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Approximation of functions by a class of Durrmeyer–Stancu type operators which includes Euler's beta function

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Abstract
In this work, we construct the genuine Durrmeyer–Stancu type operators depending on parameter α in [0, 1] as well as $\rho > 0$ and study some useful basic properties of the operators. We also obtain Grüss–Voronovskaja and quantitative Voronovskaja types approximation theorems for the aforesaid operators. Further, we present numerical and geometrical approaches to illustrate the significance of our new operators.

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1 Introduction

Let $L_B[0,1]$ denote the space of bounded Lebesgue integrable functions on $[0,1]$ and $\mathbb{N}$ the set of natural numbers. We use the symbol Π_m ($m \in \mathbb{N}$) to denote the space of polynomials of degree at most m. By taking Bernstein polynomials into account, Chen [14] and Goodman and Sharma [21] independently introduced the operators U_m (we can also call them genuine Bernstein–Durrmeyer operators) acting from $L_B[0,1]$ into Π_m, defined by

$$U_m(f,y) = (m-1)\sum_{i=1}^{m-1}\left(\int_0^1 f(t)p_{m-2,i-1}(t)\,dt\right)p_{m,i}(y) + y^m f(1) + (1-y)^m f(0)$$

for all $f \in L_B[0,1]$, where $p_{m,i}(y)$ ($m,i \in \mathbb{N}$) is considered by

$$p_{m,i}(y) = \binom{m}{i} y^i (1-y)^{m-i} \quad (0 \le y \le 1, 0 \le i \le m).$$

The above operators are limits of the Bernstein–Durrmeyer operators with Jacobi weights, $M_m^{c,d}$ for $c,d > -1$, which was studied by Păltănea [40], that is,

$$U_m(f) = \lim_{c\to -1, d\to -1} M_m^{c,d}(f) \quad \bigl(f \in C[0,1]\bigr),$$

where $C[0,1]$ denotes the space of functions which are continuous on $[0,1]$ and

$$M_m^{c,d}(f,y)=\sum_{i=0}^{m}\frac{\int_0^1 f(t)t^c(1-t)^d p_{m,i}(t)\,dt}{\int_0^1 t^c(1-t)^d p_{m,i}(t)\,dt}p_{m,i}(y).$$

Păltănea [41] presented a generalization of the operators U_m with the help of $\rho>0$, namely genuine ρ-Bernstein–Durrmeyer operators, and denoted them by U_m^ρ. For any $f\in C[0,1]$, in the same paper, he showed that the classical Bernstein operators are the limits of the operators U_m^ρ and also obtained a Voronovskaja-type result. Gonska and Păltănea [17] proved that the operators U_m^ρ preserve convexity of all orders and also obtained the degree of simultaneous approximation.

It is well known that Bernstein polynomials are one of the most widely-investigated polynomials in the theory of approximation, and so, to obtain another generalization of classical Bernstein operators, Cai et al. [13] considered the Bézier bases with shape parameter λ in $[-1,1]$ and introduced λ-Bernstein operators. Later, Kantorovich, Schurer, and Stancu variants of λ-Bernstein operators were discussed by Cai [11], Özger [36–38], and Srivastava et al. [43]. By taking λ-Bernstein polynomials into account, in a very recent past, Acu et al. [4] defined a new family of modified U_m^ρ operators and denoted the new operators by $U_{m,\lambda}^\rho$.

Chen et al. [15] recently presented a generalization of classical Bernstein operators with the help of any fixed α in $\mathbb{R}$, which they called α-Bernstein operators (linear and positive for $\alpha\in[0,1]$), and discussed the rate of convergence, Voronovskaja-type formula, and shape preserving properties of these positive linear operators. Mohiuddine et al. [26] constructed the Kantorovich variant of α-Bernstein operators. The bivariate version of α-Bernstein–Durrmeyer operators was constructed and studied by Kajla and Miclăuş [23] (also see [25] for recent work), in which they also discussed GBS operator (or generalized boolean sum operators) of α-Bernstein–Durrmeyer, while the two interesting forms of α-Baskakov–Durrmeyer were introduced by Kajla et al. [24] and Mohiuddine et al. [31]. For the classical Bernstein–Durrmeyer operators, we refer the interested reader to [16]. We also refer to [2, 3, 7, 8, 10, 12, 18, 19, 22, 27–30, 32–35, 39, 42, 45, 46] for some recent work on various Bernstein, Durrmeyer, and genuine type operators as well as statistical approximation.

We will now recall the α-Bernstein operators due to Chen et al. [15] as follows: For $g\in C[0,1]$, $\alpha\in[0,1]$ is fixed, and $m\in\mathbb{N}$, the α-Bernstein operators are defined by

$$T_{m,\alpha}(g;y)=\sum_{i=0}^{m}g(i/m)p_{m,i}^{(\alpha)}(y)\quad\bigl(y\in[0,1]\bigr),\tag{1.1}$$

where

$$p_{1,0}^{(\alpha)}(y)=1-y,\qquad p_{1,1}^{(\alpha)}(y)=y$$

and

$$\begin{aligned}p_{m,i}^{(\alpha)}(y)={}&\left[(1-\alpha)y\binom{m-2}{i}+(1-\alpha)(1-y)\binom{m-2}{i-2}\right.\\&\left.+\alpha y(1-y)\binom{m}{i}\right]y^{i-1}(1-y)^{m-i-1}\quad(m\geq 2).\end{aligned}$$

Note that $p_{m,i}^{(\alpha)}$ in relation (1.1) is called α-Bernstein polynomials of order m and the binomial coefficients

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & (0 \leq b \leq a), \\ \bar{0} & (\text{otherwise}). \end{cases}$$

For $\alpha = 1$, (1.1) is reduced to the classical Bernstein operators [9].

2 Generalized U_m^ρ operators and approximation properties

For $m \in \mathbb{N}$ and $\rho > 0$, the functional (see [41])

$$F_{m,i}^{\rho} : C[0,1] \to \mathbb{R}$$

is defined by

$$\begin{aligned} F_{m,i}^{\rho}(g) &= \int_0^1 \mu_{m,i}^{\rho}(t) g(t)\,dt \quad (i = 1, 2, \dots, m-1), \\ F_{m,0}^{\rho}(g) &= g(0), \qquad F_{m,m}^{\rho}(g) = g(1), \end{aligned} \tag{2.1}$$

where $\mu_{m,i}^{\rho}(t)$ in (2.1) is given by the formula

$$\mu_{m,i}^{\rho}(t) = \frac{t^{i\rho-1}(1-t)^{(m-i)\rho-1}}{B(i\rho, (m-i)\rho)}$$

and Euler's beta function in the last equality is defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \quad (a, b > 0).$$

Assume that θ and β are two real parameters satisfying $0 \leq \theta \leq \beta$. In view of α-Bernstein operators, for $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$ is fixed, and given a function $g \in C[0,1]$, we define the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ (or genuine (α,ρ)-Durrmeyer–Stancu operators) by

$$U_{m,\alpha}^{\beta,\theta,\rho}(g;y) = \sum_{i=0}^{m} F_{m,i}^{\beta,\theta,\rho}(g) p_{m,i}^{(\alpha)}(y), \tag{2.2}$$

where

$$F_{m,i}^{\beta,\theta,\rho}(g) = \int_0^1 \mu_{m,i}^{\rho}(t) g\left(\frac{mt+\theta}{m+\beta}\right) dt$$

for $i = 1, 2, \dots, m-1$, $F_{m,0}^{\beta,\theta,\rho}(g) = g(\frac{\theta}{m+\beta})$ and $F_{m,1}^{\beta,\theta,\rho}(g) = g(\frac{m+\theta}{m+\beta})$. Consequently, we can rewrite our operators $U_{m,\alpha}^{\beta,\theta,\rho}$ as follows:

$$\begin{aligned} U_{m,\alpha}^{\beta,\theta,\rho}(g;y) &= \sum_{i=1}^{m-1} \int_0^1 \left[\frac{t^{i\rho-1}(1-t)^{(m-i)\rho-1}}{B(i\rho,(m-i)\rho)} g\left(\frac{mt+\theta}{m+\beta}\right) dt \right] p_{m,i}^{(\alpha)}(y) \\ &\quad + g\left(\frac{\theta}{m+\beta}\right) p_{m,0}^{(\alpha)}(y) + g\left(\frac{m+\theta}{m+\beta}\right) p_{m,m}^{(\alpha)}(y). \end{aligned} \tag{2.3}$$

For the choice of $\theta = 0$ and $\beta = 0$, the operators defined by (2.3) reduce to the operators $U^{\rho}_{m,\alpha}(g;y)$ which were studied in [6]. In addition, if $\rho = 1$, then we get the genuine α-Bernstein–Durrmeyer operators $U_{m,\alpha}$ defined in [1]. If we take $\rho = 1$, $\alpha = 1$, $\theta = 0$, and $\beta = 0$, then we obtain genuine Bernstein–Durrmeyer operators. Throughout the paper, we assume that $\alpha \in [0,1]$ for which our new operators $U^{\beta,\theta,\rho}_{m,\alpha}$ are linear and positive. For interested readers who want to see the details of Stancu operators, we refer to [44].

The moments of our newly constructed operators $U^{\beta,\theta,\rho}_{m,\alpha}$ are given in the following lemma.

Lemma 1 *Let $e_i(y) = y^i$, $(i = 0,1,2,3,4)$. Then the operators $U^{\beta,\theta,\rho}_{m,\alpha}$ satisfy*

$$\begin{aligned}
U^{\beta,\theta,\rho}_{m,\alpha}(e_0;y) &= 1,\\
U^{\beta,\theta,\rho}_{m,\alpha}(e_1;y) &= \frac{my+\theta}{m+\beta},\\
U^{\beta,\theta,\rho}_{m,\alpha}(e_2;y) &= \frac{m^2y^2+2m\theta y+\theta^2}{(m+\beta)^2} + \frac{(y-y^2)m(m(1+\rho)+2\rho(1-\alpha))}{(m+\beta)^2(m\rho+1)},\\
U^{\beta,\theta,\rho}_{m,\alpha}(e_3;y) &= \frac{m^3y^3+3m\theta y(my+\theta)+\theta^3}{(m+\beta)^3} + \frac{3m(y^2-y)\theta(2\rho m^2+2m\rho(1-\alpha))}{(m+\beta)^3(m\rho+1)}\\
&\quad + \frac{6(y^2-y)\rho(1-\alpha)(1+\rho-2\rho y)}{(m+\beta)^3(m\rho+1)(m\rho+2)m} + \frac{(y^2-y)}{(m+\beta)^3(m\rho+1)(m\rho+2)}\bigl\{2y\\
&\quad - 2\rho + 3m\rho y(\rho+1) + 4\rho^2 y + \rho^2 + 3 - 6\rho^2\alpha y\bigr\},\\
U^{\beta,\theta,\rho}_{m,\alpha}(e_4;y) &= \frac{m^4y^4+4m\theta y(m^2y^2+\theta^2)+\theta^2(6m^2y^2+\theta^2)}{(m+\beta)^4}\\
&\quad + \frac{y-y^2}{(m+\beta)^4(m\rho+1)(m\rho+2)(m\rho+3)}\bigl\{6\rho^2(\rho+1)y^2m^6\\
&\quad - \rho y\bigl(12\alpha\rho^2 - y - \rho^2 y - 7\rho^2 - 18\rho - 11y - 11\bigr)m^5 + \bigl(60\alpha\rho^3y^2 - 36\alpha\rho^3 y\\
&\quad - 54\rho^3y^2 + 6\rho^2(4y-6\alpha+1) + y + 30\rho^3 y + \rho^3 + 6y^2 + 11\rho + 6y + 6\bigr)m^4\\
&\quad + 2\rho(1-\alpha)\bigl(36\rho^2\bigl(y^2-y\bigr) + 7\rho^2 - 36\rho y + 18\rho + 11\bigr)m^3\bigr\}\\
&\quad + \frac{y-y^2}{(m+\beta)^4(m\rho+1)(m\rho+2)}\bigl\{12\theta\rho(\rho+1)ym^4 + 4\theta\bigl(\rho^2(4y+1) + 3\rho\\
&\quad + 2y + 2 - 6\alpha\rho^2\bigr)m^3 + 24\theta\rho(1-\alpha)(1+\rho-2\rho y)m^2\bigr\}\\
&\quad + \frac{(y-y^2)(6\theta^2(\rho+1)m^2+12\theta^2\rho(1-\alpha)m)}{(m+\beta)^4(m\rho+1)}.
\end{aligned}$$

Proof We give a short proof for the first three parts, one can prove the rest using the same idea.

$$\begin{aligned}
U^{\beta,\theta,\rho}_{m,\alpha}(e_0;y) &= \sum_{i=0}^{m} \frac{p^{(\alpha)}_{m,i}(y)}{B(i\rho,(m-i)\rho)} \int_0^1 t^{i\rho-1}(1-t)^{(m-i)\rho-1}\,dt\\
&= \sum_{i=0}^{m} p^{(\alpha)}_{m,i}(y) = 1.
\end{aligned}$$

$$\begin{aligned}
U_{m,\alpha}^{\beta,\theta,\rho}(e_1;y) &= \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \int_0^1 \frac{mt+\theta}{m+\beta} \mu_{m,i}^{\rho}(t)\,dt \\
&= \frac{m}{m+\beta} \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \frac{\Gamma(m\rho)}{\Gamma(i\rho)\Gamma((m-i)\rho)} \frac{\Gamma(i\rho+1)\Gamma((m-i)\rho)}{\Gamma(m\rho+1)} \\
&\quad + \frac{\theta}{m+\beta} \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \int_0^1 \mu_{m,i}^{\rho}(t)\,dt \\
&= \frac{my+\theta}{m+\beta}.
\end{aligned}$$

Using the properties of Euler beta function, we have

$$\begin{aligned}
U_{m,\alpha}^{\beta,\theta,\rho}(e_2;y) &= \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \int_0^1 \left(\frac{mt+\theta}{m+\beta}\right)^2 \mu_{m,i}^{\rho}(t)\,dt \\
&= \frac{m^2}{(m+\beta)^2} \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \int_0^1 t^2 \mu_{m,i}^{\rho}(t)\,dt + \frac{2m\theta}{(m+\beta)^2} \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \int_0^1 t\mu_{m,i}^{\rho}(t)\,dt \\
&\quad + \frac{\theta^2}{(m+\beta)^2} \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) \int_0^1 \mu_{m,i}^{\rho}(t)\,dt \\
&= \frac{m^2y^2+2m\theta y+\theta^2}{(m+\beta)^2} + \frac{(y-y^2)m(m(1+\rho)+2\rho(1-\alpha))}{(m+\beta)^2(m\rho+1)}.
\end{aligned}$$

□

Corollary 1 *The central moments of* (2.3) *are as follows*:

$$\begin{aligned}
&U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y) = \frac{\theta-\beta y}{m+\beta}, \\
&U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big) \\
&\quad = \frac{1}{(m+\beta)^2(m\rho+1)} \big\{m^2\big(y-y^2\big)(\rho+1) \\
&\qquad + m\big(2(\rho-\alpha)\big(y-y^2\big) + \rho\beta\theta y(\beta y-2\theta) + \rho\theta^2\big) + \theta^2 + \beta^2y^2 - 2\beta\theta y\big\}.
\end{aligned}$$

Theorem 1 *If g is continuous on* $[0,1]$ *for any* $\alpha \in [0,1]$, *then* $U_{m,\alpha}^{\beta,\theta,\rho}(g)$ *converge uniformly to g on* $[0,1]$, *that is*,

$$\lim_{m\to\infty} \big\| U_{m,\alpha}^{\beta,\theta,\rho}(g) - g \big\| = 0.$$

Proof We obtain by Lemma 1 that

$$\lim_{m\to\infty} U_{m,\alpha}^{\beta,\theta,\rho}(e_0) = e_0, \qquad \lim_{m\to\infty} U_{m,\alpha}^{\beta,\theta,\rho}(e_1;y) = e_1$$

and similarly $\lim_{m\to\infty} \|U_{m,\alpha}^{\beta,\theta,\rho}(e_2) - e_2\| = 0$. Consequently, the Korovkin theorem gives

$$\lim_{m\to\infty} \big\| U_{m,\alpha}^{\beta,\theta,\rho}(g) - g \big\| = 0.$$

□

Lemma 2 *Let* $g \in C[0,1]$, *and let* $\|\cdot\|$ *be a uniform norm on* $[0,1]$. *Then*

$$\big\| U_{m,\alpha}^{\beta,\theta,\rho}(g) \big\| \le \|g\| \quad (m \in \mathbb{N}).$$

Proof With a view of last lemma, we have $|U_{m,\alpha}^{\beta,\theta,\rho}(g;y)| \leq U_{m,\alpha}^{\beta,\theta,\rho}(e_0;y)\|g\| = \|g\|$. □

Recall that the usual modulus of continuity for g is defined by

$$\omega(g;\sqrt{\varepsilon}) = \sup_{0<\lambda\leq\varepsilon} \sup_{y,y+\lambda\in[0,1]} \bigl|g(y+\lambda) - g(y)\bigr|.$$

Theorem 2 *Assume that* $g \in C[0,1]$ *and* $\alpha \in [0,1]$. *Then*

$$\bigl|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y)\bigr| \leq 2\omega\Bigl(g;\sqrt{\tau_{m,\alpha}^{\beta,\theta,\rho}}\Bigr) \quad \bigl(y \in [0,1]\bigr),$$

where $\tau_{m,\alpha}^{\beta,\theta,\rho} = U_{m,\alpha}^{\beta,\theta,\rho}((e_1-y)^2;y)$.

Proof From the monotonicity of the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ and taking Lemma 1 into our account, we write

$$\bigl|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y)\bigr| = \bigl|U_{m,\alpha}^{\beta,\theta,\rho}\bigl(g(t)-g(y);y\bigr)\bigr| \leq U_{m,\alpha}^{\beta,\theta,\rho}\bigl(\bigl|g(t)-g(y)\bigr|;y\bigr).$$

Since

$$\bigl|g(t)-g(y)\bigr| \leq \biggl(1 + \biggl(\frac{t-y}{\varepsilon}\biggr)^2\biggr)\omega(g;\varepsilon) \quad \bigl(y,t \in [0,1], \varepsilon > 0\bigr),$$

we fairly obtain

$$\bigl|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y)\bigr| \leq \biggl(1 + \frac{U_{m,\alpha}^{\beta,\theta,\rho}((e_1-y)^2;y)}{\varepsilon^2}\biggr)\omega(g;\varepsilon).$$

Here, the assertion of Theorem 2 is acquired by taking into account $\varepsilon = \sqrt{U_{m,\alpha}^{\beta,\theta,\rho}((e_1-y)^2;y)}$. □

Theorem 3 *Let* $g \in C^1[0,1]$. *For any* $y \in [0,1]$, *the following inequality holds*:

$$\bigl|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y)\bigr| \leq 2\sqrt{\tau_{m,\alpha}^{\beta,\theta,\rho}}\, w\Bigl(g',\sqrt{\tau_{m,\alpha}^{\beta,\theta,\rho}}\Bigr) + \bigl|g'(y)\bigr|\bigl|\nu_m^{\beta,\theta}\bigr|, \tag{2.4}$$

where $\nu_m^{\beta,\theta} = U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y)$ *and* $\tau_{m,\alpha}^{\beta,\theta,\rho} = U_{m,\alpha}^{\beta,\theta,\rho}((e_1-y)^2;y)$.

Proof One writes

$$g(t) - g(y) = (t-y)g'(y) + \int_y^t \bigl(g'(u) - g'(y)\bigr)\,du$$

for any $t \in [0,1]$ and $y \in [0,1]$. Operating $U_{m,\alpha}^{\beta,\theta,\rho}(g;y)$ on both sides of the above relation, we obtain

$$U_{m,\alpha}^{\beta,\theta,\rho}\bigl(g(t)-g(y);y\bigr) = g'(y)U_{m,\alpha}^{\beta,\theta,\rho}(t-y;y) + U_{m,\alpha}^{\beta,\theta,\rho}\biggl(\int_y^t \bigl(g'(u)-g'(y)\bigr)\,du;y\biggr).$$

We know that

$$\left|g(u)-g(y)\right| \leq w(g,\varepsilon)\left(\frac{|u-y|}{\varepsilon}+1\right) \quad \left(g \in C[0,1]\right) \tag{2.5}$$

for any $\varepsilon > 0$ and each $u \in [0,1]$. By taking (2.5) into our consideration, we obtain

$$\left|\int_y^t \left(g'(u)-g'(y)\right) du\right| \leq w\left(g',\varepsilon\right)\left(\frac{(t-y)^2}{\varepsilon}+|t-y|\right).$$

Thus,

$$\begin{aligned}\left|U_{m,\alpha}^{\beta,\theta,\rho}(g;y)-g(y)\right| &\leq \left|g'(y)\right|\left|U_{m,\alpha}^{\beta,\theta,\rho}(t-y;y)\right| \\ &\quad + w\left(g',\varepsilon\right)\left\{\frac{1}{\varepsilon}U_{m,\alpha}^{\beta,\theta,\rho}\left((t-y)^2;y\right)+U_{m,\alpha}^{\beta,\theta,\rho}(t-y;y)\right\}.\end{aligned}$$

Consequently, (2.4) follows by choosing $\varepsilon = U_{m,\alpha}^{\beta,\theta,\rho}((t-y)^2;y) = \sqrt{\tau_{m,\alpha}^{\beta,\theta,\rho}}$, which proves our result. □

3 Voronovskaja-type theorems

We obtain some Voronovskaja-type theorems including a Grüss–Voronovskaja-type theorem and a quantitative Voronovskaja-type theorem for $U_{m,\alpha}^{\beta,\theta,\rho}$. We first obtain a quantitative Voronovskaja-type theorem for our operators $U_{m,\alpha}^{\beta,\theta,\rho}$ using the Ditzian–Totik modulus of smoothness. To do this, we need the following definitions.

We first recall the Ditzian–Totik modulus of smoothness defined as follows:

$$\omega_\phi(g,\delta) := \sup_{0<|\lambda|\leq\delta}\left\{\left|g\left(y+\frac{\lambda\phi(y)}{2}\right)-g\left(y-\frac{\lambda\phi(y)}{2}\right)\right|, y \pm \frac{\lambda\phi(y)}{2} \in [0,1]\right\},$$

where $g \in C[0,1]$ and $\phi(y) = \sqrt{y(1-y)}$. The corresponding Peetre's K-functional is defined by

$$K_\phi(g,\delta) = \inf_{h\in W_\phi[0,1]}\left\{\|g-h\| + \delta\left\|\phi h'\right\| : h \in C^1[0,1], \delta > 0\right\},$$

where

$$W_\phi[0,1] = \left\{h : h \in AC_{loc}[0,1], \left\|\phi h'\right\| < \infty\right\},$$

and $AC_{loc}[0,1]$ in the last equality denotes the class of all absolutely continuous functions defined on the closed interval $[a,b] \subset [0,1]$. Then ∃ a constant $M > 0$ such that

$$K_\phi(g,\delta) \leq M\omega_\phi(g,\delta).$$

Theorem 4 *Suppose that $g, g', g'' \in C[0,1]$ and $y \in [0,1]$. Suppose also that ρ is a positive number. Then we have*

$$\left|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g''(y)\chi_{m,\beta,\theta}^{\rho,\alpha} - g(y)\right| \leq \frac{M}{m}\phi^2(y)\omega_\phi\left(g'', \frac{1}{m^{1/2}}\right)$$

for sufficiently large m, where

$$\chi_{m,\beta,\theta}^{\rho,\alpha} = \frac{m^2y^2 + 2m\theta y + \theta^2}{2(m+\beta)^2} + \frac{(y-y^2)m(m(1+\rho)+2\rho(1-\alpha))}{2(m+\beta)^2(m\rho+1)}.$$

Proof The following equality

$$g(t) - g(y) - (t-y)g'(y) = \int_y^t (t-u)g''(u)\,du$$

is satisfied for $g \in C[0,1]$. This equality implies

$$g(t) - g(y) - (t-y)g'(y) - \frac{g''(y)}{2}(t-y)^2 \le \int_y^t (t-u)\big[g''(u) - g''(y)\big]\,du. \tag{3.1}$$

If we apply the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ to each side of (3.1), we get

$$\begin{aligned}&\left|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y) - U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y);y\big)g'(y) - \frac{g''(y)}{2}U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2;y\big)\right| \\ &\quad\le U_{m,\alpha}^{\beta,\theta,\rho}\left(\left|\int_y^t |t-u|\big|g''(u)-g''(y)\big|\,du\right|;y\right).\end{aligned} \tag{3.2}$$

Let us estimate the right-hand side of (3.2) as follows:

$$\left|\int_y^t |t-u|\big|g''(u)-g''(y)\big|\,du\right| \le 2\|g''-g\|(t-y)^2 + 2\|\phi g'\|\phi^{-1}(y)|t-y|^3$$

for $g \in W_\phi[0,1]$. Then there is a constant $M > 0$ such that

$$\left.\begin{array}{l} U_{m,\alpha}^{\beta,\theta,\rho}((t-y)^2;y) \le \frac{(\rho+1)My^2(1-y)^2}{\rho m} \quad \text{and} \\ U_{m,\alpha}^{\beta,\theta,\rho}((t-y)^4;y) \le \frac{(\rho+1)^2My^4(1-y)^4}{\rho^2 m^2} \end{array}\right\} \tag{3.3}$$

hold for sufficiently large m. Using the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}&\left|U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y) - U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y);y\big)g'(y) - \frac{g''(y)}{2}\big(U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2;y\big) + U_{m,\alpha}^{\beta,\theta,\rho}(1;y)\big)\right| \\ &\quad\le 2\|g''-g\|U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2;y\big) + 2\|\phi g'\|\phi^{-1}(y)U_{m,\alpha}^{\beta,\theta,\rho}\big(|t-y|^3;y\big) \\ &\quad\le \frac{M}{m}y(1-y)\|g''-g\| + 2\|\phi g'\|\phi^{-1}(y)\big\{U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2;y\big)\big\}^{1/2}\big\{U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^4;y\big)\big\}^{1/2} \\ &\quad\le \frac{M}{m}\phi^2(y)\big\{\|g''-g\| + m^{-1/2}\|\phi g'\|\big\}\end{aligned}$$

by (3.2)–(3.3). Considering $\inf_{g\in W_\phi[0,1]}$ on the right-hand side of the last inequality, we deduce the desired result. □

The following corollary can be obtained from Theorem 4.

Corollary 2 *Let $g, g', g'' \in C[0,1]$, then*

$$\lim_{m\to\infty} m\big\{U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y) - g''(y)\chi_{m,\beta,\theta}^{\rho,\alpha}\big\} = 0.$$

The Grüss-type inequalities were defined and studied by Acu et al. [5], and Gonska and Tachev [20] for a class of sequences of positive linear operators. To obtain a Grüss–Voronovskaja-type theorem for our operators $U_{m,\alpha}^{\beta,\theta,\rho}$, we write

$$M_{m,\alpha}^{\rho}(g,h;y) = U_{m,\alpha}^{\beta,\theta,\rho}(gh;y) - U_{m,\alpha}^{\beta,\theta,\rho}(h;y)U_{m,\alpha}^{\beta,\theta,\rho}(g;y).$$

Theorem 5 *Assume that $\rho > 0$ and $g,h \in C^2[0,1]$. Then we have*

$$\lim_{m\to\infty} mM_{m,\alpha}^{\rho}(g,h;y) = \frac{(\rho+1)y(1-y)}{\rho}g'(y)h'(y)$$

for each $y \in [0,1]$.

Proof We write

$$\begin{aligned} M_{m,\alpha}^{\rho}(g,h;y) &= U_{m,\alpha}^{\beta,\theta,\rho}(gh;y) - g(y)h(y) - \big(g(y)h(y)\big)'U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y) \\ &\quad - \frac{(g(y)h(y))''}{2!}U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big) - h(y)\Bigg\{U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y) \\ &\quad - g'(y)U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y) - \frac{g''(y)}{2!}U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big)\Bigg\} \\ &\quad - U_{m,\alpha}^{\beta,\theta,\rho}(g;y)\Bigg\{U_{m,\alpha}^{\beta,\theta,\rho}(h;y) - h(y) - h'(y)U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y) \\ &\quad - \frac{h''(y)}{2!}U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big)\Bigg\} \\ &\quad + \frac{1}{2!}U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big)\big\{g(y)h''(y) + 2g'(y)h'(y) - h''(y)U_{m,\alpha}^{\beta,\theta,\rho}(g;y)\big\} \\ &\quad + U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y)\big\{g(y)h'(y) - h'(y)U_{m,\alpha}^{\beta,\theta,\rho}(g;y)\big\}. \end{aligned}$$

Since the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ converge uniformly to the function $g(y)$, we have

$$\begin{aligned} mM_{m,\alpha}^{\rho}(g,h;y) &= m\big\{U_{m,\alpha}^{\beta,\theta,\rho}(gh;y) - U_{m,\alpha}^{\beta,\theta,\rho}(g;y)U_{m,\alpha}^{\beta,\theta,\rho}(h;y)\big\} \\ &= mg'(y)h'(y)U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big) + \frac{m}{2!}h''(y)\big\{g(y) - U_{m,\alpha}^{\beta,\theta,\rho}(g;y)\big\} \\ &\quad \times U_{m,\alpha}^{\beta,\theta,\rho}\big((e_1-y)^2;y\big) + mh'(y)\big\{g(y) - U_{m,\alpha}^{\beta,\theta,\rho}(g;y)\big\}U_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y) \end{aligned}$$

by Theorem 1. We immediately prove the theorem if we pass to the limit because limits of $mU_{m,\alpha}^{\beta,\theta,\rho}(e_1-y;y)$ and $mU_{m,\alpha}^{\beta,\theta,\rho}((e_1-y)^2;y)$ are finite by Corollary 1. □

Theorem 6 *For every g in $C_B[0,1]$ (the set of all real-valued bounded and continuous functions defined on $[0,1]$) such that $g', g'' \in C_B[0,1]$. Then, for each $y \in [0,1]$ and $\rho > 0$, we have*

$$\lim_{m\to\infty} m\big\{U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y)\big\} = (\theta - \beta y)g'(y) + \frac{\rho+1}{2\rho}y(1-y)g''(y)$$

uniformly on $[0,1]$.

Proof Let $y \in [0,1]$ and $\rho > 0$. For any g in $C_B[0,1]$, it follows from Taylor's theorem that

$$g(t) = g(y) + (t-y)g'(y) + \frac{1}{2}(t-y)^2 g''(y) + (t-y)^2 r_y(t). \tag{3.4}$$

Here, $r_y(t)$ stands for the Peano form of the remainder. Note that $r_y \in C[0,1]$ and $r_y(t) \to 0$ as $t \to y$. By applying $U_{m,\alpha}^{\beta,\theta,\rho}(g;y)$ to identity (3.4), we get

$$\begin{aligned} &U_{m,\alpha}^{\beta,\theta,\rho}(g;y) - g(y) \\ &\quad = g'(y)U_{m,\alpha}^{\beta,\theta,\rho}(t-y;y) + \frac{g''(y)}{2}U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2;y\big) + U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2 r_y(t);y\big). \end{aligned} \tag{3.5}$$

Using the Cauchy–Schwarz inequality, we have

$$U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2 r_y(t);y\big) \le \sqrt{U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^4;y\big)}\sqrt{U_{m,\alpha}^{\beta,\theta,\rho}\big(r_y^2(t);y\big)}. \tag{3.6}$$

Since

$$\lim_{m\to\infty} m\big\{U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^4;y\big)\big\} = \frac{6(\rho+1)}{\rho}\big[\big(y^2-y\big)\big(2\theta-y^3\big)\big] + 4\theta\big(y^3-y^2\big) - 12\beta y^4 + 4y$$

from Lemma 1 and $\lim_{m\to\infty} U_{m,\alpha}^{\beta,\theta,\rho}(r_y^2(t);y) = 0$, it means

$$\lim_{m\to\infty} m\big\{U_{m,\alpha}^{\beta,\theta,\rho}\big((t-y)^2 r_y(t);y\big)\big\} = 0.$$

Thus we immediately obtain the desired result by applying limit to (3.5) and by considering Corollary 1. □

4 Numerical analysis

With the help of *MATHEMATICA*, we numerically examine our theoretical results with a view of convergence and error of approximation of our newly constructed operators (2.3). We first choose the parameters β, θ, ρ, α as $\beta = 0.2$, $\theta = 0.1$, $\rho = 1.5$, $\alpha = 0.9$ and the function

$$g(y) = cos(2\pi y).$$

In Fig. 1, we examine the convergence of (2.3) for different m values, and in Fig. 2, we compare the convergence of our operators with $U_{m,\alpha}^{\rho}$.

We also study the approximation properties of (2.3) by considering the following function:

$$g(y) = \frac{y|y-\frac{y}{3}|}{y^3+\frac{1}{2}} \quad \big(y \in [0,1]\big).$$

We take $m = 20$, $\alpha = 0.9$, $\beta = 1$, $\theta = 1$ $\rho = 2$ to obtain Fig. 3 to see the approximation of our operators. In Fig. 4, we give the approximations of our operators for $\alpha = 0.9$, $\beta = \theta = 1$, $\rho = 2$ and for different values of m. We give a table to compare the approximations.

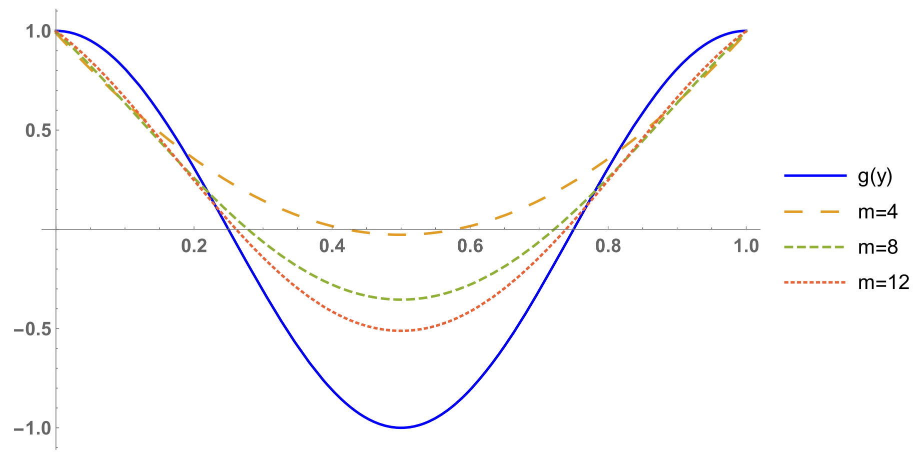

Figure 1 Convergence of operators for some m values

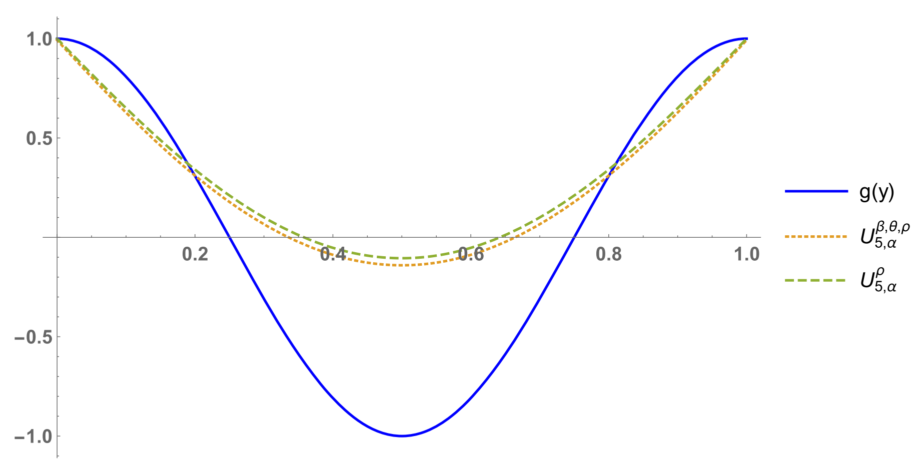

Figure 2 Comparison of operators

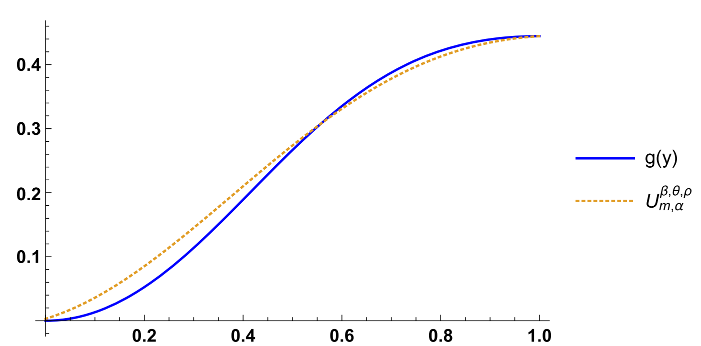

Figure 3 Approximation of operators

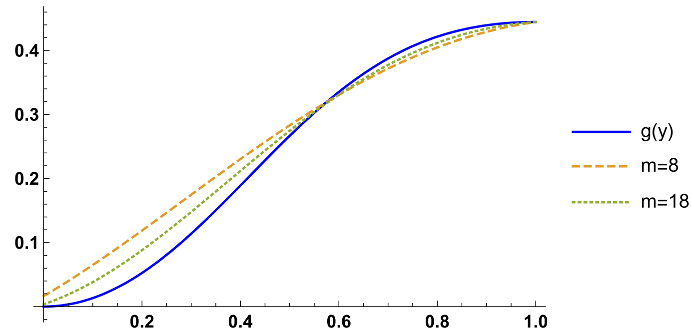

Figure 4 Convergence of operators for some m values

Table 1 Comparison of operators with maximum errors

	$m = 4$	$m = 8$	$m = 12$	$m = 16$
$\lVert g - U^{0.2,0.1,1.5}_{m,0.9}(g;y)\rVert_\infty$	0.972	0.645	0.488	0.393
$\lVert g - U^{1.5}_{m,0.9}(g;y)\rVert_\infty$	1.015	0.666	0.5	0.401
$\lVert g - U^{0.3,0.1,2.5}_{m,0.9}(g;y)\rVert_\infty$	0.869	0.569	0.426	0.341
$\lVert g - U^{2.5}_{m,0.9}(g;y)\rVert_\infty$	0.932	0.598	0.443	0.352

Table 2 Maximum error of approximation: $\lVert g - U^{\beta,\theta,\rho}_{m,\alpha}(g;y)\rVert_\infty$

$m = 10$	$\alpha = 0.9, \rho = 1$	$\alpha = 0.9, \rho = 2$	$\alpha = 0.1, \rho = 1$	$\alpha = 0.1, \rho = 2$
$\beta = \theta = 1$	0.057	0.055	0.057	0.056
$\beta = 2, \theta = 1$	0.042	0.040	0.043	0.041

It is clear from the Tables 1–2 and Figures 1–4 that our new operators are the generalization of the operators presented in the literature. They have fewer errors of approximation if we change the parameters α, β, θ, and ρ. Finally they have better approximations if we increase the values of m.

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