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Time delay induced Hopf bifurcation in a diffusive predator–prey model with prey toxicity

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Abstract
In this paper, we consider a diffusive predator–prey model with a time delay and prey toxicity. The effect of time delay on the stability of the positive equilibrium is studied by analyzing the eigenvalue spectrum. Delay-induced Hopf bifurcation is also investigated. By utilizing the normal form method and center manifold reduction for partial functional differential equations, the formulas for determining the property of Hopf bifurcation are given.

MSC: 34K18; 35B32

Keywords: Predator–prey; Delay; Stability; Hopf bifurcation

1 Introduction

Since the relationship of different biological species is very common in nature, many scholars have done a lot of works in this field [1–4]. In the real world, some biological species can release toxic substances that can affect the growth of other species. These toxic substances can even affect the living environment of human beings, so it is important to study the dynamics of the population models. In [5], Chattopadhyay studied the local and global stability of the interior equilibrium of a two-species competitive system with toxic substances. This work suggests that the toxic substance has the stabilizing effect on the model. In [6], Kar and Chaudhuri considered a two-species competing fish model with harvesting effect and toxic substances. They mainly studied the stability of the interior equilibrium. In addition, reaction–diffusion models arise in a variety of real world problems, such as in physical [7], chemical [8] and biological [9] applications. In [9], Zhang and Zhao proposed a diffusive predator–prey model with the toxic substance of the following form:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = D_1 \Delta u + ru(1-\frac{u}{K}) - \frac{muv}{\alpha+u}, & x\in\Omega, t>0, \\ \frac{\partial v(t,x)}{\partial t} = D_2 \Delta v + \epsilon v(1-\frac{hv}{u}) - \beta uv^2, & x\in\Omega, t>0, \\ \frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0, & x\in\partial\Omega, t>0, \\ u(0,x)=u_0(x)\geq 0, \qquad v(0,x)=v_0(x)\geq 0, & x\in\Omega. \end{cases} \tag{1.1}$$

All parameters are positive; u and v denote the densities of prey and predator, respectively; r and ϵ denote the growth rates of prey and predator, respectively; D_1 and D_2 are diffusion coefficients of prey and predator, respectively; K is the environmental capacity of prey. The functional response is of Holling II-type; β represents the efficiency toxic substance released by prey. They mainly studied the stability of the constant positive steady states and existence of the nonconstant positive steady states.

In the real world, time delay and asymptotic behaviors are widely studied toward the comprehension of growth process for biological species [10–13], such as gestation delay, maturation time, capturing time, and so on. Additionally, related analysis of characteristic equations also appear in the description of equilibrium models for other sciences, as, for instance, in civil engineering; see [14]. Differential equations with time delay often cause periodic oscillations, and show more abundant dynamic properties [15, 16]. In [17], the authors studied the Hopf–Hopf bifurcation in a predator–prey with predator cannibalism and time delay. In [18], the authors studied the Hopf–zero bifurcation in a delayed predator–prey model with dormancy of predators. These results all suggest that the time delay can enrich the dynamic properties of the predator–prey models.

Using the following parameters transformation:

$$
\begin{aligned}
&rt = \bar{t}, \qquad \frac{u}{K} = \bar{u}, \qquad \frac{hv}{K} = \bar{v}, \qquad d_1 = \frac{D_1}{r}, \qquad d_2 = \frac{D_2}{r}, \\
&a = \frac{K}{\alpha}, \qquad b = \frac{mK}{\alpha hr}, \qquad c = \frac{\epsilon}{rK}, \qquad s = \frac{\beta K^3}{h\epsilon},
\end{aligned}
$$

the model (1.1) becomes

$$
\begin{cases}
\frac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) - \frac{buv}{1+au}, \\
\frac{\partial v}{\partial t} = d_2 \Delta v + cv(1 - \frac{v}{u} - suv).
\end{cases}
\tag{1.2}
$$

Based on the model (1.2), we consider the following delay model:

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + u(1-u(t-\tau)) - \frac{buv}{1+au}, & x \in \Omega, t > 0, \\
\frac{\partial v(x,t)}{\partial t} = d_2 \Delta v + cv(1 - \frac{v}{u} - suv), & x \in \Omega, t > 0, \\
\frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\
u(x,t) = u_1(x,t) \geq 0, \qquad v(x,t) = v_1(x,t) \geq 0, & x \in \Omega, t \in [-\tau, 0].
\end{cases}
\tag{1.3}
$$

All parameters are positive; τ is the resource limitation of the prey logistic equation. For convenience, we denote $\Omega = (0, l\pi)$. The aim of this paper is to study the effect of time delay on the model (1.3). Compare with the model (1.1), whether some new dynamical phenomena occurs.

The organization of this paper is as follows. In Sect. 2, we study the existence of equilibria. In Sect. 3, we analyze the stability of the positive equilibrium, the existence of Hopf bifurcation, and the property of bifurcating periodic solutions. In Sect. 4, we give a numerical simulation. At last, we give a brief conclusion.

2 Existence of equilibria

The equilibria of model (1.3) are the roots of the following equations:

$$\begin{cases} u(1-u) - \frac{buv}{1+au} = 0, \\ cv(1 - \frac{v}{u} - suv) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1 *For the model* (1.3), *the following statements are true*:

(1) *The model* (1.3) *has a boundary equilibrium* $(1,0)$.
(2) *The model* (1.3) *has at least one positive equilibrium.*
(3) *Under hypothesis* $(\mathbf{H_0})$, $0 < s < a < 1$, *the model* (1.3) *has a unique positive equilibrium.*

Proof Obviously, the model (1.3) has a boundary equilibrium $(1,0)$. Now, we consider the existence of positive equilibrium denoted as (u_*, v_*). From the first equation in (2.1), we have $v_* = \frac{(1-u_*)(au_*+1)}{b}$. From the second equation in (2.1), we have $v_* = \frac{u_*}{su_*^2+1} > 0$. Then we can obtain that u_* is the positive root of the following equation:

$$h(u) = asu^4 + (1-a)su^3 + (a-s)u^2 + (1+b-a)u - 1 = 0. \tag{2.2}$$

It is not difficult to obtain that $h(0) = -1$, $\lim_{x\to\infty} h(u) = +\infty$. Thus Eq. (2.2) has at least one positive root, which means that the model (1.3) has at least one positive equilibrium. In addition, by the Descartes's rule of signs, Eq. (2.2) has a unique positive root under condition $0 < s < a < 1$, which means that the model (1.3) has a unique positive equilibrium. □

In the rest of this paper, we denote the positive equilibrium as (u_*, v_*), where $v_* = \frac{u_*}{su_*^2+1}$.

3 Stability analysis

Linearize system (1.3) at (u_*, v_*) as follows:

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = d\Delta \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_1 \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_2 \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix}, \tag{3.1}$$

where

$$L_1 = \begin{pmatrix} a_1 & -a_2 \\ ca_3 & c \end{pmatrix}, \qquad L_2 = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$a_1 = \frac{abu_*^2}{(au_*+1)^2(su_*^2+1)} > 0, \qquad a_2 = \frac{bu_*}{au_*+1} > 0, \qquad a_3 = \frac{1-su_*^2}{(su_*^2+1)^2}. \tag{3.2}$$

The characteristic equation of (3.1) is

$$\det\left(\lambda I - M_n - L_1 - L_2 e^{-\lambda\tau}\right) = 0, \tag{3.3}$$

where $I = \operatorname{diag}\{1,1\}$ and $M_n = -n^2/l^2 \operatorname{diag}\{d_1, d_2\}$, $n \in \mathbb{N}_0$. Then, we have

$$\lambda^2 + \lambda A_n + B_n + (C_n + \lambda u_*)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}, \tag{3.4}$$

where

$$
\begin{aligned}
A_n &= (d_1 + d_2)\frac{n^2}{l^2} - (a_1 + c),\\
B_n &= d_1 d_2 \frac{n^4}{l^4} - (a_1 d_2 + c d_1)\frac{n^2}{l^2} + c(a_1 + a_2 a_3),\\
C_n &= d_2 u_* \frac{n^2}{l^2} - c u_*.
\end{aligned}
$$

3.1 The case of $\tau = 0$

When $\tau = 0$, the characteristic Eq. (1.3) reduces to the following equation:

$$
\lambda^2 - T_n \lambda + D_n = 0, \quad n \in \mathbb{N}_0, \tag{3.5}
$$

where

$$
\begin{aligned}
T_n &= -A_n - u_* = -(d_1 + d_2)\frac{n^2}{l^2} + c - \tilde{c},\\
D_n &= B_n + C_n = d_1 d_2 \frac{n^4}{l^4} - (c d_1 - \tilde{c} d_2)\frac{n^2}{l^2} + c(a_2 a_3 - \tilde{c}),\\
\tilde{c} &\triangleq u_* - a_1 = \frac{u_*(2a u_* + 1 - a)}{a u_* + 1}
\end{aligned}
\tag{3.6}
$$

for $n \in \mathbb{N}_0$. The eigenvalues are given by

$$
\lambda_{1,2}^{(n)} = \frac{T_n \pm \sqrt{T_n^2 - 4D_n}}{2}, \quad n \in \mathbb{N}_0. \tag{3.7}
$$

We consider the following hypotheses:

$$
\begin{aligned}
&(\mathbf{H_1}) \quad 0 < c < \tilde{c},\\
&(\mathbf{H_2}) \quad 0 < c \le \frac{d_2}{d_1}\tilde{c}, \qquad 0 < b < \frac{2(1-u_*)^2(a u_* + 1)^2}{u_*(a u_*^2 + 1)}.
\end{aligned}
\tag{3.8}
$$

By direct calculation, we can get the following remark.

Remark 3.1 Hypothesis $(\mathbf{H_0})$ is a sufficient condition for $\tilde{c} > 0$. If Hypothesis $(\mathbf{H_1})$ holds, then $T_n < 0$ for $n \in \mathbb{N}_0$. If Hypothesis $(\mathbf{H_2})$ holds, then $D_n > 0$ for $n \in \mathbb{N}_0$.

Theorem 3.1 *When $\tau = 0$, the equilibrium (u_*, v_*) is locally asymptotically stable under Hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$.*

Proof When $\tau = 0$, by Remark 3.1, we have $T_n < 0$ and $D_n > 0$ for $n \in \mathbb{N}_0$ under Hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$. Then the eigenvalues (3.7) all have negative real parts, which can guarantee the statement in Theorem 3.1. □

3.2 The case of $\tau > 0$

To study the stability of $E_*(u_*, v_*)$ when $\tau > 0$, we suppose $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. Let $\mathrm{i}\omega$ ($\omega > 0$) be a solution of Eq. (3.4). We have

$$-\omega^2 + \mathrm{i}\omega A_n + B_n + (C_n + \mathrm{i}\omega u_*)(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0.$$

Then we have

$$\begin{cases} -\omega^2 + B_n + C_n \cos\omega\tau + \omega u_* \sin\omega\tau = 0, \\ A_n\omega - C_n \sin\omega\tau + \omega u_* \cos\omega\tau = 0, \end{cases} \tag{3.9}$$

leading to

$$\omega^4 + \left(A_n^2 - 2B_n - u_*^2\right)\omega^2 + B_n^2 - C_n^2 = 0. \tag{3.10}$$

Denote $z = \omega^2$. Then (3.10) can be changed into

$$z^2 + \left(A_n^2 - 2B_n - u_*^2\right)z + B_n^2 - C_n^2 = 0, \tag{3.11}$$

and the roots of (3.11) are

$$z_n^\pm = \frac{1}{2}\left[-\left(A_n^2 - 2B_n - u_*^2\right) \pm \sqrt{\left(A_n^2 - 2B_n - u_*^2\right)^2 - 4\left(B_n^2 - C_n^2\right)}\right].$$

By direct computation,

$$\begin{aligned} A_n^2 - 2B_n - u_*^2 &= \left(d_1^2 + d_2^2\right)\frac{n^4}{l^4} - 2(a_1 d_1 + c d_2)\frac{n^2}{l^2} - c(2a_2 a_3 - c) - \left(u_*^2 - a_1^2\right), \\ B_n - C_n &= d_1 d_2 \frac{n^4}{l^4} - \left[c d_1 + \left(d_2(a_1 + u_*)\right)\right]\frac{n^2}{l^2} + c(a_1 + a_2 a_3 + u_*), \\ B_n + C_n &= d_1 d_2 \frac{n^4}{l^4} - (c d_1 - \tilde{c} d_2)\frac{n^2}{l^2} + c(a_2 a_3 - \tilde{c}) = D_n > 0. \end{aligned}$$

Fix parameters a, b, c, s, define

$$\mathcal{D} = \{k \in \mathbb{N}_0 \mid \text{Eq. (3.11) has positive roots with } n = k\}. \tag{3.12}$$

Under $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we can obtain

$$A_0^2 - 2B_0 - u_*^2 < 0, \qquad B_0 - C_0 > 0,$$

and

$$\left(A_0^2 - 2B_0 - u_*^2\right)^2 - 4\left(B_0^2 - C_0^2\right) = 2c^2\left(u_*^2 - a_1^2\right) + \left(a_1^2 - u_*^2\right)^2 + 4a_2 a_3 c\left(u_*^2 - (a_1 + c)^2\right) + c^4 > 0.$$

This means that Eq. (3.11) has at least a pair of positive roots $z_0^\pm$. Then $\mathcal{D} \neq \emptyset$.

For $n \in \mathcal{D}$, if $z^+ > 0$, then Eq. (3.4) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_n^+$ at $\tau_n^{j,+}$, $j \in \mathbb{N}_0$. If $z^- > 0$, then Eq. (3.4) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_n^-$ at $\tau_n^{j,-}$, $j \in \mathbb{N}_0$, where

$$\begin{aligned}
&\omega_n^\pm = \sqrt{z_n^\pm}, \qquad \tau_n^{j,\pm} = \tau_n^{0,\pm} + \frac{2j\pi}{\omega_n^\pm} \quad (j = 0,1,2,\dots),\\
&\tau_n^{0,\pm} = \begin{cases} \frac{1}{\omega_n^\pm}\arccos(V_{\cos}), & V_{\sin} \geq 0,\\ \frac{1}{\omega_n^\pm}[2\pi - \arccos(V_{\cos})], & V_{\sin} < 0, \end{cases}\\
&V_{\cos} = \frac{(C_n - u_*A_n)(\omega_n^\pm)^2 - B_nC_n}{C_n^2 + u_*^2(\omega_n^\pm)^2}, \qquad V_{\sin} = \frac{\omega_n^\pm[A_nC_n - B_nu_* + u_*(\omega_n^\pm)^2]}{C_n^2 + u_*^2(\omega_n^\pm)^2}.
\end{aligned} \tag{3.13}$$

From (3.13), we have $\tau_n^{0,\pm} < \tau_n^{j,\pm}$ ($j \in \mathbb{N}$). For $k \in \mathcal{D}$, define the smallest τ so that the stability will change, $\tau_* = \min\{\tau_k^{0,\pm}$ or $\tau_k^{0,+} \mid k \in \mathcal{D}\}$.

Lemma 3.1 *Suppose* $(\mathbf{H_1})$ (*or* $(\mathbf{H_2})$) *holds. If* $(A_n^2 - 2B_n - u_*^2)^2 - 4(B_n^2 - C_n^2) > 0$, *then* $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$ *for* $\tau \in \mathcal{D}$ *and* $j \in \mathbb{N}_0$.

Proof Differentiating two sides of (3.4) with respect τ, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_n + u_*e^{-\lambda\tau}}{\lambda(C_n + \lambda u_*)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Then

$$\begin{aligned}
\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_n^{j,\pm}} &= \mathrm{Re}\left[\frac{2\lambda + A_n + u_*e^{-\lambda\tau}}{\lambda(C_n + \lambda u_*)e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right]_{\tau=\tau_n^{j,\pm}}\\
&= \left[\frac{1}{\Lambda}\left(2\omega^2 + A_n^2 - 2B_n - u_*^2\right)\right]_{\tau=\tau_n^{j,\pm}}\\
&= \pm\left[\frac{1}{\Lambda}\sqrt{\left(A_n^2 - 2B_n - u_*^2\right)^2 - 4\left(B_n^2 - C_n^2\right)}\right]_{\tau=\tau_n^{j,\pm}},
\end{aligned}$$

where $\Lambda = \omega^2u_*^2 + C_n^2 > 0$. Therefore $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$. □

Theorem 3.2 *Suppose* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. For system* (1.3), *the following statements are true*:

(1) $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau \in [0, \tau_*)$, *and unstable for* $\tau \in [\tau_*, \tau_* + \epsilon)$ *with some* ϵ.

(2) *System* (1.3) *undergoes a Hopf bifurcation at the equilibrium* $E_*(u_*, v_*)$ *when* $\tau = \tau_n^{j,+}$ (*or* $\tau = \tau_n^{j,-}$), $j \in \mathbb{N}_0$, $n \in \mathcal{D}$, *where* $\tau_n^{j,\pm}$ *is defined in* (3.13).

Remark 3.2 From Lemma (3.1), we obtain $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$ for $\tau \in \mathcal{D}$ and $j \in \mathbb{N}_0$, then the stability switch may exist.

3.3 Properties of Hopf bifurcation

Now, we will study the property of Hopf bifurcation by the method exploited in [19, 20]. For a critical value $\tau_n^{j,+}$ (or $\tau_n^{j,-}$), we denote it as $\tilde{\tau}$. Let $\tilde{u}(x,t) = u(x,\tau t) - u_*$ and $\tilde{v}(x,t) =$

$v(x,\tau t) - v_*$. Then the system (1.3) is (dropping the tilde)

$$\begin{cases} \frac{\partial u}{\partial t} = \tau[d_1\Delta u + (u+u_*)(1-u(t-1)-u_*-\frac{b(u+u_*)(v+v_*)}{1+a(u+u_*)})], \\ \frac{\partial v}{\partial t} = \tau[d_2\Delta v + c(v+v_*)(1-\frac{v+v_*}{u+u_*}-s(u+u_*)(v+v_*))]. \end{cases} \tag{3.14}$$

Denote $\tau = \tilde{\tau} + \varepsilon$, and $U = (u(x,t), v(x,t))^T$. In the phase space $\mathscr{C}_1 := C([-1,0],X)$, (3.14) can be rewritten as

$$\frac{dU(t)}{dt} = \tilde{\tau}D\Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t,\varepsilon), \tag{3.15}$$

where $L_\varepsilon(\varphi)$ and $F(\varphi,\varepsilon)$ are

$$L_\varepsilon(\phi) = \varepsilon\begin{pmatrix} a_1\phi_1(0) - a_2\phi_2(0) - u_*\phi_1(-1) \\ ca_3\phi_1(0) + c\phi_2(0) \end{pmatrix}, \tag{3.16}$$

$$F(\phi,\varepsilon) = \varepsilon D\Delta\phi + L_\varepsilon(\phi) + f(\phi,\varepsilon), \tag{3.17}$$

with

$$\begin{aligned} f(\phi,\varepsilon) &= (\tilde{\tau}+\varepsilon)\bigl(F_1(\phi,\varepsilon), F_2(\phi,\varepsilon)\bigr)^T, \\ F_1(\phi,\varepsilon) &= \bigl(\phi_1(0)+u_*\bigr)\left(1-\phi_1(-1)-u_*-\frac{b(\phi_1(0)+u_*)(\phi_2(0)+v_*)}{1+a(\phi_1(0)+u_*)}\right) \\ &\quad - a_1\phi_1(0) + a_2\phi_2(0) + u_*\phi_1(-1), \\ F_2(\phi,\varepsilon) &= c\bigl(\phi_2(0)+v_*\bigr)\left(1-\frac{\phi_2(0)+v_*}{\phi_1(0)+u_*} - s\bigl(\phi_1(0)+u_*\bigr)\bigl(\phi_2(0)+v_*\bigr)\right) \\ &\quad - ca_3\phi_1(0) - c\phi_2(0). \end{aligned}$$

respectively, for $\phi = (\phi_1,\phi_2)^T \in \mathscr{C}_1$.

Consider the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau}D\Delta U(t) + L_{\tilde{\tau}}(U_t). \tag{3.18}$$

We know that $\Lambda_n := \{\mathrm{i}\omega_n\tilde{\tau}, -\mathrm{i}\omega_n\tilde{\tau}\}$ are characteristic roots of

$$\frac{dz(t)}{dt} = -\tilde{\tau}D\frac{n^2}{l^2}z(t) + L_{\tilde{\tau}}(z_t). \tag{3.19}$$

Choose

$$\eta^n(\sigma,\tau) = \begin{cases} \tau E, & \sigma = 0, \\ 0, & \sigma \in (-1,0), \\ -\tau F, & \sigma = -1, \end{cases} \tag{3.20}$$

where

$$E = \begin{pmatrix} a_1 - d_1\frac{n^2}{l^2} & -a_2 \\ ca_3 & c - d_2\frac{n^2}{l^2} \end{pmatrix}, \qquad F = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.21}$$

Then

$$-\tilde{\tau} D\frac{n^2}{l^2}\phi(0) + L_{\tilde{\tau}}(\phi) = \int_{-1}^{0} d\eta^n(\sigma,\tau)\phi(\sigma)$$

for $\phi \in C([-1,0],\mathbb{R}^2)$.

Define the bilinear paring

$$\begin{aligned}(\psi,\varphi)_0 &= \psi(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\sigma} \psi(\xi-\sigma)\,d\eta^n(\sigma,\tilde{\tau})\varphi(\xi)\,d\xi \\ &= \psi(0)\varphi(0) + \tilde{\tau}\int_{-1}^{0} \psi(\xi+1)F\varphi(\xi)\,d\xi\end{aligned} \tag{3.22}$$

for $\varphi \in C([-1,0],\mathbb{R}^2)$, $\psi \in C([0,1],\mathbb{R}^2)$; $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm \mathrm{i}\omega_n\tilde{\tau}$, and they are also eigenvalues of A^*.

Define $p_1(\sigma) = (1,\xi)^T e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}$ ($\sigma \in [-1,0]$), $q_1(r) = (1,\eta)e^{-\mathrm{i}\omega_n\tilde{\tau}r}$ ($r \in [0,1]$), where

$$\xi = \frac{ca_3}{-c + d_2n^2/l^2 + \mathrm{i}\omega}, \qquad \eta = \frac{a_2}{c - d_2n^2/l^2 + \mathrm{i}\omega}.$$

Let $\Phi = (\Phi_1,\Phi_2)$ and $\Psi^* = (\Psi_1^*,\Psi_2^*)^T$ with

$$\begin{aligned}\Phi_1(\sigma) &= \frac{p_1(\sigma)+p_2(\sigma)}{2} = \begin{pmatrix}\mathrm{Re}(e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}) \\ \mathrm{Re}(\xi\,\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\sigma})\end{pmatrix}, \\ \Phi_2(\sigma) &= \frac{p_1(\sigma)-p_2(\sigma)}{2\mathrm{i}} = \begin{pmatrix}\mathrm{Im}(e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}) \\ \mathrm{Im}(\xi\,\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\sigma})\end{pmatrix}\end{aligned}$$

for $\sigma \in [-1,0]$, and

$$\begin{aligned}\Psi_1^*(r) &= \frac{q_1(r)+q_2(r)}{2} = \begin{pmatrix}\mathrm{Re}(e^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}}) \\ \mathrm{Re}(\eta\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}})\end{pmatrix}, \\ \Psi_2^*(r) &= \frac{q_1(r)-q_2(r)}{2\mathrm{i}} = \begin{pmatrix}\mathrm{Im}(e^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}}) \\ \mathrm{Im}(\eta\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}})\end{pmatrix}\end{aligned}$$

for $r \in [0,1]$. Then we can compute by (3.22)

$$D_1^* := (\Psi_1^*,\Phi_1)_0, \qquad D_2^* := (\Psi_1^*,\Phi_2)_0, \qquad D_3^* := (\Psi_2^*,\Phi_1)_0, \qquad D_4^* := (\Psi_2^*,\Phi_2)_0.$$

Define $(\Psi^*,\Phi) = (\Psi_j^*,\Phi_k) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^*\end{pmatrix}$ and $\Psi = (\Psi_1,\Psi_2)^T = (\Psi^*,\Phi)^{-1}\Psi^*$. Then $(\Psi,\Phi)_0 = I_2$. In addition, define $f_n := (\beta_n^1,\beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix}\cos\frac{n}{l}x \\ 0\end{pmatrix}, \qquad \beta_n^2 = \begin{pmatrix}0 \\ \cos\frac{n}{l}x\end{pmatrix}.$$

We also define

$$c\cdot f_n = c_1\beta_n^1 + c_2\beta_n^2, \quad \text{for } c = (c_1,c_2)^T \in \mathscr{C}_1$$

and

$$\langle u,v\rangle := \frac{1}{l\pi}\int_0^{l\pi} u_1\overline{v_1}\,dx + \frac{1}{l\pi}\int_0^{l\pi} u_2\overline{v_2}\,dx$$

for $u=(u_1,u_2)^T$, $v=(v_1,v_2)^T$, $u,v\in X$ and $\langle\varphi,f_0\rangle = (\langle\varphi,f_0^1\rangle,\langle\varphi,f_0^2\rangle)^T$.

Rewrite Eq. (3.14) as

$$\frac{dU(t)}{dt} = A_{\tilde\tau}U_t + R(U_t,\varepsilon), \tag{3.23}$$

where

$$R(U_t,\varepsilon) = \begin{cases} 0, & \theta\in[-1,0); \\ F(U_t,\varepsilon), & \theta = 0. \end{cases} \tag{3.24}$$

The solution is

$$U_t = \Phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n + h(x_1,x_2,\varepsilon), \tag{3.25}$$

where $\binom{x_1}{x_2} = (\Psi,\langle U_t,f_n\rangle)$, and $h(x_1,x_2,\varepsilon)\in P_S\mathscr{C}_1$, $h(0,0,0)=0$, $Dh(0,0,0)=0$. Then

$$U_t = \Phi\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1,x_2,0). \tag{3.26}$$

Let $z = x_1 - \mathrm{i}x_2$. Then

$$\Phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n = (\Phi_1,\Phi_2)\begin{pmatrix} \frac{z+\overline{z}}{2} \\ \frac{\mathrm{i}(z-\overline{z})}{2} \end{pmatrix} f_n = \frac{1}{2}(p_1 z + \overline{p_1 z})f_n,$$

and

$$h(x_1,x_2,0) = h\left(\frac{z+\overline{z}}{2},\frac{\mathrm{i}(z-\overline{z})}{2},0\right).$$

Equation (3.26) is

$$\begin{aligned} U_t &= \frac{1}{2}(p_1 z + \overline{p_1 z})f_n + h\left(\frac{z+\overline{z}}{2},\frac{\mathrm{i}(z-\overline{z})}{2},0\right) \\ &= \frac{1}{2}(p_1 z + \overline{p_1 z})f_n + W(z,\overline{z}), \end{aligned} \tag{3.27}$$

where

$$W(z,\overline{z}) = h\left(\frac{z+\overline{z}}{2},\frac{\mathrm{i}(z-\overline{z})}{2},0\right).$$

From [19], z satisfies

$$\dot z = \mathrm{i}\omega_n\tilde\tau z + g(z,\overline{z}), \tag{3.28}$$

where

$$g(z,\overline{z}) = \bigl(\Psi_1(0) - \mathrm{i}\Psi_2(0)\bigr)\bigl\langle F(U_t,0),f_n\bigr\rangle. \tag{3.29}$$

Let

$$W(z,\overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{3.30}$$

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots, \tag{3.31}$$

then

$$\begin{aligned}
u_t(0) &= \frac{1}{2}(z+\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
v_t(0) &= \frac{1}{2}(\xi + \overline{\xi}\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
u_t(-1) &= \frac{1}{2}\bigl(ze^{-\mathrm{i}\omega_n\tilde{\tau}} + \overline{z}e^{\mathrm{i}\omega_n\tilde{\tau}}\bigr)\cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} \\
&\quad + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots,
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_1(U_t,0) = \frac{1}{\tilde{\tau}}F_1 &= -u_t(0)u_t(-1) + \alpha_1 u_t^2(0) + \alpha_2 u_t(0)v_t(0) + \alpha_3 u_t^3(0) \\
&\quad + \alpha_4 u_t^2(0)v_t(0) + O(4),
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\overline{F}_2(U_t,0) = \frac{1}{\tilde{\tau}}F_2 &= \beta_1 u_t^2(0) + \beta_2 u_t(0)v_t(0) + \beta_3 v_t^2(0) + \beta_4 u_t^3(0) + \beta_5 u_t^2(0)v_t(0) \\
&\quad + \beta_6 u_t(0)v_t^2(0) + O(4),
\end{aligned} \tag{3.33}$$

with

$$\begin{aligned}
&\alpha_1 = \frac{abv_*}{(au_*+1)^3}, \qquad \alpha_2 = -\frac{b}{(au_*+1)^2}, \\
&\alpha_3 = -\frac{a^2bv_*}{(au_*+1)^4}, \qquad \alpha_4 = \frac{ab}{(au_*+1)^3}, \\
&\beta_1 = -\frac{cv_*^2}{u_*^3}, \qquad \beta_2 = -\frac{2cv_*(su_*^2-1)}{u_*^2}, \qquad \beta_3 = -\frac{c(su_*^2+1)}{u_*}, \\
&\beta_4 = \frac{cv_*^2}{u_*^4}, \qquad \beta_5 = -\frac{2cv_*}{u_*^3}, \qquad \beta_6 = -\frac{c(su_*^2-1)}{u_*^2}.
\end{aligned} \tag{3.34}$$

Hence,

$$\begin{aligned}
\overline{F}_1(U_t,0) &= \cos^2\left(\frac{nx}{l}\right)\left(\frac{z^2}{2}\chi_{20} + z\overline{z}\chi_{11} + \frac{\overline{z}^2}{2}\overline{\chi}_{20}\right) \\
&\quad + \frac{z^2\overline{z}}{2}\left(\chi_1\cos\frac{nx}{l} + \chi_2\cos^3\frac{nx}{l}\right) + \cdots, \\
\overline{F}_2(U_t,0) &= \cos^2\left(\frac{nx}{l}\right)\left(\frac{z^2}{2}\varsigma_{20} + z\overline{z}\varsigma_{11} + \frac{\overline{z}^2}{2}\overline{\varsigma}_{20}\right) \\
&\quad + \frac{z^2\overline{z}}{2}\left(\varsigma_1\cos\frac{nx}{l} + \varsigma_2\cos^3\frac{nx}{l}\right) + \cdots,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\langle F(U_t,0),f_n\rangle &= \tilde{\tau}\left(\langle\overline{F}_1(U_t,0),f_n^1\rangle,\langle\overline{F}_2(U_t,0),f_n^2\rangle\right)^T \\
&= \frac{z^2}{2}\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\Gamma + z\overline{z}\tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varsigma_{11}\end{pmatrix}\Gamma + \frac{\overline{z}^2}{2}\tilde{\tau}\begin{pmatrix}\overline{\chi}_{20}\\ \overline{\varsigma}_{20}\end{pmatrix}\Gamma + \frac{z^2\overline{z}}{2}\tilde{\tau}\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix} + \cdots
\end{aligned} \tag{3.36}$$

with

$$\begin{aligned}
\Gamma &= \frac{1}{l\pi}\int_0^{l\pi}\cos^3\left(\frac{nx}{l}\right)dx, \\
\kappa_1 &= \frac{\chi_1}{l\pi}\int_0^{l\pi}\cos^2\left(\frac{nx}{l}\right)dx + \frac{\chi_2}{l\pi}\int_0^{l\pi}\cos^4\left(\frac{nx}{l}\right)dx, \\
\kappa_2 &= \frac{\varsigma_1}{l\pi}\int_0^{l\pi}\cos^2\left(\frac{nx}{l}\right)dx + \frac{\varsigma_2}{l\pi}\int_0^{l\pi}\cos^4\left(\frac{nx}{l}\right)dx,
\end{aligned}$$

and

$$\begin{aligned}
\chi_{20} &= \frac{1}{2}\left(\alpha_1 + \xi\alpha_2 - e^{-\mathrm{i}\tilde{\tau}\omega_n}\right), \qquad \chi_{11} = -\frac{1}{4}\left(e^{-\mathrm{i}\tilde{\tau}\omega_n} + e^{\mathrm{i}\tilde{\tau}\omega_n} - \left(2\alpha_1 + \alpha_2(\overline{\xi} + \xi)\right)\right), \\
\chi_1 &= W_{11}^{(1)}(0)\left(2\alpha_1 + \alpha_2\xi - e^{-\mathrm{i}\tilde{\tau}\omega_n}\right) + W_{11}^{(2)}(0)\alpha_2 - W_{11}^{(1)}(-1) - \frac{1}{2}W_{20}^{(1)}(-1) \\
&\quad + W_{20}^{(1)}(0)\left(\frac{1}{2}\left(2\alpha_1 + \alpha_2\overline{\xi} - e^{\mathrm{i}\tilde{\tau}\omega_n}\right)\right) + W_{20}^{(2)}(0)\frac{\alpha_2}{2}, \\
\chi_2 &= \frac{1}{4}\left(3\alpha_3 + \alpha_4(\overline{\xi} + 2\xi)\right), \qquad \varsigma_{20} = \frac{1}{2}\left(\beta_1 + \xi(\beta_2 + \beta_3\xi)\right), \\
\varsigma_{11} &= \frac{1}{4}\left(2\beta_1 + \beta_2\left(W_{11}^{(1)}(0) + \xi\right) + 2\beta_3W_{11}^{(1)}(0)\xi\right), \\
\varsigma_2 &= \frac{1}{4}\left(3\beta_4 + \beta_5(\overline{\xi} + 2\xi) + \beta_6\xi(2\overline{\xi} + \xi)\right), \\
\varsigma_1 &= W_{11}^{(1)}(0)(2\beta_1 + \beta_2\xi) + W_{11}^{(2)}(0)(\beta_2 + 2\beta_3\xi) \\
&\quad + W_{20}^{(1)}(0)\left(\beta_1 + \frac{\beta_2\overline{\xi}}{2}\right) + W_{20}^{(2)}(0)\left(\frac{\beta_2}{2} + \beta_3\overline{\xi}\right).
\end{aligned}$$

Denote

$$\Psi_1(0) - \mathrm{i}\Psi_2(0) := (\gamma_1\gamma_2).$$

Notice that

$$\Gamma = \frac{1}{l\pi}\int_0^{l\pi}\cos^3\frac{nx}{l}\,dx = 0, \quad n = 1,2,3,\dots,$$

and we have

$$\begin{aligned}
&\bigl(\Psi_1(0) - \mathrm{i}\Psi_2(0)\bigr)\langle F(U_t,0),f_n\rangle \\
&\quad = \frac{z^2}{2}(\gamma_1\chi_{20} + \gamma_2\varsigma_{20})\Gamma\tilde{\tau} + z\bar{z}(\gamma_1\chi_{11} + \gamma_2\varsigma_{11})\Gamma\tilde{\tau} + \frac{\bar{z}^2}{2}(\gamma_1\overline{\chi}_{20} + \gamma_2\overline{\varsigma}_{20})\Gamma\tilde{\tau} \\
&\quad\quad + \frac{z^2\bar{z}}{2}\tilde{\tau}[\gamma_1\kappa_1 + \gamma_2\kappa_2] + \cdots.
\end{aligned} \tag{3.37}$$

Then by (3.29), (3.31) and (3.37), we have $g_{20} = g_{11} = g_{02} = 0$, for $n = 1,2,3,\dots$.

If $n = 0$, $\Gamma = \frac{1}{l\pi}\int_0^{l\pi}\cos^3\frac{nx}{l}\,dx = 1$, then we have

$$g_{20} = \gamma_1\tilde{\tau}\chi_{20} + \gamma_2\tilde{\tau}\varsigma_{20}, \qquad g_{11} = \gamma_1\tilde{\tau}\chi_{11} + \gamma_2\tilde{\tau}\varsigma_{11}, \qquad g_{02} = \gamma_1\tilde{\tau}\overline{\chi}_{20} + \gamma_2\tilde{\tau}\overline{\varsigma}_{20}.$$

And for $n \in \mathbb{N}_0$, $g_{21} = \tilde{\tau}(\gamma_1\kappa_1 + \gamma_2\kappa_2)$. Next, we just need to compute $W_{11}(\theta) := (W_{11}^{(1)}(\theta), W_{11}^{(2)}(\theta))^T$ and $W_{20}(\theta) := (W_{20}^{(1)}(\theta), W_{20}^{(2)}(\theta))^T$.

By (3.30), we can obtain

$$\dot{W}(z,\bar{z}) = W_{20}z\dot{z} + W_{11}\dot{z}\bar{z} + W_{11}z\dot{\bar{z}} + W_{02}\bar{z}\dot{\bar{z}} + \cdots,$$

$$A_{\tilde{\tau}}W(z,\bar{z}) = A_{\tilde{\tau}}W_{20}\frac{z^2}{2} + A_{\tilde{\tau}}W_{11}z\bar{z} + A_{\tilde{\tau}}W_{02}\frac{\bar{z}^2}{2} + \cdots.$$

From [19], we have

$$\dot{W}(z,\bar{z}) = A_{\tilde{\tau}}W + H(z,\bar{z}),$$

where

$$\begin{aligned}
H(z,\bar{z}) &= H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \\
&= F(U_t,0) - \Phi\bigl(\Psi,\langle F(U_t,0),f_n\rangle\bigr)_0\cdot f_n.
\end{aligned} \tag{3.38}$$

Hence, we have

$$(2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{20} = H_{20}, \qquad -A_{\tilde{\tau}}W_{11} = H_{11}, \qquad (-2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{02} = H_{02}, \tag{3.39}$$

that is,

$$W_{20} = (2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{20}, \qquad W_{11} = -A_{\tilde{\tau}}^{-1}H_{11}, \qquad W_{02} = (-2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{02}. \tag{3.40}$$

For $-1 \le \theta < 0$, we have

$$\begin{aligned}
H(z,\bar{z}) &= -\Phi(\theta)\Psi(\theta)\langle F(U_t,\theta),f_n\rangle\cdot f_n \\
&= -\left(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2\mathrm{i}}\right)\begin{pmatrix}\Phi_1(\theta)\\ \Phi_2(\theta)\end{pmatrix}\langle F(U_t,\theta),f_n\rangle\cdot f_n
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\big[p_1(\theta)\big(\Phi_1(\theta) - \mathrm{i}\Phi_2(\theta)\big) + p_2(\theta)\big(\Phi_1(\theta) + \mathrm{i}\Phi_2(\theta)\big)\big]\langle F(U_t,\theta), f_n\rangle \cdot f_n \\
&= -\frac{1}{2}\bigg[\big(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}\big)\frac{z^2}{2} + \big(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}\big)z\overline{z} \\
&\quad + \big(p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20}\big)\frac{\overline{z}^2}{2}\bigg] + \cdots .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
H_{20}(\theta) &= \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}) \cdot f_0, & n = 0, \end{cases} \\
H_{11}(\theta) &= \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_0, & n = 0, \end{cases} \\
H_{02}(\theta) &= \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20}) \cdot f_0, & n = 0, \end{cases}
\end{aligned}
$$

and

$$
H(z,\overline{z})(0) = F(U_t,0) - \Phi\big(\Psi, \langle F(U_t,0), f_n\rangle\big)_0 \cdot f_n,
$$

where

$$
\begin{aligned}
H_{20}(0) &= \begin{cases} \tilde{\tau}\binom{\chi_{20}}{\varsigma_{20}}\cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau}\binom{\chi_{20}}{\varsigma_{20}} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\overline{g}_{02}) \cdot f_0, & n = 0, \end{cases} \\
H_{11}(0) &= \begin{cases} \tilde{\tau}\binom{\chi_{11}}{\varsigma_{11}}\cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau}\binom{\chi_{11}}{\varsigma_{11}} - \frac{1}{2}(p_1(0)g_{11} + p_2(0)\overline{g}_{11}) \cdot f_0, & n = 0. \end{cases}
\end{aligned}
\tag{3.41}
$$

By the definition of $A_{\tilde{\tau}}$ and (3.39), we have

$$
\dot{W}_{20} = A_{\tilde{\tau}} W_{20} = 2\mathrm{i}\omega_n\tilde{\tau} W_{20} + \frac{1}{2}\big(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}\big) \cdot f_n, \quad -1 \le \theta < 0.
$$

That is,

$$
W_{20}(\theta) = \frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}\left(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)\right) \cdot f_n + E_1 e^{2\mathrm{i}\omega_n\tilde{\tau}\theta},
$$

where

$$
E_1 = \begin{cases} W_{20}(0), & n = 1,2,3,\dots, \\ W_{20}(0) - \frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)) \cdot f_0, & n = 0. \end{cases}
$$

By the definition of $A_{\tilde{\tau}}$ and (3.39), we have that for $-1 \leq \theta < 0$,

$$\begin{aligned}
&-\left(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)\right) \cdot f_0 + 2\mathrm{i}\omega_n\tilde{\tau}E_1 - A_{\tilde{\tau}}\left(\frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)\right) \cdot f_0\right) \\
&\quad - A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}\left(\frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)\right) \cdot f_n + E_1 e^{2\mathrm{i}\omega_n\tilde{\tau}\theta}\right) \\
&= \tilde{\tau}\begin{pmatrix}\chi_{20} \\ \varsigma_{20}\end{pmatrix} - \frac{1}{2}\left(p_1(0)g_{20} + p_2(0)\overline{g}_{02}\right) \cdot f_0.
\end{aligned}$$

As

$$A_{\tilde{\tau}}p_1(0) + L_{\tilde{\tau}}(p_1 \cdot f_0) = \mathrm{i}\omega_0 p_1(0) \cdot f_0$$

and

$$A_{\tilde{\tau}}p_2(0) + L_{\tilde{\tau}}(p_2 \cdot f_0) = -\mathrm{i}\omega_0 p_2(0) \cdot f_0,$$

we have

$$2\mathrm{i}\omega_n E_1 - A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}E_1 e^{2\mathrm{i}\omega_n} = \tilde{\tau}\begin{pmatrix}\chi_{20} \\ \varsigma_{20}\end{pmatrix}\cos^2\left(\frac{nx}{l}\right), \quad n \in \mathbb{N}_0.$$

That is,

$$E_1 = \tilde{\tau}\begin{pmatrix} 2\mathrm{i}\omega_n\tilde{\tau} + d_1\frac{n^2}{l^2} - a_1 + u_* e^{-2\mathrm{i}\omega_n\tilde{\tau}} & a_2 \\ -ca_3 & 2\mathrm{i}\omega_n\tilde{\tau} + d_2\frac{n^2}{l^2} - c \end{pmatrix}^{-1}\begin{pmatrix}\chi_{20} \\ \varsigma_{20}\end{pmatrix}\cos^2\left(\frac{nx}{l}\right).$$

Similarly, from (3.40), we have

$$-\dot{W}_{11} = \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}\right) \cdot f_n, \quad -1 \leq \theta < 0.$$

That is,

$$W_{11}(\theta) = \frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}\left(p_1(\theta)\overline{g}_{11} - p_1(\theta)g_{11}\right) + E_2.$$

Then, we have

$$E_2 = \tilde{\tau}\begin{pmatrix} d_1\frac{n^2}{l^2} - a_1 + u_* & a_2 \\ -ca_3 & d_2\frac{n^2}{l^2} - c \end{pmatrix}^{-1}\begin{pmatrix}\chi_{11} \\ \varsigma_{11}\end{pmatrix}\cos^2\left(\frac{nx}{l}\right).$$

Thus, we can obtain

$$\begin{aligned}
&c_1(0) = \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{1}{2}g_{21}, \qquad \mu_2 = -\frac{\operatorname{Re}(c_1(0))}{\operatorname{Re}(\lambda'(\tau_n^j))}, \\
&T_2 = -\frac{1}{\omega_n\tilde{\tau}}\left[\operatorname{Im}\left(c_1(0)\right) + \mu_2\operatorname{Im}\left(\lambda'\left(\tau_n^j\right)\right)\right], \qquad \beta_2 = 2\operatorname{Re}\left(c_1(0)\right).
\end{aligned} \tag{3.42}$$

Theorem 3.3 *For any critical value $\tau_n^{j,\pm}$, the bifurcating periodic solutions exists for $\tau > \tau_n^{j,\pm}$ (or $\tau < \tau_n^{j,\pm}$) when $\mu_2 > 0$ (or $\mu_2 < 0$), and are orbitally asymptotically stable (or unstable) when $\beta_2 < 0$ (or $\beta_2 > 0$).*

4 A numerical simulation

In this section, we give a numerical simulation done with Matlab. The numerical simulation of the systems is implemented by finite-difference methods. In the model (1.3), we fix the following parameters:

$$s = 1, \qquad a = 2.5, \qquad b = 3, \qquad d_1 = 0.5, \qquad d_2 = 1, \qquad c = 0.1, \qquad l = 2.$$

The model (1.3) has a unique positive equilibrium $(u_*, v_*) \approx (0.4691, 0.3845)$. By direct computation, we have $\mathcal{D} = \{0, 1\} \neq \emptyset$, and $\tau_* \approx 2.0053$. By Theorem 3.2, we know that (u_*, v_*) is locally asymptotically stable when $\tau \in [0, \tau_*)$ (shown in Fig. 1). Hopf bifurcation occurs when $\tau = \tau_*$. By Theorem 3.3, we have

$$\mu_2 \approx 0.6816 > 0, \qquad \beta_2 \approx -0.0942 < 0, \qquad \text{and} \qquad T_2 \approx 7.0182 > 0.$$

Hence, the locally asymptotically stable bifurcating periodic solutions exists for $\tau > 2.0053$, and the periods of bifurcating periodic solutions increase (shown in Fig. 2).

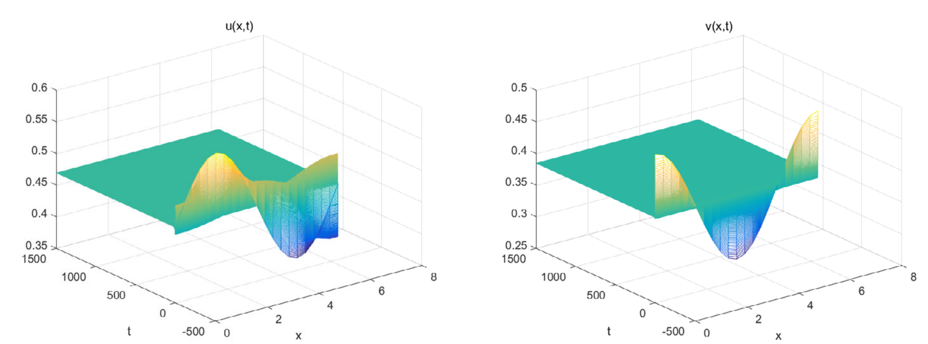

Figure 1 Numerical simulations of system (1.3) for $\tau = 1.5$

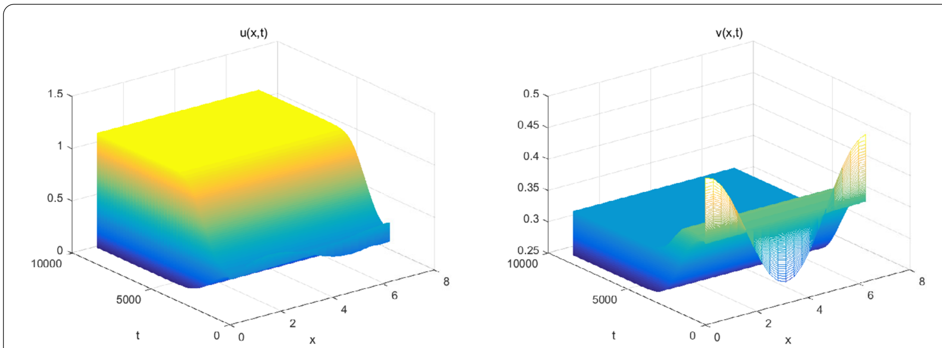

Figure 2 Numerical simulations of system (1.3) for $\tau = 2.1$

5 Conclusion

In this paper, we considered a delayed diffusive predator–prey model with toxic substances released by prey. We mainly analyzed the effect of the time delay on the stability of the positive equilibrium, and time delay induced Hopf bifurcation. We gave some parameters that determine the properties of Hopf bifurcation, namely bifurcation direction and the stability of the bifurcating periodic solution. Compared with the model (1.1), time delay is an important factor in relationship between prey and predator, since it may affect the stability of the positive equilibrium and induce Hopf bifurcation.

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Availability of data and materials
Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The idea of this research was introduced by RY. All authors contributed to the main results and numerical simulations. All authors read and approved the final manuscript.

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