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A new construction of Lupaş operators and its approximation properties

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Abstract
The aim of this paper is to study a new generalization of Lupaş-type operators whose construction depends on a real-valued function ρ by using two sequences u_m and v_m of functions. We prove that the new operators provide better weighted uniform approximation over $[0,\infty)$. In terms of weighted moduli of smoothness, we obtain degrees of approximation associated with the function ρ. Also, we prove Voronovskaya-type theorem, quantitative estimates for the local approximation.

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1 Introduction

The Weierstrass approximation theorem is the basis of approximation theory introduced by Weierstrass [15], which states that each continuous function defined on $[a,b]$ can be approximated uniformly by some polynomial. In 1912, Bernstein [3] established a constructive proof of the Weierstrass theorem by using Korovkin's theorem [9].

On the other hand, Cárdenas et al. [4] defined the Bernstein-type operators by $B_m(go\rho^{-1})o\rho$ and also presented a better degree of approximation depending on ρ. This type of approximation operators generalizes the Korovkin set from $\{e_0,e_1,e_2\}$ to $\{e_0,\rho,\rho^2\}$. In 2014, Aral et al. [1] also proposed a new modification of Szász–Mirakyan type operators to investigate approximation properties of the announced operators acting on functions defined on unbounded intervals $[0,\infty)$. For various other generalizations of Szász–Mirakyan type operators, one can go through these papers [12–14] of Srivastava et al.

Very recently, for $m \geq 1$, $z \geq 0$, and suitable functions g defined on $[0,\infty)$, Hatice et al. [8] introduced a new modification of Lupaş operators [11] using a suitable function ρ as follows:

$$L_m^{\rho}(g;z) = 2^{-m\rho(z)} \sum_{j=0}^{\infty} \frac{(m\rho(z))_j}{2^j j!} \left(go\rho^{-1}\right)\left(\frac{j}{m}\right), \tag{1.1}$$

where ρ satisfies following properties:

(ρ_1) ρ is a continuously differentiable function on $[0,\infty)$,

(ρ_2) $\rho(0) = 0$ and $\inf_{z\in[0,\infty)} \rho'(z) \geq 1$, and $(m\rho(z))_j$ is the rising factorial defined as follows:

$$\begin{aligned} &(m\rho(z))_0 = 1, \\ &(m\rho(z))_j = (m\rho(z))(m\rho(z)+1)(m\rho(z)+2)\cdots(m\rho(z)+j-1), \quad j \geq 0. \end{aligned}$$

If we put $\rho(z) = z$ in (1.1), then it reduces to the classical Lupaş operators defined in [11].

Very recently, a new construction of Szász–Mirakjan operators was given by Aral et al. [2] by using ρ and two sequences of functions α_m, β_m defined on a subinterval of $[0,\infty)$:

$$\tilde{S}_m^{\rho}(g;z) = e^{-\alpha_m(z)} \sum_{j=0}^{\infty} \frac{(\beta_m(z))^j}{j!} (go\rho^{-1}) \left(\frac{j}{m}\right). \tag{1.2}$$

Inspired by the idea used by Aral et al. in [2], in this paper we define a new construction of Lupaş operator (1.1) which depends on $\alpha_m(z)$ and $\beta_m(z)$, where $\alpha_m(z)$ and $\beta_m(z)$ are sequences of functions defined on $\tilde{E} \subset [0,\infty)$.

The paper is organized as follows. In Sect. 2, the construction of the announced operator is presented and its moments and central moments are calculated. In Sect. 3, we study convergence properties by using weighted space. In Sect. 4, we obtain the rate of convergence of new constructed operators associated with the weighted modulus of continuity. In Sect. 5, we prove Voronovskaya-type asymptotic formula. Finally, in Sect. 6, we give some approximation results related to $\mathcal{K}$-functional, also we define a Lipschitz-type functions.

2 The construction of Lupaş-type operators

Let g be a continuous functions on $[0,\infty)$ and $\tilde{E} \subset [0,\infty)$. For given $m_0 \in \mathbb{N}$, define $\mathbb{N}_1 = \{m \in \mathbb{N} : m \geq m_0\}$.

Let $\alpha_m, \beta_m : \tilde{E} \to \mathbb{R}$ such that

$$\beta_m(z) - \alpha_m(z) \geq 0 \quad \text{for any } z \in \tilde{E} \text{ and } m \in \mathbb{N}_1, \tag{2.1}$$

where α_m, β_m are positive functions defined on $\tilde{E}$.

Then we consider the new operators in the following form:

$$L_m^{\rho}(g;z) = 2^{-\alpha_m(z)} \sum_{j=0}^{\infty} \frac{(\beta_m(z))_j}{2^j j!} (go\rho^{-1}) \left(\frac{j}{m}\right), \tag{2.2}$$

where ρ is a function which satisfies the conditions (ρ_1) and (ρ_2).

We will impose some assumptions on these operators, to show the sequence of operators (2.2) is an approximation process.

We suppose that, for $z \in \tilde{E}$,

$$L_m^{\rho}(1;z) = 1 + u_m(z),$$

where $u_m : \tilde{E} \to \mathbb{R}$. From (2.2), we obtain

$$\begin{aligned} L_m^\rho(1;z) &= 2^{-\alpha_m(z)} \sum_{j=0}^{\infty} \frac{(\beta_n(z))_j}{2^j j!} \\ &= 2^{\beta_m(z)-\alpha_m(z)}. \end{aligned}$$

Thus, we get

$$2^{\beta_m(z)-\alpha_m(z)} = 1 + u_m(z). \tag{2.3}$$

Secondly, we assume that

$$L_m^\rho(\rho;z) = \rho(z) + \nu_m(z), \tag{2.4}$$

where $\nu_m : \tilde{E} \to \mathbb{R}$. From (2.2), we infer

$$\begin{aligned} L_m^\rho(\rho;z) &= 2^{-\alpha_m(z)} \sum_{j=0}^{\infty} \frac{(\beta_m(z))_j}{2^j j!} \frac{j}{m} \\ &= \frac{\beta_m(z)}{m} 2^{\beta_m(z)-\alpha_m(z)}. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$\frac{\beta_m(z)}{m} 2^{\beta_m(z)-\alpha_m(z)} = \rho(z) + \nu_m(z). \tag{2.6}$$

Now combining (2.3) and (2.6), we get

$$\beta_m(z) = m \frac{\rho(z) + \nu_m(z)}{1 + u_m(z)}, \tag{2.7}$$

and from relation (2.3), we can write

$$\beta_m(z) - \alpha_m(z) = \ln_2\bigl(1 + u_m(z)\bigr)$$

and

$$\alpha_m(z) = m \frac{\rho(z) + \nu_m(z)}{1 + u_m(z)} - \ln_2\bigl(1 + u_m(z)\bigr), \tag{2.8}$$

where $u_m(z) > -1$ for any $z \in \tilde{E}$ and $m \in \mathbb{N}_1$.

Therefore, as a consequence, operators (2.2) become

$$L_m^\rho(g;z) = 2^{-m\frac{\rho(z)+\nu_m(z)}{1+u_m(z)}} \bigl(1 + u_m(z)\bigr) \sum_{j=0}^{\infty} \frac{(m\frac{\rho(z)+\nu_m(z)}{1+u_m(z)})_j}{2^j j!} \bigl(go\rho^{-1}\bigr)\left(\frac{j}{m}\right) \tag{2.9}$$

for $m \in \mathbb{N}_1$ and for any $z \in \tilde{E}$.

We can recover some linear positive operators which are already in the literature. From operators (2.9) and for the particular choices of the functions u_m, ν_m, and ρ:

(i) If we take $u_m(z) = v_m(z) = 0$, operators (2.9) turn out to be operators (1.1).
(ii) If we take $u_m(z) = v_m(z) = 0$, $\rho(z) = z$, operators (2.9) turn out to be the classical Lupaş operators given in [11] by

$$L_m(g;z) = 2^{-mz} \sum_{j=0}^{\infty} \frac{(mz)_j}{2^j j!} g\left(\frac{j}{m}\right).$$

Now, in order to obtain weighted approximation processes, we assume that the following inequalities hold:

$$\left|u_m(z)\right| \le u_m \quad \text{and} \quad \left|v_m(z)\right| \le v_m, \quad z \in \tilde{E} \tag{2.10}$$

such that

$$\lim_{m\to\infty} u_m = \lim_{m\to\infty} v_m = 0. \tag{2.11}$$

From (2.10) and (2.11) it is clear that $(L_m^{\rho}(g;z))_{m\ge m_0}$ is an approximation process on $\tilde{E} \subset [0,\infty)$.

Now, we give some lemmas which are required to prove our main results.

Lemma 2.1 *For the operators $L_m^{\rho}(g;z)$ and for all $z \in \tilde{E}$, we have*:
1. $L_m^{\rho}(1;z) = 1 + u_m(z)$,
2. $L_m^{\rho}(\rho;z) = \rho(z) + v_m(z)$,
3. $L_m^{\rho}(\rho^2;z) = \frac{(\rho(z)+v_m(z))^2}{1+u_m(z)} + \frac{2}{m}(\rho(z) + v_m(z))$,
4. $L_m^{\rho}(\rho^3;z) = \frac{(\rho(z)+v_m(z))^3}{(1+u_m(z))^2} + \frac{6(\rho(z)+v_m(z))^2}{m(1+u_m(z))} + \frac{6}{m^2}(\rho(z) + v_m(z))$,
5. $L_m^{\rho}(\rho^4;z) = \frac{(\rho(z)+v_m(z))^4}{(1+u_m(z))^3} + \frac{12(\rho(z)+v_m(z))^3}{m(1+u_m(z))^2} + \frac{36(\rho(z)+v_m(z))^2}{m^2(1+u_m(z))} + \frac{26}{m^3}(\rho(z) + v_m(z))$.

Lemma 2.2 *By using Lemma 2.1 and by the linearity of operators L_m^{ρ}, we can acquire the central moments as follows*:
1. $L_m^{\rho}(\rho(\zeta) - \rho(z);z) = v_m(z) - \rho(z)u_m(z)$,
2. $L_m^{\rho}((\rho(\zeta) - \rho(u))^2;u) = \frac{(\rho(z)+v_m(z))^2}{1+u_m(z)} + \frac{2(\rho(z)+v_m(z))(1-m\rho(z))}{m} + (1 + u_m(z))\rho^2(z)$,
3. $L_m^{\rho}((\rho(\zeta) - \rho(u))^3;u) =$ $\frac{(\rho(z)+v_m(z))^3}{(1+u_m(z))^2} + \frac{3(\rho(z)+v_m(z))^2(2-n\rho(z))}{m(1+u_m(z))} + \frac{(\rho(z)+v_m(z))(6-6n\rho(z)+3m^2\rho^2(z))}{m^2} - (1 + u_m(z))\rho^3(z)$,
4. $L_m^{\rho}((\rho(\zeta) - \rho(u))^4;u) = \frac{(\rho(z)+v_m(z))^4}{(1+u_m(z))^3} + \frac{(\rho(z)+v_m(z))^3(12-4n\rho(z))}{m(1+u_m(z))^2} +$ $\frac{(\rho(z)+v_m(z))^2(36-24n\rho(z)+6m^2\rho^2(z))}{m^2(1+u_m(z))} + \frac{(\rho(z)+v_m(z))(26-24n\rho(z)+12m^2\rho^2(z)-4m^3\rho^3(z))}{m^3} + (1 + u_m(z))\rho^4(z)$.

Remark 2.3 If we put $u_m(z) = v_m(z) = 0$ in Lemma 2.1 and Lemma 2.2, we have the same result proved in Lemma 2 and Lemma 3 of [8].

3 Direct result in weighted space

In this section, by using weighted space, we discuss some convergence properties for the operators L_m^{ρ}.

Let $\Phi(z) = 1 + \rho^2(z)$ be a weight function and $\mathcal{B}_\Phi[0,\infty)$ be the weighted spaces defined as follows:

$$\mathcal{B}_\Phi[0,\infty) = \{g : [0,\infty) \to \mathbb{R} \,|\, |g(z)| \le \mathcal{K}_f \Phi(z), z \ge 0\},$$

where $\mathcal{K}_g$ is a constant and $\mathcal{B}_\Phi[0,\infty)$ is a normed linear space equipped with the norm

$$\|g\|_\Phi = \sup_{z\in[0,\infty)} \frac{|g(z)|}{\Phi(z)}.$$

Also, the subspaces $\mathcal{C}_\Phi[0,\infty)$, $\mathcal{U}_\Phi[0,\infty)$ and $\mathcal{U}_\Phi[0,\infty)$ of $\mathcal{B}_\Phi[0,\infty)$ are defined as

$$\mathcal{C}_\Phi[0,\infty) = \{g \in \mathcal{B}_\Phi[0,\infty) : g \text{ is continuous on } [0,\infty)\},$$

$$\mathcal{C}^*_\Phi[0,\infty) = \left\{g \in \mathcal{C}_\Phi[0,\infty) : \lim_{z\to\infty} \frac{g(z)}{\Phi(z)} = \mathcal{K}_g = \text{Constant}\right\},$$

$$\mathcal{U}_\Phi[0,\infty) = \left\{g \in \mathcal{C}_\Phi[0,\infty) : \frac{g(z)}{\Phi(z)} \text{ is uniformly continuous on } [0,\infty)\right\}.$$

It is obvious that $\mathcal{C}^*_\Phi[0,\infty) \subset \mathcal{U}_\Phi[0,\infty) \subset \mathcal{C}_\Phi[0,\infty) \subset \mathcal{B}_\Phi[0,\infty)$.

In [6], Gadjiev proved the following results for the weighted Korovkin-type theorems.

Lemma 3.1 ([6]) *For $m \geq 1$, $\mathcal{G}_m : \mathcal{B}_\Phi[0,\infty) \to \mathcal{B}_\Phi[0,\infty)$ satisfying*

$$\left|\mathcal{G}_m(\Phi;z)\right| \leq \mathcal{K}_m\Phi(z), \quad z \geq 0,$$

holds, where $\mathcal{K}_m > 0$ is a constant depending on m.

Theorem 3.2 ([6]) *For $m \geq 1$, $\mathcal{G}_m : \mathcal{B}_\Phi[0,\infty) \to \mathcal{B}_\Phi[0,\infty)$ satisfies*

$$\lim_{m\to\infty} \left\|\mathcal{G}_m\rho^i - \rho^i\right\|_\Phi = 0, \quad i = 0,1,2.$$

Then, for any function $g \in C^*_\Phi[0,\infty)$, we obtain

$$\lim_{m\to\infty} \left\|\mathcal{G}_m(g) - g\right\|_\Phi = 0.$$

Therefore, our result follows.

Theorem 3.3 *For each $g \in C^*_\Phi[0,\infty)$, the following relation*

$$\lim_{m\to\infty} \sup_{z\in\tilde{E}} \frac{|L^\rho_m(g;z) - g(z)|}{\Phi(z)} = 0$$

holds, provided that conditions (2.10) and (2.11) are fulfilled.

Proof Let $g \in C^*_\Phi[0,\infty)$. Then $|g(z)| \leq \mathcal{H}_g\Phi(z)$, $z \geq 0$. L^ρ_m being linear and positive, it is monotone. Thus

$$L^\rho_m(\Phi;z) = 1 + u_m(z) + \frac{(\rho(z) + v_m(z))^2}{1 + u_m(z)} + \frac{2}{m}\big(\rho(z) + v_m(z)\big)$$

implies that the operator L^ρ_m maps the space $C_\Phi[0,\infty)$ into $B_\Phi[0,\infty)$.

By (2.10) and Lemma 2.1, we have

$$\lim_{m\to\infty} \sup_{z\in\tilde{E}} \frac{|L^\rho_m(\rho^r) - \rho^r|}{\Phi(z)} = 0, \quad r = 0,1,2. \tag{3.1}$$

As we know each $L_m^\rho(g;z)$ is defined on $\tilde{E}$. Now, by considering the following sequence of operators, we extend it on $[0,\infty)$:

$$\mathcal{A}_m = \begin{cases} L_m^\rho(g;z), & \text{if } z \in \tilde{E}, \\ g(z), & \text{if } z \in [0,\infty) \setminus \tilde{E}. \end{cases}$$

Obviously,

$$\left\| \mathcal{A}_m(g) - g \right\|_\Phi = \sup_{z \in \tilde{E}} \frac{L_m^\rho(g;z) - g(z)}{\Phi(z)}. \tag{3.2}$$

By applying 3.2 to the operators $\mathcal{G}_m = \mathcal{A}_m$ the claim will be proved.

Hence, we have to prove that

$$\left\| \mathcal{A}_m(\rho^r) - \rho^r \right\|_\Phi \to 0 \quad \text{as } m \to \infty, r = 0,1,2.$$

Since

$$\left\| \mathcal{A}_m(\rho^r) - \rho^r \right\|_\Phi = \sup_{z \in \tilde{E}} \frac{|L_m^\rho(\rho^r)(z) - \rho^r(z)|}{\Phi(z)},$$

by using (3.1), we have

$$\lim_{m\to\infty} \left\| \mathcal{A}_m(g) - g \right\|_\Phi = 0.$$

By using (3.2), we get the desired result. □

4 Rate of convergence

In this section, by using weighted modulus of continuity $\omega_\rho(g;\delta)$, we determine the rate of convergence for L_m^ρ which was recently considered by Holhoş [7] as follows:

$$\omega_\rho(g;\delta) = \sup_{z,\zeta \in [0,\infty), |\rho(\zeta)-\rho(z)| \le \delta} \frac{|g(\zeta) - g(z)|}{\Phi(\zeta) + \Phi(z)}, \quad \delta > 0, \tag{4.1}$$

where $g \in \mathcal{C}_\Phi[0,\infty)$, with the following properties:

(i) $\omega_\rho(g;0) = 0$,

(ii) $\omega_\rho(g;\delta) \ge 0$, $\delta \ge 0$ for $g \in \mathcal{C}_\Phi[0,\infty)$,

(iii) $\lim_{\delta\to 0} \omega_\rho(g;\delta) = 0$ for each $g \in U_\Phi[0,\infty)$.

Theorem 4.1 ([7]) *Let us consider a sequence of positive linear operators* $\mathcal{G}_m : \mathcal{C}_\Phi[0,\infty) \to \mathcal{B}_\Phi[0,\infty)$ *with*

$$\left\| \mathcal{G}_m(\rho^0) - \rho^0 \right\|_{\Phi^0} = a_m, \tag{4.2}$$

$$\left\| \mathcal{G}_m(\rho) - \rho \right\|_{\Phi^{\frac{1}{2}}} = b_m, \tag{4.3}$$

$$\left\| \mathcal{G}_m(\rho^2) - \rho^2 \right\|_{\Phi} = c_m, \tag{4.4}$$

$$\left\| \mathcal{G}_m(\rho^3) - \rho^3 \right\|_{\Phi^{\frac{3}{2}}} = d_m, \tag{4.5}$$

where the sequences a_m, b_m, c_m, and d_m converge to zero as $m \to \infty$. Then

$$\left\| \mathcal{G}_m(g) - g \right\|_{\Phi^{\frac{3}{2}}} \leq (7 + 4a_m + 2c_m)\omega_\rho(g;\delta_m) + \|g\|_\Phi a_m \tag{4.6}$$

for all $g \in \mathcal{C}_\Phi[0,\infty)$, where

$$\delta_m = 2\sqrt{(a_m + 2b_m + c_m)(1 + a_m)} + a_m + 3b_m + 3c_m + d_m.$$

Theorem 4.2 *Let us have, for each $g \in \mathcal{C}_\Phi[0,\infty)$,*

$$\begin{aligned}\left\| L_m^\rho(g) - g \right\|_{\Phi^{\frac{3}{2}}} \leq{}& \left(7 + 4u_m + 2\left(v_m^2 + 2v_m + \frac{2}{m} + \frac{2v_m}{m}\right)\right)\omega_\rho(g;\delta_m) \\ &+ \|g\|_\Phi u_m,\end{aligned}$$

where

$$\begin{aligned}\delta_m ={}& 2\sqrt{\left(u_m + 4v_m + v_m^2 + \frac{2}{m} + \frac{2v_m}{m}\right)(1 + u_m)} \\ &+ v_m^3 + 6v_m^2 + 12v_m + \frac{6v_m^2}{m} + \frac{18v_m}{m} + \frac{6v_m}{m^2} + \frac{12}{m} + \frac{6}{m^2}.\end{aligned}$$

Proof If we calculate the sequences (a_m), (b_m), (c_m), and (d_m), then by using Lemma 2.1, clearly we have

$$\left\| L_m^\rho(\rho^0) - \rho^0 \right\|_{\Phi^0} = \sup_{z \in \tilde{E}} u_m(z) \leq u_m = a_m,$$

$$\left\| L_m^\rho(\rho) - \rho \right\|_{\Phi^{\frac{1}{2}}} = \sup_{z \in \tilde{E}} \frac{v_m(z)}{\sqrt{1 + \rho^2}} \leq v_m = b_m,$$

and

$$\left\| L_m^\rho(\rho^2) - \rho^2 \right\|_\Phi \leq v_m^2 + 2v_n + \frac{2}{m} + \frac{2v_m}{m} = c_m.$$

Finally,

$$\left\| L_m^\rho(\rho^3) - \rho^3 \right\|_{\Phi^{\frac{3}{2}}} \leq v_m^3 + 3v_m^2 + 3v_m + \frac{6v_m^2}{m} + \frac{12v_m}{m} + \frac{6}{m} + \frac{6}{m^2} + \frac{6v_m}{m^2} = d_m.$$

Thus conditions (4.1)–(4.5) are satisfied. Now, by Theorem 4.1, we obtain the desired result. □

Remark 4.3 From property (*iii*) of $\omega_\rho(g;\delta)$ and Theorem 4.2, we have

$$\lim_{m \to \infty} \left\| L_m^\rho(g) - g \right\|_{\Phi^{\frac{3}{2}}} = 0 \quad \text{for } g \in U_\Phi[0,\infty).$$

5 Voronovskaya-type theorem

In this section, we establish Voronovskaya-type result for L_m^ρ.

Theorem 5.1 *Let $g \in \mathcal{C}_\Phi[0,\infty)$, $z \in [0,\infty)$ and suppose that $(go\rho^{-1})'$ and $(go\rho^{-1})''$ exist at $\rho(z)$. If $(go\rho^{-1})''$ is bounded on $[0,\infty)$ and*

$$\lim_{m\to\infty} mz_m(z) = \ell_1, \qquad \lim_{m\to\infty} m\nu_m(z) = \ell_2,$$

then we have

$$\lim_{m\to\infty} m\big[L_m^\rho(g;z) - g(z)\big] = \rho(z)\ell_1 + \ell_2 - \rho(z)\ell_1\big(go\rho^{-1}\big)'\rho(z) + \rho(z)\big(go\rho^{-1}\big)''\rho(z).$$

Proof By using the Taylor expansion of $(go\rho^{-1})$ at $\rho(z) \in [0,\infty)$, there exists a point ζ lying between z and z, we have

$$\begin{aligned} g(\zeta) &= \big(go\rho^{-1}\big)\big(\rho(\zeta)\big) \\ &= \big(go\rho^{-1}\big)\big(\rho(z)\big) + \big(go\rho^{-1}\big)'\big(\rho(z)\big)\big(\rho(\zeta) - \rho(z)\big) \\ &\quad + \frac{(go\rho^{-1})''(\rho(z))(\rho(\zeta) - \rho(z))^2}{2} + \lambda_z(\zeta)\big(\rho(\zeta) - \rho(z)\big)^2, \end{aligned} \tag{5.1}$$

where

$$\lambda_z(\zeta) = \frac{(go\rho^{-1})''(\rho(\zeta)) - (go\rho^{-1})''(\rho(z))}{2}. \tag{5.2}$$

Therefore, the assumption on g and (5.2) ensure that

$$\big|\lambda_z(\zeta)\big| \le \mathcal{K} \quad \text{for all } \zeta \in [0,\infty)$$

and $\lim_{\zeta\to z} \lambda_z(\zeta) = 0$. By applying operators (2.9) to (5.1), we obtain

$$\begin{aligned} \big[L_m^\rho(g;z) - g(z)\big] &= \big(go\rho^{-1}\big)'\big(\rho(z)\big)L_m^\rho\big(\big(\rho(\zeta) - \rho(z)\big);z\big) \\ &\quad + \frac{(go\rho^{-1})''(\rho(z))L_m^\rho((\rho(\zeta) - \rho(y))^2;z)}{2} \\ &\quad + L_m^\rho\big(\lambda^z(\zeta)\big(\big(\rho(\zeta) - \rho(z)\big)^2;z\big)\big). \end{aligned} \tag{5.3}$$

From Lemmas 2.1 and 2.2, we obtain

$$\lim_{m\to\infty} mL_m^\rho\big(\big(\rho(\zeta) - \rho(z)\big);z\big) = \ell_2 - \rho(z)\ell_1 \tag{5.4}$$

and

$$\lim_{m\to\infty} mL_m^\rho\big(\big(\rho(\zeta) - \rho(z)\big)^2;z\big) = 2\rho(z). \tag{5.5}$$

Since from (5.2), for every $\epsilon > 0$, $\lim_{\zeta\to z}\lambda_z(\zeta) = 0$. Let $\delta > 0$ such that $|\lambda_z(\zeta)| < \epsilon$ for every $\zeta \geq 0$. From the Cauchy–Schwarz inequality, we get immediately

$$\lim_{m\to\infty} mL_m^{\rho}\left(\left|\lambda_z(\zeta)\right|\left(\rho(\zeta)-\rho(z)\right)^2;z\right) \leq \epsilon \lim_{m\to\infty} mL_m^{\rho}\left(\left(\rho(\zeta)-\rho(z)\right)^2;z\right) + \frac{L}{\delta^2}\lim_{m\to\infty} L_m^{\rho}\left(\left(\rho(\zeta)-\rho(z)\right)^4;z\right).$$

Since

$$\lim_{m\to\infty} mL_m^{\rho}\left(\left(\rho(\zeta)-\rho(z)\right)^4;z\right) = 0, \tag{5.6}$$

we obtain

$$\lim_{m\to\infty} mL_m^{\rho}\left(\left|\lambda_z(\zeta)\right|\left(\rho(\zeta)-\rho(z)\right)^2;z\right) = 0. \tag{5.7}$$

Thus, taking into account equations (5.4), (5.5), and (5.7) to equation (5.3), this proves the theorem. □

6 Local and global approximation

In order to prove local approximation theorems for the operators, let $\mathcal{C}_B[0,\infty)$ be the space of real-valued continuous and bounded functions g with the norm $\|\cdot\|$ given by

$$\|g\| = \sup_{0\leq z<\infty}\left|g(z)\right|.$$

We begin by considering the $\mathcal{K}$-functional

$$\mathcal{K}_2(g,\delta) = \inf_{r\in W^2}\left\{\|g-r\| + \delta\left\|g''\right\|\right\},$$

where $\delta > 0$ and $W^2 = \{s \in \mathcal{C}_B[0,\infty) : r', r'' \in \mathcal{C}_B[0,\infty)\}$. Then, in view of the known result [5], there exists an absolute constant $\mathcal{D} > 0$ such that

$$L(g,\delta) \leq \mathcal{D}\omega_2(g,\sqrt{\delta}). \tag{6.1}$$

For $g \in \mathcal{C}_B[0,\infty)$, the second order modulus of smoothness is defined as

$$\omega_2(g,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}}\ \sup_{z\in[0,\infty)}\left|g(z+2h)-2g(z+h)+g(z)\right|,$$

and the usual modulus of continuity is defined as

$$\omega(g,\delta) = \sup_{0<h\leq\delta}\ \sup_{z\in[0,\infty)}\left|g(z+h)-g(z)\right|.$$

Theorem 6.1 *There exists an absolute constant $\mathcal{D} > 0$ such that*

$$\left|L_m^{\rho}(g;z)-g(z)\right| \leq \mathcal{D}\mathcal{K}\left(g,\delta_m(z)\right),$$

where $g \in \mathcal{C}_B[0,\infty)$ *and*

$$\delta_m(z) = \left\{ \frac{(\rho(z) + v_m(z))^2}{1 + u_m(z)} + \frac{2(\rho(z) + v_m(z))(1 - m\rho(z))}{m} + \bigl(1 + u_m(z)\bigr)\rho^2(z) \right\}.$$

Proof Let $r \in W^2$ and $z, \zeta \in [0,\infty)$. By using Taylor's formula we have

$$r(\zeta) = r(z) + \bigl(ro\rho^{-1}\bigr)'\bigl(\rho(z)\bigr)\bigl(\rho(\zeta) - \rho(z)\bigr) + \int_{\rho(z)}^{\rho(\zeta)} \bigl(\rho(\zeta) - v\bigr)\bigl(ro\rho^{-1}\bigr)''(v)\,dv. \tag{6.2}$$

By using the equality

$$\bigl(ro\rho^{-1}\bigr)''\bigl(\rho(z)\bigr) = \frac{r''(z)}{(\rho'(z))^2} - r''(z)\frac{\rho''(z)}{(\rho'(z))^3}, \tag{6.3}$$

putting $v = \rho(z)$ in the last term in equality (6.2), we get

$$\begin{aligned} \int_{\rho(z)}^{\rho(\zeta)} \bigl(\rho(\zeta) - v\bigr)\bigl(ro\rho^{-1}\bigr)''(v)\,dv &= \int_z^{\zeta} \bigl(\rho(\zeta) - \rho(z)\bigr)\left[\frac{r''(z)\rho'(z) - r'(z)\rho''(v)}{(\rho'(z))^2}\right] dz \\ &= \int_{\rho(z)}^{\rho(\zeta)} \bigl(\rho(\zeta) - v\bigr)\frac{r''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2} dv \\ &\quad - \int_{\rho(z)}^{\rho(\zeta)} \bigl(\rho(\zeta) - v\bigr)\frac{r'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3} dv. \end{aligned} \tag{6.4}$$

By applying $\mathcal{S}^{*\mu,\lambda}_{m,\rho}$ to (6.2) and also by using Lemma 2.1 and (6.4), we deduce

$$\begin{aligned} L_m^{\rho}(r;z) = r(z) &+ L_m^{\rho}\left(\int_{\rho(z)}^{\rho(\zeta)} \bigl(\rho(\zeta) - v\bigr)\frac{r''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2} dv; z\right) \\ &- L_m^{\rho}\left(\int_{\rho(z)}^{\rho(\zeta)} \bigl(\rho(\zeta) - v\bigr)\frac{r'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3} dv; z\right). \end{aligned}$$

By using conditions (ρ_1) and (ρ_2), we get

$$\bigl|L_m^{\rho}(r;z) - r(z)\bigr| \leq \mathcal{M}^{\rho}_{m,2}(z)\bigl(\|r''\| + \|r'\|\|\rho''\|\bigr),$$

where

$$\mathcal{M}^{\rho}_{m,2}(z) = L_m^{\rho}\bigl(\bigl(\rho(\zeta) - \rho(z)\bigr)^2; z\bigr).$$

For all $g \in \mathcal{C}_B[0,\infty)$, we have

$$\begin{aligned} \bigl|L_m^{\rho}(r;z)\bigr| &\leq \|go\rho^{-1}\|\bigl(1 + u_m(z)\bigr) \\ &\leq \|g\| L_m^{\rho}(1;z) = \|g\|. \end{aligned} \tag{6.5}$$

Hence we have

$$\bigl|L_m^{\rho}(g;z) - g(z)\bigr| \leq \bigl|L_m^{\rho}(g - r;z)\bigr| + \bigl|L_m^{\rho}(r;z) - r(z)\bigr| + \bigl|r(z) - g(z)\bigr|$$

$$\begin{aligned}&\leq 2\|g-r\|\\&\quad+\left\{\frac{(\rho(z)+v_m(z))^2}{1+u_m(z)}+\frac{2(\rho(z)+v_m(z))(1-m\rho(z))}{m}\right.\\&\quad\left.+\big(1+u_m(z)\big)\rho^2(z)\right\}\big(\|r''\|+\|r'\|\|\rho''\|\big).\end{aligned}$$

If we choose $\mathcal{D}=\max\{2,\|\rho''\|\}$, then

$$\begin{aligned}\big|L_m^\rho(g;z)-g(z)\big|&\leq\mathcal{D}\left(2\|g-r\|+\left\{\frac{(\rho(z)+v_m(z))^2}{1+u_m(z)}\right.\right.\\&\quad\left.\left.+\frac{2(\rho(z)+v_m(z))(1-m\rho(z))}{m}+\big(1+u_m(z)\big)\rho^2(z)\right\}\|r''\|_{W^2}\right).\end{aligned}$$

Taking infimum over all $r\in W^2$, we obtain

$$\big|L_m^\rho(g;z)-g(z)\big|\leq\mathcal{D}\mathcal{K}\big(g,\delta_m(z)\big).\qquad\square$$

Let $0<\alpha\leq 1$, ρ be a function with conditions (ρ_1), (ρ_2) and $\mathrm{Lip}_{\mathcal{M}}(\rho(y);\alpha)$, $\mathcal{H}\geq 0$ satisfying

$$\big|g(\xi)-g(z)\big|\leq\mathcal{H}\big|\rho(\xi)-\rho(z)\big|^\alpha,\quad y,\xi\geq 0.$$

Moreover, $\mathcal{Y}\subset[0,\infty)$ is a bounded subset and the function $g\in\mathrm{Lip}_{\mathcal{M}}(\rho(z);\alpha)$, $0<\alpha\leq 1$ on $\mathcal{Y}$ if

$$\big|g(\zeta)-g(z)\big|\leq\mathcal{H}_{\alpha,g}\big|\rho(\zeta)-\rho(z)\big|^\alpha,\quad z\in\mathcal{Y}\text{ and }\zeta\geq 0,$$

where $\mathcal{H}_{\alpha,g}$ is a constant.

Theorem 6.2 *Let ρ be a function satisfying conditions (ρ_1), (ρ_2). Then, for any $g\in\mathrm{Lip}_{\mathcal{H}}(\rho(z);\alpha)$, $0<\alpha\leq 1$ and for every $z\in(0,\infty)$, $m\in\mathbb{N}$, we have*

$$\big|L_m^\rho(g;z)-g(z)\big|\leq\mathcal{H}\big(\delta_m(z)\big)^{\frac{\alpha}{2}},\tag{6.6}$$

where

$$\delta_m(z)=\left\{\frac{(\rho(z)+v_m(z))^2}{1+z_m(z)}+\frac{2(\rho(z)+v_m(z))(1-m\rho(z))}{m}+\big(1+z_m(z)\big)\rho^2(z)\right\}.$$

Proof Assume that $\alpha=1$. Then, for $g\in\mathrm{Lip}_{\mathcal{M}}(\alpha;1)$ and $z\in(0,\infty)$, we have

$$\begin{aligned}\big|L_m^\rho(g;z)-g(z)\big|&\leq L_m^\rho\big(\big|g(\zeta)-g(z)\big|;z\big)\\&\leq\mathcal{H}L_m^\rho\big(\big|\rho(\zeta)-g(z)\big|;z\big).\end{aligned}$$

By applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\big|L_m^\rho(g;z)-g(z)\big|&\leq\mathcal{H}\big[L_m^\rho\big(\big(\rho(\zeta)-\rho(z)\big)^2;z\big)\big]^{\frac{1}{2}}\\&\leq\mathcal{H}\sqrt{\delta_m(z)}.\end{aligned}$$

Let us assume that $\alpha \in (0,1)$. Then, for $g \in \mathrm{Lip}_{\mathcal{H}}(\alpha;1)$ and $z \in (0,\infty)$, we have

$$\begin{aligned}\left|L_m^\rho(g;z) - g(z)\right| &\leq L_m^\rho\left(\left|g(\zeta) - g(z)\right|;z\right) \\ &\leq \mathcal{H}L_m^\rho\left(\left|\rho(\zeta) - g(z)\right|^\alpha;z\right).\end{aligned}$$

By taking $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, $g \in \mathrm{Lip}_{\mathcal{H}}(\rho(z);\alpha)$, and applying Hölder's inequality, we have

$$\left|L_m^\rho(g;z) - g(z)\right| \leq \mathcal{H}\left[L_m^\rho(\left|\left(\rho(\zeta) - \rho(z)\right|;z\right)\right]^\alpha.$$

Finally, by applying the Cauchy–Schwarz inequality, we get

$$\left|L_m^\rho(g;z) - g(z)\right| \leq \mathcal{H}\left(\delta_m(z)\right)^{\frac{\alpha}{2}}. \qquad \square$$

Theorem 6.3 *Let ρ be a function satisfying conditions (ρ_1), (ρ_2) and $\mathcal{Y}$ be a bounded subset of $[0,\infty)$. Then, for any $g \in \mathrm{Lip}_{\mathcal{H}}(\rho(z);\alpha)$, $0 < \alpha \leq 1$, on $\mathcal{Y}$ $\alpha \in (0,1]$, we have*

$$\left|L_m^\rho(g;z) - g(z)\right| \leq \mathcal{H}_{\alpha,g}\left\{\left(\delta_m(z)\right)^{\frac{\alpha}{2}} + 2\left[\rho'(z)\right]^\alpha d^\alpha(z,\mathcal{Y})\right\}, \quad z \in [0,\infty), m \in \mathbb{N},$$

where $d(z,\mathcal{Y}) = \inf\{\|z - x\| : x \in \mathcal{Y}\}$ and $\mathcal{H}_{\alpha,g}$ is a constant depending on α and g, and

$$\delta_m(z) = \left\{\frac{1}{(m+1)^2}\rho^2(z) + \frac{2m-1}{(m+1)^2}\rho(z) + \frac{1}{3(m+1)^2}\right\}.$$

Proof Let $\overline{\mathcal{Y}}$ be the closure of $\mathcal{Y}$ in $[0,\infty)$. Then there exists a point $z_0 \in \overline{\mathcal{Y}}$ such that $d(z,\mathcal{Y}) = |z - z_0|$.

Using the monotonicity of L_m^ρ and the hypothesis of g, we obtain

$$\begin{aligned}\left|L_m^\rho(g;z) - g(z)\right| &\leq L_m^\rho\left(\left|g(\zeta) - g(z_0)\right|;z\right) + L_m^\rho\left(\left|g(z) - g(z_0)\right|;z\right) \\ &\leq \mathcal{H}_{\alpha,g}\left\{L_m^\rho\left(\left|\rho(\zeta) - \rho(z)\right|^\alpha;z\right) + 2\left|\rho(z) - \rho(z_0)\right|^\alpha\right\}.\end{aligned}$$

By using Hölder's inequality for $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, as well as the fact $|\rho(z) - \rho(z_0)| = \rho'(z)|\rho(z) - \rho(z_0)|$ in the last inequality, we get

$$\left|L_m^\rho(g;z) - g(z)\right| \leq \mathcal{H}_{\alpha,g}\left\{\left[L_m^\rho\left(\left(\rho(\zeta) - \rho(z)\right)^2;z\right)\right]^{\frac{1}{2}} + 2\left[\rho'(z)\left|\rho(z) - \rho(z_0)\right|\right]^\alpha\right\}.$$

Hence, by Lemma 2.2 we get the proof. $\square$

Now, we recall the local approximation given in [10] for $g \in \mathcal{C}_B[0,\infty)$ as follows:

$$\widetilde{\omega}_\alpha^\rho(g;z) = \sup_{\zeta \neq z, \zeta \in (0,\infty)} \frac{|g(\zeta) - g(z)|}{|\zeta - z|^\alpha}, \quad z \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{6.7}$$

Then we get the next result.

Theorem 6.4 *Let $g \in \mathcal{C}_B[0,\infty)$ and $\alpha \in (0,1]$. Then, for all $z \in [0,\infty)$, we have*

$$\left|L_m^\rho(g;z) - g(z)\right| \leq \widetilde{\omega}_\alpha^\rho(g;z)\left(\delta_m(z)\right)^{\frac{\alpha}{2}},$$

where

$$\delta_m(z) = \left\{ \frac{(\rho(z) + \nu_m(z))^2}{1 + z_m(z)} + \frac{2(\rho(z) + \nu_m(z))(1 - m\rho(z))}{m} + \bigl(1 + z_m(z)\bigr)\rho^2(z) \right\}.$$

Proof We know that

$$\bigl|L_m^\rho(g;z) - g(z)\bigr| \leq L_m^\rho\bigl(\bigl|g(\zeta) - g(z)\bigr|;z\bigr).$$

From equation (6.7), we have

$$\bigl|L_m^\rho(g;z) - g(z)\bigr| \leq \widetilde{\omega}_\alpha^\rho(g;z) L_m^\rho\bigl(\bigl|\rho(\zeta) - \rho(z)\bigr|^\alpha;z\bigr).$$

By applying Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned} \bigl|L_m^\rho(g;z) - g(z)\bigr| &\leq \widetilde{\omega}_\alpha^\rho(g;z)\bigl[L_{m,q}^\rho\bigl(\bigl(\rho(\zeta) - \rho(z)\bigr)^2;z\bigr)\bigr]^{\frac{\alpha}{2}} \\ &\leq \widetilde{\omega}_\alpha^\rho(g;z)\bigl(\delta_m(z)\bigr)^{\frac{\alpha}{2}}, \end{aligned}$$

which proves the desired result. □

Conclusion. Here, a new construction of the generalized Lupaş operators is constructed. We have investigated convergence properties, order of approximation, Voronovskaja-type results, and quantitative estimates for the local approximation. The constructed operators have better flexibility and rate of convergence which depend on the selection of the function ρ and the sequences u_m, v_m.

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The data and material used to support the findings of this study are included within the article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors carried out the whole manuscript. All authors read and approved the final manuscript.

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