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Numerical method of highly nonlinear and nonautonomous neutral stochastic differential delay equations with Markovian switching

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Abstract
In this paper, we establish a partially truncated Euler–Maruyama scheme for highly nonlinear and nonautonomous neutral stochastic differential delay equations with Markovian switching. We investigate the strong convergence rate and almost sure exponential stability of the numerical solutions under the generalized Khasminskii-type condition.

Keywords: Partially truncated Euler–Maruyama method; Neutral stochastic differential delay equations; Markovian switching; Highly nonlinear and nonautonomous equations

1 Introduction

Stochastic differential equations play an important role in various fields, such as biology, chemistry, and finance [3, 20, 27]. In practice, parameters and forms in stochastic systems may change when something unexpected happens. At this point, we can use stochastic differential equations with Markovian switching. Mao and Yuan [24] studied stochastic differential equations with Markovian switching in depth. Many stochastic systems not only depend on the present and past states, but also contain derivatives with delays and the function itself, which can be described by neutral stochastic differential delay equations (NSDDEs) [20]. Kolmanovskii et al. [12] established a fundamental theory for neutral stochastic differential delay equations with Markovian switching (NSDDEwMSs) and discussed some important properties of the solutions.

In many cases the true solutions of the equations cannot be found. So it is very useful to study explicit forms of the numerical solutions. The Euler–Maruyama (EM) method for stochastic differential delay equations with Markovian switching (SDDEwMSs) was investigated in [25] and [37]. Wu and Mao [34] showed the convergence of EM method for neutral stochastic functional differential equations. However, Hutzenthaler et al. [9] showed that pth moments of the EM approximations diverge to infinity for any $p \in [1, \infty)$ when the coefficients grow superlinearly. Many implicit methods were established to estimate the solutions of the equations with superlinearly growing coefficients [2, 4, 8, 11, 26, 30, 32, 33]. Due to the advantages of explicit numerical solutions, such as less computation, plenty of modified EM methods have been studied to approximate the solutions of superlinear stochastic differential equations. The tamed EM scheme was proposed in [10] to estimate the solutions of stochastic differential equations with one-sided Lipschitz drift coefficient and global Lipschitz diffusion coefficient. Sabanis [28, 29] developed tamed EM schemes for nonlinear stochastic differential equations. More detail on the other explicit numerical methods can be found in [1, 16, 18]. In addition, Mao initialized the truncated EM method in [21] and obtained the convergence rate in [22]. Then Guo et al. [7] discussed the convergence rate of the truncated EM method for stochastic differential delay equations. The truncated EM method for time-changed nonautonomous stochastic differential equations was shown in [19]. To get the asymptotic behaviors easily, Guo et al. [6] proposed the partially truncated EM method. In [38], the partially truncated EM method for stochastic differential delay equations was proposed. Cong et al. [5] used the partially truncated EM method to get the convergence rate and almost sure exponential stability of highly nonlinear SDDEwMSs. Tan and Yuan in [33] showed the convergence rates of the theta-method for nonlinear neutral stochastic differential delay equations driven by Brownian motion and Poisson jumps, but the stability was not analyzed as time goes to infinity. In [39], the convergence of the EM method for NSDDEwMSs was proved, but the convergence rate was not given. To our best knowledge, there are few papers concerning with numerical solutions of highly nonlinear and nonautonomous NSDDEwMSs. Therefore, in this paper, we give the strong convergence rate of the partially truncated EM method for highly nonlinear and nonautonomous NSDDEwMSs.

Moreover, many scholars are interested in the asymptotic behaviors of the stochastic systems [3, 5, 6, 20, 24, 31]. The almost surely asymptotic stability of NSDDEwMSs was discussed in [23]. Then Li and Mao [15] established LaSalle-type stability theorem for NSDDEwMSs. Liu et al. [17] showed the mean square polynomial stability of the EM method and the backward EM method for stochastic differential equations. The almost sure exponential stability of EM approximations for stochastic differential delay equations was investigated By means of the semimartingale convergence theorem [36]. The exponential mean square stability of the split-step theta method for NSDDEs was investigated in [40]. Lan and Yuan [14] studied the exponential stability of the exact solutions and θ-EM $(1/2 < \theta \leq 1)$ approximations to NSDDEwMSs. Lan [13] gave the asymptotic mean-square and almost sure exponential stability of the modified truncated EM method for NSDDEs under local Lipschitz condition and nonlinear growth condition. However, there is little literature studying the almost sure exponential stability of the partially truncated EM method for highly nonlinear and nonautonomous NSDDEwMSs. The second goal of this paper is to fill this gap.

This paper is organized as follows. We introduce some useful notations and establish the partially truncated EM scheme for NSDDEwMSs in Sect. 2. In Sect. 3, we discuss the strong convergence rate. In Sect. 4, we show the almost sure exponential stability of numerical solutions. Section 5 contains two examples to illustrate that our main result covers a large class of highly nonlinear and nonautonomous NSDDEwMSs.

2 Mathematical preliminaries

Unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. For $x \in \mathbb{R}^n$, let $|x|$ denote its Euclidean norm. If A is a matrix, denote by $|A| = \sqrt{\text{trace}(A^T A)}$ its trace norm. By $A \leq 0$ and $A < 0$ we mean that A is nonpositive and negative definite, respectively. For real numbers a, b, we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\lfloor a \rfloor$ be the largest integer that does not exceed a. Let $\mathbb{R}_+ = [0, +\infty)$ and $\tau > 0$. By $\mathscr{C}([-\tau, 0]; \mathbb{R}^n)$ we denote the family of continuous functions ν from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\nu\| = \sup_{-\tau \leq \tilde{\theta} \leq 0} |\nu(\tilde{\theta})|$. If H is a set, then $\mathbb{I}_H$ denotes its indicator function, that is, $\mathbb{I}_H(\omega) = 1$ if $\omega \in H$ and $\mathbb{I}_H(\omega) = 0$ if $\omega \notin H$. Let C stand for a generic positive real constant different in different cases.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. For $p > 0$, let $\mathscr{L}^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_0$-measurable $\mathscr{C}([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ such that $\mathbb{E}\|\xi\|^p < \infty$. Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space.

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij} + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij} + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, and γ_{ij} is the transition rate from i to j with $\gamma_{ij} > 0$ if $i \neq j$, whereas $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We suppose that the Markov chain r is independent of the Brownian motion B. As is well known [31], almost every sample path of r is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$, that is, there is a sequence of stopping times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k \to \infty$ almost surely such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) \mathbb{I}_{[\tau_k, \tau_{k+1})}(t),$$

where $\mathbb{I}$ is the indicator function defined as before. Hence r is constant on each interval $[\tau_k, \tau_{k+1})$:

$$r(t) = r(\tau_k), \quad t \in [\tau_k, \tau_{k+1}), k = 0, 1, 2, \ldots.$$

In this paper, we consider highly nonlinear and nonautonomous neutral stochastic differential delay equations with Markovian switching of the form

$$\begin{aligned} &d\big[x(t) - D\big(x(t - \tau), r(t)\big)\big] \\ &\quad = f\big(t, x(t), x(t - \tau), r(t)\big)\, dt + g\big(t, x(t), x(t - \tau), r(t)\big)\, dB(t), \quad t \geq 0, \end{aligned} \tag{2.1}$$

with initial data

$$x_0 = \xi \in \mathscr{L}^p_{\mathcal{F}_0}\big([-\tau, 0]; \mathbb{R}^n\big) \quad \text{and} \quad r(0) = r_0, \tag{2.2}$$

where r_0 is $\mathbb{S}$-valued $\mathcal{F}_0$-measurable random variable. Here $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n \times m}$, and $D : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$. They are all Borel-measurable functions.

We suppose that the drift and diffusion coefficients can be decomposed as

$$\begin{aligned} f(t,x,y,i) &= \tilde{F}(t,x,y,i) + F(t,x,y,i), \\ g(t,x,y,i) &= \tilde{G}(t,x,y,i) + G(t,x,y,i). \end{aligned} \tag{2.3}$$

To estimate the partially truncated EM method for (2.1), we need the following lemma [24].

Lemma 2.1 *Given $\Delta > 0$, let $r_k^\Delta = r(k\Delta)$ for $k \geq 0$. Then $\{r_k^\Delta, k = 0,1,2,\ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$\mathbb{P}(\Delta) = \bigl(\mathbb{P}_{ij}(\Delta)\bigr)_{N\times N} = e^{\Delta\Gamma}. \tag{2.4}$$

Then we impose two standard necessary hypotheses on the initial data and neutral term.

Assumption 2.2 There exist constants $K_1 > 0$ and $\alpha \in (0,1]$ such that

$$\bigl|\xi(\bar{t}) - \xi(\bar{s})\bigr| \leq K_1|\bar{t} - \bar{s}|^\alpha, \quad -\tau \leq \bar{s} < \bar{t} \leq 0. \tag{2.5}$$

Assumption 2.3 (The contractive mapping) $D(0,i) = 0$, and there exists a constant $K_2 \in (0,1)$ such that

$$\bigl|D(x,i) - D(y,i)\bigr| \leq K_2|x-y| \tag{2.6}$$

for all $x, y \in \mathbb{R}^n$ and $i \in \mathbb{S}$.

By Assumption 2.3 we have $|D(x,i)| \leq K_2|x|$ for all $x \in \mathbb{R}^n$ and $i \in \mathbb{S}$.

Since γ_{ij} is independent of x, the paths of r could be generated before approximating x. The discrete Markovian chain $\{r_k^\Delta, k = 0,1,2,\ldots\}$ can be generated as follows: Compute the one-step transition probability matrix $\mathbb{P}(\Delta)$. Let $r_0^\Delta = i_0$ and generate a random number ξ_1 uniformly distributed in $[0,1]$. Define

$$r_1^\Delta = \begin{cases} i_1 & \text{if } i_1 \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} \mathbb{P}_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} \mathbb{P}_{i_0,j}(\Delta), \\ N & \text{if } \sum_{j=1}^{N-1} \mathbb{P}_{i_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set $\sum_{j=1}^{0} \mathbb{P}_{i_0,j}(\Delta) = 0$ as usual. Then independently generate a new random number ξ_2 uniformly distributed in $[0,1]$ as well. Define

$$r_2^\Delta = \begin{cases} i_2 & \text{if } i_2 \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} \mathbb{P}_{r_1^\Delta,j}(\Delta) \leq \xi_2 < \sum_{j=1}^{i_2} \mathbb{P}_{r_1^\Delta,j}(\Delta), \\ N & \text{if } \sum_{j=1}^{N-1} \mathbb{P}_{r_1^\Delta,j}(\Delta) \leq \xi_2. \end{cases}$$

Repeating this procedure, we can obtain a trajectory of $\{r_k^\Delta, k = 1,2,\ldots\}$. The procedure can be applied independently to get more trajectories. After generating the discrete Markov chain $\{r_k^\Delta, k = 0,1,2,\ldots\}$, we can now define the partially truncated EM approximate solution for NSDDEwMSs (2.1) with initial data (2.2).

To define the partially truncated EM scheme, we first choose a strictly increasing continuous function $\varphi(w):\mathbb{R}_+\to\mathbb{R}_+$ such that $\varphi(w)\to\infty$ as $w\to\infty$ and

$$\sup_{0\le t\le T}\sup_{|x|\vee|y|\le w}\big(\big|F(t,x,y,i)\big|\vee\big|G(t,x,y,i)\big|\big)\le\varphi(w),\quad \forall w\ge 1. \tag{2.7}$$

Let φ^{-1} denote the inverse function of φ. Hence φ^{-1} is a strictly increasing continuous function from $[\varphi(1),\infty)$ to $\mathbb{R}_+$. Then we also choose $K_0\ge 1\vee\varphi(1)$ and a strictly decreasing function $h:(0,1]\to(0,\infty)$ such that

$$\lim_{\Delta\to 0}h(\Delta)=\infty,\qquad \Delta^{\frac{1}{4}}h(\Delta)\le K_0,\quad \forall\Delta\in(0,1]. \tag{2.8}$$

For a given step size $\Delta\in(0,1]$, define the truncated mapping π_Δ from $\mathbb{R}^n$ to the closed ball $\{x\in\mathbb{R}^n:|x|\le\varphi^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x)=\big(|x|\wedge\varphi^{-1}\big(h(\Delta)\big)\big)\frac{x}{|x|}, \tag{2.9}$$

where we let $\frac{x}{|x|}=0$ for $x=0$. Then we can define the truncated functions

$$F_\Delta(t,x,y,i)=F\big(t,\pi_\Delta(x),\pi_\Delta(y),i\big),\qquad G_\Delta(t,x,y,i)=G\big(t,\pi_\Delta(x),\pi_\Delta(y),i\big)$$

for $x,y\in\mathbb{R}^n$. Thus we obtain that

$$\begin{aligned}&f_\Delta(t,x,y,i)=\tilde{F}(t,x,y,i)+F_\Delta(t,x,y,i),\\&g_\Delta(t,x,y,i)=\tilde{G}(t,x,y,i)+G_\Delta(t,x,y,i).\end{aligned}$$

Moreover, we can easily get that for any $x,y\in\mathbb{R}^n$,

$$\big|F_\Delta(t,x,y,i)\big|\vee\big|G_\Delta(t,x,y,i)\big|\le\varphi\big(\varphi^{-1}\big(h(\Delta)\big)\big)=h(\Delta). \tag{2.10}$$

Let us now establish our discrete-time truncated EM numerical solutions to approximate the true solution. For some positive integer M, we take step size $\Delta=\tau/M$. It is easy to see that Δ becomes sufficiently small by choosing M sufficiently large. Define $t_k=k\Delta$ for $k=-M,-M+1,-M+2,\dots,-1,0,1,2,\dots$. Set $X_\Delta(t_k)=\xi(t_k)$ for $k=-M,-M+1,-M+2,\dots,-1,0$ and then form

$$\begin{aligned}X_\Delta(t_{k+1})={}&X_\Delta(t_k)+D\big(X_\Delta(t_{k+1-M}),r_{k+1}^\Delta\big)-D\big(X_\Delta(t_{k-M}),r_k^\Delta\big)\\&+f_\Delta\big(t_k,X_\Delta(t_k),X_\Delta(t_{k-M}),r_k^\Delta\big)\Delta+g_\Delta\big(t_k,X_\Delta(t_k),X_\Delta(t_{k-M}),r_k^\Delta\big)\Delta B_k\end{aligned} \tag{2.11}$$

for $k=0,1,2,\dots$, where $\Delta B_k=B(t_{k+1})-B(t_k)$. To form continuous-time step approximations, define

$$\mu(t)=\sum_{k=0}^{\infty}t_k\mathbb{I}_{[t_k,t_{k+1})}(t),\qquad \bar{r}(t)=\sum_{k=0}^{\infty}r_k^\Delta\mathbb{I}_{[t_k,t_{k+1})}(t), \tag{2.12}$$

where $\mathbb{I}$ is the indicator function. As usual, there are two kinds of continuous-time step approximations. The first one whose sample paths are not continuous is

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k)\mathbb{I}_{[t_k,t_{k+1})}(t). \tag{2.13}$$

The other one with continuous sample paths is

$$\begin{aligned} x_\Delta(t) = \xi(0) &+ D\big(\bar{x}_\Delta(t-\tau),\bar{r}(t)\big) - D\big(\xi(-\tau),r_0^\Delta\big) \\ &+ \int_0^t f_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,ds \\ &+ \int_0^t g_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,dB(s), \end{aligned} \tag{2.14}$$

which is continuous in t. Is easy to see that $X_\Delta(t_k) = \bar{x}_\Delta(t_k) = x_\Delta(t_k)$. Namely, they coincide at t_k.

3 Strong convergence rate

In this section, we estimate the strong convergence rate of the partially truncated EM method for (2.1). Now, to achieve this goal, we have to impose the following assumptions on the coefficients.

Assumption 3.1 There exist constants $K_3 > 0$ and $\beta \geq 0$ such that

$$\big|\tilde{F}(t,x,y,i) - \tilde{F}(t,\bar{x},\bar{y},i)\big| \vee \big|\tilde{G}(t,x,y,i) - \tilde{G}(t,\bar{x},\bar{y},i)\big| \leq K_3\big(|x-\bar{x}| + |y-\bar{y}|\big) \tag{3.1}$$

and

$$\begin{aligned} &\big|F(t,x,y,i) - F(t,\bar{x},\bar{y},i)\big| \vee \big|G(t,x,y,i) - G(t,\bar{x},\bar{y},i)\big| \\ &\quad\leq K_3\big(1 + |x|^\beta + |y|^\beta + |\bar{x}|^\beta + |\bar{y}|^\beta\big)\big(|x-\bar{x}| + |y-\bar{y}|\big) \end{aligned} \tag{3.2}$$

for all $t \in [0,T]$, $x,y,\bar{x},\bar{y} \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

By Assumption 3.1 we get that there exists a constant $\bar{K}_3 > 0$ such that

$$\big|\tilde{F}(t,x,y,i)\big| \vee \big|\tilde{G}(t,x,y,i)\big| \leq \bar{K}_3\big(1 + |x| + |y|\big) \tag{3.3}$$

and

$$\big|F(t,x,y,i)\big| \vee \big|G(t,x,y,i)\big| \leq \bar{K}_3\big(1 + |x|^{\beta+1} + |y|^{\beta+1}\big) \tag{3.4}$$

for all $t \in [0,T]$, $x,y \in \mathbb{R}^n$, and $i \in \mathbb{S}$, where $\bar{K}_3 = 4K_3 + \sup_{t\in[0,T],i\in\mathbb{S}}[\tilde{F}(t,0,0,i) + \tilde{G}(t,0,0,i) + F(t,0,0,i) + G(t,0,0,i)]$. We also derive from Assumption 3.1 that

$$\begin{aligned} &\big|f(t,x,y,i) - f(t,\bar{x},\bar{y},i)\big| \vee \big|g(t,x,y,i) - g(t,\bar{x},\bar{y},i)\big| \\ &\quad\leq K_3\big(1 + |x|^\beta + |y|^\beta + |\bar{x}|^\beta + |\bar{y}|^\beta\big)\big(|x-\bar{x}| + |y-\bar{y}|\big) \end{aligned} \tag{3.5}$$

for all $t \in [0,T]$, $x,y,\bar{x},\bar{y} \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

Before stating the next assumption, we introduce functions $\bar{V}_i$, $i = 1, 2, 3$, such that for any $x, y \in \mathbb{R}^n$,

$$0 \le \bar{V}_i(x,y) \le K_{\bar{V}i}\big(1 + |x|^{l_i} + |y|^{l_i}\big), \quad i = 1, 2, 3,$$

for some $K_{\bar{V}i} > 0$ and $l_i \ge 1$. Denote $l_v = \max\{l_1, l_2, l_3\}$.

Assumption 3.2 There exist constants $K_4 > 0$ and $\bar{q} > 2$ such that

$$\begin{aligned}&\big(x - D(y,i) - \bar{x} + D(\bar{y},i)\big)^T\big(F(t,x,y,i) - F(t,\bar{x},\bar{y},i)\big)\\&\quad + \frac{\bar{q}-1}{2}\big|G(t,x,y,i) - G(t,\bar{x},\bar{y},i)\big|^2 \le K_4|x - \bar{x}|^2 + \big|\bar{V}_1(y,\bar{y})\big||y - \bar{y}|^2\end{aligned} \tag{3.6}$$

for all $t \in [0, T]$, $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

By Assumption 3.2 we obtain that for any $q \in (2, \bar{q})$,

$$\begin{aligned}&\big(x - D(y,i) - \bar{x} + D(\bar{y},i)\big)^T\big(f(t,x,y,i) - f(t,\bar{x},\bar{y},i)\big)\\&\qquad + \frac{q-1}{2}\big|g(t,x,y,i) - g(t,\bar{x},\bar{y},i)\big|^2\\&\quad \le (\bar{K}_4 + K_4)|x - \bar{x}|^2 + \big(\bar{K}_4 + \big|\bar{V}_1(y,\bar{y})\big|\big)|y - \bar{y}|^2,\end{aligned} \tag{3.7}$$

where $\bar{K}_4 = 2K_3 + \frac{K_3^2(q-1)(\bar{q}-1)}{\bar{q}-q}$. The proof is trivial, so we omit it.

Assumption 3.3 There exist constants $K_5 > 0$ and $\bar{p} > \bar{q}$ such that

$$\begin{aligned}&\big(x - D(y,i)\big)^T F(t,x,y,i) + \frac{\bar{p}-1}{2}\big|G(t,x,y,i)\big|^2\\&\quad \le K_5\big(1 + |x|^2\big) + \big|\bar{V}_2(y,0)\big||y|^2\end{aligned} \tag{3.8}$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

By Assumption 3.3 we derive that for any $p \in [2, \bar{p})$,

$$\begin{aligned}&\big(x - D(y,i)\big)^T f(t,x,y,i) + \frac{p-1}{2}\big|g(t,x,y,i)\big|^2\\&\quad \le (\bar{K}_5 + K_5)\big(1 + |x|^2\big) + \big(\bar{K}_5 + \big|\bar{V}_2(y,0)\big|\big)|y|^2,\end{aligned} \tag{3.9}$$

where $\bar{K}_5 = 3\bar{K}_3 + \frac{3\bar{K}_3^2(p-1)(\bar{p}-1)}{2(\bar{p}-p)}$.

Assumption 3.4 There exist constants $K_6 > 0$, $K_7 > 0$, $\theta \in (0, 1]$, and $\sigma \in (0, 1]$ such that

$$\begin{aligned}&\big|f(t_1,x,y,i) - f(t_2,x,y,i)\big| \le K_6\big(1 + |x|^{\beta+1} + |y|^{\beta+1}\big)|t_1 - t_2|^{\theta},\\&\big|g(t_1,x,y,i) - g(t_2,x,y,i)\big| \le K_7\big(1 + |x|^{\beta+1} + |y|^{\beta+1}\big)|t_1 - t_2|^{\sigma}\end{aligned} \tag{3.10}$$

for all $t_1, t_2 \in [0, T]$, $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$, where β is as in Assumption 3.1.

The following lemma gives that the p-moment of the true solution is bounded. This lemma can be proved similarly to the proof of Theorem 2.4 presented in [12] by means of the technique used in Theorem 2.1 of [35].

Lemma 3.5 *Let Assumptions 3.1 and 3.3 hold. Then neutral stochastic differential delay equations with Markovian switching (2.1) with initial data (2.2) has a unique solution $x(t)$ on $t \geq -\tau$. In addition, this solution has the property that*

$$\sup_{-\tau \leq t \leq T} \mathbb{E}\left|x(t)\right|^p < \infty, \quad \forall T > 0. \tag{3.11}$$

To get the strong convergence rate, we impose another assumption.

Assumption 3.6 There exist constants $K_8 > 0$ and $\bar{p} > \bar{q}$ such that

$$\begin{aligned}&\left(x - D(y,i)\right)^T F_\Delta(t,x,y,i) + \frac{\bar{p}-1}{2}\left|G_\Delta(t,x,y,i)\right|^2 \\ &\quad \leq K_8\left(1 + |x|^2\right) + \left|\bar{V}_3(y,0)\right||y|^2\end{aligned} \tag{3.12}$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

By Assumption 3.6 we can show that for any $p \in [2,\bar{p})$,

$$\begin{aligned}&\left(x - D(y,i)\right)^T f_\Delta(t,x,y,i) + \frac{p-1}{2}\left|g_\Delta(t,x,y,i)\right|^2 \\ &\quad \leq (\bar{K}_8 + K_8)\left(1 + |x|^2\right) + \left(\bar{K}_8 + \left|\bar{V}_3(y,0)\right|\right)|y|^2,\end{aligned} \tag{3.13}$$

where $\bar{K}_8 = 3\bar{K}_3 + \frac{3\bar{K}_3^2(p-1)(\bar{p}-1)}{2(\bar{p}-p)}$.

Remark 3.7 When $D(\cdot,\cdot) = 0$, we can derive that for any functions satisfying Assumption 3.3,

$$x^T F_\Delta(t,x,y,i) + \frac{\bar{p}-1}{2}\left|G_\Delta(t,x,y,i)\right|^2 \leq \tilde{K}_8\left(1 + |x|^2\right) + \left|\bar{V}_2(y,0)\right|^2|y|^2 \tag{3.14}$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$, where $\tilde{K}_8 = 2K_5([1/\varphi^{-1}(h(1))] \vee 1)$. In other words, Assumption 3.6 can be eliminated if there is no neutral term.

Remark 3.8 In fact, there are plenty of functions such that $D(y,i)$, $F(t,x,y,i)$, and $G(t,x,y,i)$ satisfy Assumption 3.3 and the corresponding $F_\Delta(t,x,y,i)$ and $G_\Delta(t,x,y,i)$ satisfy Assumption 3.6. For example, when $i = 1$, define $D(y,1) = -\frac{1}{6}y$, $f(t,x,y,1) = -2y^3 + (t(1-t))^{\frac{1}{3}}y - 10x + 2y$, $g(t,x,y,1) = (t(1-t))^{\frac{1}{3}}|y|^{\frac{3}{2}}$ for $t \in [0,1]$ and $x, y \in \mathbb{R}^1$. Thus $F(t,x,y,1) = -2y^3$ and $G(t,x,y,1) = (t(1-t))^{\frac{1}{3}}|y|^{\frac{3}{2}}$. We can easily prove that Assumptions 3.3 and 3.6 are satisfied. A detailed proof is presented in Sect. 5.

Lemma 3.9 *Let Assumptions 2.3, 3.1, and 3.6 hold. Then for any $p \in [2,\bar{p})$, we have*

$$\sup_{0<\Delta\leq 1}\sup_{0\leq t\leq T} \mathbb{E}\left|x_\Delta(t)\right|^p \leq C, \quad \forall T > 0. \tag{3.15}$$

Proof For any $\Delta \in (0,1]$ and $t \in [0,T]$, by Itô's formula we derive that

$$\begin{aligned}
&\mathbb{E}\left|x_\Delta(t) - D\big(\bar{x}_\Delta(t-\tau),\bar{r}(t)\big)\right|^p - \left|\xi(0) - D\big(\xi(-\tau), r_0^\Delta\big)\right|^p \\
&\quad\le \mathbb{E}\int_0^t p\left|x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^{p-2}\Big[\big(x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big)^T \\
&\qquad\cdot f_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big) \\
&\qquad+ \frac{p-1}{2}\left|g_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^2\Big]\,ds \\
&\quad\le \mathbb{E}\int_0^t p\left|x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^{p-2}\big(x_\Delta(s) - \bar{x}_\Delta(s)\big)^T \\
&\qquad\cdot f_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,ds \\
&\qquad+ \mathbb{E}\int_0^t p\left|x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^{p-2}\Big[\big(\bar{x}_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big)^T \\
&\qquad\cdot f_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big) \\
&\qquad+ \frac{p-1}{2}\left|g_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^2\Big]\,ds \\
&\quad=: A_1 + A_2.
\end{aligned} \tag{3.16}$$

Let us first estimate A_1:

$$\begin{aligned}
A_1 &\le p\mathbb{E}\int_0^t \left|x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^{p-2}\big(x_\Delta(s) - \bar{x}_\Delta(s)\big)^T \\
&\qquad\cdot \tilde{F}\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,ds \\
&\qquad+ p\mathbb{E}\int_0^t \left|x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^{p-2}\big(x_\Delta(s) - \bar{x}_\Delta(s)\big)^T \\
&\qquad\cdot F_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,ds \\
&=: A_{11} + A_{12}.
\end{aligned} \tag{3.17}$$

By Assumptions 2.3 and 3.1 and Young's inequality we derive that

$$\begin{aligned}
A_{11} &\le (p-2)\mathbb{E}\int_0^t \left|x_\Delta(s) - D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^p\,ds \\
&\qquad+ \frac{p}{2}\mathbb{E}\int_0^t \left|x_\Delta(s) - \bar{x}_\Delta(s)\right|^{\frac{p}{2}}\left|\tilde{F}\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^{\frac{p}{2}}\,ds \\
&\le C\int_0^t \big(1 + \mathbb{E}\left|x_\Delta(s)\right|^p + \mathbb{E}\left|\bar{x}_\Delta(s)\right|^p + \mathbb{E}\left|\bar{x}_\Delta(s-\tau)\right|^p\big)\,ds.
\end{aligned} \tag{3.18}$$

Moreover, for any $t \in [0,T]$, there always is an integer $k \ge 0$ such that $t \in [t_k, t_{k+1})$. By Hölder's inequality and BDG's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left|x_\Delta(t) - \bar{x}_\Delta(t)\right|^{\frac{p}{2}} \\
&\quad= \mathbb{E}\left|x_\Delta(t) - x_\Delta(t_k)\right|^{\frac{p}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\mathbb{E}\left|\int_{t_k}^{t} f_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,ds\right|^{\frac{p}{2}}\\
&\quad + C\mathbb{E}\left|\int_{t_k}^{t} g_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\,dB(s)\right|^{\frac{p}{2}}\\
&\le C\Delta^{\frac{p}{2}-1}\mathbb{E}\int_{t_k}^{t}\big|\tilde{F}\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^{\frac{p}{2}}\,ds\\
&\quad + C\Delta^{\frac{p}{2}-1}\mathbb{E}\int_{t_k}^{t}\big|F_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^{\frac{p}{2}}\,ds\\
&\quad + C\Delta^{\frac{p}{4}-1}\mathbb{E}\int_{t_k}^{t}\big|\tilde{G}\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^{\frac{p}{2}}\,ds\\
&\quad + C\Delta^{\frac{p}{4}-1}\mathbb{E}\int_{t_k}^{t}\big|G_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^{\frac{p}{2}}\,ds\\
&\le C\Delta^{\frac{p}{4}}h^{\frac{p}{2}}(\Delta) + C\Delta^{\frac{p}{4}}\Big(1+\sup_{0\le s\le t}\mathbb{E}\big|\bar{x}_\Delta(s)\big|^{\frac{p}{2}}+\sup_{0\le s\le t}\mathbb{E}\big|\bar{x}_\Delta(s-\tau)\big|^{\frac{p}{2}}\Big).
\end{aligned}
\tag{3.19}
$$

Thus, by (2.8), (2.10), and (3.19) and Young's inequality we get

$$
\begin{aligned}
A_{12} &\le (p-2)\mathbb{E}\int_0^t\big|x_\Delta(s)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^p\,ds\\
&\quad + \frac{p}{2}\mathbb{E}\int_0^t\big|x_\Delta(s)-\bar{x}_\Delta(s)\big|^{\frac{p}{2}}\big|F_\Delta\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^{\frac{p}{2}}\,ds\\
&\le (p-2)\mathbb{E}\int_0^t\big|x_\Delta(s)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^p\,ds\\
&\quad + \frac{p}{2}h^{\frac{p}{2}}(\Delta)\int_0^t\mathbb{E}\big|x_\Delta(s)-\bar{x}_\Delta(s)\big|^{\frac{p}{2}}\,ds\\
&\le (p-2)\mathbb{E}\int_0^t\big|x_\Delta(s)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^p\,ds\\
&\quad + Ch^{\frac{p}{2}}(\Delta)\Delta^{\frac{p}{4}}\int_0^t\Big(1+h^{\frac{p}{2}}(\Delta)+\sup_{0\le l\le s}\mathbb{E}\big|\bar{x}_\Delta(l)\big|^{\frac{p}{2}}+\sup_{0\le l\le s}\mathbb{E}\big|\bar{x}_\Delta(l-\tau)\big|^{\frac{p}{2}}\Big)\,ds\\
&\le C\int_0^t\Big(1+\sup_{0\le l\le s}\mathbb{E}\big|x_\Delta(l)\big|^p+\sup_{0\le l\le s}\mathbb{E}\big|\bar{x}_\Delta(l)\big|^p+\sup_{0\le l\le s}\mathbb{E}\big|\bar{x}_\Delta(l-\tau)\big|^p\Big)\,ds.
\end{aligned}
\tag{3.20}
$$

Now, we are handling A_2. By Assumptions 2.3 and 3.6 and Hölder's inequality we get

$$
\begin{aligned}
A_2 &\le \mathbb{E}\int_0^t p\big|x_\Delta(s)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^{p-2}\big[(\bar{K}_8+K_8)\big(1+\big|\bar{x}_\Delta(s)\big|^2\big)\\
&\quad + \big(\bar{K}_8+\big|\bar{V}_3\big(\bar{x}_\Delta(s-\tau),0\big)\big|\big)\big|\bar{x}_\Delta(s-\tau)\big|^2\big]\,ds\\
&\le C\mathbb{E}\int_0^t\big|x_\Delta(s)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^p\,ds + C\mathbb{E}\int_0^t\big(1+\big|\bar{x}_\Delta(s)\big|^p\\
&\quad + \big|\bar{x}_\Delta(s-\tau)\big|^p\big)\,ds + C\mathbb{E}\int_0^t\big|\bar{V}_3\big(\bar{x}_\Delta(s-\tau),0\big)\big|^{\frac{p}{2}}\big|\bar{x}_\Delta(s-\tau)\big|^p\,ds\\
&\le C\int_0^t\big(1+\mathbb{E}\big|x_\Delta(s)\big|^p+\mathbb{E}\big|\bar{x}_\Delta(s)\big|^p+\mathbb{E}\big|\bar{x}_\Delta(s-\tau)\big|^p\big)\,ds
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&+ C\int_0^t \left(\mathbb{E}\left|\bar{V}_3\left(\bar{x}_\Delta(s-\tau),0\right)\right|^p + \mathbb{E}\left|\bar{x}_\Delta(s-\tau)\right|^{2p}\right) ds \\
&\le C\int_0^t \left(1 + \mathbb{E}\left|x_\Delta(s)\right|^p + \mathbb{E}\left|\bar{x}_\Delta(s)\right|^p + \mathbb{E}\left|\bar{x}_\Delta(s-\tau)\right|^p\right) ds \\
&+ C\int_0^t \mathbb{E}\left|\bar{x}_\Delta(s-\tau)\right|^{l_{\nu*}p}\, ds,
\end{aligned}
$$

where $l_{\nu*} = l_\nu \vee 2$. Inserting (3.17), (3.18), (3.20), and (3.21) into (3.16) yields that

$$
\begin{aligned}
&\mathbb{E}\left|x_\Delta(t) - D\left(\bar{x}_\Delta(t-\tau),\bar{r}(t)\right)\right|^p \\
&\quad \le C\left(1 + \int_0^t \sup_{0\le l\le s} \mathbb{E}\left|x_\Delta(l)\right|^p ds + \int_0^t \sup_{0\le l\le s} \mathbb{E}\left|x_\Delta(l-\tau)\right|^{l_{\nu*}p} ds\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l) - D\left(\bar{x}_\Delta(l-\tau),\bar{r}(l)\right)\right|^p \\
&\quad \le C\left(1 + \int_0^t \sup_{0\le l\le s} \mathbb{E}\left|x_\Delta(l)\right|^p ds + \int_0^t \sup_{0\le l\le s} \mathbb{E}\left|x_\Delta(l-\tau)\right|^{l_{\nu*}p} ds\right).
\end{aligned}
\tag{3.22}
$$

Moreover, for any $c_0 > 0$,

$$
\begin{aligned}
\sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l)\right|^p &= \sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l) - D\left(\bar{x}_\Delta(l-\tau),\bar{r}(l)\right) + D\left(\bar{x}_\Delta(l-\tau),\bar{r}(l)\right)\right|^p \\
&\le (1+c_0)^{p-1} \sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l) - D\left(\bar{x}_\Delta(l-\tau),\bar{r}(l)\right)\right|^p \\
&\quad + \left(\frac{1+c_0}{c_0}\right)^{p-1} K_2^p\left(\|\xi\|^p + \sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l)\right|^p\right).
\end{aligned}
\tag{3.23}
$$

Then we can take c_0 large enough such that $(\frac{1+c_0}{c_0})^{p-1}K_2^p < 1$ for any $K_2 \in (0,1)$. Thus

$$
\sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l)\right|^p \le c_1 \sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l) - D\left(\bar{x}_\Delta(l-\tau),\bar{r}(l)\right)\right|^p + c_2\|\xi\|^p,
\tag{3.24}
$$

where

$$
c_1 = \frac{c_0^{p-1}(1+c_0)^{p-1}}{c_0^{p-1} - (1+c_0)^{p-1}K_2^p} \quad \text{and} \quad c_2 = \frac{(1+c_0)^{p-1}K_2^p}{c_0^{p-1} - (1+c_0)^{p-1}K_2^p}.
\tag{3.25}
$$

An application of Gronwall's inequality yields that

$$
\sup_{0\le l\le t} \mathbb{E}\left|x_\Delta(l)\right|^p \le C\left(1 + \int_0^t \sup_{0\le l\le s} \mathbb{E}\left|x_\Delta(l-\tau)\right|^{l_{\nu*}p} ds\right).
\tag{3.26}
$$

The following technique is similar to that in Theorem 2.1 of [35]. Define

$$
p_i = \left(\lfloor T/\tau\rfloor + 2 - i\right)p l_{\nu*}^{\lfloor T/\tau\rfloor+1-i}, \quad i = 1,2,\ldots,\lfloor T/\tau\rfloor + 1.
$$

We can observe that

$$
p_{i+1}l_{\nu*} < p_i \quad \text{and} \quad p_{\lfloor T/\tau\rfloor+1} = p, \quad i = 1,2,\ldots,\lfloor T/\tau\rfloor.
$$

By (3.26) and $\xi \in \mathscr{L}^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ we derive that

$$\sup_{0\leq l\leq \tau} \mathbb{E}\big|x_\Delta(l)\big|^{p_1} \leq C.$$

Then Hölder's inequality leads to

$$\sup_{0\leq l\leq 2\tau} \mathbb{E}\big|x_\Delta(l)\big|^{p_2} \leq C\biggl(1+\int_0^{2\tau} \sup_{0\leq l\leq s}\bigl(\mathbb{E}\big|x_\Delta(l-\tau)\big|^{p_1}\bigr)^{\frac{l_{v*}p_2}{p_1}}\,ds\biggr) \leq C.$$

The desired result follows by repeating this procedure. We complete the proof. □

Lemma 3.10 *Let Assumptions 2.3, 3.1, and 3.6 hold. Then for any $\Delta \in (0,1]$ and $t \in [0,T]$, we have*

$$\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^p \leq C\Delta^{\frac{p}{2}}h^p(\Delta). \tag{3.27}$$

Therefore

$$\lim_{\Delta\to 0} \mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^p = 0. \tag{3.28}$$

Proof Fix any $\Delta \in (0,1]$. For any $t \in [0,T]$, there is an integer $k \geq 0$ such that $t \in [t_k, t_{k+1})$. In the same way as in the proof of (3.19), we have

$$\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^p \leq C\Delta^{\frac{p}{2}}\bigl(1+h^p(\Delta)+\mathbb{E}\big|\bar{x}_\Delta(t)\big|^p+\mathbb{E}\big|\bar{x}_\Delta(t-\tau)\big|^p\bigr).$$

Then Lemma 3.9 gives that

$$\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^p \leq C\Delta^{\frac{p}{2}}h^p(\Delta).$$

We complete the proof. □

Lemma 3.11 *Let Assumptions 2.3, 3.1, and 3.6 hold. For any real number $L > \|\xi\|$, define the stopping time*

$$\tau_{\Delta,L} = \inf\bigl\{t \geq 0 : \big|x_\Delta(t)\big| \geq L\bigr\}. \tag{3.29}$$

Then we have

$$\mathbb{P}(\tau_{\Delta,L} \leq T) \leq \frac{C}{L^p}. \tag{3.30}$$

Proof By Itô's formula and Assumption 3.6 we get

$$
\begin{aligned}
&\mathbb{E}\bigl|x_\Delta(t\wedge\tau_{\Delta,L})-D\bigl(\bar{x}_\Delta(t\wedge\tau_{\Delta,L}-\tau),\bar{r}(t\wedge\tau_{\Delta,L})\bigr)\bigr|^p-\bigl|\xi(0)-D\bigl(\xi(-\tau),r_0^\Delta\bigr)\bigr|^p\\
&\quad\le\mathbb{E}\int_0^{t\wedge\tau_{\Delta,L}}p\bigl|x_\Delta(s)-D\bigl(\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{p-2}\biggl[\bigl(x_\Delta(s)-D\bigl(\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr)^T\\
&\qquad\cdot f_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)+\frac{p-1}{2}\bigl|g_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^2\biggr]\,ds\\
&\quad\le\mathbb{E}\int_0^{t\wedge\tau_{\Delta,L}}p\bigl|x_\Delta(s)-D\bigl(\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{p-2}\biggl[\bigl(\bar{x}_\Delta(s)-D\bigl(\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr)^T\\
&\qquad\cdot f_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)+\frac{p-1}{2}\bigl|g_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^2\biggr]\,ds\\
&\qquad+\mathbb{E}\int_0^{t\wedge\tau_{\Delta,L}}p\bigl|x_\Delta(s)-D\bigl(\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{p-2}\bigl(x_\Delta(s)-\bar{x}_\Delta(s)\bigr)^T\\
&\qquad\cdot f_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\,ds\\
&\quad\le C\int_0^t\mathbb{E}\bigl|x_\Delta(s\wedge\tau_{\Delta,L})-D\bigl(\bar{x}_\Delta(s\wedge\tau_{\Delta,L}-\tau),\bar{r}(s\wedge\tau_{\Delta,L})\bigr)\bigr|^p\,ds\\
&\qquad+C\int_0^t\bigl(1+\mathbb{E}\bigl|\bar{x}_\Delta(s)\bigr|^p+\mathbb{E}\bigl|\bar{x}_\Delta(s-\tau)\bigr|^p\bigr)\,ds\\
&\qquad+C\mathbb{E}\int_0^{t\wedge\tau_{\Delta,L}}\bigl|\bar{V}_3\bigl(\bar{x}_\Delta(s-\tau),0\bigr)\bigr|^{\frac{p}{2}}\bigl|\bar{x}_\Delta(s-\tau)\bigr|^p\,ds\\
&\qquad+C\mathbb{E}\int_0^t\bigl|x_\Delta(s)-\bar{x}_\Delta(s)\bigr|^{\frac{p}{2}}\bigl|f_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{\frac{p}{2}}\,ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_{\Delta,L}}\bigl|\bar{V}_3\bigl(\bar{x}_\Delta(s-\tau),0\bigr)\bigr|^{\frac{p}{2}}\bigl|\bar{x}_\Delta(s-\tau)\bigr|^p\,ds\\
&\quad\le\frac{1}{2}\int_0^t\mathbb{E}\bigl|\bar{V}_3\bigl(\bar{x}_\Delta(s\wedge\tau_{\Delta,L}-\tau),0\bigr)\bigr|^p\,ds+\frac{1}{2}\int_0^t\mathbb{E}\bigl|\bar{x}_\Delta(s-\tau)\bigr|^{2p}\,ds\\
&\quad\le C\int_0^t\bigl(1+\mathbb{E}\bigl|\bar{x}_\Delta(s-\tau)\bigr|^{l_v p}+\mathbb{E}\bigl|\bar{x}_\Delta(s-\tau)\bigr|^{2p}\bigr)\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_0^t\bigl|x_\Delta(s)-\bar{x}_\Delta(s)\bigr|^{\frac{p}{2}}\bigl|f_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{\frac{p}{2}}\,ds\\
&\quad\le C\mathbb{E}\int_0^t\bigl|x_\Delta(s)-\bar{x}_\Delta(s)\bigr|^{\frac{p}{2}}\bigl|\tilde{F}\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{\frac{p}{2}}\,ds\\
&\qquad+C\mathbb{E}\int_0^t\bigl|x_\Delta(s)-\bar{x}_\Delta(s)\bigr|^{\frac{p}{2}}\bigl|F_\Delta\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^{\frac{p}{2}}\,ds\\
&\quad\le C\mathbb{E}\int_0^t\bigl(\bigl|x_\Delta(s)\bigr|^{\frac{p}{2}}+\bigl|\bar{x}_\Delta(s)\bigr|^{\frac{p}{2}}\bigr)\bigl(1+\bigl|\bar{x}_\Delta(s)\bigr|^{\frac{p}{2}}+\bigl|\bar{x}_\Delta(s-\tau)\bigr|^{\frac{p}{2}}\bigr)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ Ch^{\frac{p}{2}}(\Delta) \int_0^t \mathbb{E}\big|x_\Delta(s) - \bar{x}_\Delta(s)\big|^{\frac{p}{2}}\, ds \\
&\le C\big(1 + \Delta^{\frac{p}{4}} h^p(\Delta)\big) \le C,
\end{aligned}
$$

where (2.8), (2.10), (3.3), Young's inequality, and Lemma 3.9 were used. Then we obtain that

$$
\begin{aligned}
&\mathbb{E}\big|x_\Delta(t \wedge \tau_{\Delta,L}) - D\big(\bar{x}_\Delta(t \wedge \tau_{\Delta,L} - \tau), \bar{r}(t \wedge \tau_{\Delta,L})\big)\big|^p \\
&\quad \le C\bigg(1 + \int_0^t \mathbb{E}\big|\bar{x}_\Delta(s-\tau)\big|^{l_v * p}\, ds \\
&\qquad + \int_0^t \mathbb{E}\big|x_\Delta(s \wedge \tau_{\Delta,L}) - D\big(\bar{x}_\Delta(s \wedge \tau_{\Delta,L} - \tau), \bar{r}(s \wedge \tau_{\Delta,L})\big)\big|^p\, ds\bigg),
\end{aligned}
$$

where $l_v * = l_v \vee 2$. Using the same technique as in Lemma 3.9 gives that

$$
\mathbb{E}\big|x_\Delta(T \wedge \tau_{\Delta,L}) - D\big(\bar{x}_\Delta(T \wedge \tau_{\Delta,L} - \tau), \bar{r}(T \wedge \tau_{\Delta,L})\big)\big|^p \le C. \tag{3.31}
$$

We can get from (2.6) that

$$
\begin{aligned}
&\mathbb{I}_{\{\tau_{\Delta,L} \le T\}}\big|x_\Delta(\tau_{\Delta,L}) - D\big(\bar{x}_\Delta(\tau_{\Delta,L} - \tau), \bar{r}(\tau_{\Delta,L})\big)\big| \\
&\quad \ge \mathbb{I}_{\{\tau_{\Delta,L} \le T\}}\big(\big|x_\Delta(\tau_{\Delta,L})\big| - \big|D\big(\bar{x}_\Delta(\tau_{\Delta,L} - \tau), \bar{r}(\tau_{\Delta,L})\big)\big|\big) \\
&\quad \ge L - K_2 L.
\end{aligned} \tag{3.32}
$$

Hence we derive from (3.31) and (3.32) that

$$
\begin{aligned}
\mathbb{P}(\tau_{\Delta,L} \le T) &\le \frac{\mathbb{E}(\mathbb{I}_{\{\tau_{\Delta,L} \le T\}}|x_\Delta(\tau_{\Delta,L}) - D(\bar{x}_\Delta(\tau_{\Delta,L} - \tau), \bar{r}(\tau_{\Delta,L}))|^p)}{(1-K_2)^p L^p} \\
&\le \frac{\mathbb{E}|x_\Delta(T \wedge \tau_{\Delta,L}) - D(\bar{x}_\Delta(T \wedge \tau_{\Delta,L} - \tau), \bar{r}(T \wedge \tau_{\Delta,L}))|^p}{(1-K_2)^p L^p} \\
&\le \frac{C}{(1-K_2)^p L^p}.
\end{aligned} \tag{3.33}
$$

Then the desired result follows. We complete the proof. □

The following lemma can be proved in a similar way as Lemma 3.11 was, so we omit the proof.

Lemma 3.12 *Let Assumptions 2.3, 3.1, and 3.3 hold. For any real number $L > \|\xi\|$, define the stopping time*

$$
\tau_L = \inf\big\{t \ge 0 : \big|x(t)\big| \ge L\big\}. \tag{3.34}
$$

Then we have

$$
\mathbb{P}(\tau_L \le T) \le \frac{C}{L^p}. \tag{3.35}
$$

Lemma 3.13 *Let Assumptions 2.2, 2.3, 3.1–3.4, and 3.6 hold. Assume that $q \in [2,\bar{q})$ and $p > (\beta + l_v + 2)q$. Let $L > \|\xi\|$ be a real number, and let $\Delta \in (0,1]$ be sufficiently small such that $\varphi^{-1}(h(\Delta)) \geq L$. Then we have*

$$\mathbb{E}\big|x(T\wedge\rho_{\Delta,L}) - x_\Delta(T\wedge\rho_{\Delta,L})\big|^q \leq C\big(\Delta^{\frac{q}{2}}h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big), \tag{3.36}$$

where $\rho_{\Delta,L} := \tau_L \wedge \tau_{\Delta,L}$ with τ_L, $\tau_{\Delta,L}$ defined as before.

Proof For simplicity, we write $\rho_{\Delta,L} = \rho$. Denote $e_\Delta(t) = x(t) - D(x(t-\tau), r(t)) - x_\Delta(t) + D(\bar{x}_\Delta(t-\tau), \bar{r}(t))$. For $0 \leq s \leq t\wedge\rho$, we can observe that

$$\big|x(s)\big| \vee \big|x(s-\tau)\big| \vee \big|\bar{x}_\Delta(s)\big| \vee \big|\bar{x}_\Delta(s-\tau)\big| \leq L \leq \varphi^{-1}\big(h(\Delta)\big).$$

Recalling the definition of F_Δ and G_Δ, we have

$$\begin{aligned}
&F_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) = F\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big), \\
&G_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) = G\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)
\end{aligned}$$

for $0 \leq s \leq t\wedge\rho$. Hence we derive that

$$\begin{aligned}
&f_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) \\
&\quad = \tilde{F}\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) + F_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) \\
&\quad = \tilde{F}\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) + F\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) \\
&\quad = f\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big).
\end{aligned}$$

Similarly,

$$g_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big) = g\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big).$$

By Itô's formula we get

$$\begin{aligned}
&\mathbb{E}\big|e_\Delta(t\wedge\rho)\big|^q \\
&\quad \leq \mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\Big[e_\Delta^T(s)\big(f\big(s, x(s), x(s-\tau), r(s)\big) - f_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big) \\
&\qquad + \frac{q-1}{2}\big|g\big(s, x(s), x(s-\tau), r(s)\big) - g_\Delta\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big|^2\Big]\,ds \\
&\quad \leq \mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\Big[e_\Delta^T(s)\big(f\big(s, x(s), x(s-\tau), r(s)\big) - f\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big) \\
&\qquad + \frac{q-1}{2}\big|g\big(s, x(s), x(s-\tau), r(s)\big) - g\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big|^2\Big]\,ds \\
&\quad \leq \mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\Big[\big(x(s) - D\big(x(s-\tau), r(s)\big) - \bar{x}_\Delta(s) + D\big(\bar{x}_\Delta(s-\tau), r(s)\big)\big)^T \\
&\qquad \cdot \big(f\big(s, x(s), x(s-\tau), r(s)\big) - f\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q-1}{2}\big|g\big(s,x(s),x(s-\tau),r(s)\big)-g\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^2\bigg]\,ds\\
&+\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(\bar{x}_\Delta(s)-x_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)-D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big)^T\\
&\cdot\big(f\big(s,x(s),x(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big)\,ds\\
\le{}&\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\bigg[\big(x(s)-D\big(x(s-\tau),r(s)\big)-\bar{x}_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big)^T\\
&\cdot\big(f\big(s,x(s),x(s-\tau),r(s)\big)-f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big)\\
&+\frac{q-1}{2}\big|g\big(s,x(s),x(s-\tau),r(s)\big)-g\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^2\bigg]\,ds\\
&+\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(x(s)-\bar{x}_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(x(s-\tau),r(s)\big)\big)^T\\
&\cdot\big(f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big)\,ds\\
&+\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(\bar{x}_\Delta(s)-x_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)-D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big)^T\\
&\cdot\big(f\big(s,x(s),x(s-\tau),r(s)\big)-f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big)\,ds\\
&+\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(\bar{x}_\Delta(s)-x_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)-D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big)^T\\
&\cdot\big(f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big)\,ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{q-1}{2}\big|g\big(s,x(s),x(s-\tau),r(s)\big)-g\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^2\\
&\quad\le\frac{\bar{q}-1}{2}\big|g\big(s,x(s),x(s-\tau),r(s)\big)-g\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big|^2\\
&\qquad+\frac{(q-1)(\bar{q}-1)}{2(\bar{q}-q)}\big|g\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-g\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathbb{E}\big|e_\Delta(t\wedge\rho)\big|^q\\
&\quad\le\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\bigg[\big(x(s)-D\big(x(s-\tau),r(s)\big)-\bar{x}_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big)^T\\
&\qquad\cdot\big(f\big(s,x(s),x(s-\tau),r(s)\big)-f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big)\\
&\qquad+\frac{q-1}{2}\big|g\big(s,x(s),x(s-\tau),r(s)\big)-g\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big|^2\bigg]\,ds\\
&\qquad+\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(x(s)-\bar{x}_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(x(s-\tau),r(s)\big)\big)^T\\
&\qquad\cdot\big(f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big)\,ds\\
&\qquad+\mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(\bar{x}_\Delta(s)-x_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)
\end{aligned}
$$

$$
\begin{aligned}
& - D\big(\bar{x}_\Delta(s-\tau), r(s)\big)\big)^T \qquad (3.37)\\
& \cdot \big(f\big(s, x(s), x(s-\tau), r(s)\big) - f\big(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)\big)\big)\,ds \\
& + \mathbb{E}\int_0^{t\wedge\rho} q\big|e_\Delta(s)\big|^{q-2}\big(\bar{x}_\Delta(s) - x_\Delta(s) + D\big(\bar{x}_\Delta(s-\tau), \bar{r}(s)\big) - D\big(\bar{x}_\Delta(s-\tau), r(s)\big)\big)^T \\
& \cdot \big(f\big(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)\big) - f\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big)\,ds \\
& + \mathbb{E}\int_0^{t\wedge\rho} \frac{(q-1)(\bar{q}-1)}{2(\bar{q}-q)} q\big|e_\Delta(s)\big|^{q-2} \\
& \cdot \big|g\big(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)\big) - g\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big|^2\,ds \\
=: {} & B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned}
$$

By Hölder's inequality, Assumptions 2.2 and 3.2, and Lemmas 3.9 and 3.10 we get

$$
\begin{aligned}
B_1 \le {} & C\mathbb{E}\int_0^{t\wedge\rho} \big|e_\Delta(s)\big|^q\,ds + C\mathbb{E}\int_0^{t\wedge\rho} \big|x(s) - \bar{x}_\Delta(s)\big|^q\,ds \\
& + C\mathbb{E}\int_0^{t\wedge\rho} \big(\bar{K}_4 + \big|\bar{V}_1\big(x(s-\tau), \bar{x}_\Delta(s-\tau)\big)\big|\big)^{\frac{q}{2}} \big|x(s-\tau) - \bar{x}_\Delta(s-\tau)\big|^q\,ds \\
\le {} & C\bigg(\mathbb{E}\int_0^{t\wedge\rho} \big|e_\Delta(s)\big|^q\,ds + \int_0^t \mathbb{E}\big|x(s\wedge\rho) - x_\Delta(s\wedge\rho)\big|^q\,ds \\
& + \int_0^T \mathbb{E}\big|x_\Delta(s) - \bar{x}_\Delta(s)\big|^q\,ds + \int_{-\tau}^0 \big|\xi(s) - \xi\big(\lfloor s/\Delta\rfloor\Delta\big)\big|^q\,ds \\
& + \int_0^T \big(\mathbb{E}\big|\bar{V}_1\big(x(s-\tau), \bar{x}_\Delta(s-\tau)\big)\big|^q\big)^{\frac{1}{2}} \big(\mathbb{E}\big|x_\Delta(s-\tau) - \bar{x}_\Delta(s-\tau)\big|^{2q}\big)^{\frac{1}{2}}\,ds \\
& + \mathbb{E}\int_0^{t\wedge\rho} \big|\bar{V}_1\big(x(s-\tau), \bar{x}_\Delta(s-\tau)\big)\big|^{\frac{q}{2}} \big|x(s-\tau) - x_\Delta(s-\tau)\big|^q\,ds\bigg) \qquad (3.38)\\
\le {} & C\bigg(\mathbb{E}\int_0^{t\wedge\rho} \big|e_\Delta(s)\big|^q\,ds + \int_0^t \mathbb{E}\big|x(s\wedge\rho) - x_\Delta(s\wedge\rho)\big|^q\,ds + \Delta^{\frac{q}{2}} h^q(\Delta) + \Delta^{q\alpha} \\
& + \int_0^t \big(\mathbb{E}\big|\bar{V}_1\big(x(s\wedge\rho-\tau), \bar{x}_\Delta(s\wedge\rho-\tau)\big)\big|^q\big)^{\frac{1}{2}} \\
& \times \big(\mathbb{E}\big|x(s\wedge\rho-\tau) - x_\Delta(s\wedge\rho-\tau)\big|^{2q}\big)^{\frac{1}{2}}\,ds\bigg) \\
\le {} & C\bigg(\mathbb{E}\int_0^{t\wedge\rho} \big|e_\Delta(s)\big|^q\,ds + \int_0^t \mathbb{E}\big|x(s\wedge\rho) - x_\Delta(s\wedge\rho)\big|^q\,ds + \Delta^{\frac{q}{2}} h^q(\Delta) + \Delta^{q\alpha} \\
& + \int_0^t \big(\mathbb{E}\big|x(s\wedge\rho-\tau) - x_\Delta(s\wedge\rho-\tau)\big|^{2q}\big)^{\frac{1}{2}}\,ds\bigg).
\end{aligned}
$$

As for B_2, we derive from Assumptions 2.3 and 3.4 that

$$
\begin{aligned}
B_2 \le {} & C\mathbb{E}\int_0^{t\wedge\rho} \big|x(s) - \bar{x}_\Delta(s) + D\big(\bar{x}_\Delta(s-\tau), r(s)\big) - D\big(x(s-\tau), r(s)\big)\big|^q\,ds \\
& + C\mathbb{E}\int_0^{t\wedge\rho} \big|f\big(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)\big) - f\big(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)\big)\big|^q\,ds \\
& + C\mathbb{E}\int_0^{t\wedge\rho} \big|e_\Delta(s)\big|^q\,ds
\end{aligned}
$$

$$\begin{aligned}
&\le C\mathbb{E}\int_0^{t\wedge\rho}\big(\big|x(s)-\bar{x}_\Delta(s)\big|^q+\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(x(s-\tau),r(s)\big)\big|^q\big)\,ds\\
&\quad+C\mathbb{E}\int_0^{t\wedge\rho}\big(\big|f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q\\
&\quad+\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\big)\,ds\\
&\quad+C\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds\\
&\le C\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+C\mathbb{E}\int_0^{t\wedge\rho}\big(\big|x(s)-\bar{x}_\Delta(s)\big|^q+\big|x(s-\tau)-\bar{x}_\Delta(s-\tau)\big|^q\big)\,ds\\
&\quad+C\mathbb{E}\int_0^{t\wedge\rho}\big(1+\big|\bar{x}_\Delta(s)\big|^{q\beta+q}+\big|\bar{x}_\Delta(s-\tau)\big|^{q\beta+q}\big)\Delta^{q\theta}\,ds\\
&\quad+C\mathbb{E}\int_0^{t\wedge\rho}\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\big)\,ds.
\end{aligned}\tag{3.39}$$

From (3.38) we get

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\rho}\big(\big|x(s)-\bar{x}_\Delta(s)\big|^q+\big|x(s-\tau)-\bar{x}_\Delta(s-\tau)\big|^q\big)\,ds\\
&\le C\int_0^t\mathbb{E}\big|x(s\wedge\rho)-x_\Delta(s\wedge\rho)\big|^q\,ds+C\big(\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\alpha}\big),
\end{aligned}\tag{3.40}$$

and we have

$$\mathbb{E}\int_0^{t\wedge\rho}\big(1+\big|\bar{x}_\Delta(s)\big|^{q\beta+q}+\big|\bar{x}_\Delta(s-\tau)\big|^{q\beta+q}\big)\Delta^{q\theta}\,ds\le C\Delta^{q\theta}.\tag{3.41}$$

Moreover, let j be the integer part of T/Δ. Then

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\rho}\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\,ds\\
&=\sum_{k=0}^{j}\mathbb{E}\int_{t_k}^{t_{k+1}}\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\\
&\quad-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(t_k)\big)\big|^q\mathbb{I}_{[0,t\wedge\rho]}(s)\,ds\\
&\le 2^{q-1}\sum_{k=0}^{j}\mathbb{E}\int_{t_k}^{t_{k+1}}\big(\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q\\
&\quad+\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(t_k)\big)\big|^q\big)\mathbb{I}_{[0,t\wedge\rho]}(s)\mathbb{I}_{\{r(s)\neq r(t_k)\}}\,ds\\
&\le C\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}}\mathbb{E}\big(\mathbb{E}\big[\big(1+\big|\bar{x}_\Delta(s)\big|^q+\big|\bar{x}_\Delta(s-\tau)\big|^q+h^q(\Delta)\big)\mathbb{I}_{\{r(s)\neq r(t_k)\}}|r(t_k)\big]\big)\,ds,
\end{aligned}\tag{3.42}$$

where in the last step, we used the fact that $\bar{x}_\Delta(s)$ and $\bar{x}_\Delta(s-\tau)$ are conditionally independent of $\mathbb{I}_{\{r(s)\neq r(t_k)\}}$ with respect to the σ-algebra generated by $r(t_k)$. Applying the Markov property yields that

$$\begin{aligned}
&\mathbb{E}\big(\mathbb{I}_{\{r(s)\neq r(t_k)\}}|r(t_k)\big)\\
&\quad=\sum_{i\in\mathbb{S}}\mathbb{I}_{\{r(t_k)=i\}}\mathbb{P}\big(r(s)\neq i|r(t_k)=i\big)\\
&\quad=\sum_{i\in\mathbb{S}}\mathbb{I}_{\{r(t_k)=i\}}\sum_{j\neq i}\big(\gamma_{ij}(s-t_k)+o(s-t_k)\big)\\
&\quad\leq\max_{0\leq i\leq N}\big(-\gamma_{ii}\Delta+o(\Delta)\big)\sum_{i\in\mathbb{S}}\mathbb{I}_{\{r(t_k)=i\}}\\
&\quad\leq C\Delta+o(\Delta).
\end{aligned}\tag{3.43}$$

By Lemma 3.9 we have

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\rho}\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\,ds\\
&\quad\leq\big(C\Delta+o(\Delta)\big)\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}}\big(1+\mathbb{E}\big|\bar{x}_\Delta(s)\big|^q+\mathbb{E}\big|\bar{x}_\Delta(s-\tau)\big|^q+h^q(\Delta)\big)\,ds\\
&\quad\leq h^q(\Delta)\big(C\Delta+o(\Delta)\big).
\end{aligned}\tag{3.44}$$

Inserting (3.40), (3.41), and (3.44) into (3.39) gives that

$$\begin{aligned}
B_2\leq{}&C\bigg(\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+\int_0^t\mathbb{E}\big|x(s\wedge\rho)-x_\Delta(s\wedge\rho)\big|^q\,ds\\
&+\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\alpha}+\Delta^{q\theta}+o(\Delta)\bigg).
\end{aligned}\tag{3.45}$$

In addition, we obtain from Assumptions 2.2 and 3.1 and Lemmas 3.5, 3.9, and 3.10 that

$$\begin{aligned}
B_3\leq{}&C\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds\\
&+C\mathbb{E}\int_0^{t\wedge\rho}\big|\bar{x}_\Delta(s)-x_\Delta(s)+D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)-D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q\,ds\\
&+C\mathbb{E}\int_0^{t\wedge\rho}\big|f\big(s,x(s),x(s-\tau),r(s)\big)-f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q\,ds\\
\leq{}&C\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+C\int_0^T\mathbb{E}\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\,ds\\
&+C\mathbb{E}\int_0^{t\wedge\rho}\big(1+\big|x(s)\big|^{q\beta}+\big|x(s-\tau)\big|^{q\beta}+\big|\bar{x}_\Delta(s)\big|^{q\beta}+\big|\bar{x}_\Delta(s-\tau)\big|^{q\beta}\big)\\
&\cdot\big(\big|x(s)-\bar{x}_\Delta(s)\big|^q+\big|x(s-\tau)-\bar{x}_\Delta(s-\tau)\big|^q\big)\,ds+C\int_0^T\mathbb{E}\big|\bar{x}_\Delta(s)-x_\Delta(s)\big|^q\,ds\\
\leq{}&C\bigg(\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+\int_0^T\mathbb{E}\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\,ds\\
&+\int_0^t\mathbb{E}\big|x(s\wedge\rho)-x_\Delta(s\wedge\rho)\big|^q\,ds+\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\alpha}\bigg).
\end{aligned}\tag{3.46}$$

Furthermore, let j be the integer part of T/Δ. Then by Assumption 2.3 and Lemma 3.9 we have

$$\begin{aligned}
&\sup_{0\le s\le T}\mathbb{E}\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q \\
&\quad\le \max_{0\le k\le j}\Big(\sup_{t_k\le s\le t_{k+1}}\mathbb{E}\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\Big) \\
&\quad\le 2\max_{0\le k\le j}\Big(\sup_{t_k\le s\le t_{k+1}}\mathbb{E}\big[\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\mathbb{I}_{\{r(s)\neq r(t_k)\}}\big]\Big) \\
&\quad\le C\max_{0\le k\le j}\Big(\sup_{t_k\le s\le t_{k+1}}\mathbb{E}\big[\big(\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q \\
&\qquad+\big|D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\big)\mathbb{I}_{\{r(s)\neq r(t_k)\}}\big]\Big) \\
&\quad\le C\max_{0\le k\le j}\Big(1+\sup_{t_k\le s\le t_{k+1}}\mathbb{E}\big|\bar{x}_\Delta(s-\tau)\big|^q\Big)\mathbb{E}(\mathbb{I}_{\{r(s)\neq r(t_k)\}}) \\
&\quad\le C\mathbb{E}(\mathbb{I}_{\{r(s)\neq r(t_k)\}}).
\end{aligned}$$

By (3.43) we get

$$\mathbb{E}(\mathbb{I}_{\{r(s)\neq r(t_k)\}})=\mathbb{E}\big[\mathbb{E}\big(\mathbb{I}_{\{r(s)\neq r(t_k)\}}|r(t_k)\big)\big]\le C\Delta+o(\Delta).$$

Hence, for any $s\in[0,T]$, we derive that

$$\begin{aligned}
&\mathbb{E}\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q \\
&\quad\le \sup_{0\le s\le T}\mathbb{E}\big|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q \\
&\quad\le C\Delta+o(\Delta).
\end{aligned}\tag{3.47}$$

Inserting (3.47) into (3.46) gives that

$$\begin{aligned}
B_3\le{}& C\Big(\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+\int_0^t\mathbb{E}\big|x(s\wedge\rho)-x_\Delta(s\wedge\rho)\big|^q\,ds \\
&+\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\alpha}+o(\Delta)\Big).
\end{aligned}\tag{3.48}$$

Similarly to B_2 and B_3, we easily derive that

$$\begin{aligned}
B_4\le{}& C\Big(\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+\mathbb{E}\int_0^{t\wedge\rho}\big|\bar{x}_\Delta(s)-x_\Delta(s)\big|^q\,ds \\
&+\mathbb{E}\int_0^{t\wedge\rho}\big|D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)-D\big(\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q\,ds \\
&+\mathbb{E}\int_0^{t\wedge\rho}\big|f\big(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)\big|^q\,ds \\
&+\mathbb{E}\int_0^{t\wedge\rho}\big|f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\big)-f\big(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\big|^q\,ds\Big) \\
\le{}& C\Big(\mathbb{E}\int_0^{t\wedge\rho}\big|e_\Delta(s)\big|^q\,ds+\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\theta}+o(\Delta)\Big)
\end{aligned}\tag{3.49}$$

and

$$
\begin{aligned}
B_5 &\le C\biggl(\mathbb{E}\int_0^{t\wedge\rho}\bigl|e_\Delta(s)\bigr|^q\,ds \\
&\quad+\mathbb{E}\int_0^{t\wedge\rho}\bigl|g\bigl(s,\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\bigr)-g\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\bigr)\bigr|^q\,ds \\
&\quad+\mathbb{E}\int_0^{t\wedge\rho}\bigl|g\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),r(s)\bigr)-g\bigl(\mu(s),\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau),\bar{r}(s)\bigr)\bigr|^q\,ds\biggr) \\
&\le C\biggl(\mathbb{E}\int_0^{t\wedge\rho}\bigl|e_\Delta(s)\bigr|^q\,ds+\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\sigma}+o(\Delta)\biggr).
\end{aligned}
\tag{3.50}
$$

Substituting (3.38), (3.45), (3.48), (3.49), and (3.50) into (3.37) yields that

$$
\begin{aligned}
&\mathbb{E}\bigl|e_\Delta(t\wedge\rho)\bigr|^q \\
&\quad\le C\biggl(\int_0^t\mathbb{E}\bigl|e_\Delta(s\wedge\rho)\bigr|^q\,ds+\int_0^t\sup_{0\le l\le s}\mathbb{E}\bigl|x(l\wedge\rho)-x_\Delta(l\wedge\rho)\bigr|^q\,ds \\
&\qquad+\bigl(\Delta^{\frac{q}{2}}h^q(\Delta)\vee\Delta^{q(\alpha\wedge\theta\wedge\sigma)}\bigr)+\int_0^t\bigl(\mathbb{E}\bigl|x(s\wedge\rho-\tau)-x_\Delta(s\wedge\rho-\tau)\bigr|^{2q}\bigr)^{\frac{1}{2}}\,ds\biggr).
\end{aligned}
$$

Using Gronwall's inequality gives that

$$
\begin{aligned}
&\mathbb{E}\bigl|e_\Delta(t\wedge\rho)\bigr|^q \\
&\quad\le C\biggl(\int_0^t\sup_{0\le l\le s}\mathbb{E}\bigl|x(l\wedge\rho)-x_\Delta(l\wedge\rho)\bigr|^q\,ds+\bigl(\Delta^{\frac{q}{2}}h^q(\Delta)\vee\Delta^{q(\alpha\wedge\theta\wedge\sigma)}\bigr) \\
&\qquad+\int_0^t\Bigl(\sup_{0\le l\le s}\mathbb{E}\bigl|x(l\wedge\rho-\tau)-x_\Delta(l\wedge\rho-\tau)\bigr|^{2q}\Bigr)^{\frac{1}{2}}\,ds\biggr).
\end{aligned}
\tag{3.51}
$$

Therefore

$$
\begin{aligned}
&\sup_{0\le l\le t}\mathbb{E}\bigl|e_\Delta(l\wedge\rho)\bigr|^q \\
&\quad\le C\biggl(\int_0^t\sup_{0\le l\le s}\mathbb{E}\bigl|x(l\wedge\rho)-x_\Delta(l\wedge\rho)\bigr|^q\,ds+\bigl(\Delta^{\frac{q}{2}}h^q(\Delta)\vee\Delta^{q(\alpha\wedge\theta\wedge\sigma)}\bigr) \\
&\qquad+\int_0^t\Bigl(\sup_{0\le l\le s}\mathbb{E}\bigl|x(l\wedge\rho-\tau)-x_\Delta(l\wedge\rho-\tau)\bigr|^{2q}\Bigr)^{\frac{1}{2}}\,ds\biggr).
\end{aligned}
\tag{3.52}
$$

Let $y(t)=x(t)-D(x(t-\tau),r(t))$ and $y_\Delta(t)=x_\Delta(t)-D(\bar{x}_\Delta(t-\tau),\bar{r}(t))$. Thus $e_\Delta(t)=y(t)-y_\Delta(t)$. Then for any $c_3,c_4,c_5>0$, we have

$$
\begin{aligned}
&\bigl|x(t)-x_\Delta(t)\bigr|^q \\
&\quad\le(1+c_3)^{q-1}\bigl|y(t)-y_\Delta(t)\bigr|^q+\biggl(\frac{1+c_3}{c_3}\biggr)^{q-1}\bigl|D\bigl(x(t-\tau),r(t)\bigr)-D\bigl(\bar{x}_\Delta(t-\tau),\bar{r}(t)\bigr)\bigr|^q \\
&\quad\le(1+c_3)^{q-1}\bigl|e_\Delta(t)\bigr|^q+\biggl(\frac{(1+c_3)(1+c_4)}{c_3}\biggr)^{q-1}\bigl|D\bigl(x(t-\tau),r(t)\bigr)-D\bigl(\bar{x}_\Delta(t-\tau),r(t)\bigr)\bigr|^q \\
&\qquad+\biggl(\frac{(1+c_3)(1+c_4)}{c_3c_4}\biggr)^{q-1}\bigl|D\bigl(\bar{x}_\Delta(t-\tau),r(t)\bigr)-D\bigl(\bar{x}_\Delta(t-\tau),\bar{r}(t)\bigr)\bigr|^q
\end{aligned}
$$

$$\begin{aligned}
&\le (1+c_3)^{q-1}\left|e_\Delta(t)\right|^q + \left(\frac{(1+c_3)(1+c_4)}{c_3}\right)^{q-1} K_2^q \left|x(t-\tau)-\bar{x}_\Delta(t-\tau)\right|^q \\
&\quad + \left(\frac{(1+c_3)(1+c_4)}{c_3c_4}\right)^{q-1} \left|D\big(\bar{x}_\Delta(t-\tau),r(t)\big)-D\big(\bar{x}_\Delta(t-\tau),\bar{r}(t)\big)\right|^q \\
&\le (1+c_3)^{q-1}\left|e_\Delta(t)\right|^q + \left(\frac{(1+c_3)(1+c_4)(1+c_5)}{c_3}\right)^{q-1} K_2^q \left|x(t-\tau)-x_\Delta(t-\tau)\right|^q \\
&\quad + \left(\frac{(1+c_3)(1+c_4)(1+c_5)}{c_3c_5}\right)^{q-1} K_2^q \left|x_\Delta(t-\tau)-\bar{x}_\Delta(t-\tau)\right|^q \\
&\quad + \left(\frac{(1+c_3)(1+c_4)}{c_3c_4}\right)^{q-1} \left|D\big(\bar{x}_\Delta(t-\tau),r(t)\big)-D\big(\bar{x}_\Delta(t-\tau),\bar{r}(t)\big)\right|^q.
\end{aligned}$$

Choose c_3 sufficiently large and choose c_4, c_5 sufficiently small such that $c_6 := (\frac{(1+c_3)(1+c_4)(1+c_5)}{c_3})^{q-1}K_2^q < 1$. Then let $c_7 = (\frac{(1+c_3)(1+c_4)(1+c_5)}{c_3c_5})^{q-1}K_2^q$ and $c_8 = (\frac{(1+c_3)(1+c_4)}{c_3c_4})^{q-1}$. Hence we derive from (3.47) that

$$\begin{aligned}
&\sup_{0\le s\le t} \mathbb{E}\left|x(s)-x_\Delta(s)\right|^q \\
&\quad \le (1+c_3)^{q-1} \sup_{0\le s\le t} \mathbb{E}\left|e_\Delta(s)\right|^q + c_6 \sup_{0\le s\le t} \mathbb{E}\left|x(s-\tau)-x_\Delta(s-\tau)\right|^q \\
&\qquad + c_7 \sup_{0\le s\le t} \mathbb{E}\left|x_\Delta(s-\tau)-\bar{x}_\Delta(s-\tau)\right|^q \\
&\qquad + c_8 \sup_{0\le s\le t} \mathbb{E}\left|D\big(\bar{x}_\Delta(s-\tau),r(s)\big)-D\big(\bar{x}_\Delta(s-\tau),\bar{r}(s)\big)\right|^q \\
&\quad \le (1+c_3)^{q-1} \sup_{0\le s\le t} \mathbb{E}\left|e_\Delta(s)\right|^q + c_6 \sup_{0\le s\le t} \mathbb{E}\left|x(s)-x_\Delta(s)\right|^q \\
&\qquad + c_7 \sup_{0\le s\le t} \mathbb{E}\left|x_\Delta(s)-\bar{x}_\Delta(s)\right|^q + c_6 \sup_{-\tau\le s\le 0} \mathbb{E}\left|\xi(s)-\xi\big(\lfloor s/\Delta\rfloor\Delta\big)\right|^q \\
&\qquad + C\big(\Delta+o(\Delta)\big) \\
&\quad \le (1+c_3)^{q-1} \sup_{0\le s\le t} \mathbb{E}\left|e_\Delta(s)\right|^q + c_6 \sup_{0\le s\le t} \mathbb{E}\left|x(s)-x_\Delta(s)\right|^q \\
&\qquad + C\big(\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\alpha}+o(\Delta)\big).
\end{aligned}$$

Therefore

$$\sup_{0\le s\le t} \mathbb{E}\left|x(s)-x_\Delta(s)\right|^q \le \frac{(1+c_3)^{q-1}}{1-c_6} \sup_{0\le s\le t} \mathbb{E}\left|e_\Delta(s)\right|^q + C\big(\Delta^{\frac{q}{2}}h^q(\Delta)+\Delta^{q\alpha}+o(\Delta)\big).$$

Then we have

$$\begin{aligned}
&\sup_{0\le l\le t} \mathbb{E}\left|x(l\wedge\rho)-x_\Delta(l\wedge\rho)\right|^q \\
&\quad \le C\bigg(\int_0^t \sup_{0\le l\le s} \mathbb{E}\left|x(l\wedge\rho)-x_\Delta(l\wedge\rho)\right|^q\,ds + \big(\Delta^{\frac{q}{2}}h^q(\Delta)\vee\Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big) \\
&\qquad + \int_0^t \Big(\sup_{0\le l\le s} \mathbb{E}\left|x(l\wedge\rho-\tau)-x_\Delta(l\wedge\rho-\tau)\right|^{2q}\Big)^{\frac{1}{2}}\,ds\bigg).
\end{aligned}$$

An application of Gronwall's inequality gives that

$$\begin{aligned}
&\sup_{0\le s\le t} \mathbb{E}\big|x(s\wedge\rho) - x_\Delta(s\wedge\rho)\big|^q \\
&\quad\le C\left(\Delta_f^q + \int_0^t \Big(\sup_{0\le l\le s} \mathbb{E}\big|x(l\wedge\rho-\tau) - x_\Delta(l\wedge\rho-\tau)\big|^{2q}\Big)^{\frac{1}{2}}\,ds\right),
\end{aligned}$$

where $\Delta_f = \Delta^{\frac{1}{2}} h(\Delta) \vee \Delta^{(\alpha\wedge\theta\wedge\sigma)}$. Then we use the same technique as in Lemma 3.9 to get the convergence rate. Define

$$q_i = \big(\lfloor T/\tau\rfloor + 2 - i\big) q 2^{\lfloor T/\tau\rfloor+1-i}, \quad i = 1,2,\dots,\lfloor T/\tau\rfloor + 1.$$

We find that

$$2q_{i+1} < q_i \quad \text{and} \quad q_{\lfloor T/\tau\rfloor+1} = q, \quad i = 1,2,\dots,\lfloor T/\tau\rfloor.$$

Note that $|x(s-\tau) - x_\Delta(s-\tau)| = 0$ for $s \in [0,\tau]$. Then we derive that

$$\sup_{0\le s\le\tau} \mathbb{E}\big|x(s\wedge\rho) - x_\Delta(s\wedge\rho)\big|^{q_1} \le C\Delta_f^{q_1}.$$

Then by Hölder's inequality we obtain that

$$\begin{aligned}
&\sup_{0\le s\le 2\tau} \mathbb{E}\big|x(s\wedge\rho) - x_\Delta(s\wedge\rho)\big|^{q_2} \\
&\quad\le C\left(\Delta_f^{q_2} + \int_0^{2\tau} \big(\mathbb{E}\big|x(s\wedge\rho-\tau) - x_\Delta(s\wedge\rho-\tau)\big|^{2q_2\frac{q_1}{2q_2}}\big)^{\frac{q_2}{q_1}}\,ds\right) \le C\Delta_f^{q_2}.
\end{aligned}$$

By induction we could get the desired result. We complete the proof. □

Theorem 3.14 *Let Assumptions 2.2, 2.3, 3.1–3.4, and 3.6 hold. Let $q \in [2,\bar{q})$ and $p > (\beta + l_v + 2)q$. For any sufficiently small $\Delta \in (0,1]$, assume that*

$$h(\Delta) \ge \varphi\Big(\big(\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big)^{\frac{-1}{p-q}}\Big). \tag{3.53}$$

Then for every such small Δ, we have

$$\mathbb{E}\big|x(T) - x_\Delta(T)\big|^q \le C\big(\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big) \tag{3.54}$$

and

$$\mathbb{E}\big|x(T) - \bar{x}_\Delta(T)\big|^q \le C\big(\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big) \tag{3.55}$$

for any $T > 0$.

Proof Let τ_L, $\tau_{\Delta,L}$, and $\rho_{\Delta,L}$ be as before. Denote $z_\Delta(t) = x(t) - x_\Delta(t)$. We write $\rho_{\Delta,L} = \rho$ for simplicity. Obviously,

$$\mathbb{E}\big|z_\Delta(T)\big|^q = \mathbb{E}\big(\big|z_\Delta(T)\big|^q \mathbb{I}_{\{\rho>T\}}\big) + \mathbb{E}\big(\big|z_\Delta(T)\big|^q \mathbb{I}_{\{\rho\le T\}}\big). \tag{3.56}$$

Let $\delta > 0$ be arbitrary. By Young's inequality we get

$$u^q v = (\delta u^p)^{\frac{q}{p}} \left(\frac{v^{p/(p-q)}}{\delta^{q/(p-q)}} \right)^{\frac{p-q}{p}} \leq \frac{q\delta}{p} u^p + \frac{p-q}{p\delta^{q/(p-q)}} v^{p/(p-q)}, \quad \forall u, v > 0.$$

Hence

$$\mathbb{E}\big(|z_\Delta(T)|^q \mathbb{I}_{\{\rho \leq T\}}\big) \leq \frac{q\delta}{p} \mathbb{E}|z_\Delta(T)|^p + \frac{p-q}{p\delta^{q/(p-q)}} \mathbb{P}\{\rho \leq T\}. \tag{3.57}$$

Applying Lemmas 3.5 and 3.9 gives that

$$\mathbb{E}|z_\Delta(T)|^p \leq C. \tag{3.58}$$

By Lemmas 3.11 and 3.12 we have

$$\mathbb{P}(\rho \leq T) \leq \mathbb{P}(\tau_L \leq T) + \mathbb{P}(\tau_{\Delta,L} \leq T) \leq \frac{C}{L^p}. \tag{3.59}$$

Inserting (3.58) and (3.59) into (3.57) yields that

$$\mathbb{E}\big(|z_\Delta(T)|^q \mathbb{I}_{\{\rho \leq T\}}\big) \leq \frac{Cq\delta}{p} + \frac{C(p-q)}{pL^p \delta^{q/(p-q)}}. \tag{3.60}$$

Choose $\delta = \Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}$ and $L = (\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)})^{\frac{-1}{p-q}}$. Then we have

$$\mathbb{E}|z_\Delta(T)|^q \leq \mathbb{E}|z_\Delta(T \wedge \rho)|^q + C\big(\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big). \tag{3.61}$$

By condition (3.53) we obtain that

$$\varphi^{-1}\big(h(\Delta)\big) \geq \big(\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big)^{\frac{-1}{p-q}} = L.$$

We derive from Lemma 3.13 that

$$\mathbb{E}|z_\Delta(T)|^q \leq C\big(\Delta^{\frac{q}{2}} h^q(\Delta) \vee \Delta^{q(\alpha\wedge\theta\wedge\sigma)}\big). \tag{3.62}$$

Hence we get the desired result (3.54). Then combing Lemma 3.10 and (3.54) gives (3.55). We complete the proof. □

4 Stability

In this section, we investigate the almost sure exponential stability of the partially truncated EM method for neutral stochastic differential delay equations with Markovian switching. In order to achieve this aim, we need to assume that Assumption 3.1 holds on $t \in [0, \infty)$. Let $\tilde{F}(t, 0, 0, i) = F(t, 0, 0, i) = 0$ and $\tilde{G}(t, 0, 0, i) = G(t, 0, 0, i) = 0$ for all $t \in [0, \infty)$ and $i \in \mathbb{S}$, which means $f(t, 0, 0, i) = g(t, 0, 0, i) = 0$.

Assumption 4.1 There exist constants $\Lambda \geq 0$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ satisfying $\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4$ such that

$$2\big(x - D(y, i)\big)^T \tilde{F}(t, x, y, i) + (1 + \Lambda)\big|\tilde{G}(t, x, y, i)\big|^2 \leq -\lambda_1 |x|^2 + \lambda_2 |y|^2 \tag{4.1}$$

and

$$2\big(x - D(y,i)\big)^T F_\Delta(t,x,y,i) + \big(1 + \varLambda^{-1}\big)\big|G_\Delta(t,x,y,i)\big|^2 \le \lambda_3|x|^2 + \lambda_4|y|^2 \tag{4.2}$$

for all $t \in [0,\infty)$, $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

Remark 4.2 In fact, there are many functions such that $D(y,i)$, $\tilde{F}(t,x,y,i)$, and $\tilde{G}(t,x,y,i)$ satisfying (4.1) and the corresponding $F_\Delta(t,x,y,i)$ and $G_\Delta(t,x,y,i)$ satisfying (4.2). The example and proof will be given in Sect. 5.

In the rest of this paper, we set $\varLambda = 0$ and $\varLambda^{-1}|G_\Delta(t,x,y,i)|^2 = 0$ if there is no term $G_\Delta(t,x,y,i)$. Also, when the linearly growing term $\tilde{G}(t,x,y,i)$ is absent, we set $\varLambda = \infty$ and $\varLambda|\tilde{G}(t,x,y,i)|^2 = 0$.

By Assumption 4.1 we obtain that

$$\begin{aligned} &2\big(x - D(y,i)\big)^T f_\Delta(t,x,y,i) + \big|g_\Delta(t,x,y,i)\big|^2 \\ &\quad \le -(\lambda_1 - \lambda_3)|x|^2 + (\lambda_2 + \lambda_4)|y|^2 \end{aligned} \tag{4.3}$$

for all $t \in [0,\infty)$, $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

Theorem 4.3 *Let Assumptions 2.3, 3.1, and 4.1 hold on $t \in [0,\infty)$. Then the partially truncated EM numerical solution (2.11) is almost surely exponentially stable. Precisely, let $\lambda > 0$ be the unique root of*

$$(\lambda_2 + \lambda_4)e^{\lambda\tau} + \lambda\big(K_2 + K_2^2\big)e^{\lambda\tau} + (-\lambda_1 + \lambda_3) + \lambda(1 + K_2) = 0, \tag{4.4}$$

and let $\varepsilon \in (0, \frac{\lambda}{2})$ be arbitrary. Then there exists a $\Delta^* > 0$ such that for any $\Delta < \Delta^*$, we have

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log\big|X_\Delta(t_k)\big| \le -\frac{\lambda}{2} + \varepsilon \quad a.s. \tag{4.5}$$

Proof Define

$$Y\big(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta\big) = X_\Delta(t_k) - D\big(X_\Delta(t_{k-M}), r_k^\Delta\big). \tag{4.6}$$

Then (2.11) becomes

$$\begin{aligned} &Y\big(t_{k+1}, X_\Delta(t_{k+1}), X_\Delta(t_{k+1-M}), r_{k+1}^\Delta\big) \\ &\quad = Y\big(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta\big) + f_\Delta\big(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta\big)\Delta \\ &\qquad + g_\Delta\big(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta\big)\Delta B_k. \end{aligned} \tag{4.7}$$

We write $Y_k = Y(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta)$, $f_{\Delta,k} = f_\Delta(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta)$, and $g_{\Delta,k} = g_\Delta(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta)$ for simplicity. Hence we have

$$|Y_{k+1}|^2 = |Y_k|^2 + \big(2{Y_k}^T f_{\Delta,k} + |g_{\Delta,k}|^2 + |f_{\Delta,k}|^2\Delta\big)\Delta + m_{\Delta,k}, \tag{4.8}$$

where

$$m_{\Delta,k} = |g_{\Delta,k}\Delta B_k|^2 - |g_{\Delta,k}|^2\Delta + 2{f_{\Delta,k}}^T(g_{\Delta,k}\Delta B_k) + 2{Y_k}^T(g_{\Delta,k}\Delta B_k). \tag{4.9}$$

By (3.2) we have

$$\left|F_\Delta(t,x,y,i)\right|^2 \le 18K_3^2\left(|x|^2 + |y|^2\right) \quad \text{if } |x| \vee |y| \le 1$$

and

$$\left|F_\Delta(t,x,y,i)\right|^2 \le h^2(\Delta) \le h^2(\Delta)\left(|x|^2 + |y|^2\right) \quad \text{if } |x| \vee |y| \ge 1.$$

Thus

$$\begin{aligned}
|f_{\Delta,k}|^2\Delta &\le 2\left(20K_3^2 + h^2(\Delta)\right)\Delta\left(\left|X_\Delta(t_k)\right|^2 + \left|X_\Delta(t_{k-M})\right|^2\right) \\
&\le 2\left(20K_3^2\Delta + K_0^2\Delta^{\frac{1}{2}}\right)\left(\left|X_\Delta(t_k)\right|^2 + \left|X_\Delta(t_{k-M})\right|^2\right) \\
&\le 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\left(\left|X_\Delta(t_k)\right|^2 + \left|X_\Delta(t_{k-M})\right|^2\right).
\end{aligned}$$

Using (4.3) yields that

$$\begin{aligned}
|Y_{k+1}|^2 &\le |Y_k|^2 + \left(-(\lambda_1 - \lambda_3)\left|X_\Delta(t_k)\right|^2 + (\lambda_2 + \lambda_4)\left|X_\Delta(t_{k-M})\right|^2\right. \\
&\quad \left. + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\left(\left|X_\Delta(t_k)\right|^2 + \left|X_\Delta(t_{k-M})\right|^2\right)\right)\Delta + m_{\Delta,k} \\
&= |Y_k|^2 + \left(-\lambda_1 + \lambda_3 + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\right)\Delta\left|X_\Delta(t_k)\right|^2 \\
&\quad + \left(\lambda_2 + \lambda_4 + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\right)\Delta\left|X_\Delta(t_{k-M})\right|^2 + m_{\Delta,k}.
\end{aligned} \tag{4.10}$$

Let

$$\begin{aligned}
&P_{\Delta,1} = -\lambda_1 + \lambda_3 + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}, \\
&P_{\Delta,2} = \lambda_2 + \lambda_4 + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}.
\end{aligned}$$

Therefore, for any positive constant $J > 1$, we derive that

$$\begin{aligned}
&J^{(k+1)\Delta}|Y_{k+1}|^2 - J^{k\Delta}|Y_k|^2 \\
&\quad \le J^{(k+1)\Delta}P_{\Delta,1}\Delta\left|X_\Delta(t_k)\right|^2 + J^{(k+1)\Delta}P_{\Delta,2}\Delta\left|X_\Delta(t_{k-M})\right|^2 \\
&\qquad + \left(J^{(k+1)\Delta} - J^{k\Delta}\right)|Y_k|^2 + J^{(k+1)\Delta}m_{\Delta,k} \\
&\quad \le J^{(k+1)\Delta}P_{\Delta,1}\Delta\left|X_\Delta(t_k)\right|^2 + J^{(k+1)\Delta}P_{\Delta,2}\Delta\left|X_\Delta(t_{k-M})\right|^2 \\
&\qquad + \left(J^{(k+1)\Delta} - J^{k\Delta}\right)2\left(\left|X_\Delta(t_k)\right|^2 + \left|X_\Delta(t_{k-M})\right|^2\right) + J^{(k+1)\Delta}m_{\Delta,k} \\
&\quad \le \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,1}\Delta\right]J^{(k+1)\Delta}\left|X_\Delta(t_k)\right|^2 \\
&\qquad + \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,2}\Delta\right]J^{(k+1)\Delta}\left|X_\Delta(t_{k-M})\right|^2 + J^{(k+1)\Delta}m_{\Delta,k},
\end{aligned}$$

which means that

$$\begin{aligned}
&J^{k\Delta}|Y_k|^2 - |Y_0|^2 \\
&\quad \le \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,1}\Delta\right] \sum_{i=0}^{k-1} J^{(i+1)\Delta} \left|X_\Delta(t_i)\right|^2 \\
&\qquad + \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,2}\Delta\right] \sum_{i=0}^{k-1} J^{(i+1)\Delta} \left|X_\Delta(t_{i-M})\right|^2 + \sum_{i=0}^{k-1} J^{(i+1)\Delta} m_{\Delta,i}.
\end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned}
&\sum_{i=0}^{k-1} J^{(i+1)\Delta} \left|X_\Delta(t_{i-M})\right|^2 \\
&\quad = \sum_{i=-M}^{-1} J^{(i+1+M)\Delta} \left|X_\Delta(t_i)\right|^2 + \sum_{i=0}^{k-1} J^{(i+1+M)\Delta} \left|X_\Delta(t_i)\right|^2 - \sum_{i=k-M}^{k-1} J^{(i+1+M)\Delta} \left|X_\Delta(t_i)\right|^2.
\end{aligned}$$

Thus

$$J^{k\Delta}|Y_k|^2 + \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,2}\Delta\right] \sum_{i=k-M}^{k-1} J^{(i+1+M)\Delta} \left|X_\Delta(t_i)\right|^2 \le U_k, \tag{4.12}$$

where

$$\begin{aligned}
U_k &= |Y_0|^2 + \Big(\left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,1}\Delta\right] \\
&\quad + \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,2}\Delta\right] J^{M\Delta}\Big) \sum_{i=0}^{k-1} J^{(i+1)\Delta} \left|X_\Delta(t_i)\right|^2 \\
&\quad + \left[2\left(1 - J^{-\Delta}\right) + P_{\Delta,2}\Delta\right] \sum_{i=-M}^{-1} J^{(i+1+M)\Delta} \left|X_\Delta(t_i)\right|^2 + \sum_{i=0}^{k-1} J^{(i+1)\Delta} m_{\Delta,i}.
\end{aligned}$$

Let us now introduce the function

$$\begin{aligned}
Q(J) &= (2 + P_{\Delta,2}\Delta) J^{(M+1)\Delta} - 2J^{M\Delta} + (2 + P_{\Delta,1}\Delta) J^{\Delta} - 2 \\
&= \left[\left(\lambda_2 + \lambda_4 + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\right)\Delta + 2\right] J^{(M+1)\Delta} - 2J^{M\Delta} \\
&\quad + \left[\left(-\lambda_1 + \lambda_3 + 2\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\right)\Delta + 2\right] J^{\Delta} - 2.
\end{aligned} \tag{4.13}$$

Define

$$\Delta_1^* = \left(\frac{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4}{4(20K_3^2 + K_0^2)}\right)^2.$$

When $\Delta < \Delta_1^*$, we can observe that

$$Q(1) = \left[-(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) + 4\left(20K_3^2 + K_0^2\right)\Delta^{\frac{1}{2}}\right]\Delta < 0.$$

Moreover, choose $\Delta_2^* > 0$ such that for any $\Delta < \Delta_2^*$,

$$2 + \bigl(-\lambda_1 + \lambda_3 + 2\bigl(20K_3^2 + K_0^2\bigr)\Delta^{\frac{1}{2}}\bigr)\Delta > 0.$$

Hence we can derive that for any $J > 1$,

$$\begin{aligned} Q'(J) &= \bigl[2M\bigl(J^\Delta - 1\bigr) + \bigl(\bigl(\lambda_2 + \lambda_4 + 2\bigl(20K_3^2 + K_0^2\bigr)\Delta^{\frac{1}{2}}\bigr)(M+1)\Delta + 2\bigr)J^\Delta\bigr]\Delta J^{M\Delta-1} \\ &\quad + \bigl[2 + \bigl(-\lambda_1 + \lambda_3 + 2\bigl(20K_3^2 + K_0^2\bigr)\Delta^{\frac{1}{2}}\bigr)\Delta\bigr]\Delta J^{\Delta-1} > 0. \end{aligned}$$

Therefore there exists a unique constant $J_\Delta^* > 1$ such that

$$Q\bigl(J_\Delta^*\bigr) = 0$$

for any $\Delta < \Delta_1^* \wedge \Delta_2^*$. Choosing $J = J_\Delta^*$ for any $\Delta < \Delta_1^* \wedge \Delta_2^*$ yields that

$$\begin{aligned} U_k &= |Y_0|^2 + \bigl[2\bigl(1 - J^{-\Delta}\bigr) + P_{\Delta,2}\Delta\bigr]\sum_{i=-M}^{-1} J^{(i+1+M)\Delta}\bigl|X_\Delta(t_i)\bigr|^2 \\ &\quad + \sum_{i=0}^{k-1} J^{(i+1)\Delta} m_{\Delta,i}. \end{aligned} \tag{4.14}$$

Note that the initial sequence $X_\Delta(t_i) < \infty$ for any $i = -M, -M+1, \ldots, 0$ and that $\sum_{i=0}^{k-1} J^{(i+1)\Delta} m_{\Delta,i}$ is a martingale. Applying the discrete-type semimartingale convergence theorem gives that for any $\Delta < \Delta_1^* \wedge \Delta_2^*$,

$$\lim_{k\to\infty} U_k < \infty \quad \text{a.s.}$$

By (4.12) we obtain that

$$\begin{aligned} &\limsup_{k\to\infty}\bigl(J_\Delta^{*\,k\Delta}|Y_k|^2\bigr) \\ &\quad \le \limsup_{k\to\infty}\Biggl(J_\Delta^{*\,k\Delta}|Y_k|^2 + \bigl[2\bigl(1 - J^{-\Delta}\bigr) + P_{\Delta,2}\Delta\bigr]\sum_{i=k-M}^{k-1} J_\Delta^{*\,(i+1+M)\Delta}\bigl|X_\Delta(t_i)\bigr|^2\Biggr) \\ &\quad \le \lim_{k\to\infty} U_k < \infty \quad \text{a.s.} \end{aligned} \tag{4.15}$$

In addition, for any $c_0^* > 0$,

$$\begin{aligned} &\sup_{k\ge0}\bigl(J_\Delta^{*\,k\Delta}\bigl|X_\Delta(t_k)\bigr|^2\bigr) \\ &\quad = \sup_{k\ge0}\bigl(J_\Delta^{*\,k\Delta}\bigl|X_\Delta(t_k) - D\bigl(X_\Delta(t_{k-M}), r_k^\Delta\bigr) + D\bigl(X_\Delta(t_{k-M}), r_k^\Delta\bigr)\bigr|^2\bigr) \\ &\quad \le \bigl(1 + c_0^*\bigr)\sup_{k\ge0}\bigl(J_\Delta^{*\,k\Delta}\bigl|X_\Delta(t_k) - D\bigl(X_\Delta(t_{k-M}), r_k^\Delta\bigr)\bigr|^2\bigr) \\ &\qquad + \frac{1 + c_0^*}{c_0^*}\sup_{k\ge0}\bigl(J_\Delta^{*\,k\Delta}\bigl|D\bigl(X_\Delta(t_{k-M}), r_k^\Delta\bigr)\bigr|^2\bigr) \end{aligned}$$

$$\begin{aligned}
&\le \left(1+c_0^*\right)\sup_{k\ge 0}\left(J_\Delta^{*\,k\Delta}\left|X_\Delta(t_k)-D\left(X_\Delta(t_{k-M}),r_k^\Delta\right)\right|^2\right)\\
&\quad+\frac{1+c_0^*}{c_0^*}K_2^2\Big[\sup_{-M\le k\le 0}J_\Delta^{*\,(k+M)\Delta}\left|X_\Delta(t_k)\right|^2+\sup_{k\ge 0}J_\Delta^{*\,(k+M)\Delta}\left|X_\Delta(t_k)\right|^2\Big]\\
&\le \left(1+c_0^*\right)\sup_{k\ge 0}\left(J_\Delta^{*\,k\Delta}\left|X_\Delta(t_k)-D\left(X_\Delta(t_{k-M}),r_k^\Delta\right)\right|^2\right)+\frac{1+c_0^*}{c_0^*}K_2^2J_\Delta^{*\,\tau}\|\xi\|^2\\
&\quad+\frac{1+c_0^*}{c_0^*}K_2^2J_\Delta^{*\,\tau}\sup_{k\ge 0}\left(J_\Delta^{*\,k\Delta}\left|X_\Delta(t_k)\right|^2\right).
\end{aligned}$$

Then we take c_0^* sufficiently large such that $\frac{1+c_0^*}{c_0^*}K_2^2J_\Delta^{*\,\tau}<1$ for any $K_2\in(0,1)$. Hence

$$\sup_{k\ge 0}\left(J_\Delta^{*\,k\Delta}\left|X_\Delta(t_k)\right|^2\right)\le c_1^*\sup_{k\ge 0}\left(J_\Delta^{*\,k\Delta}|Y_k|^2\right)+c_2^*\|\xi\|^2, \tag{4.16}$$

where

$$c_1^*=\frac{c_0^*(1+c_0^*)}{c_0^*-(1+c_0^*)K_2^2J_\Delta^{*\,\tau}},\qquad c_2^*=\frac{(1+c_0^*)K_2^2J_\Delta^{*\,\tau}}{c_0^*-(1+c_0^*)K_2^2J_\Delta^{*\,\tau}}.$$

Therefore

$$\lim_{k\to\infty}\sup\left(J_\Delta^{*\,k\Delta}\left|X_\Delta(t_k)\right|^2\right)<\infty\quad\text{a.s.} \tag{4.17}$$

By (4.13) we get that

$$\begin{aligned}
&\left[\lambda_2+\lambda_4+2\left(20K_3^2+K_0^2\right)\Delta^{\frac12}\right]J_\Delta^{*\,\tau}+2J_\Delta^{*\,\tau}\left(1-J^{-\Delta}\right)\frac{1}{\Delta}\\
&\quad+\left[-\lambda_1+\lambda_3+2\left(20K_3^2+K_0^2\right)\Delta^{\frac12}\right]+2\left(1-J^{-\Delta}\right)\frac{1}{\Delta}=0.
\end{aligned}\tag{4.18}$$

Choose the constant ϑ such that $J=e^{\vartheta}$. Hence $1-J^{-\Delta}=1-e^{-\vartheta\Delta}$. Define

$$\begin{aligned}
\bar{Q}_\Delta(\vartheta)&=\left[\lambda_2+\lambda_4+2\left(20K_3^2+K_0^2\right)\Delta^{\frac12}\right]e^{\vartheta\tau}+2e^{\vartheta\tau}\left(1-e^{-\vartheta\Delta}\right)\frac{1}{\Delta}\\
&\quad+\left[-\lambda_1+\lambda_3+2\left(20K_3^2+K_0^2\right)\Delta^{\frac12}\right]+2\left(1-e^{-\vartheta\Delta}\right)\frac{1}{\Delta}.
\end{aligned}\tag{4.19}$$

Let $\vartheta_\Delta^*=\log J_\Delta^*$. Then we have

$$\bar{Q}_\Delta\left(\vartheta_\Delta^*\right)=0. \tag{4.20}$$

Since

$$\lim_{\Delta\to 0}\left(1-e^{-\vartheta\Delta}\right)\frac{1}{\Delta}=\vartheta,$$

we derive that

$$\lim_{\Delta\to 0}\bar{Q}_\Delta(\vartheta)=(\lambda_2+\lambda_4)e^{\vartheta\tau}+2\vartheta e^{\vartheta\tau}+(-\lambda_1+\lambda_3)+2\vartheta. \tag{4.21}$$

By the definition of λ we get from (4.20) and (4.21) that

$$\lim_{\Delta\to 0}\vartheta^*_\Delta=\lambda,$$

which means that for any $\varepsilon\in(0,\frac{\lambda}{2})$, there exists $\Delta^*_3>0$ such that for any $\Delta<\Delta^*_3$, we have

$$\vartheta^*_\Delta>\lambda-2\varepsilon.$$

We derive from (4.17) and the definition of ϑ^*_Δ that

$$\limsup_{k\to\infty} e^{\vartheta^*_\Delta k\Delta}\bigl|X_\Delta(t_k)\bigr|^2<\infty.$$

Then for any $\Delta<\Delta^*_1\wedge\Delta^*_2\wedge\Delta^*_3=:\Delta^*$, we have

$$\limsup_{k\to\infty}\frac{1}{k\Delta}\log\bigl|X_\Delta(t_k)\bigr|\le-\frac{\lambda}{2}+\varepsilon\quad\text{a.s.}$$

which is the desired result. We complete the proof. □

5 Example

Example 5.1 Consider a nonlinear and nonautonomous neutral stochastic differential delay equations with Markovian switching

$$\begin{aligned}&d\bigl[x(t)-D\bigl(x(t-\tau),r(t)\bigr)\bigr]\\&\quad=f\bigl(t,x(t),x(t-\tau),r(t)\bigr)\,dt+g\bigl(t,x(t),x(t-\tau),r(t)\bigr)\,dB(t),\quad t\ge 0,\end{aligned}\tag{5.1}$$

with the initial data x_0 satisfying Assumption 2.2. Here $B(t)$ is a scalar Brownian motion. Moreover r is a Markovian chain on the state space $\mathbb{S}=\{1,2\}$ with generator

$$\Gamma=\begin{pmatrix}-2&2\\1&-1\end{pmatrix}.$$

In addition, for all $t\in[0,1]$, $x,y\in\mathbb{R}^1$, and $i\in\mathbb{S}$, let

$$D(y,i)=\begin{cases}-\frac{1}{6}y & \text{if } i=1,\\-\frac{1}{12}y & \text{if } i=2,\end{cases}\qquad g(t,x,y,i)=\begin{cases}(t(1-t))^{\frac{1}{3}}|y|^{\frac{3}{2}} & \text{if } i=1,\\(t(1-t))^{\frac{1}{4}}|y|^{\frac{5}{2}} & \text{if } i=2,\end{cases}$$

$$f(t,x,y,i)=\begin{cases}-2y^3+(t(1-t))^{\frac{1}{3}}y-10x+2y & \text{if } i=1,\\-4y^5+(t(1-t))^{\frac{1}{4}}y-20x+2y & \text{if } i=2.\end{cases}$$

We easily see that

$$\tilde F(t,x,y,i)=\begin{cases}(t(1-t))^{\frac{1}{3}}y-10x+2y & \text{if } i=1,\\(t(1-t))^{\frac{1}{4}}y-20x+2y & \text{if } i=2,\end{cases}\qquad \tilde G(t,x,y,i)=\begin{cases}0 & \text{if } i=1,\\0 & \text{if } i=2,\end{cases}$$

$$F(t,x,y,i)=\begin{cases}-2y^3 & \text{if } i=1,\\-4y^5 & \text{if } i=2,\end{cases}\qquad G(t,x,y,i)=\begin{cases}(t(1-t))^{\frac{1}{3}}|y|^{\frac{3}{2}} & \text{if } i=1,\\(t(1-t))^{\frac{1}{4}}|y|^{\frac{5}{2}} & \text{if } i=2.\end{cases}$$

Obviously, Assumptions 2.3 and 3.1 hold with $K_3 = 20$ and $\beta = 4$. Now we verify Assumptions 3.2–3.4 and 3.6. For Assumption 3.2, we get

$$\begin{aligned}
&\big(x - D(y,1) - \bar{x} + D(\bar{y},1)\big)^T\big(F(t,x,y,1) - F(t,\bar{x},\bar{y},1)\big) \\
&\quad + \frac{\bar{q}-1}{2}\big|G(t,x,y,1) - G(t,\bar{x},\bar{y},1)\big|^2 \\
&\quad \le -2(x-\bar{x})\big(y^3 - \bar{y}^3\big) - \frac{1}{3}(y-\bar{y})\big(y^3 - \bar{y}^3\big) + \frac{\bar{q}-1}{2}\big||y|^{\frac{3}{2}} - |\bar{y}|^{\frac{3}{2}}\big|^2 \\
&\quad \le |x-\bar{x}|^2 + \big((\bar{q}-1) \vee 5\big)\big(1 + |y|^4 + |\bar{y}|^4\big)|y-\bar{y}|^2, \\
&\big(x - D(y,2) - \bar{x} + D(\bar{y},2)\big)^T\big(F(t,x,y,2) - F(t,\bar{x},\bar{y},2)\big) \\
&\quad + \frac{\bar{q}-1}{2}\big|G(t,x,y,2) - G(t,\bar{x},\bar{y},2)\big|^2 \\
&\quad \le -4(x-\bar{x})\big(y^5 - \bar{y}^5\big) - \frac{1}{3}(y-\bar{y})\big(y^5 - \bar{y}^5\big) + \frac{\bar{q}-1}{2}\big||y|^{\frac{5}{2}} - |\bar{y}|^{\frac{5}{2}}\big|^2 \\
&\quad \le 2|x-\bar{x}|^2 + \big((\bar{q}-1) \vee 40\big)\big(1 + |y|^8 + |\bar{y}|^8\big)|y-\bar{y}|^2.
\end{aligned}$$

Therefore Assumption 3.2 is satisfied. For Assumption 3.3, we derive that

$$\begin{aligned}
&\big(x - D(y,1)\big)^T F(t,x,y,1) + \frac{\bar{p}-1}{2}\big|G(t,x,y,1)\big|^2 \\
&\quad \le -2xy^3 - \frac{1}{3}y^4 + \frac{\bar{p}-1}{2}|y|^3 \le \big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 12\big)\big(1 + |y|^4\big)|y|^2, \\
&\big(x - D(y,2)\big)^T F(t,x,y,2) + \frac{\bar{p}-1}{2}\big|G(t,x,y,2)\big|^2 \\
&\quad \le -4xy^5 - \frac{1}{3}y^6 + \frac{\bar{p}-1}{2}|y|^5 \le 2\big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 40\big)\big(1 + |y|^8\big)|y|^2.
\end{aligned}$$

Hence Assumption 3.3 is satisfied. Moreover, Assumption 3.4 holds with $\theta = \sigma = \frac{1}{3} \wedge \frac{1}{4} = \frac{1}{4}$ for $i \in \mathbb{S}$. To verify Assumption 3.6, we need to consider four cases.

Case 1: If $(|x| \vee |y|) \le \varphi^{-1}(h(\Delta))$, then we have

$$\begin{aligned}
&\big(x - D(y,1)\big)^T F_\Delta(t,x,y,1) + \frac{\bar{p}-1}{2}\big|G_\Delta(t,x,y,1)\big|^2 \\
&\quad = \left(x + \frac{1}{6}y\right)\big(-2y^3\big) + \frac{\bar{p}-1}{2}\big|\big(t(1-t)\big)^{\frac{1}{3}}|y|^{\frac{3}{2}}\big|^2 \\
&\quad \le \big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 12\big)\big(1 + |y|^4\big)|y|^2, \\
&\big(x - D(y,2)\big)^T F_\Delta(t,x,y,2) + \frac{\bar{p}-1}{2}\big|G_\Delta(t,x,y,2)\big|^2 \\
&\quad = \left(x + \frac{1}{12}y\right)\big(-4y^5\big) + \frac{\bar{p}-1}{2}\big|\big(t(1-t)\big)^{\frac{1}{4}}|y|^{\frac{5}{2}}\big|^2 \\
&\quad \le 2\big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 40\big)\big(1 + |y|^8\big)|y|^2.
\end{aligned}$$

Case 2: If $(|x| \wedge |y|) > \varphi^{-1}(h(\Delta))$, then we have

$$\begin{aligned}
&\big(x - D(y,1)\big)^T F_\Delta(t,x,y,1) + \frac{\bar{p}-1}{2}\big|G_\Delta(t,x,y,1)\big|^2 \\
&\quad = \left(x + \frac{1}{6}y\right)\left(-2\left(\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right)^3\right) + \frac{\bar{p}-1}{2}\left|\big(t(1-t)\big)^{\frac{1}{3}}\left|\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right|^{\frac{3}{2}}\right|^2 \\
&\quad \le -2\left(\frac{\varphi^{-1}(h(\Delta))}{|y|}\right)^3 xy^3 - \frac{1}{3}\left(\frac{\varphi^{-1}(h(\Delta))}{|y|}\right)^3 y^4 + \frac{\bar{p}-1}{2}\left(\frac{\varphi^{-1}(h(\Delta))}{|y|}\right)^3 |y|^3 \\
&\quad \le \big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 12\big)\big(1 + |y|^4\big)|y|^2, \\
&\big(x - D(y,2)\big)^T F_\Delta(t,x,y,2) + \frac{\bar{p}-1}{2}\big|G_\Delta(t,x,y,2)\big|^2 \\
&\quad = \left(x + \frac{1}{12}y\right)\left(-4\left(\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right)^5\right) + \frac{\bar{p}-1}{2}\left|\big(t(1-t)\big)^{\frac{1}{4}}\left|\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right|^{\frac{5}{2}}\right|^2 \\
&\quad \le -4\left(\frac{\varphi^{-1}(h(\Delta))}{|y|}\right)^5 xy^5 - \frac{1}{3}\left(\frac{\varphi^{-1}(h(\Delta))}{|y|}\right)^5 y^6 + \frac{\bar{p}-1}{2}\left(\frac{\varphi^{-1}(h(\Delta))}{|y|}\right)^5 |y|^5 \\
&\quad \le 2\big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 40\big)\big(1 + |y|^8\big)|y|^2.
\end{aligned}$$

Case 3: If $|y| > \varphi^{-1}(h(\Delta))$ and $|x| < \varphi^{-1}(h(\Delta))$, then we derive that

$$\begin{aligned}
&\big(x - D(y,1)\big)^T F_\Delta(t,x,y,1) + \frac{\bar{p}-1}{2}\big|G_\Delta(t,x,y,1)\big|^2 \\
&\quad = \left(x + \frac{1}{6}y\right)\left(-2\left(\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right)^3\right) + \frac{\bar{p}-1}{2}\left|\big(t(1-t)\big)^{\frac{1}{3}}\left|\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right|^{\frac{3}{2}}\right|^2 \\
&\quad \le \big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 12\big)\big(1 + |y|^4\big)|y|^2, \\
&\big(x - D(y,2)\big)^T F_\Delta(t,x,y,2) + \frac{\bar{p}-1}{2}\big|G_\Delta(t,x,y,2)\big|^2 \\
&\quad = \left(x + \frac{1}{12}y\right)\left(-4\left(\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right)^5\right) + \frac{\bar{p}-1}{2}\left|\big(t(1-t)\big)^{\frac{1}{4}}\left|\varphi^{-1}\big(h(\Delta)\big)\frac{y}{|y|}\right|^{\frac{5}{2}}\right|^2 \\
&\quad \le 2\big(1 + |x|^2\big) + \big((\bar{p}-1) \vee 40\big)\big(1 + |y|^8\big)|y|^2.
\end{aligned}$$

Case 4: If $|y| < \varphi^{-1}(h(\Delta))$ and $|x| > \varphi^{-1}(h(\Delta))$, then the proof is similar to the previous case.

Combing the four cases, we get that Assumption 3.6 is satisfied as well. Then we choose $\varphi(\cdot)$ and $h(\cdot)$. We can observe that

$$\sup_{0 \le t \le T} \sup_{|x| \vee |y| \le w} \big(\big|F(t,x,y,i)\big| \vee \big|G(t,x,y,i)\big|\big) \le 4w^5, \quad \forall w \ge 1,$$

which means that $\varphi(w) = 4w^5$. Let $h(\Delta) = K_0 \Delta^{-\frac{1}{8}}$. Then by Theorem 3.14, when $\alpha = \frac{1}{4}$, we obtain that

$$\mathbb{E}\big|x(T) - x_\Delta(T)\big|^2 \le C\Delta^{\frac{1}{2}} \quad \text{and} \quad \mathbb{E}\big|x(T) - \bar{x}_\Delta(T)\big|^2 \le C\Delta^{\frac{1}{2}}.$$

Since the explicit solution of (5.1) cannot be calculated, we regard the partially truncated EM scheme with step size 2^{-14} as the true solution in the numerical experiments. Figure 1

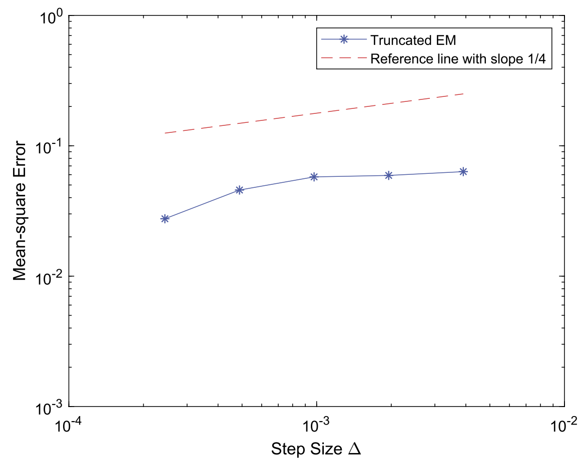

Figure 1 The convergence order of the truncated EM scheme for (5.1)

presents the $\mathscr{L}^2$-errors defined by

$$\left(\mathbb{E}\left|x(T)-x_\Delta(T)\right|^2\right)^{\frac{1}{2}} \approx \left(\frac{1}{1000}\sum_{i=1}^{1000}\left|\left(x(T)\right)_i-\left(x_\Delta(T)\right)_i\right|^2\right)^{\frac{1}{2}}$$

with step sizes 2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7} at $T = 1$. 1000 sample paths were simulated in the numerical experiments. We can observe that the convergence order of partially truncated EM method for (5.1) is approximately $\frac{1}{4}$, which is close to our result.

Example 5.2 Consider a nonlinear and nonautonomous neutral stochastic differential delay equations with Markovian switching

$$\begin{aligned} &d\left[x(t)-D\left(x(t-\tau),r(t)\right)\right] \\ &\quad = f\left(t,x(t),x(t-\tau),r(t)\right)dt+g\left(t,x(t),x(t-\tau),r(t)\right)dB(t), \quad t\geq 0, \end{aligned} \tag{5.2}$$

with the initial data x_0 satisfying Assumption 2.2. Here $B(t)$ and the Markovian chain are the same as Example 5.1. In addition, for all $t\in[0,\infty)$, $x,y\in\mathbb{R}^1$, and $i\in\mathbb{S}$, let

$$D(y,i)=\begin{cases}\frac{1}{6}\sin y & \text{if } i=1,\\ \frac{1}{12}\sin y & \text{if } i=2,\end{cases} \qquad g(t,x,y,i)=\begin{cases}|\sin(t(1-t))|^{\frac{1}{3}}|x|^{\frac{3}{2}} & \text{if } i=1,\\ |\sin(t(1-t))|^{\frac{1}{4}}|x|^{\frac{5}{2}} & \text{if } i=2,\end{cases}$$

$$f(t,x,y,i)=\begin{cases}-2x^3+|\sin(t(1-t))|^{\frac{1}{3}}y-10x+2y & \text{if } i=1,\\ -4x^5+|\sin(t(1-t))|^{\frac{1}{4}}y-20x+2y & \text{if } i=2.\end{cases}$$

It is easy to see that

$$\tilde{F}(t,x,y,i)=\begin{cases}|\sin(t(1-t))|^{\frac{1}{3}}y-10x+2y & \text{if } i=1,\\ |\sin(t(1-t))|^{\frac{1}{4}}y-20x+2y & \text{if } i=2,\end{cases} \qquad \tilde{G}(t,x,y,i)=\begin{cases}0 & \text{if } i=1,\\ 0 & \text{if } i=2,\end{cases}$$

$$F(t,x,y,i)=\begin{cases}-2x^3 & \text{if } i=1,\\ -4x^5 & \text{if } i=2,\end{cases}\qquad G(t,x,y,i)=\begin{cases}|\sin(t(1-t))|^{\frac{1}{3}}|x|^{\frac{3}{2}} & \text{if } i=1,\\ |\sin(t(1-t))|^{\frac{1}{4}}|x|^{\frac{5}{2}} & \text{if } i=2.\end{cases}$$

Obviously, $D(y,i)$ satisfies Assumption 2.3 for $i\in\mathbb{S}$. Now let us check Assumption 4.1. There is no $\tilde{G}(t,x,y,i)$ term, so we have $\Lambda=\infty$. Then we derive that

$$\begin{aligned}&2\bigl(x-D(y,1)\bigr)^T\tilde{F}(t,x,y,1)\le -16|x|^2+5|y|^2,\\ &2\bigl(x-D(y,2)\bigr)^T\tilde{F}(t,x,y,2)\le -35|x|^2+6|y|^2.\end{aligned}$$

Then like in verifying Assumption 3.6, we need to consider the second inequality in four cases.

Case 1: If $(|x|\vee|y|)\le\varphi^{-1}(h(\Delta))$, then we get

$$\begin{aligned}&2\bigl(x-D(y,1)\bigr)^T F_\Delta(t,x,y,1)+\bigl|G_\Delta(t,x,y,1)\bigr|^2\\ &\quad=2\left(x-\frac{1}{6}\sin y\right)\bigl(-2x^3\bigr)+\bigl|\bigl|\sin\bigl(t(1-t)\bigr)\bigr|^{\frac{1}{3}}|x|^{\frac{3}{2}}\bigr|^2\\ &\quad\le -|x|^2\left(-2|x|+\frac{1}{2}\right)^2+\frac{1}{4}|x|^2\le\frac{1}{4}|x|^2+\frac{1}{4}|y|^2,\\ &2\bigl(x-D(y,2)\bigr)^T F_\Delta(t,x,y,2)+\bigl|G_\Delta(t,x,y,2)\bigr|^2\\ &\quad=2\left(x-\frac{1}{12}\sin y\right)\bigl(-4x^5\bigr)+\bigl|\bigl|\sin\bigl(t(1-t)\bigr)\bigr|^{\frac{1}{4}}|x|^{\frac{5}{2}}\bigr|^2\\ &\quad\le -2|x|^2\left(-2|x|^2+\frac{1}{4}\right)^2+\frac{1}{8}|x|^2\le\frac{1}{8}|x|^2+\frac{1}{8}|y|^2.\end{aligned}$$

Case 2: If $(|x|\wedge|y|)>\varphi^{-1}(h(\Delta))$, then we have

$$\begin{aligned}&2\bigl(x-D(y,1)\bigr)^T F_\Delta(t,x,y,1)+\bigl|G_\Delta(t,x,y,1)\bigr|^2\\ &\quad=2\left(x-\frac{1}{6}\sin y\right)\left(-2\left(\varphi^{-1}\bigl(h(\Delta)\bigr)\frac{x}{|x|}\right)^3\right)+\left|\bigl|\sin\bigl(t(1-t)\bigr)\bigr|^{\frac{1}{3}}\left|\varphi^{-1}\bigl(h(\Delta)\bigr)\frac{x}{|x|}\right|^{\frac{3}{2}}\right|^2\\ &\quad\le -4\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^3|x|^4+\frac{2}{3}\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^3|x|^3+\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^3|x|^3\\ &\quad\le -\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^3|x|^2\left(-2|x|+\frac{1}{2}\right)^2+\frac{1}{4}|x|^2\le\frac{1}{4}|x|^2+\frac{1}{4}|y|^2,\\ &2\bigl(x-D(y,2)\bigr)^T F_\Delta(t,x,y,2)+\bigl|G_\Delta(t,x,y,2)\bigr|^2\\ &\quad=2\left(x-\frac{1}{12}\sin y\right)\left(-4\left(\varphi^{-1}\bigl(h(\Delta)\bigr)\frac{x}{|x|}\right)^5\right)+\left|\bigl|\sin\bigl(t(1-t)\bigr)\bigr|^{\frac{1}{4}}\left|\varphi^{-1}\bigl(h(\Delta)\bigr)\frac{x}{|x|}\right|^{\frac{5}{2}}\right|^2\\ &\quad\le -8\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^5x^6+\frac{2}{3}\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^5|x|^5+\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^5|x|^5\\ &\quad\le -2\left(\frac{\varphi^{-1}(h(\Delta))}{|x|}\right)^5|x|^2\left(-2|x|^2+\frac{1}{4}\right)^2+\frac{1}{8}|x|^2\le\frac{1}{8}|x|^2+\frac{1}{8}|y|^2.\end{aligned}$$

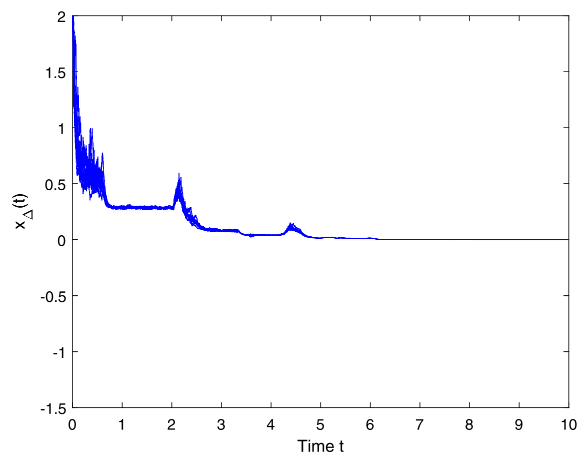

Figure 2 10 sample paths of $X_\Delta(t)$ for (5.2)

Case 3: If $|x| > \varphi^{-1}(h(\Delta))$ and $|y| < \varphi^{-1}(h(\Delta))$, then we derive that

$$\begin{aligned}
&2\big(x - D(y,1)\big)^T F_\Delta(t,x,y,1) + \big|G_\Delta(t,x,y,1)\big|^2 \\
&\quad = 2\left(x - \frac{1}{6}\sin y\right)\left(-2\left(\varphi^{-1}\big(h(\Delta)\big)\frac{x}{|x|}\right)^3\right) + \left|\big|\sin\big(t(1-t)\big)\big|^{\frac{1}{3}}\left|\varphi^{-1}\big(h(\Delta)\big)\frac{x}{|x|}\right|^{\frac{3}{2}}\right|^2 \\
&\quad \le \frac{1}{4}|x|^2 + \frac{1}{4}|y|^2, \\
&2\big(x - D(y,2)\big)^T F_\Delta(t,x,y,2) + \big|G_\Delta(t,x,y,2)\big|^2 \\
&\quad = 2\left(x - \frac{1}{12}\sin y\right)\left(-4\left(\varphi^{-1}\big(h(\Delta)\big)\frac{x}{|x|}\right)^5\right) + \left|\big|\sin\big(t(1-t)\big)\big|^{\frac{1}{4}}\left|\varphi^{-1}\big(h(\Delta)\big)\frac{x}{|x|}\right|^{\frac{5}{2}}\right|^2 \\
&\quad \le \frac{1}{8}|x|^2 + \frac{1}{8}|y|^2.
\end{aligned}$$

Case 4: If $|x| < \varphi^{-1}(h(\Delta))$ and $|y| > \varphi^{-1}(h(\Delta))$, then the proof is similar to the above process. Therefore Assumption 4.1 holds. Moreover, we easily to see that Assumption 3.1 is satisfied on $t \in [0,\infty)$. Then by Theorem 4.3 the partially truncated EM numerical solution is almost surely exponentially stable. Figure 2 shows the almost sure exponential stability of the partially truncated EM method for (5.2) with 10 sample paths.

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