

$$1-\frac{(x-y)^+}{x(1-y)}\le 1-\left(\frac{(x-y)^+}{x(1-y)}\right)^{\alpha-1}\le(\alpha-1)\left(1-\frac{(x-y)^+}{x(1-y)}\right).$$

Using this fact and (2.4), we obtain

$$x(1-y)-(x-y)^+\le\frac{\Gamma(\alpha)G_\alpha(x,y)}{x^{\alpha-2}(1-y)^{\alpha-2}}\le(\alpha-1)\bigl(x(1-y)-(x-y)^+\bigr).$$

Hence estimates (2.2) follow from

$$x(1-y)-(x-y)^+=\min(x,y)\bigl(1-\max(x,y)\bigr).\qquad\square$$

Remark 2.3 In [1, Lemma 2.2], the authors stated that for all $x,y\in[0,1]$,
(i) $x^{\alpha-1}(1-x)y(1-y)^{\alpha-1}\le\Gamma(\alpha)G_\alpha(x,y)\le(\alpha-1)y(1-y)^{\alpha-1}$,
(ii) $x^{\alpha-1}(1-x)y(1-y)^{\alpha-1}\le\Gamma(\alpha)G_\alpha(x,y)\le(\alpha-1)x^{\alpha-1}(1-x)$.
Note that since for all $x,y\in[0,1]$,

$$xy\le\min(x,y)\quad\text{and}\quad(1-x)(1-y)\le\bigl(1-\max(x,y)\bigr),$$

we get

$$x^{\alpha-1}(1-x)y(1-y)^{\alpha-1}\le\Gamma(\alpha)H_\alpha(x,y)\le\min\bigl(x^{\alpha-1}(1-x),y(1-y)^{\alpha-1}\bigr).$$

Combining this fact with (2.2), we immediately obtain inequalities (i) and (ii).

Therefore estimates (2.2) improve those stated in [1, Lemma 2.2].

Lemma 2.4 *Let* $q \in C((0,1),[0,\infty))$ *and assume that* $0 < M_{q,\alpha} < \infty$. *Then*

$$M_{q,\alpha+1} \le M \le M_{q,\alpha},$$

where M *is the constant defined by* (1.5).

Proof Let

$$E = \left\{ a > 0 : \int_0^1 G_\alpha(x,y)h(y)q(y)\,dy \le ah(x), x \in [0,1] \right\},$$

where $h(x) := x^{\alpha-1}(1-x)$, $x \in [0,1]$.

By (2.2) we obtain

$$\begin{aligned}
&\int_0^1 G_\alpha(x,y)h(y)q(y)\,dy \\
&\quad \le \frac{1}{\Gamma(\alpha-1)} x^{\alpha-2} \int_0^1 y^{\alpha-1}(1-y)^{\alpha-1} \min(x,y)\bigl(1-\max(x,y)\bigr)q(y)\,dy \\
&\quad \le M_{q,\alpha} h(x).
\end{aligned}$$

It follows that $E \neq \emptyset$ and $M \le M_{q,\alpha}$, where $M := \inf E$.

On the other hand, using again (2.2) and that $\min(x,y)(1-\max(x,y)) \ge xy(1-x)(1-y)$ for $x,y \in [0,1]$, we deduce that for any $a \in E$,

$$\begin{aligned}
ah(x) &\ge \frac{1}{\Gamma(\alpha)} x^{\alpha-2} \int_0^1 y^{\alpha-1}(1-y)^{\alpha-1} \min(x,y)\bigl(1-\max(x,y)\bigr)q(y)\,dy \\
&\ge \frac{1}{\Gamma(\alpha)} x^{\alpha-2} \int_0^1 y^{\alpha-1}(1-y)^{\alpha-1} xy(1-x)(1-y)q(y)\,dy \\
&= h(x)M_{q,\alpha+1}.
\end{aligned}$$

Hence for each $a \in E$,

$$a \ge M_{q,\alpha+1}.$$

Therefore $M \ge M_{q,\alpha+1}$, that is, $M \in [M_{q,\alpha+1}, M_{q,\alpha}]$. □

Remark 2.5 From Lemma 2.4 it is obvious that if $M_{q,\alpha} < 1$, then $M := \inf E < 1$. Note that the inequality $M_{q,\alpha} < 1$ can be verified for a large class of functions q, including the singular cases. For example, let $B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ for $a > 0$ and $b > 0$.

Then by using MATLAB we obtain

(i) If $q \in C((0,1))$ with $q > 0$ and $\|q\|_\infty \le 1$, then

$$M_{q,\alpha} \le \frac{B(\alpha,\alpha)}{\Gamma(\alpha-1)} < 1.$$

(ii) If $q(x) := (1-x)^{-\frac{\alpha}{2}}$, then

$$M_{q,\alpha} = \frac{B(\alpha, \frac{\alpha}{2})}{\Gamma(\alpha-1)} < 1.$$

(iii) If $q(x) := x^{-\frac{\alpha}{3}}(1-x)^{-\frac{\alpha}{2}}$, then

$$M_{q,\alpha} = \frac{B(\frac{2\alpha}{3}, \frac{\alpha}{2})}{\Gamma(\alpha-1)} < 1.$$

3 Existence and uniqueness

We need the following useful lemma.

Lemma 3.1 *Let $2 < \alpha < 3$, and let φ be a function such that $x \to (1-x)^{\alpha-1}\varphi(x) \in C((0,1)) \cap L^1((0,1))$. Then the unique continuous solution of the problem*

$$\begin{cases} \mathcal{D}^\alpha v(x) = -\varphi(x), & x \in (0,1), \\ v(0) = v'(0) = v(1) = 0, \end{cases} \tag{3.1}$$

is given by

$$V\varphi(x) := \int_0^1 G_\alpha(x,y)\varphi(y)\,dy.$$

Proof Let φ be a function such that $x \to (1-x)^{\alpha-1}\varphi(x) \in C((0,1)) \cap L^1((0,1))$. Since by Lemma 2.2, $G_\alpha(x,y)$ belongs to $C([0,1]\times[0,1])$ with

$$0 \le G_\alpha(x,y) \le \frac{1}{\Gamma(\alpha-1)}(1-y)^{\alpha-1},$$

we deduce by the dominated convergence theorem that $V\varphi \in C([0,1])$

and $V\varphi(0) = V\varphi(1) = 0$. Therefore $I^{3-\alpha}(V|\varphi|)$ is bounded on $[0,1]$. By Fubini's theorem we obtain

$$\begin{aligned} I^{3-\alpha}(V\varphi)(x) &= \frac{1}{\Gamma(3-\alpha)}\int_0^x (x-y)^{2-\alpha} V\varphi(y)\,dy \\ &= \int_0^1 K(x,r)\varphi(r)\,dr, \end{aligned}$$

where $K(x,r) := \frac{1}{\Gamma(3-\alpha)}\int_0^x (x-y)^{2-\alpha} G_\alpha(y,r)\,dy$.
Simple calculation gives

$$K(x,r) = \frac{1}{2}x^2(1-r)^{\alpha-1} - \frac{1}{2}\big((x-r)^+\big)^2.$$

Hence, for $x \in (0,1)$, we have

$$I^{3-\alpha}(V\varphi)(x) = \frac{x^2}{2}\int_0^1 (1-r)^{\alpha-1}\varphi(r)\,dr - \frac{1}{2}\int_0^x (x-r)^2\varphi(r)\,dr.$$

This implies that

$$\frac{d^3}{dx^3}\big(I^{3-\alpha}(V\varphi)\big)(x) = -\varphi(x).$$

Now, since for each $y \in (0,1)$,

$$\lim_{x\to 0}\frac{G_\alpha(x,y)}{x} = 0 \quad \text{and} \quad 0 \le \frac{G_\alpha(x,y)}{x} \le \frac{1}{\Gamma(\alpha-1)}(1-y)^{\alpha-1},$$

by the dominated convergence theorem we obtain $(V\varphi)'(0) = 0$.

To prove the uniqueness, let $v, w \in C([0,1])$ be two solutions of problem (3.1) and set $\theta := v - w$. Then $\theta \in C([0,1])$, and we have

$$\begin{cases} \mathcal{D}^\alpha \theta(x) = 0, & x \in (0,1), \\ \theta(0) = \theta'(0) = \theta(1) = 0. \end{cases}$$

By [3, Corollary 2.1] there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\theta(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + c_3 x^{\alpha-3}.$$

Applying the boundary conditions, we obtain $c_3 = c_2 = c_1 = 0$, that is, $v = w$. □

Remark 3.2 The conclusion of Lemma 3.1 remains true for $\alpha = 3$.

Proof of Theorem 1.4 Assume that (A1) and (A2) hold and $M < 1$, where M is given by (1.5). Let us prove that problem (1.1) has a unique solution v in $C_h([0,1])$. In addition, for any $v_0 \in C_h([0,1])$, the iterative sequence $v_k(x) := \int_0^1 G_\alpha(x,y) f(y, v_{k-1}(y))\,dy$ converges to v with respect to the h-norm, and we have

$$\|v_k - v\|_h \le \frac{M^k}{1-M}\|v_1 - v_0\|_h.$$

To this end, define the operator T by

$$Tv(x) = \int_0^1 G_\alpha(x,y) f\big(y, v(y)\big)\,dy, x \in [0,1], v \in C_h\big([0,1]\big). \tag{3.2}$$

We claim that T is a contraction operator from $(C_h([0,1]), \|\cdot\|_h)$ into itself.
Let $v \in C_h([0,1])$, and let $\sigma > 0$ be such that $|v(x)| \le \sigma h(x)$ for all $x \in [0,1]$.
Since by Lemma 2.2(ii), $0 \le G_\alpha(x,y) \le \frac{1}{\Gamma(\alpha-1)}(1-y)^{\alpha-2}$, it follows from $(A2)$ that

$$\begin{aligned} \big|G_\alpha(x,y) f\big(y,v(y)\big)\big| &\le \frac{1}{\Gamma(\alpha-1)}(1-y)^{\alpha-2}\big(\big|f\big(y,v(y)\big) - f(y,0)\big| + \big|f(y,0)\big|\big) \\ &\le \frac{1}{\Gamma(\alpha-1)}(1-y)^{\alpha-2}\big(q(y)\big|v(y)\big| + \big|f(y,0)\big|\big) \\ &\le \frac{1}{\Gamma(\alpha-1)}\big(\sigma y^{\alpha-1}(1-y)^{\alpha-1} q(y) + (1-y)^{\alpha-2}\big|f(y,0)\big|\big). \end{aligned}$$

Since $G_\alpha(x,y)$ is continuous on $[0,1]\times[0,1]$, by (A1)–(A2) and the dominated convergence theorem we deduce that $Tv\in C([0,1])$.

Furthermore, from Lemma 2.2(ii) we have

$$0\le G_\alpha(x,y)\le \frac{1}{\Gamma(\alpha-1)}h(x)(1-y)^{\alpha-2}. \tag{3.3}$$

Hence by using (3.3) and similar arguments as before we obtain

$$\left|Tv(x)\right|\le\left[\sigma M_{q,\alpha}+\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-y)^{\alpha-2}\left|f(y,0)\right|dy\right]h(x),$$

and thus $T(C_h([0,1]))\subset C_h([0,1])$.

Next, for any $v,w\in C_h([0,1])$, by using $(A2)$ we obtain that for $x\in[0,1]$,

$$\begin{aligned}\left|Tv(x)-Tw(x)\right|&\le\int_0^1 G_\alpha(x,y)\left|f\big(y,v(y)\big)-f\big(y,w(y)\big)\right|dy\\&\le\int_0^1 G_\alpha(x,y)q(y)\left|v(y)-w(y)\right|dy\\&\le\|v-w\|_h\int_0^1 G_\alpha(x,y)q(y)h(y)\,dy\\&\le M\|v-w\|_h h(x).\end{aligned}$$

Hence

$$\|Tv-Tw\|_h\le M\|v-w\|_h.$$

Since $M<1$, T becomes a contraction operator in $C_h([0,1])$. So there exists a unique $v\in C_h([0,1])$ satisfying

$$v(x)=\int_0^1 G_\alpha(x,y)f\big(y,v(y)\big)\,dy,\quad x\in(0,1). \tag{3.4}$$

It remains to prove that v is a solution of problem (1.1). Indeed, it is clear that $x\to(1-x)^{\alpha-1}f(x,v(x))\in C((0,1))$. Next, by using $(A2)$ and $v\in C_h([0,1])$ we obtain

$$\begin{aligned}\left|(1-x)^{\alpha-1}f\big(x,v(x)\big)\right|&\le(1-x)^{\alpha-1}\big(\left|f\big(x,v(x)\big)-f(x,0)\right|+\left|f(x,0)\right|\big)\\&\le(1-x)^{\alpha-1}\big(q(x)\left|v(x)\right|+\left|f(x,0)\right|\big)\\&\le\sigma x^{\alpha-1}(1-x)^{\alpha-1}q(x)+(1-x)^{\alpha-2}\left|f(x,0)\right|.\end{aligned}$$

Therefore by (A1) and (A2) it follows that $x\to(1-x)^{\alpha-1}f(x,v(x))\in L^1((0,1))$. Hence from Lemma 3.1 we derive that v is a solution of problem (1.1).

Finally, it is well known that for any $v_0\in C_h([0,1])$, the iterative sequence $v_k(x):=\int_0^1 G_\alpha(x,y)f(y,v_{k-1}(y))\,dy$ converges to v, and we have

$$\|v_k-v\|_h\le\frac{M^k}{1-M}\|v_1-v_0\|_h. \qquad\square$$

Example 3.3 Let $2 < \alpha \leq 3$. Consider the problem

$$\begin{cases} \mathcal{D}^{\alpha} v(x) + q(x)\cos v = 0, \quad x \in (0,1), \\ v(0) = v'(0) = v(1) = 0, \end{cases} \tag{3.5}$$

where $q \in C((0,1))$ with $q > 0$ and $\|q\|_{\infty} \leq 1$. Let $f(x,v) := q(x)\cos v$ for $(x,v) \in (0,1) \times \mathbb{R}$. We have $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ and

$$\int_0^1 (1-x)^{\alpha-2} \bigl|f(x,0)\bigr| \, dx \leq \|q\|_{\infty} \int_0^1 (1-x)^{\alpha-2} \, dx < \infty.$$

So assumption (A1) is verified.

On the other hand, since $v \to \cos v$ is a Lipschitz function, we obtain

$$\bigl|f(x,v) - f(x,w)\bigr| \leq q(x)|v-w|, \quad x \in (0,1), v, w \in \mathbb{R}.$$

By Lemma 2.4 and Remark 2.5(i) we have

$$0 < M \leq M_{q,\alpha} \leq \frac{\|q\|_{\infty}}{\Gamma(\alpha-1)} \int_0^1 x^{\alpha-1}(1-x)^{\alpha-1} \, dx < 1.$$

Hence by Theorem 1.4 problem (3.5) has a unique solution $v \in C_h([0,1])$.

Example 3.4 Let $2 < \alpha \leq 3$ and consider the singular problem

$$\begin{cases} \mathcal{D}^{\alpha} v(x) + (1-x)^{-\frac{\alpha}{2}}(1 + \sin v) = 0, \quad x \in (0,1), \\ v(0) = v'(0) = v(1) = 0. \end{cases} \tag{3.6}$$

In this case, we have $f(x,v) = (1-x)^{-\frac{\alpha}{2}}(1+\sin v)$ for $(x,v) \in (0,1) \times \mathbb{R}$.

So $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 (1-x)^{\alpha-2}|f(x,0)| \, dx = \int_0^1 (1-x)^{\frac{\alpha}{2}-2} \, dx < \infty$, that is, assumption (A1) is satisfied.

On the other hand, we clearly have

$$\bigl|f(x,v) - f(x,w)\bigr| \leq q(x)|v-w|, \quad x \in (0,1), v, w \in \mathbb{R},$$

where $q(x) := (1-x)^{-\frac{\alpha}{2}}$.

From Lemma 2.4 and Remark 2.5(ii) we deduce that

$$0 < M \leq M_{q,\alpha} = \frac{1}{\Gamma(\alpha-1)} \int_0^1 x^{\alpha-1}(1-x)^{\frac{\alpha}{2}-1} \, dx < 1.$$

Hence by Theorem 1.4 this problem has a unique solution $v \in C_h([0,1])$. In particular, for $\alpha = \frac{5}{2}$, the unique solution is approximated (see Fig. 1) by the iterative sequence $v_k(x) := \int_0^1 G_{\frac{5}{2}}(x,y)(1-y)^{-\frac{5}{4}}(1+\sin(v_{k-1}(y))) \, dy$ with $v_0(x) = x^{\frac{3}{2}}(1-x)$, $x \in [0,1]$.

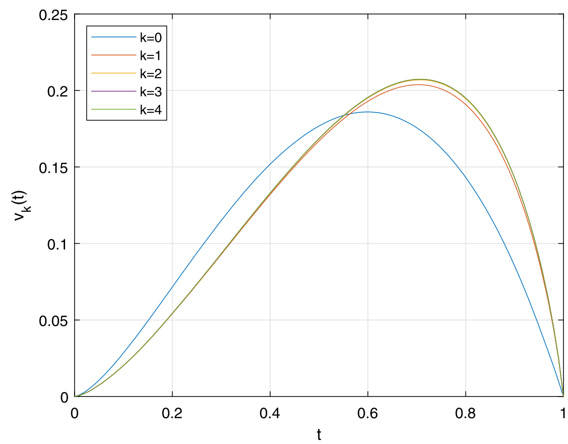

Figure 1 The approximation of the solution of problem (3.6)

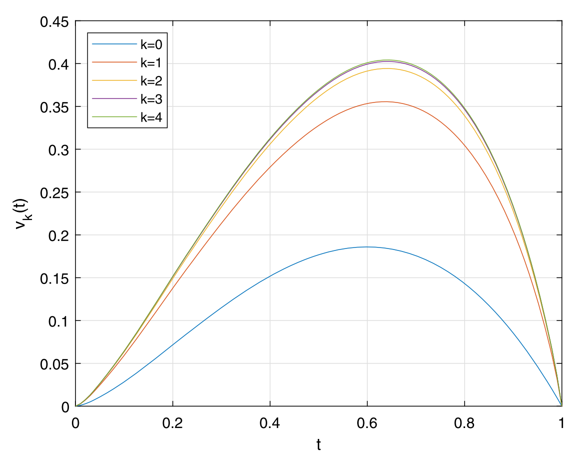

Figure 2 The approximation of the solution of problem (3.7)

Example 3.5 Consider the problem

$$\begin{cases} \mathcal{D}^{\frac{5}{2}} v(x) + x^{-\frac{5}{6}}(1-x)^{-\frac{5}{4}}(1+v) = 0, \quad x \in (0,1), \\ v(0) = v'(0) = v(1) = 0. \end{cases} \tag{3.7}$$

As in Example 3.4, we verify that assumptions (A1) and (A2) are satisfied. Therefore by Theorem 1.4 problem (3.7) has a unique solution $v \in C_h([0,1])$, and the iterative sequence defined by $v_0(x) := x^{\frac{3}{2}}(1-x)$, $x \in [0,1]$, and

$$v_k(x) := \int_0^1 G_{\frac{5}{2}}(x,y) y^{-\frac{5}{6}}(1-y)^{-\frac{5}{4}}\big(1 + v_{k-1}(y)\big)\, dy$$

converges to v (see Fig. 2).

Acknowledgements
The authors express their thanks to the referees for their valuable suggestions and comments, which improved the presentation.

Funding
The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding this Research group NO (RG-1435-043).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 October 2020 Accepted: 13 December 2020 Published online: 07 January 2021

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