Zhang et al. *Advances in Difference Equations* (2020) 2020:655
https://doi.org/10.1186/s13662-020-03118-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A nonlinear generalized subdivision scheme of arbitrary degree with a tension parameter

Zeze Zhang[1*], Hongchan Zheng[1], Jie Zhou[2] and Lulu Pan[1]

*Correspondence:
zhangzz@mail.nwpu.edu.cn
[1]School of Mathematics and Statistics, Northwestern Polytechnical University, 710072 Xi'an, Shaanxi, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, a nonlinear generalized subdivision scheme of arbitrary degree with a tension parameter is presented, which refines 2D point-normal pairs. The construction of the scheme is built upon the stationary linear generalized subdivision scheme of arbitrary degree with a tension parameter, by replacing the weighted binary arithmetic average in the linear scheme with the circle average. For such a nonlinear scheme, we investigate its smoothness and get that it can reach C^1 with suitable choices of the tension parameter when degree $m \geq 3$. Besides, the nonlinear scheme can reconstruct the circle and the selection of parameters and initial normal vectors can effectively control the shape of the limit curves.

Keywords: Linear and nonlinear subdivision schemes; Circle average; Convergence analysis; Smoothness analysis; Tension parameter

1 Introduction
Subdivision schemes have received extensive attention recently due to their simplicity and high efficiency. They are broadly classified into two main categories: linear and nonlinear schemes, according to whether the new points are the linear combinations of old points in the iterative process. Generally speaking, linear subdivision schemes are simple to be implemented and easy to be analyzed, but possibly there are artifacts and undesired inflexions on the limit curves, while nonlinear subdivision schemes can eliminate artifacts and preserve shape. So more and more attention has been paid to the nonlinear subdivision schemes. The construction of the nonlinear scheme mainly includes the algebraic method, which generally involves different nonlinear average (see [1–3]), and the geometric method (see [4–6]).

Recently Lipovetsky and Dyn [7, 8] proposed a nonlinear modified Lane–Riesenfeld algorithm and a nonlinear modified 4-point scheme by the circle average. Li et al. [9] presented a nonlinear 4-point interpolatory scheme and a nonlinear 3-point approximating scheme with a free parameter respectively, following the ideas presented by Lipovetsky and Dyn in [7]. In this paper, we present a new family of nonlinear schemes with a tension parameter that generalizes the nonlinear modified Lane–Riesenfeld algorithm presented in [7] and the nonlinear 3-point approximating scheme with a parameter introduced in [9].

In addition, the shape of the limit curve can be effectively controlled with the selection of parameters and initial normal vectors.

The outline of this paper is organized as follows. Section 2 provides a short survey of required background, including a summary on the stationary linear generalized subdivision scheme of arbitrary degree with a tension parameter introduced in [10] and a brief review on the circle average. In Sect. 3, we modify the stationary linear generalized subdivision scheme and provide a nonlinear subdivision scheme. The convergence and smoothness are analyzed in Sect. 4. In Sect. 5, some examples are given to illustrate the performance of the nonlinear scheme. Finally, we conclude the paper with a short summary and further research work in Sect. 6.

2 Preliminaries

In this section, we recall the definition of the circle average and present the algorithm of the stationary linear generalized scheme of arbitrary degree with a tension parameter (LGP scheme) introduced in [10].

2.1 The circle average

Given a real weight $t \in [0,1]$ and in a 2D space two point-normal pairs (PNPs) $P_0 = (p_0, n_0)$, $P_1 = (p_1, n_1)$, each consisting of a point and a normal unit vector, the circle average produces a new pair $P_t = P_0 \circledcirc_t P_1 = (p_t, n_t)$. The point p_t is on an auxiliary arc $\widehat{p_0 p_1}$, at arc distance $t\theta$ from p_0, where θ is the angle between n_0 and n_1, and $0 \leq \theta \leq \pi$. The selection of the auxiliary arc depends upon whether the normal vectors n_0 and n_1 are in different half-planes (relative to $p_0 p_1$). The normal n_t is the geodesic average of n_0 and n_1, and the length of the line segment $[p_0, p_1]$ is denoted by $|p_0 p_1|$.

For the selection criterion of the auxiliary arc $\widehat{p_0 p_1}$ and for specific details on the circle average, refer to [7].

2.2 The stationary linear generalized scheme of arbitrary degree with a tension parameter

In this subsection, we recall the stationary linear generalized subdivision scheme of arbitrary degree with a tension parameter (LGP scheme) introduced in [10].

Let $\mathbf{f} = \{p_1, p_2, \ldots, p_n\}$ be the initial control polygon and u be the tension parameter. The LGP scheme of degree 1 is as follows:

$$\begin{cases} p_{2i}^1 = p_i, \\ p_{2i+1}^1 = \frac{u}{1+u} p_i + \frac{1}{1+u} p_{i+1}, \end{cases}$$

and the LGP scheme of degree m ($m \geq 2$) is defined by $p_i^m = \frac{1}{2}(p_i^{m-1} + p_{i+1}^{m-1})$. The algorithm of the LGP scheme is presented in Algorithm 1, which can be seen as a generalization of the Lane–Riesenfeld algorithm.

3 The nonlinear generalized subdivision scheme of arbitrary degree with a tension parameter

In this section, we give the rule of the nonlinear generalized subdivision scheme of arbitrary degree with a tension parameter (NGP scheme). The NGP scheme refining 2D PNPs is obtained from the LGP scheme. We substitute the weighted binary arithmetic average

Algorithm 1 The refinement step of the LGP algorithm

Input: The tension parameter $u(u \in \mathbb{R}_+)$ and the data to be refined $\mathbf{f} = \{p_i\}_{i\in\mathbb{Z}}$. The degree $m(m \in \mathbb{N}_0)$ of the LGP scheme.

Output: The refinement data $S(\mathbf{f})$.

$p_{2i}^0 \leftarrow p_i$
$p_{2i+1}^0 \leftarrow p_i$
for $j \in \mathbb{N}_0$ **do**
    for $i \in \mathbb{Z}$ **do**
        $p_{2i}^{j,0} \leftarrow p_i^{j-1}$
        $p_{2i+1}^{j,0} \leftarrow \frac{u}{1+u} p_i^{j-1} + \frac{1}{1+u} p_{i+1}^{j-1}$
    end for(i)
    for $k = 1, 2, \ldots, m-1$ **do**
        for $i \in \mathbb{Z}$ **do**
            $p_i^{j,k} \leftarrow \frac{1}{2}(p_i^{j,k-1} + p_{i+1}^{j,k-1})$
        end for(i)
    end for(k)
    for $i \in \mathbb{Z}$ **do**
        $p_i^j \leftarrow p_i^{j,m-1}$
    end for(i)
end for(j)

Algorithm 2 The refinement step of the NGP algorithm

Input: The tension parameter $u(u \in \mathbb{R}_+)$ and the data to be refined $\mathbf{P} = \{P_i = (p_i, n_i)\}_{i\in\mathbb{Z}}$. The degree $m(m \in \mathbb{N}_0)$ of the NGP subdivision scheme.

Output: The refinement data $T(\mathbf{P})$.

$P_{2i}^0 \leftarrow P_i$
$P_{2i+1}^0 \leftarrow P_i$
for $j \in \mathbb{N}_0$ **do**
    for $i \in \mathbb{Z}$ **do**
        $P_{2i}^{j,0} \leftarrow P_i^{j-1}$
        $P_{2i+1}^{j,0} \leftarrow P_i^{j-1} \circledcirc_{\frac{1}{1+u}} P_{i+1}^{j-1}$
    end for(i)
    for $k = 1, 2, \ldots, m-1$ **do**
        for $i \in \mathbb{Z}$ **do**
            $P_i^{j,k} \leftarrow P_i^{j,k-1} \circledcirc_{\frac{1}{2}} P_{i+1}^{j,k-1}$
        end for(i)
    end for(k)
    for $i \in \mathbb{Z}$ **do**
        $P_i^j \leftarrow P_i^{j,m-1}$
    end for(i)
end for(j)

in the LGP scheme by the circle average and obtain the NGP scheme, which is presented in Algorithm 2.

Remark 1 When $u = 1$, the NGP scheme reduces to the modified Lane–Riesenfeld scheme introduced in [7], which is C^1 for $m \geq 2$. In particular, the NGP scheme of degree 3 becomes the nonlinear 3-point approximating scheme with a tension parameter introduced in [9], when $u = \frac{1}{2w} - 1$.

Remark 2 As the weighted circle averages of the two point-normal pairs are located on a circle, the NGP scheme can reconstruct the circle when the initial data is sampled from a circle (see Lemma 2.1 of [7]).

4 The convergence and smoothness analysis

In this section, we discuss the convergence and smoothness of the NGP scheme. If $\mathbf{p}$ is a sequence of points, we use the symbol Δp for the sequence of differences: $\Delta p_i = p_{i+1} - p_i$. Further we define $d(\mathbf{p}) = \sup_i \|p_{i+1} - p_i\|$ and $\|\mathbf{p}\|_\infty = \sup_i \|p_i\|$.

4.1 Convergence analysis

In this subsection, we analyze the convergence of the proposed nonlinear scheme. The convergence analysis is based on the proximity condition and the displacement-safe condition, i.e., if a nonlinear scheme is displacement-safe for manifold data with a contractivity factor $\eta \in (0,1)$, then the nonlinear scheme is convergent for any input manifold(see Theorem 3.6 of [11]).

First, we introduce some notation related to the NGP algorithm. For $k = 0, 1, \ldots, m-1$ and $j \in \mathbb{N}_0$, define $P_i^{j,k} = (p_i^{j,k}, n_i^{j,k})$, $e^{j,k} = \max_{i\in\mathbb{Z}}\{|p_i^{j,k} p_{i+1}^{j,k}|\}$, and $\theta^{j,k} = \max_{i\in\mathbb{Z}}\{\theta(n_i^{j,k}, n_{i+1}^{j,k})\}$. For $k = 0, 1, \ldots, m-2$, and $j \in \mathbb{N}_0$, define $\mu^{j,k} = \frac{1}{2\cos\frac{\theta^{j,k}}{4}}$. We also define, for $j \in \mathbb{N}_0$,

$$\mu^{j,m-1} = \max_{i\in\mathbb{Z}} \left\{ \frac{\sin(\frac{\theta(n_{i-1}^j, n_i^j)}{2(1+u)})}{\sin\frac{\theta(n_{i-1}^j, n_i^j)}{2}}, \frac{\sin(\frac{u\theta(n_{i-1}^j, n_i^j)}{2(1+u)})}{\sin\frac{\theta(n_{i-1}^j, n_i^j)}{2}} \right\}, \quad e^j = e^{j,m-1}, \theta^j = \theta^{j,m-1}, \mu^j = \mu^{j,m-1}.$$

Lemma 1 (Contractivity) *There exists $J \in \mathbb{N}_0$ such that the NGP algorithm is contractive in the refinement levels above J, equivalently $e^{j+1} \leq \eta e^j$ with $\eta \in (0,1)$ for $j \geq J$.*

Proof Consider the pairs $\{p_{2i+1}^{j,0}\}_{i\in\mathbb{Z}}$ inserted in the refinement step of the NGP algorithm for $k = 0$. By the definition of the circle average, from Fig. 1(a), we get

$$e^{j+1,0} \leq \mu^j e^j, \tag{1}$$

as

$$\left|p_{2i-2}^{j+1,0} p_{2i-1}^{j+1,0}\right| = \frac{|p_{i-1}^j p_i^j|}{\sin\frac{\theta(n_{i-1}^j, n_i^j)}{2}} \sin\frac{\theta(n_{i-1}^j, n_i^j)}{2+2u} \leq \frac{e^j \sin\frac{\theta(n_{i-1}^j, n_i^j)}{2+2u}}{\sin\frac{\theta(n_{i-1}^j, n_i^j)}{2}} \leq e^j \mu^j,$$

$$\left|p_{2i-1}^{j+1,0} p_{2i}^{j+1,0}\right| = \frac{|p_{i-1}^j p_i^j|}{\sin\frac{\theta(n_{i-1}^j, n_i^j)}{2}} \sin\frac{u\theta(n_{i-1}^j, n_i^j)}{2+2u} \leq \frac{e^j \sin\frac{u\theta(n_{i-1}^j, n_i^j)}{2+2u}}{\sin\frac{\theta(n_{i-1}^j, n_i^j)}{2}} \leq e^j \mu^j.$$

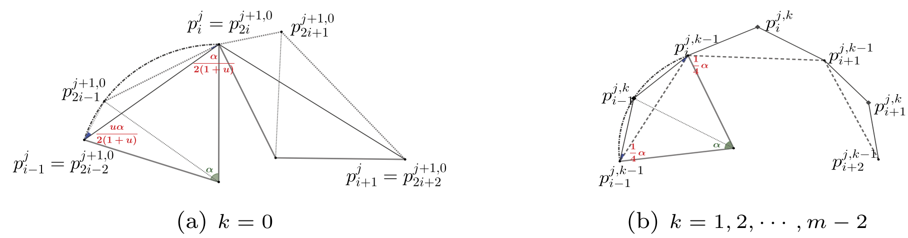

Figure 1 Geometrical illustration for the proof of Lemma 1

In any smoothing step of the NGP algorithm for $k = 1, 2, \ldots, m-1$, by the triangle inequality, from Fig. 1(b), we have

$$\left|p_i^{j,k} p_{i+1}^{j,k}\right| \leq \left|p_i^{j,k} p_{i+1}^{j,k-1}\right| + \left|p_{i+1}^{j,k-1} p_{i+1}^{j,k}\right| \leq \frac{e^{j,k-1}}{2\cos\frac{\theta^{j,k-1}}{4}} + \frac{e^{j,k-1}}{2\cos\frac{\theta^{j,k-1}}{4}} = \left(2\mu^{j,k-1}\right)e^{j,k-1}.$$

Hence

$$e^{j,k} \leq \left(2\mu^{j,k-1}\right)e^{j,k-1}, \quad k = 1, 2, \ldots, m-1, \tag{2}$$

which, together with (1), yields

$$\begin{aligned} e^{j+1} = e^{j+1,m-1} &\leq \left(2\mu^{j+1,m-2}\right)e^{j+1,m-2} \leq \left(2\mu^{j+1,m-2}\right)\left(2\mu^{j+1,m-3}\right)e^{j+1,m-3} \\ &\leq \cdots \leq \prod_{k=0}^{m-2}\left(2\mu^{j+1,k}\right)e^{j+1,0} \leq \mu^j \prod_{k=0}^{m-2}\left(2\mu^{j+1,k}\right)e^j. \end{aligned} \tag{3}$$

So we get $e^{j+1} \leq \eta_{j+1} e^j$ from (3) by setting $\eta_{j+1} = \mu^j \prod_{k=0}^{m-2}(2\mu^{j+1,k})$.

Let $B_1 = \max\{\frac{u}{1+u}, \frac{1}{1+u}\}$, we deduce that there exists j^* such that $\mu^j \leq B$ for $j \geq j^*$ with $\frac{1}{2} \leq B_1 < B < 1$, due to the fact $\lim_{j\to\infty} \mu^j = B_1$. Hence $\eta_{j+1} \leq B \prod_{k=0}^{m-2} \frac{1}{\cos\frac{\theta^{j+1,k}}{4}}$ for $j \geq j^*$. By the subdivision of the normals, we have

$$\theta^{j+1,0} \leq B_1 \theta^j, \quad \theta^{j+1,k} \leq \theta^{j+1,k-1}. \tag{4}$$

Thus

$$\theta^{j+1} \leq \theta^{j+1,0} \leq B_1 \theta^j. \tag{5}$$

Furthermore, based on (4) and (5), we obtain $\theta^{j,k} \leq \theta^{j,0} \leq B_1\theta^{j-1} \leq \theta^{j-1}$ for $k = 0, 1, \ldots, m-1$, and $\eta_{j+1} = \mu^j \prod_{k=0}^{m-2}(\frac{1}{\cos\frac{\theta^{j+1,k}}{4}}) \leq B(\frac{1}{\cos\frac{\theta^j}{4}})^{m-1}$. It follows from (5) that $\frac{1}{\cos\frac{\theta^j}{4}}$ is monotone decreasing with j. Let j^{**} be the minimal j for $(\frac{1}{\cos\frac{\theta^j}{4}})^{m-1} < \frac{1}{B}$ and $J = \max(j^*, j^{**})$, then $\eta_j \leq \eta_J < 1$ for $j \geq J$. Define $\eta = \eta_J$, we get $e^{j+1} \leq \eta e^j$ with $\eta \in (0,1)$ for $j \geq J$, which completes the proof. □

Lemma 2 (Safe displacement) *The NGP scheme is displacement safe in refinement levels above j^* described in Lemma 1, namely satisfies $|p_{2i}^{j+1} - p_i^j| \leq ce^j$ with $c \in \mathbb{R}_+$ for $j \geq j^*$.*

Proof By the triangle inequality, along with $p_{2i}^{j+1,0} = p_i^j$ and $p_{2i}^{j+1} = p_{2i}^{j+1,m-1}$, we obtain

$$|p_{2i}^{j+1}p_i^j| \le \sum_{k=0}^{m-2} |p_{2i}^{j+1,k}p_{2i}^{j+1,k+1}|.$$

It follows from (1) and (2) that, for $k = 0, 1, \dots, m-2$,

$$\begin{aligned} |p_i^{j+1,k}p_i^{j+1,k+1}| &\le \frac{e^{j+1,k}}{2\cos\frac{\theta^{j+1,k}}{4}} \le \cdots \\ &\le \frac{e^{j+1,0}}{2\cos\frac{\theta^{j+1,0}}{4}\prod_{n=1}^{k}\cos\frac{\theta^{j+1,n}}{4}} \le \frac{\mu^j e^j}{2\prod_{n=0}^{k}\cos\frac{\theta^{j+1,n}}{4}}. \end{aligned}$$

Since $\theta^j \le \theta^0 \le \pi$, we have $\frac{\theta^{j+1,k}}{4} \le \frac{\theta^j}{4} \le \frac{\pi}{4} < \frac{\pi}{3}$ and $\max_{i\in\mathbb{Z}} |p_{2i}^{j+1,k}p_{2i}^{j+1,k+1}| \le \frac{\mu^j e^j}{2(\cos\frac{\pi}{3})^{k+1}} \le 2^k\mu^j e^j$ for $k = 0, 1, \dots, m-2$. So

$$\max_{i\in\mathbb{Z}} |p_{2i}^{j+1}p_i^j| \le \mu^j e^j \sum_{k=0}^{m-2} 2^k \le B2^{m-1}e^j. \tag{6}$$

Set $c = B2^{m-1}$, we easily get $|p_{2i}^{j+1} - p_i^j| \le ce^j (c > 0)$ from (6), which completes the proof. □

By Lemma 1 and Lemma 2, we conclude the convergence of the points. It remains to prove the convergence of the normals. The convergence of the normals is an immediate consequence of [12], which is a special case of Corollary 3.3 in [12] with $\alpha_1 = u$ and $\alpha_2 = \alpha_2 = \cdots = \alpha_m = 1$.

With the above results, we can now obtain the following theorem.

Theorem 3 *The NGP scheme of degree m* $(m \ge 1)$ *is convergent.*

4.2 Smoothness analysis

In this subsection, we study the smoothness of the NGP scheme. To conveniently study the smoothness, we introduce some additional notation and rewrite Algorithm 2 as Algorithm 3, which includes one nonlinear 3-point approximating refinement step and $m-3$ smoothing steps.

For $j \in \mathbb{N}_0$ and $k = 0, 1, \dots, m-3$, define $\tilde{e}^{j,k} = \max_{i\in\mathbb{Z}}\{|p_i^{j,k}p_{i+1}^{j,k}|\}$, $\tilde{\theta}^{j,k} = \max_{i\in\mathbb{Z}}\{\theta(n_i^{j,k}, n_{i+1}^{j,k})\}$. And for $j \in \mathbb{N}_0$ and $k = 0, 1, \dots, m-4$, define $\tilde{\mu}^{j,k} = \frac{1}{2\cos\frac{\theta^{j,k}}{4}}$. We also define, for $j \in \mathbb{N}_0$,

$$\tilde{e}^j = \tilde{e}^{j,m-3}, \qquad \tilde{\theta}^j = \tilde{\theta}^{j,m-3}, \qquad \mathcal{P}^j = \{p_i^{j,m-3} : i \in \mathbb{Z}\}.$$

For given two PNPs, $P_i = (p_i, n_i)$, $i = 0, 1$, we denote by e the length of the segment $[p_0, p_1]$, θ the angle between n_0 and n_1, and $\Delta_t = |p_t q_t|$, where p_t is the point of $P_0 \circledcirc_t P_1$ and q_t denotes the linear average $q_t = (1-t)p_0 + tp_1$. And $q_i^{j,k} (i \in \mathbb{Z})$ denote the points obtained by the kth step of the jth iteration of the LGP scheme operating on $\mathcal{P}^{j-1}$. Set $\Delta_i^{j,k} = |p_i^{j,k}q_i^{j,k}|$, $\Delta^{j,k} = \sup_i \Delta_i^{j,k}$, and $\Delta^j = \Delta^{j,m-3}$.

In the triangle $p_t p_0 q_t$, we have the following result by applying the cosine theorem.

Lemma 4 *With the above notation,* $\Delta_t \le \chi_t e\theta$, *where* $t = \frac{1}{2(1+u)}$ *and* $\chi_t = \sqrt{\frac{\pi|t|^3}{3!} + |t|\frac{(1-t)^2}{4}}$.

Algorithm 3 The refinement step of the NGP algorithm

Input: The parameter $u(u \in \mathbb{R}_+)$ and the data to be refined $\mathbf{P} = \{P_i = (p_i, n_i)\}_{i\in\mathbb{Z}}$. The degree $m(m \in \mathbb{N}_0)$ of the NGP subdivision scheme.

Output: The refinement data $T(\mathbf{P})$.

$P^0_{2i} \leftarrow P_i$
$P^0_{2i+1} \leftarrow P_i$
for $j \in \mathbb{N}_0$ **do**
  for $i \in \mathbb{Z}$ **do**
    $P^{j,0}_{2i} \leftarrow P^{j-1}_i \circledcirc_{\frac{1}{2}} P^{j-1}_{i+1}$
    $S_L \leftarrow P^{j-1}_{i+1} \circledcirc_{\frac{1}{2(1+u)}} P^{j-1}_i$
    $S_R \leftarrow P^{j-1}_{i+1} \circledcirc_{\frac{1}{2(1+u)}} P^{j-1}_{i+2}$
    $P^{j,0}_{2i+1} \leftarrow S_L \circledcirc_{\frac{1}{2}} S_R$
  end for(i)
  for $k = 1, 2, \ldots, m-3$ **do**
    for $i \in \mathbb{Z}$ **do**
      $P^{j,k}_i \leftarrow P^{j,k-1}_i \circledcirc_{\frac{1}{2}} P^{j,k-1}_{i+1}$
    end for(i)
  end for(k)
  for $i \in \mathbb{Z}$ **do**
    $P^j_i \leftarrow P^{j,m-3}_i$
  end for(i)
end for(j)

In the rest of this subsection, we prove several lemmas about the NGP scheme, which show that an alternative proximity condition (16) holds.

Lemma 5 *There exists $\widetilde{j}^* \in \mathbb{N}_0$ such that $\tilde{\theta}^{j+1,0} \le \frac{1}{2}\tilde{\theta}^j$ and $\tilde{e}^{j+1,0} \le \frac{3}{4}\tilde{e}^j$ for $j \ge \widetilde{j}^*$.*

Proof Let $S_L = (s_L, n_L)$, $S_R = (s_R, n_R)$ be the intermediate pairs inserted in the refinement step of the NGP scheme for $k = 0$ (see Fig. 2(a)). Since

$$\begin{aligned}\theta\left(n^{j+1,0}_{2i-1}, n_R\right) &= \frac{1}{2}\theta(n_L, n_R) \le \frac{\theta(n_L, n^j_{i+1}) + \theta(n^j_{i+1}, n_R)}{2}\\ &= \frac{\theta(n^j_i, n^j_{i+1}) + \theta(n^j_{i+1}, n^j_{i+2})}{4(1+u)} \le \frac{1}{2(1+u)}\tilde{\theta}^j\end{aligned}$$

and $\theta(n^{j+1,0}_{2i}, n_R) = \frac{u\theta(n^j_{i+1}, n^j_{i+2})}{2(1+u)} \le \frac{u}{2(1+u)}\tilde{\theta}^j$, we have $\theta(n^{j+1,0}_{2i-1}, n^{j+1,0}_{2i}) \le \frac{1}{2}\tilde{\theta}^j$. Similarly,

$$\theta\left(n^{j+1,0}_{2i-2}, n^{j+1,0}_{2i-1}\right) \le \frac{1}{2}\tilde{\theta}^j.$$

Thus

$$\tilde{\theta}^{j+1,0} \le \frac{1}{2}\tilde{\theta}^j. \tag{7}$$

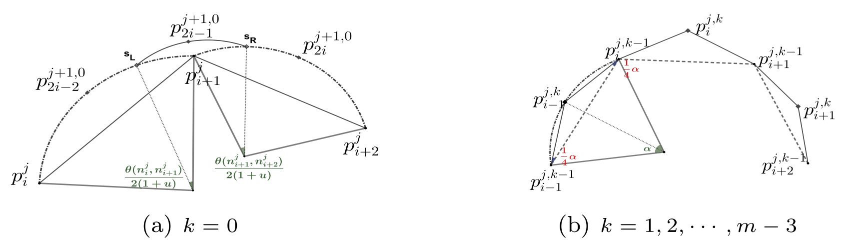

Figure 2 The setup of Lemma 5, 6

It follows from (7) together with

$$\tilde{\theta}^{j+1,k} \le \tilde{\theta}^{j+1,k-1} \tag{8}$$

for $k = 1, 2, \ldots, m-3$ (see Fig. 2(b)) that

$$\tilde{\theta}^{j} = \tilde{\theta}^{j,m-3} \le \tilde{\theta}^{j,m-4} \le \cdots \le \tilde{\theta}^{j,0} \le \frac{1}{2}\tilde{\theta}^{j-1} \le \cdots \le \left(\frac{1}{2}\right)^{j}\tilde{\theta}^{0}. \tag{9}$$

Hence $\theta(n_L, n_R) \le \frac{1}{1+u}\tilde{\theta}^{j} \le (\frac{1}{2})^{j}\frac{\tilde{\theta}^{0}}{1+u}$. Also, since

$$\left|p_{2i-1}^{j+1,0}p_{2i}^{j+1,0}\right| \le \left|p_{2i-1}^{j+1,0}s_R\right| + \left|p_{2i}^{j+1,0}s_R\right| = \frac{|s_L s_R|}{2\cos\frac{\theta(n_L,n_R)}{4}} + \frac{|p_{i+1}^{j}p_{i+2}^{j}|}{\sin\frac{\theta(n_{i+1}^{j},n_{i+2}^{j})}{2}}\sin\frac{u\theta(n_{i+1}^{j},n_{i+2}^{j})}{4(1+u)}$$

and

$$\begin{aligned} |s_L s_R| &\le \left|s_L p_{i+1}^{j}\right| + \left|p_{i+1}^{j}s_R\right| = \frac{|p_i^j p_{i+1}^j|\sin\frac{\theta(n_i^j,n_{i+1}^j)}{4(1+u)}}{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{2}} + \frac{|p_{i+1}^j p_{i+2}^j|\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{4(1+u)}}{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{2}} \\ &\le \tilde{e}^{j}\left[\frac{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{4(1+u)}}{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{2}} + \frac{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{4(1+u)}}{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{2}}\right], \end{aligned}$$

we obtain

$$\begin{aligned} \left|p_{2i-1}^{j+1,0}p_{2i}^{j+1,0}\right| &\le \tilde{e}^{j}\left[\frac{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{4(1+u)}}{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{2}} + \frac{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{4(1+u)}}{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{2}}\right]\frac{1}{2\cos\frac{\tilde{\theta}^{j}}{4(1+u)}} + \frac{\tilde{e}^{j}\sin\frac{u\theta(n_{i+1}^j,n_{i+2}^j)}{4(1+u)}}{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{2}} \\ &\le \tilde{e}^{j}\underbrace{\left[\max_{i\in\mathbb{Z}}\left(\frac{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{4(1+u)}}{\sin\frac{\theta(n_i^j,n_{i+1}^j)}{2}}\right)\frac{1}{\cos(\frac{1}{4(1+u)}(\frac{1}{2})^{j}\tilde{\theta}^{0})} + \max_{i\in\mathbb{Z}}\frac{\sin\frac{u\theta(n_{i+1}^j,n_{i+2}^j)}{4(1+u)}}{\sin\frac{\theta(n_{i+1}^j,n_{i+2}^j)}{2}}\right]}_{\vartheta_j}. \end{aligned}$$

Due to $\lim_{j\to\infty}\vartheta_j = \frac{1}{2} < 1$, there exists $\widetilde{j^{*}} \in \mathbb{N}_0$ such that $\vartheta_j \le \vartheta_{\widetilde{j^{*}}} = \frac{3}{4} < 1$ for $j \ge \widetilde{j^{*}}$, which implies that $\tilde{e}^{j+1,0} \le \frac{3}{4}\tilde{e}^{j}$ for $j \ge \widetilde{j^{*}}$. □

Lemma 6 *For all* $j \in \mathbb{N}_0$, $\tilde{\theta}^{j} \le (\frac{1}{2})^{j}\tilde{\theta}^{0}$ *and* $\tilde{e}^{\widetilde{J}+j} \le (\frac{\pi}{2})^{m-3}(\frac{3}{4})^{j}\tilde{e}^{\widetilde{J}}$, *where* $\widetilde{J} = \max\{\widetilde{j^{*}}, \widetilde{j^{**}}\}$.

Proof From (9) in Lemma 5, we easily get, for all $j \in \mathbb{N}_0$,

$$\tilde{\theta}^j \le \left(\frac{1}{2}\right)^j \tilde{\theta}^0. \tag{10}$$

Since for $k = 0, 1, \ldots, m-4$ (see Fig. 2),

$$\left|p_i^{j,k+1} p_{i+1}^{j,k+1}\right| \le \left|p_i^{j,k+1} p_{i+1}^{j,k}\right| + \left|p_{i+1}^{j,k} p_{i+1}^{j,k+1}\right| \le \frac{\tilde{e}^{j,k}}{2\cos\frac{\tilde{\theta}^{j,k}}{4}} + \frac{\tilde{e}^{j,k}}{2\cos\frac{\tilde{\theta}^{j,k}}{4}} = 2\widetilde{\mu}^{j,k}\tilde{e}^{j,k},$$

it is clear that $\tilde{e}^{j,k+1} \le 2\widetilde{\mu}^{j,k}\tilde{e}^{j,k}$. Thus

$$\tilde{e}^{j+1} \le 2\widetilde{\mu}^{j+1,m-4}\tilde{e}^{j+1,m-4} \le \cdots \le \prod_{k=0}^{m-4}\left(2\widetilde{\mu}^{j+1,k}\right)\tilde{e}^{j+1,0} \le \frac{3}{4}\prod_{k=0}^{m-4}\left(2\widetilde{\mu}^{j+1,k}\right)\tilde{e}^{j}. \tag{11}$$

From (8) we easily get

$$\frac{3}{4}\prod_{k=0}^{m-4}\left(2\widetilde{\mu}^{j+1,k}\right) = \frac{3}{4}\prod_{k=0}^{m-4}\frac{1}{\cos\frac{\tilde{\theta}^{j+1,k}}{4}} \le \frac{3}{4}\left(\frac{1}{\cos\frac{\tilde{\theta}^{j}}{4}}\right)^{m-3}. \tag{12}$$

Let $\widetilde{j}^{**}$ be the minimal j for which $(\frac{1}{\cos\frac{\tilde{\theta}^j}{4}})^{m-3} < \frac{4}{3}$ and $\widetilde{J} = \max\{\widetilde{j}^*, \widetilde{j}^{**}\}$, where $\widetilde{j}^*$ is described in Lemma 5, then $\frac{3}{4}(\frac{1}{\cos\frac{\tilde{\theta}^j}{4}})^{m-3} < 1$ for $j \ge \widetilde{J}$. We get $\tilde{e}^{\widetilde{J}+j} \le \widetilde{\eta}_{\widetilde{J}+j}\tilde{e}^{\widetilde{J}+j-1}$ from (11) and (12) by setting $\widetilde{\eta}_j = \frac{3}{4}(\frac{1}{\cos\frac{\tilde{\theta}^{j-1}}{4}})^{m-3}$. Thus

$$\tilde{e}^{\widetilde{J}+j} \le \widetilde{\eta}_{\widetilde{J}+j}\widetilde{\eta}_{\widetilde{J}+j-1}\cdots\widetilde{\eta}_{\widetilde{J}+1}\tilde{e}^{\widetilde{J}} = \prod_{l=\widetilde{J}+1}^{\widetilde{J}+j}\widetilde{\eta}_l\tilde{e}^{\widetilde{J}}. \tag{13}$$

Clearly it follows from (7) that

$$\begin{aligned}\prod_{l=\widetilde{J}+1}^{\widetilde{J}+j}\widetilde{\eta}_l &= \prod_{l=\widetilde{J}+1}^{\widetilde{J}+j}\frac{3}{4}\left(\frac{1}{\cos\frac{\tilde{\theta}^{l-1}}{4}}\right)^{m-3} = \left(\frac{3}{4}\right)^j\left(\prod_{l=\widetilde{J}}^{\widetilde{J}+j-1}\frac{1}{\cos\frac{\tilde{\theta}^{l}}{4}}\right)^{m-3} \\ &\le \left(\frac{3}{4}\right)^j\left(\prod_{l=\widetilde{J}}^{\widetilde{J}+j-1}\frac{1}{\cos\frac{\tilde{\theta}^{0}}{2^{l+4}}}\right)^{m-3} \le \left(\frac{3}{4}\right)^j\left(\prod_{l=0}^{\infty}\frac{1}{\cos\frac{\pi}{2^{l+2}}}\right)^{m-3} = \left(\frac{\pi}{2}\right)^{m-3}\left(\frac{3}{4}\right)^j.\end{aligned}$$

Combining this with (13) leads to the claim of the lemma. □

To study the convergence of the nonlinear scheme, we recall the following result (see Lemma 3.3 in [8]).

Lemma 7 *In the above notation, for $k \ge 1$, we have*

$$\Delta_i^{j,k} \le \Delta^{j,k-1} + \chi_{\frac{1}{2}}\tilde{e}^{j,k-1}\tilde{\theta}^{j,k-1}. \tag{14}$$

With the above lemmas, we obtain the following results.

Lemma 8 *Let S be the linear scheme corresponding to the NGP scheme, and $\triangle^j = \triangle^{j,m-3}$. Then, for $j \geq \widetilde{J}$,*

$$\Delta^j = \left\|\mathcal{P}^j - S\mathcal{P}^{j-1}\right\| \leq \left[\chi_t + (m-3)\chi_{\frac{1}{2}}\right]\tilde{e}^{j-1}\tilde{\theta}^{j-1}. \tag{15}$$

Lemma 9 *Let S be the linear scheme corresponding to the NGP scheme. Then*

$$\left\|\mathcal{P}^{\widetilde{J}+j} - S\mathcal{P}^{\widetilde{J}+j-1}\right\| \leq \frac{8}{3}\left[\chi_t + (m-3)\chi_{\frac{1}{2}}\right]\left(\frac{\pi}{2}\right)^{m-3}\left(\frac{3}{8}\right)^j \tilde{e}^{\widetilde{J}}\tilde{\theta}^{\widetilde{J}}. \tag{16}$$

Remark 3 Condition (16) in Lemma 9 is an alternative proximity condition.

With the above lemmas in this subsection, we can obtain the following theorem.

Theorem 10 *The NGP scheme of degree m ($m \geq 3$) is C^1 when $\sqrt{2}-1 < u < \sqrt{2}+1$.*

Proof The proof is analogous to the proofs of Theorem 3.8 and Theorem 3.13 in [8]. Throughout this proof, δ denotes the difference operator. Let $d_j = \|\delta(\frac{\delta\mathcal{P}^{\widetilde{J}+j}}{2^{-(\widetilde{J}+j)}})\|$, it is clear that

$$d_j \leq \left\|\delta\left(\frac{\delta S\mathcal{P}^{\widetilde{J}+j-1}}{2^{-(\widetilde{J}+j)}}\right)\right\| + \left\|\delta\left(\frac{\delta(\mathcal{P}^{\widetilde{J}+j} - S\mathcal{P}^{\widetilde{J}+j-1})}{2^{-(\widetilde{J}+j)}}\right)\right\|.$$

Using the fact introduced in [10] that, for $m \geq 3$,

$$\left\|\delta\left(\frac{\delta S\mathcal{P}^{j-1}}{2^{-j}}\right)\right\| \leq B_1\left\|\delta\left(\frac{\delta\mathcal{P}^{j-1}}{2^{-(j-1)}}\right)\right\|, \quad B_1 = \max\left\{\frac{1}{1+u}, \frac{u}{1+u}\right\}, \tag{17}$$

we have

$$\begin{aligned} d_j &\leq B_1\left\|\delta\left(\frac{\delta\mathcal{P}^{\widetilde{J}+j-1}}{2^{-(\widetilde{J}+j-1)}}\right)\right\| + 2^{\widetilde{J}+j}\left\|\delta\left(\delta\left(\mathcal{P}^{\widetilde{J}+j} - S\mathcal{P}^{\widetilde{J}+j-1}\right)\right)\right\| \\ &\leq B_1\left\|\delta\left(\frac{\delta\mathcal{P}^{\widetilde{J}+j-1}}{2^{-(\widetilde{J}+j-1)}}\right)\right\| + \frac{2^{\tilde{J}+5}}{3}\left(\frac{\pi}{2}\right)^{m-3}\left(\frac{3}{4}\right)^j\left[\chi_t + (m-3)\chi_{\frac{1}{2}}\right]\tilde{e}^{\widetilde{J}}\tilde{\theta}^{\widetilde{J}} \\ &= B_1 d_{j-1} + \left(\frac{3}{4}\right)^j K, \end{aligned} \tag{18}$$

where $t = \frac{1}{2(1+u)}$ and $K = \frac{2^{\tilde{J}+5}}{3}(\frac{\pi}{2})^{m-3}[\chi_t + (m-3)\chi_{\frac{1}{2}}]\tilde{e}^{\widetilde{J}}\tilde{\theta}^{\widetilde{J}}$. Clearly, it follows from (18) that

$$\begin{aligned} d_j &\leq B_1 d_{j-1} + \left(\frac{3}{4}\right)^j K \leq B_1^2 d_{j-2} + B_1\left(\frac{3}{4}\right)^{j-1} K + \left(\frac{3}{4}\right)^j K \\ &\leq \cdots \leq B_1^j d_0 + B_1^{j-1}\frac{3}{4}K + B_1^{j-2}\left(\frac{3}{4}\right)^2 K + \cdots + \left(\frac{3}{4}\right)^j K \\ &= B_1^j\left(d_0 + \frac{3K}{4B_1 - 3}\right) - \frac{3K}{4B_1 - 3}\left(\frac{3}{4}\right)^j, \end{aligned} \tag{19}$$

which is essential for the proof of C^1 smoothness. When $u \in (\sqrt{2}-1, \sqrt{2}+1)$, it follows that $2B_1^2 < 1$, which is required in the proof of the theorem. Similar to the proof of Theorem 3.8 and Theorem 3.13 in [8], we complete the proof according to (17) and (19). □

Remark 4 The NGP scheme of arbitrary degree m reduces to the nonlinear one introduced in [7] for $u = 1$, which is C^1 for $m \geq 2$.

5 Examples

In this section, we present some examples to illustrate the performance of the nonlinear subdivision scheme presented in this paper.

Figure 3 and Fig. 4 illustrate limit curves generated by the LGP scheme of degree 3 and the NGP scheme of degree 3 with different tension parameters, which have the same control polygon. The resulting curves generated by the LGP scheme of degree 3 progressively tend to shrink towards the control polygon with the increasing of the tension parameter u, but cannot reconstruct the circle in Fig. 3, while the limit curves generated by the NGP scheme of degree 3 with different tension parameters can reproduce the circle in Fig. 4.

The limit curves generated by the LGP scheme of degree 3 and the NGP scheme of degree 3 with the parameter $u = \frac{1}{2}$ and four different initial normals at P are shown in Fig. 5. It shows that the selection of the initial normals effectively controls the shape of the limit curves generated by the NGP scheme. Figure 6 shows that some examples illustrating

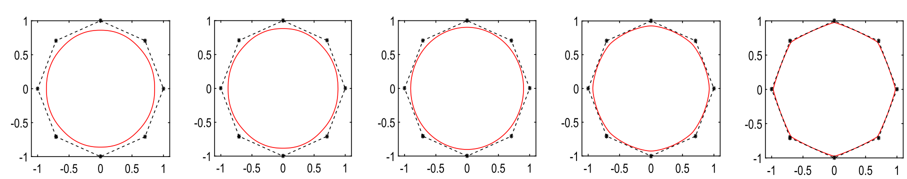

Figure 3 Limit curves generated by the LGP scheme [10] of degree 3 with $u = \frac{1}{10}, \frac{1}{2}, 1, 2, 10$(from left to right)

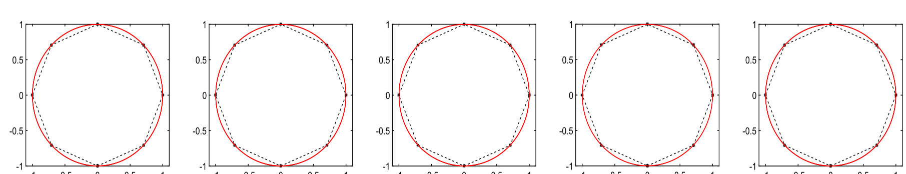

Figure 4 Limit curves generated by the NGP scheme of degree 3 with $u = \frac{1}{10}, \frac{1}{2}, 1, 2, 10$ after six iterations, respectively(from left to right)

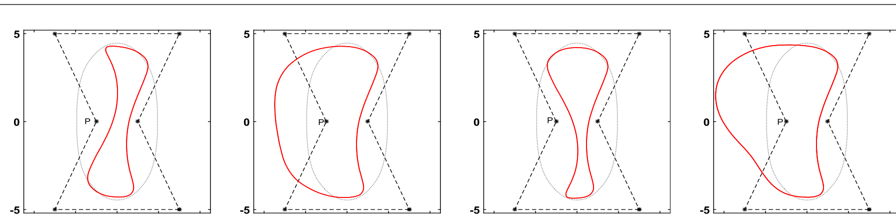

Figure 5 Limit curves generated by the LGP scheme of degree 3(dash-dot) and the NGP scheme of degree 3(solid) with $u = \frac{1}{2}$, starting from the same polygon with four different initial normals at P

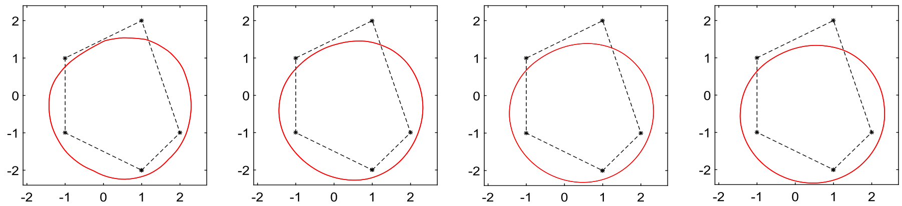

Figure 6 Limit curves generated by the NGP scheme of degree 2, 3, 4, 5 with the parameter $u = \frac{1}{2}$ (from left to right), respectively

the shape of the resulting curve are produced by the same control polygon, but with the tension parameter $u = \frac{1}{2}$ and different degrees 2, 3, 4, 5 of the NGP scheme. It shows that the resulting curve becomes more and more smooth with increase in the degree of the NGP scheme.

6 Conclusion

In this paper, by suitably using the circle average, we have presented a nonlinear generalized subdivision scheme of arbitrary degree m with a tension parameter based on the stationary linear generalized subdivision scheme of arbitrary degree with a tension parameter. The scheme can be seen as the generalization of the nonlinear modified Lane–Riesenfeld algorithm presented in [7] and the nonlinear 3-point approximating scheme with a parameter introduced in [9], and can reach C^1 for $m \geq 3$. The nonlinear scheme can reconstruct the circle by the circle average, and suitable choices of parameters and initial normal vectors can effectively control the shape of the limit curve. It can be seen that the limit curves become smoother with increase in the degree of the NGP scheme in Fig. 6. So we may focus on proving higher order of smoothness of the NGP scheme in the future.

Acknowledgements
The authors thank all the reviewers for their careful reading and valuable comments.

Funding
None.

Availability of data and materials
The datasets used during the current study are available from the corresponding author on reasonable request.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to this work equally. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Northwestern Polytechnical University, 710072 Xi'an, Shaanxi, P.R. China. [2]School of Science, Xi'an Polytechnic University, 710072 Xi'an, Shaanxi, P.R. China.

Received: 30 January 2020 Accepted: 12 November 2020 Published online: 23 November 2020

References

1. Schaefer, S., Vouga, E., Goldman, R.: Nonlinear subdivision through nonlinear averaging. Comput. Aided Geom. Des. **25**(3), 162–180 (2008)

2. Amat, S., Liandrat, J.: On a nonlinear 4-point quaternary approximating subdivision scheme eliminating the Gibbs phenomenon. SeMA J. **62**(1), 15–25 (2013)
3. Amat, S., Donat, R., Liandrat, J., Trillo, J.C.: Analysis of a new nonlinear subdivision scheme. Applications in image processing. Found. Comput. Math. **6**(2), 193–225 (2006)
4. Yang, X.: Normal based subdivision scheme for curve design. Comput. Aided Geom. Des. **23**(3), 243–260 (2006)
5. Deng, C., Wang, G.: Incenter subdivision scheme for curve interpolation. Comput. Aided Geom. Des. **27**(1), 48–59 (2010)
6. Li, Y., Deng, C.: A variation of incenter subdivision scheme. J. Comput.-Aided Des. Comput. Graph. **24**(12), 1542–1548 (2012) (in Chinese)
7. Lipovetsky, E., Dyn, N.: A weighted binary average for subdivision schemes of point-normal pairs. Comput. Aided Geom. Des. **48**, 36–48 (2016)
8. Lipovetsky, E., Dyn, N.: C^1 analysis of some 2D subdivision schemes refining point-normal pairs with the circle average. Comput. Aided Geom. Des. **69**, 45–54 (2019)
9. Li, C., Zheng, H., Lin, Z.: Nonlinear subdivision schemes with free parameters based on circle average. J. Comput.-Aided Des. Comput. Graph. **8**(31), 1330–1340 (2019) (in Chinese)
10. Fang, M., Ma, W., Wang, G.: A generalized curve subdivision scheme of arbitrary order with a tension parameter. Comput. Aided Geom. Des. **27**, 720–733 (2010)
11. Dyn, N., Sharon, N.: Manifold-valued subdivision schemes based on geodesic inductive averaging. J. Comput. Appl. Math. **311**, 54–67 (2017)
12. Dyn, N., Sharon, N.: A global approach to the refinement of manifold data. Math. Comput. **86**(303), 375–395 (2017)