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New post quantum analogues of Ostrowski-type inequalities using new definitions of left–right (p,q)-derivatives and definite integrals

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Abstract

The main objective of this paper is to introduce a new more elegant notion of left–right (p,q)-derivative and definite integrals. To show the significance of these concepts, we discuss some of their basic properties. A new generalized (p,q)-integral identity is also obtained. Utilizing this identity as an auxiliary result we then obtain our main results using the concept of n-polynomial convex functions.

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1 Introduction

Integral inequalities play a significant role in both pure and applied sciences because of their wide applications in mathematics and physics, as well as many other natural and human social sciences, while convexity theory has remained an important tool in the establishment of the theory of integral inequalities. The Hermite–Hadamard inequality [13], as a member of the family of integral inequalities, is a classical inequality that has long fascinated numerous mathematical researchers, which can be stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}$$

if $f:[a,b]\mapsto\mathbb{R}$ is convex.

This inequality provides us a necessary and sufficient condition for a function to be convex, it also gives us an estimate of the integral average for a continuous convex function on an interval. Recently, the improvements, generalizations, and variants for the Hermite–Hadamard inequality have been the subject of much research. The left-hand side of the Hermite–Hadamard inequality can be estimated by the Ostrowski [29] inequality, which

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reads as

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a)$$

with the best possible constant $1/4$ if $f : [a,b] \mapsto \mathbb{R}$ is differentiable, where $\|f'\|_\infty = \max\{|f(x)| | x \in [a,b]\}$.

In recent years, several successful attempts have been made in obtaining the variants and applications of Ostrowski inequality. For example, Dragomir and Rassias [12] provided many interesting results on its applications in numerical integration.

In the past few years, a variety of novel approaches have been utilized by researchers in generalizing the classical inequalities. One of those approaches is utilizing the concepts of quantum calculus instead of ordinary calculus. It is very well known to everyone that quantum calculus is calculus without limits. In quantum calculus we often establish the q-analogues of classical mathematical objects which can be recaptured by taking $q \to 1^-$. Historically the subject of quantum calculus can be traced back to Euler and Jacobi, but in recent decades it has experienced a rapid development [16]. Consequently, new generalizations of the classical concepts of quantum calculus have been introduced and investigated in the literature. Tariboon and Ntouyas [34] introduced the quantum calculus concepts on finite intervals, obtained several q-analogues of classical mathematical objects, and opened a new venue of research. For instance, they have obtained the q-analogue of Hölder's integral inequality, q-analogue of Hermite–Hadamard's inequality, q-trapezoid inequality, q-Ostrowski inequality, q-Cauchy–Bunyakovsky–Schwarz inequality, and q-analogue of Grüss–Čebyšev inequality, etc. This motivated other researchers and, as a result, numerous novel results pertaining to quantum analogues of classical mathematical results have been introduced in the literature. For example Noor et al. [28] and Sudsutad et al. [33] obtained some more q-analogues of trapezoid-like inequalities for first order q-differentiable convex functions, and Liu and Zhuang [25] derived some new q-analogues of trapezoid-like inequalities involving second order q-differentiable convex functions. Alp et al. [2] obtained a corrected q-analogue of Hermite–Hadamard's inequality. Zhang et al. [38] obtained a new generalized q-integral identity and obtained several new q-analogues for first order q-differentiable convex functions. Noor et al. [27] utilized the concepts of quantum calculus and obtained some new q-analogues of the Ostrowski-type inequality. These new analogues reduce to the original results if $q \to 1^-$. For some details from the application point of view of quantum differential and integral operators, see [1, 3–10, 14, 15, 17–21, 24, 30–32, 37]. Recently Kunt and Baidar [22] introduced some new concepts of quantum calculus, namely left–right quantum derivatives and definite integrals. Using these new definitions, the authors have obtained some new q-analogues of classical integral inequalities. Another significant generalization of quantum calculus is the post-quantum calculus. In quantum calculus we deal with a q-number with one base q, however, post-quantum calculus includes p- and q-numbers with two independent variables p and q. This was first considered by Chakarabarti and Jagannathan [11]. Tunc and Gov [36] introduced the concepts of (p,q)-derivatives and (p,q)-integrals on finite intervals as follows.

The main purpose of the article is to establish some new post-quantum estimates of the Ostrowski inequality by the use of new modified definitions of left–right (p,q)-derivatives and definite integrals. But before we start to proceed towards the main results, we need

to recall some basic concepts and previously known results from the quantum and post-quantum calculus.

Definition 1.1 ([36]) Let $0 < q < p \le 1$, $\mathcal{K} \subseteq \mathbb{R}$ be an interval such that $a \in \mathcal{K}$, and $f : \mathcal{K} \to \mathbb{R}$ be a continuous function. Then the left-(p,q)-derivative $_a\mathcal{D}_{p,q}f(x)$ of f at $x \in \mathcal{K} \setminus \{a\}$ is defined by

$$_{a^+}\mathcal{D}_{p,q}f(x) = \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)}.$$

Definition 1.2 ([36]) Let $0 < q < p \le 1$, $\mathcal{K} \subseteq \mathbb{R}$ be an interval such that $a \in \mathcal{K}$, and $f : \mathcal{K} \to \mathbb{R}$ be a continuous function. Then the left-(p,q)-integral $\int_a^x f(t)_a \, \mathrm{d}_{p,q}t$ on $\mathcal{K}$ is defined by

$$\int_a^x f(t)_{a^+} \, \mathrm{d}_{p,q}t = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right).$$

Recently, several researchers have utilized these concepts in obtaining some new (p,q)-analogues of classical inequalities. For example, Kunt et al. [23] and Luo et al. [26] obtained new refinements of the Hermite–Hadamard inequality using the concepts of post-quantum calculus.

2 New definitions

We now define some new concepts, examples, and their basic properties.

Definition 2.1 Let $f : I \to \mathbb{R}$ be a continuous function and let $t \in I$ and $0 < q < p \le 1$. Then the right (p,q)-derivative on I of function f at t is defined as

$$_{b^-}D_{p,q}f(t) = \frac{f(px + (1-p)b)) - f(qx + (1-q)b)}{(p-q)(b-t)}, \quad t \ne b.$$

Example 2.2 Let $f : [a,b] \to \mathbb{R}$, $f(t) = (b-t)^n$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
_{b^-}D_{p,q}f(t) &= {}_{b^-}D_{p,q}(b-t)^n \\
&= \frac{(b-(pt+(1-p)b))^n - (b-(qt+(1-q)b))^n}{(p-q)(b-t)} \\
&= \frac{p^n(b-t)^n - q^n(b-t)^n}{(p-q)(b-t)} \\
&= \frac{p^n - q^n}{p-q}(b-t)^n \\
&= [n]_{p,q}(b-t)^n.
\end{aligned}$$

Theorem 2.3 *Let $f, g : [a,b] \to \mathbb{R}$ be arbitrary functions and $\lambda \in \mathbb{R}$, then*

   I. $_{b^-}D_{p,q}[f(t) + g(t)] = {}_{b^-}D_{p,q}f(t) + {}_{b^-}D_{p,q}g(t)$;

  II. $_{b^-}D_{p,q}\lambda f(t) = \lambda_{b^-}D_{p,q}f(t)$;

 III.

$$_{b^-}D_{p,q}(fg)(t) = g(pt + (1-p)b)_{b^-}D_{p,q}f(t) + f(qt + (1-q)b)_{b^-}D_{p,q}g(t)$$

$$= f\big(pt + (1-p)b\big){}_{b^-}D_{p,q}g(t) + g\big(qt + (1-q)b\big){}_{b^-}D_{p,q}f(t);$$

IV. ${}_{b^-}D_{p,q}(f/g)(t) = \frac{g(pt+(1-p)b){}_{b^-}D_{p,q}f(t)+f(qt+(1-q)b){}_{b^-}D_{p,q}g(t)}{g(pt+(1-p)b)g(qt+(1-q)b)}.$

Proof We leave the details of the proof of parts I and II as these are obvious.

III. By Definition 2.1, we have

$${}_{b^-}D_{p,q}(fg)(t)$$
$$= \frac{f(pt + (1-p)b)g(pt + (1-p)b) - f(qt + (1-q)b)g(pt + (1-p)b)}{(p-q)(b-t)}$$
$$+ \frac{f(qt + (1-q)b)g(pt + (1-p)b) - f(qt + (1-q)b)g(qt + (1-q)b)}{(p-q)(b-t)}$$
$$= g\big(pt + (1-p)b\big){}_{b^-}D_{p,q}f(t) + f\big(qt + (1-q)b\big){}_{b^-}D_{p,q}g(t).$$

The second equation can be obtained in a similar way by interchanging the functions f and g.

IV. By Definition 2.1, we have

$${}_{b^-}D_{p,q}(f/g)(t)$$
$$= \frac{(f/g)(pt + (1-p)b) - (f/g)(qt + (1-q)b)}{(p-q)(b-t)}$$
$$= \frac{f(pt + (1-p)b)g(qt + (1-q)b) - f(qt + (1-q)b)g(pt + (1-p)b)}{g(pt + (1-p)b)g(qt + (1-q)b)(p-q)(b-t)}$$
$$= \frac{g(pt + (1-p)b){}_{b^-}D_{p,q}f(t) + f(qt + (1-q)b){}_{b^-}D_{p,q}g(t)}{g(pt + (1-p)b)g(qt + (1-q)b)}.$$

This completes the proof. □

We now define right-(p,q)-quantum integral as right-(p,q)-antiderivative of $\mathcal{F}(t)$ by using the following shifting operator:

$$\Delta_{p,q}\mathcal{F}(t) = \mathcal{F}\left(\frac{q}{p}t + \left(1 - \frac{q}{p}\right)b\right), \tag{2.1}$$

where $\mathcal{F}(t)$ is the (p,q)-antiderivative of f.

Applying mathematical induction to (2.1), we have

$$\Delta_{p,q}^{n}\mathcal{F}(t) = \begin{bmatrix} \mathcal{F}(\frac{q^n}{p^n}t + (1 - \frac{q^n}{p^n})b), & n \in \mathbb{N} \\ \mathcal{F}(t), & n = 0 \end{bmatrix}. \tag{2.2}$$

From Definition 2.1, we have

$$f(t) = \frac{\mathcal{F}(pt + (1-p)b) - \mathcal{F}(qt + (1-q)b)}{(p-q)(b-t)}.$$

Making the use of $u = pt + (1 - p)b$, we have

$$f\left(\frac{u - (1 - p)b}{p}\right) = \frac{\mathcal{F}(u) - \mathcal{F}(\frac{q}{p}u + (1 - \frac{q}{p})b)}{(\frac{p-q}{p})(b - u)}$$

$$= \frac{1 - \Delta_{p,q}}{(\frac{p-q}{p})(b - u)}\mathcal{F}(t).$$

Hence

$$\mathcal{F}(t) = \frac{1}{1 - \Delta_{p,q}}\left(1 - \frac{q}{p}\right)(b - u)f\left(\frac{u - (1 - p)b}{p}\right).$$

Applying the formula of expansion of geometric series to (2.1), we obtain

$$\mathcal{F}(t) = \left(1 - \frac{q}{p}\right)\sum_{n=0}^{\infty}\Delta_{p,q}^{n}(b - u)f\left(\frac{u - (1 - p)b}{p}\right)$$

$$= \left(1 - \frac{q}{p}\right)\sum_{n=0}^{\infty}\left(b - \left(\frac{q^{n}}{p^{n}}u + \left(1 - \frac{q^{n}}{p^{n}}\right)b\right)\right)$$

$$\times f\left(\frac{1}{p}\left(\frac{q^{n}}{p^{n}}u + \left(1 - \frac{q}{p}\right)b\right) + \left(1 - \frac{1}{p}\right)b\right)$$

$$= (p - q)(b - u)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(\frac{q^{n}}{p^{n+1}}u + \left(1 - \frac{q^{n}}{p^{n+1}}\right)b\right).$$

Thus

$$\mathcal{F}(t) = (p - q)(b - t)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(\frac{q^{n}}{p^{n+1}}t + \left(1 - \frac{q^{n}}{p^{n+1}}\right)b\right).$$

We now define right (p, q)-integral on a finite interval as:

Definition 2.4 Let $f : I \to \mathbb{R}$ be a continuous function. Then for $0 < q < p \leq 1$, the right-(p, q)-integral of $f(t)$ on I is defined as

$$\int_{a}^{b} f(t)_{b^{-}}\, \mathrm{d}_{p,q}t = (p - q)(b - t)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(\frac{q^{n}}{p^{n+1}}a + \left(1 - \frac{q^{n}}{p^{n+1}}\right)b\right). \tag{2.3}$$

For any $c \in (a, b)$, we have

$$\int_{a}^{c} f(t)_{b^{-}}\, \mathrm{d}_{p,q}t = \int_{a}^{b} f(t)_{b^{-}}\, \mathrm{d}_{p,q}t - \int_{c}^{b} f(t)_{b^{-}}\, \mathrm{d}_{p,q}t.$$

If we take $b = 0$ in (2.3), then

$$\int_{a}^{0} f(t)_{0^{-}}\, \mathrm{d}_{p,q}t = (p - q)(-t)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(\frac{q^{n}}{p^{n+1}}a\right),$$

which is the right-(p, q)-integral of $f(t)$ on $[a, 0]$.

Theorem 2.5 *Let $f, g : [a, b] \to \mathbb{R}$ be arbitrary functions and $\lambda \in \mathbb{R}$, then we have*

I. $\int_a^b [f(t) + g(t)]_{b^-} \, \mathrm{d}_{p,q} t = \int_a^b f(t)_{b^-} \, \mathrm{d}_{p,q} t + \int_a^b g(t)_{b^-} \, \mathrm{d}_{p,q} t$;

II. $\int_a^b \lambda f(t) = \lambda \int_a^b f(t)_{b^-} \, \mathrm{d}_{p,q} t$;

III. $_{b^-}\mathcal{D}_{p,q} \int_s^b f(t)_{b^-} \, \mathrm{d}_{p,q} t = f(s)$;

IV.

$$\int_s^u {}_{b^-}\mathcal{D}_{p,q} f(t)_{b^-} \, \mathrm{d}_{p,q} t = f(s) - f(u); \tag{2.4}$$

V. $\int_a^b f(qt + (1-q)b)_{b^-}\mathcal{D}_{p,q} g(pt)_{b^-} \, \mathrm{d}_{p,q} t = (fg)|_a^b - \int_a^b g(pt + (1-p)b)_{b^-} \mathrm{D}_{p,q} f(t)_{b^-} \, \mathrm{d}_{p,q} t$ or
$\int_a^b f(pt + (1-p)b)_{b^-}\mathcal{D}_{p,q} g(qt)_{b^-} \, \mathrm{d}_{p,q} t = (fg)|_a^b - \int_a^b g(qt + (1-q)b)_{b^-} \mathrm{D}_{p,q} f(t)_{b^-} \, \mathrm{d}_{p,q} t$.

Proof The proofs of claims I and II are obvious.

III. Using Definitions 2.1 and 2.4, we have

$${}_{b^-}\mathcal{D}_{p,q} \int_s^b f(t)_{b^-} \, \mathrm{d}_{p,q} t$$

$$= {}_{b^-}\mathcal{D}_{p,q} \left[(p-q)(b-s) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} s + \left(1 - \frac{q^n}{p^{n+1}}\right) b \right) \right]$$

$$= (p-q) \left[\frac{(b - (ps + (1-p)b)) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}(ps + (1-p)b) + (1 - \frac{q^n}{p^{n+1}})b\right)}{(p-q)(b-s)} \right.$$

$$\left. - \frac{(b - (qs + (1-q)b)) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}(qs + (1-q)b) + (1 - \frac{q^n}{p^{n+1}})b\right)}{(p-q)(b-s)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{p^n} f\left(\frac{q^n}{p^n} s + \left(1 - \frac{q^n}{p^n}\right) b \right) - \sum_{n=0}^{\infty} \frac{q^{n+1}}{p^{n+1}} f\left(\frac{q^{n+1}}{p^{n+1}} s + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right) b \right)$$

$$= f(s).$$

IV. Using Definitions 2.1 and 2.4, we have

$$\int_s^b {}_{b^-}\mathcal{D}_{p,q} f(t)_{b^-} \, \mathrm{d}_{p,q} t$$

$$= \int_s^u \frac{f(pt + (1-p)b) - f(qt + (1-q)b)}{(p-q)(b-t)} \, \mathrm{d}_{p,q} t$$

$$= \frac{1}{p-q} \left[\int_s^u \frac{f(pt + (1-p)b)}{(b-t)} \, \mathrm{d}_{p,q} t - \int_s^u \frac{f(qt + (1-q)b)}{(b-t)} \, \mathrm{d}_{p,q} t \right]$$

$$= \frac{1}{p-q} \left[(p-q)(b-s) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \frac{f(\frac{q^n}{p^{n+1}}(ps + (1-p)b) + (1 - \frac{q^n}{p^{n+1}})b)}{(b-s)} \right.$$

$$\left. - (p-q)(b-s) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \frac{f(\frac{q^n}{p^{n+1}}(ps + (1-p)b) + (1 - \frac{q^n}{p^{n+1}})b)}{(b-s)} \right]$$

$$= \sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n} s + \left(1 - \frac{q^n}{p^n}\right) b \right) - \sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{p^{n+1}} s + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right) b \right)$$

$$= f(b) - f(s).$$

V. From claim III of Theorem 2.3, we have

$$f\big(qt + (1-q)b\big)_{b^-}D_{p,q}g(t) = {}_{b^-}D_{p,q}(fg)(t) - g\big(pt + (1-p)b\big)_{b^-}D_{p,q}f(t)$$

By integrating over $[a, b]$ and using (2.4), we have

$$\int_a^b f\big(qt + (1-q)b\big)_{b^-}D_{p,q}g(t)_{b^-}\,\mathrm{d}_{p,q}t = (fg)|_a^b - \int_a^b g\big(pt + (1-p)b\big)_{b^-}D_{p,q}f(t)_{b^-}\,\mathrm{d}_{p,q}t.$$

This completes the proof.   □

We now derive (p, q)-analogue of Hermite–Hadamard's inequality.

Theorem 2.6 *Let $f : I \to \mathbb{R}$ be a convex and (p, q)-integrable function with $0 < q < p \leq 1$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^b f(t)_{a^+}\,\mathrm{d}_{p,q}t + \int_a^b f(t)_{b^-}\,\mathrm{d}_{p,q}t\right] \leq \frac{f(a) + f(b)}{2}. \tag{2.5}$$

Proof It is obvious that

$$\int_0^1 f\big(tb + (1-t)a\big)_{0^+}\,\mathrm{d}_{p,q}t$$

$$= (p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)$$

$$= \frac{1}{b-a}\left[(p-q)(b-a)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)_{a^+}\,\mathrm{d}_{p,q}t. \tag{2.6}$$

Similarly,

$$\int_0^1 f\big(ta + (1-t)b\big)_{0^+}\,\mathrm{d}_{p,q}t$$

$$= (p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}a + \left(1 - \frac{q^n}{p^{n+1}}\right)b\right)$$

$$= \frac{1}{b-a}\left[(p-q)(b-a)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}a + \left(1 - \frac{q^n}{p^{n+1}}\right)b\right)\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)_{b^-}\,\mathrm{d}_{p,q}t. \tag{2.7}$$

Since f is convex on I, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\big[f\big(tb + (1-t)a\big) + f\big(ta + (1-t)b\big)\big] \leq \frac{f(a) + f(b)}{2}, \tag{2.8}$$

for all $t \in I$.

Taking (p, q)-integral of (2.8) and using (2.6) and (2.7), we obtain the required inequality. $\qquad\square$

3 A key lemma

The following auxiliary result will play a significant role in the development of our next results.

Lemma 3.1 *Let* $f : [a, b] \to \mathbb{R}$ *be a* (p, q)-*differentiable function on* (a, b) *with* $a < b$. *If* ${}_a D_{p,q} f$ *is integrable on* $[a, b]$ *and* $0 < q < p \le 1$, *then*

$$
\begin{aligned}
f(x) &- \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_{a^+} \, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_{b^-} \, d_{p,q}x \\
&= \frac{q(x-a)^2}{b-a} \int_0^1 t_{a^+} D_{p,q} f\big(tx + (1-t)a\big)_{0^+} \, d_{p,q}t \\
&\quad + \frac{q(b-x)^2}{b-a} \int_0^1 t_{b^-} D_{p,q} f\big(tx + (1-t)b\big)_{0^+} \, d_{p,q}t.
\end{aligned}
\tag{3.1}
$$

Proof Let

$$
S_1 = \int_0^1 t_{a^+} D_{p,q} f\big(tx + (1-t)a\big)_{0^+} \, d_{p,q}t,
$$

$$
S_2 = \int_0^1 t_{b^-} D_{p,q} f\big(tx + (1-t)b\big)_{0^+} \, d_{p,q}t,
$$

A direct computation gives

$$
\begin{aligned}
S_1 &= \int_0^1 t_{a^+} D_{p,q} f\big(tx + (1-t)a\big)_{0^+} \, d_{p,q}t \\
&= \int_0^1 \frac{f(tpx + (1-tp)a) - f(tqx + (1-tq)a)}{(p-q)(x-a)}{}_{0^{+}z} \, d_{p,q}t \\
&= \frac{1}{x-a}\left[\sum_{n=0}^\infty \frac{q^n}{p^{n+1}} f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)a \right) \right. \\
&\qquad\left. - \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} f\left(x\frac{q^{n+1}}{p^{n+1}} + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right)a \right) \right] \\
&= \frac{1}{x-a}\left[\sum_{n=0}^\infty \frac{q^n}{p^{n+1}} f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)a \right) \right. \\
&\qquad\left. - \frac{p}{q}\sum_{n=1}^\infty \frac{q^n}{p^{n+1}} f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)a \right) \right] \\
&= \frac{1}{x-a}\left[\left(1 - \frac{p}{q}\right) \sum_{n=1}^\infty \frac{q^n}{p^{n+1}} f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)a \right) + \frac{f(x)}{q} \right] \\
&= \frac{1}{x-a}\left[\frac{f(x)}{q} - \left(\frac{p-q}{q}\right) \sum_{n=1}^\infty \frac{q^n}{p^{n+1}} f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)a \right) \right] \\
&= \frac{1}{x-a}\left[\frac{f(x)}{q} - \left(\frac{1}{pq}\right) \int_0^p f\big(tx + (1-t)a\big)_{a^+} \, d_{p,q}t \right]
\end{aligned}
$$

$$= \frac{1}{x-a}\left[\frac{f(x)}{q} - \left(\frac{1}{pq(x-a)}\right)\int_a^{xp+(1-p)a} f(x)_{a^+}\, \mathrm{d}_{p,q}x\right].$$

Similarly,

$$\mathcal{S}_2 = \int_0^1 t_{b^-}\mathrm{D}_{p,q}f\bigl(tx + (1-t)b\bigr)_{0^+}\, \mathrm{d}_{p,q}t$$

$$= \int_0^1 \frac{f(tpx + (1-tp)b) - f(tqx + (1-tq)b)}{(p-q)(b-x)}{}_{0^+}\, \mathrm{d}_{p,q}t$$

$$= \frac{1}{b-x}\left[\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)b\right)\right.$$

$$\left. - \sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(x\frac{q^{n+1}}{p^{n+1}} + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right)b\right)\right]$$

$$= \frac{1}{b-x}\left[\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)b\right)\right.$$

$$\left. - \frac{p}{q}\sum_{n=1}^\infty \frac{q^n}{p^{n+1}}f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)b\right)\right]$$

$$= \frac{1}{b-x}\left[\left(1 - \frac{p}{q}\right)\sum_{n=1}^\infty \frac{q^n}{p^{n+1}}f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)b\right) + \frac{f(x)}{q}\right]$$

$$= \frac{1}{b-x}\left[\frac{f(x)}{q} - \left(\frac{p-q}{q}\right)\sum_{n=1}^\infty \frac{q^n}{p^{n+1}}f\left(px\frac{q^n}{p^{n+1}} + \left(1 - p\frac{q^n}{p^{n+1}}\right)b\right)\right]$$

$$= \frac{1}{b-x}\left[\frac{f(x)}{q} - \left(\frac{1}{pq}\right)\int_0^p f\bigl(tx + (1-t)b\bigr)_{b^-}\, \mathrm{d}_{p,q}t\right]$$

$$= \frac{1}{b-x}\left[\frac{f(x)}{q} - \left(\frac{1}{pq(b-x)}\right)\int_{xp+(1-p)b}^b f(x)_{b^-}\, \mathrm{d}_{p,q}x\right].$$

Thus we have

$$f(x) - \frac{1}{p(b-a)}\int_a^{px+(1-p)a} f(x)_{a^+}\, \mathrm{d}_{p,q}x - \frac{1}{p(b-a)}\int_{px+(1-p)b}^b f(x)_{b^-}\, \mathrm{d}_{p,q}x$$

$$= \frac{q(x-a)^2}{b-a}\mathcal{S}_1 + \frac{q(b-x)^2}{b-a}\mathcal{S}_2,$$

which leads to the desired identity (3.1).   □

Remark 3.2 By taking $p \to 1$, we obtain equality (3.1) of [22].

In order to prove our next results, we need the definition of n-polynomial convex functions which was introduced and studied by Toplu et al. [35].

Definition 3.3 ([35]) Let $n \in \mathbb{N}$. A nonnegative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be an n-polynomial convex function if for every $x, y \in I$ and $t \in [0, 1]$, we have

$$f\bigl(tx + (1-t)y\bigr) \le \frac{1}{n}\sum_{s=1}^n \bigl[1 - (1-t)^s\bigr]f(x) + \frac{1}{n}\sum_{s=1}^n \bigl[1 - t^s\bigr]f(y).$$

Theorem 3.4 *Let $f : [a, b] \to \mathbb{R}$ be continuous and (p, q)-differentiable function on (a, b) with $a < b$ and $_{a^+}D_{p,q}f$ and $_{b^-}D_{p,q}f$ be (p, q)-integrable. If $|_{a^+}D_{p,q}f|$ and $|_{b^-}D_{p,q}f|$ are n-polynomial convex functions and $|_{a^+}D_{p,q}f|, |_{b^-}D_{p,q}f| \le M$, then we have*

$$
\left| f(x) - \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_{a^+} \, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_{b^-} \, d_{p,q}x \right|
$$

$$
\le \frac{qM(x-a)^2}{n(b-a)} \sum_{s=1}^\infty \left(\frac{2}{p+q} - \frac{p-q}{p^{s+2}-q^{s+2}} - (p-q) \sum_{n=0}^\infty \frac{q^{2n}}{p^{2n+2}} \left(1 - \frac{q^n}{p^{n+1}} \right)^s \right)
$$

$$
+ \frac{qM(b-x)^2}{n(b-a)} \sum_{s=1}^\infty \left(\frac{2(p+q-1)}{p+q} - (p-q) \sum_{n=0}^\infty \frac{q^n}{p^{(n+1)}} \left(\frac{q^n}{p^{n+1}} \right)^s \right.
$$

$$
\left. - (p-q) \sum_{n=0}^\infty \frac{q^{n(s+2)}}{p^{(n+1)(s+2)}} \right).
$$

Proof Using Lemma 3.1 and the fact that $|_{a^+}D_{p,q}f|$ and $|_{b^-}D_{p,q}f|$ are n-polynomial convex functions, we have

$$
\left| f(x) - \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_{a^+} \, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_{b^-} \, d_{p,q}x \right|
$$

$$
\le \frac{q(x-a)^2}{b-a} \int_0^1 t \left|_{a^+}D_{p,q}f\big(tx + (1-t)a\big)\right|_{0^+} d_{p,q}t
$$

$$
+ \frac{q(b-x)^2}{b-a} \int_0^1 t \left|_{b^-}D_{p,q}f\big(tx + (1-t)b\big)\right|_{0^+} d_{p,q}t
$$

$$
\le \frac{q(x-a)^2}{b-a} \int_0^1 t \left[\frac{|_{a^+}D_{p,q}f(x)|}{n} \sum_{s=1}^n [1 - (1-t)^s] \right.
$$

$$
+ \frac{|_{a^+}D_{p,q}f(a)|}{n} \sum_{s=1}^n [1 - t^s] \Bigg]_{0^+} d_{p,q}t
$$

$$
+ \frac{q(b-x)^2}{b-a} \int_0^1 t \left[\frac{|_{b^-}D_{p,q}f(x)|}{n} \sum_{s=1}^n [1 - (1-t)^s] \right.
$$

$$
+ \frac{|_{b^-}D_{p,q}f(b)|}{n} \sum_{s=1}^n [1 - t^s] \Bigg]_{0^+} d_{p,q}t
$$

$$
= \frac{q(x-a)^2}{n(b-a)} \left[|_{a^+}D_{p,q}f(x)| \sum_{s=1}^n \int_0^1 t[1 - (1-t)^s]_{0^+} \, d_{p,q}t \right.
$$

$$
+ |_{a^+}D_{p,q}f(a)| \sum_{s=1}^n \int_0^1 t[1 - t^s]_{0^+} \, d_{p,q}t \Bigg]
$$

$$
+ \frac{q(b-x)^2}{n(b-a)} \left[|_{b^-}D_{p,q}f(x)| \sum_{s=1}^n \int_0^1 t[1 - (1-t)^s]_{0^+} \, d_{p,q}t \right.
$$

$$
+ |_{b^-}D_{p,q}f(b)| \sum_{s=1}^n \int_0^1 t[1 - t^s]_{0^+} \, d_{p,q}t \Bigg]
$$

$$
\leq \frac{qM(x-a)^2}{n(b-a)} \sum_{s=1}^{\infty} \left(\frac{2}{p+q} - \frac{p-q}{p^{s+2}-q^{s+2}} - (p-q) \sum_{n=0}^{\infty} \frac{q^{2n}}{p^{2n+2}} \left(1 - \frac{q^n}{p^{n+1}}\right)^s \right)
$$

$$
+ \frac{qM(b-x)^2}{n(b-a)} \sum_{s=1}^{\infty} \left(\frac{2(p+q-1)}{p+q} \right.
$$

$$
\left. - (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{(n+1)}} \left(\frac{q^n}{p^{n+1}} \right)^s - (p-q) \sum_{n=0}^{\infty} \frac{q^{n(s+2)}}{p^{(n+1)(s+2)}} \right).
$$

This completes the proof. □

Theorem 3.5 *Let $f : [a,b] \to \mathbb{R}$ be continuous and (p,q)-differentiable function on (a,b) with $a < b$ and $_{a^+}D_{p,q}f$ and $_{b^-}D_{p,q}f|^{e_2}$ be (p,q)-integrable. If $|_{a^+}D_{p,q}f|^{e_2}$ and $|_{b^-}D_{p,q}f|^{e_2}$ are n-polynomial convex functions and $|_{a^+}D_{p,q}f|^{e_2}, |_{b^-}D_{p,q}f|^{e_2} \leq M$, then for $e_1, e_2 > 1, e_1^{-1} + e_2^{-1} = 1$, we have*

$$
\left| f(x) - \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_{a^+}\, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_{b^-}\, d_{p,q}x \right|
$$

$$
\leq \frac{qM[(b-x)^2 + (b-x)^2}{(b-a)}
$$

$$
\times \left((p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right)^{e_1} \right)^{\frac{1}{e_1}} \left(\frac{2}{n} \right)^{\frac{1}{e_2}} \sum_{s=1}^{n} \left((p-q) \sum_{n=0}^{\infty} \frac{q^n s + 1}{p^{(s+1)(n+1)}} \right).
$$

Proof Using Lemma 3.1, Hölder's integral inequality, and the fact that $|_{a^+}D_{p,q}f|^{e_2}$ and $|_{b^-}D_{p,q}f|^{e_2}$ are n-polynomial convex functions, we have

$$
\left| f(x) - \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_{a^+}\, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_{b^-}\, d_{p,q}x \right|
$$

$$
\leq \frac{q(x-a)^2}{b-a} \left(\int_0^1 t^{e_1}{}_{0^+}\, d_{p,q}t \right)^{\frac{1}{e_1}} \left(\int_0^1 \left| {}_{a^+}D_{p,q}f\left(tx+(1-t)a\right) \right|^{e_2}{}_{0^+}\, d_{p,q}t \right)^{\frac{1}{e_1}}
$$

$$
+ \frac{q(b-x)^2}{b-a} \left(\int_0^1 t^{e_1}{}_0\, d_{p,q}t \right)^{\frac{1}{e_1}} \left(\int_0^1 \left| {}_{b^-}D_{p,q}f\left(tx+(1-t)b\right) \right|^{e_2}{}_{0^+}\, d_{p,q}t \right)^{\frac{1}{e_2}}
$$

$$
\leq \frac{q(x-a)^2}{b-a} \left(\frac{p-q}{p^{e_1+1}-q^{e_1+1}} \right)^{\frac{1}{e_1}}
$$

$$
\times \left(\frac{|_{a^+}D_{p,q}f(x)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 \left[1-(1-t)^s\right]_{0^+}\, d_{p,q}t \right.
$$

$$
\left. + \frac{|_{a^+}D_{p,q}f(a)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 \left[1-t^s\right]_{0^+}\, d_{p,q}t \right)^{\frac{1}{e_2}}
$$

$$
+ \frac{q(b-x)^2}{b-a} \left((p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right)^{e_1} \right)^{\frac{1}{e_1}}
$$

$$
\times \left(\frac{|_{b^-}D_{p,q}f(x)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 \left[1-(1-t)^s\right]_{0^+}\, d_{p,q} \right.
$$

$$+ \frac{|_{b^-}D_{p,q}f(b)|^{e_2}}{n} \sum_{s=1}^{n} \left[1 - t^s\right]_{0^+} d_{p,q} \Bigg)^{\frac{1}{e_2}}$$

$$\leq \frac{qM[(b-x)^2 + (b-x)^2]}{(b-a)}$$

$$\times \left((p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right)^{e_1}\right)^{\frac{1}{e_1}} \left(\frac{2}{n}\right)^{\frac{1}{e_2}} \sum_{s=1}^{n} \left((p-q) \sum_{n=0}^{\infty} \frac{q^n s + 1}{p^{(s+1)(n+1)}}\right).$$

This completes the proof. □

Theorem 3.6 *Let $f : [a,b] \to \mathbb{R}$ be continuous and (p,q)-differentiable function on (a,b) with $a < b$ and $_{a^+}D_{p,q}f$ and $_{b^-}D_{p,q}f$ be (p,q)-integrable. If $|_{a^+}D_{p,q}f|^{e_2}$ and $|_{b^-}D_{p,q}f|^{e_2}$ are n-polynomial convex functions and $|_{a^+}D_{p,q}f|^{e_2}, |_{b^-}D_{p,q}f|^{e_2} \leq M$, then for $e_2 > 1$, we have*

$$\left| f(x) - \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_a\, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_a\, d_{p,q}x \right|$$

$$\leq \frac{qM(x-a)^2}{n(b-a)} \left(\frac{1}{p+q}\right)^{1-\frac{1}{e_2}}$$

$$\times \left[\sum_{s=1}^{\infty} \left(\frac{2}{p+q} - \frac{p-q}{p^{s+2} - q^{s+2}} - (p-q) \sum_{n=0}^{\infty} \frac{q^{2n}}{p^{2n+2}} \left(1 - \frac{q^n}{p^{n+1}}\right)^s\right) \right]^{\frac{1}{e_2}}$$

$$+ \frac{qM(b-x)^2}{n(b-a)} \left(\frac{p+q-1}{p+q}\right)^{1-\frac{1}{e_2}}$$

$$\times \left[\sum_{s=1}^{\infty} \left(\frac{2(p+q-1)}{p+q} - (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{(n+1)}} \left(\frac{q^n}{p^{n+1}}\right)^s - (p-q) \sum_{n=0}^{\infty} \frac{q^{n(s+2)}}{p^{(n+1)(s+2)}}\right) \right]^{\frac{1}{e_2}}.$$

Proof Using Lemma 3.1, power-mean integral inequality, and the fact that $|_{a^+}D_{p,q}f|^{e_2}$ and $|_{b^-}D_{p,q}f|^{e_2}$ are n-polynomial convex functions, we have

$$\left| f(x) - \frac{1}{p(b-a)} \int_a^{px+(1-p)a} f(x)_a\, d_{p,q}x - \frac{1}{p(b-a)} \int_{px+(1-p)b}^b f(x)_a\, d_{p,q}x \right|$$

$$\leq \frac{q(x-a)^2}{b-a} \left(\int_0^1 t_{0^+}\, d_{p,q}t\right)^{1-\frac{1}{e_2}} \left(\int_0^1 t\big|_{a^+}D_{p,q}f\big(tx + (1-t)a\big)\big|^{e_2}_{0^+}\, d_{p,q}t\right)^{\frac{1}{e_2}}$$

$$+ \frac{q(b-x)^2}{b-a} \left(\int_0^1 t_{0^+}\, d_{p,q}t\right)^{1-\frac{1}{e_2}} \left(\int_0^1 t\big|_{b^-}D_{p,q}f\big(tx + (1-t)b\big)\big|^{e_2}_{0^+}\, d_{p,q}t\right)^{\frac{1}{e_2}}$$

$$\leq \frac{q(x-a)^2}{b-a} \left(\frac{1}{p+q}\right)^{1-\frac{1}{e_2}}$$

$$\times \left(\frac{|_{a^+}D_{p,q}f(x)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 t\left[1 - (1-t)^s\right]_{0^+} d_{p,q}t\right.$$

$$\left. + \frac{|_{a^+}D_{p,q}f(a)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 t\left[1 - t^s\right]_{0^+} d_{p,q}t\right)^{\frac{1}{e_2}}$$

$$+ \frac{q(x-a)^2}{b-a} \left(\frac{p+q-1}{p+q} \right)^{1-\frac{1}{e_2}}$$

$$\times \left(\frac{|_{b^-}D_{p,q}f(x)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 t\big[1-(1-t)^s\big]_{0^+} \, d_{p,q}t \right.$$

$$+ \frac{|_{b^-}D_{p,q}f(a)|^{e_2}}{n} \sum_{s=1}^{n} \int_0^1 t\big[1-t^s\big]_{0^+} \, d_{p,q}t \Bigg)^{\frac{1}{e_2}}$$

$$\leq \frac{qM(x-a)^2}{n(b-a)} \left(\frac{1}{p+q} \right)^{1-\frac{1}{e_2}}$$

$$\times \left[\sum_{s=1}^{\infty} \left(\frac{2}{p+q} - \frac{p-q}{p^{s+2}-q^{s+2}} - (p-q) \sum_{n=0}^{\infty} \frac{q^{2n}}{p^{2n+2}} \left(1 - \frac{q^n}{p^{n+1}} \right)^s \right) \right]^{\frac{1}{e_2}}$$

$$+ \frac{qM(b-x)^2}{n(b-a)} \left(\frac{p+q-1}{p+q} \right)^{1-\frac{1}{e_2}}$$

$$\times \left[\sum_{s=1}^{\infty} \left(\frac{2(p+q-1)}{p+q} - (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{(n+1)}} \left(\frac{q^n}{p^{n+1}} \right)^s - (p-q) \sum_{n=0}^{\infty} \frac{q^{n(s+2)}}{p^{(n+1)(s+2)}} \right) \right]^{\frac{1}{e_2}}.$$

This completes the proof. □

4 Conclusion

We have introduced new concepts of left–right (p,q)-derivatives and definite integrals, respectively. We have discussed some basic properties of these newly introduced concepts. Using them we have derived a new (p,q)-integral identity. With the help of this identity, we have derived some new (p,q)-analogues of Ostrowski-type inequalities, essentially utilizing the concept of n-polynomial convex functions. We hope that the ideas and techniques of this paper will inspire interested readers.

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