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A novel fractional structure of a multi-order quantum multi-integro-differential problem

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Abstract

In the present research manuscript, we formulate a new generalized structure of the nonlinear Caputo fractional quantum multi-integro-differential equation in which such a multi-order structure of quantum integrals is considered for the first time. In fact, in the light of this type of boundary value problem equipped with the multi-integro-differential setting, one can simply study different cases of the existing usual integro-differential problems in the literature. In this direction, we utilize well-known analytical techniques to derive desired criteria which guarantee the existence of solutions for the proposed multi-order quantum multi-integro-differential problem. Further, some numerical examples are considered to examine our theoretical and analytical findings using the proposed methods.

MSC: Primary 34A08; secondary 34A12; 34A45

Keywords: Boundary value problem; Multi-integro-differential equation; Quantum calculus; The Caputo quantum derivative

1 Introduction

As years and even decades go by, the human beings need to be acquainted with different natural phenomena more and more. One possible way to achieve this purpose is to apply the logical techniques and tools available in mathematics, and particularly the mathematical operators, in the modeling of different processes. Various fractional operators have been formulated by different researchers, and their applicability is becoming increasingly apparent to researchers every day. In consequence, it is necessary that we derive and investigate various models of processes from all aspects by utilizing the fractional operators in boundary value problems. Some instances of the application of these operators can be found in applied sciences such as electrical circuits, medicine, biomathematics, etc. [1–6]. Moreover, the importance of this field implies that the researchers are interested in finding different aspects of the structure of the general fractional BVPs and some dynamical properties of their solutions. In this context, a lot of researchers have been studying many modern and general fractional models and relevant dynamical behaviors of this type of fractional BVPs (see, for example, [7–18]).

In 1910, Jackson [19] formulated a new field of the fractional calculus entitled the quantum fractional calculus or simply q-calculus. Shortly afterwards, Adams worked on the newly-defined quantum calculus and published some papers about q-difference equations [20–22]. At the same time, Carman and Starcher also continued this novel branch of the fractional calculus [23, 24]. In the subsequent step, Trjitzinsky investigated analytic theory of linear quantum differential equations and also nonlinear quantum differential systems [25, 26]. After the World War II, Abdi [27] studied certain quantum differential equations in 1962. Finally, Miller [28] combined quantum differential equations with Lie theory and investigated new theoretical results in this regard. By continuing this trend in the subsequent years, numerous researchers extended this field and obtained many interesting findings on the fractional quantum differential equations and inclusions (for more details, see [29–48]).

In 2013, Zhou and Liu [49], with the aid of Mönch's fixed point theorem along with an analytical technique based on the measure of weak noncompactness, turned to the following fractional quantum boundary problem:

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_q^{\sigma}\varpi(z) + \hat{h}_*(z,\varpi(z)) = 0, \quad z \in [0,1], \\ \varpi(0) = 0, \qquad {}^{\mathcal{C}}\mathfrak{D}_q^{2}\varpi(0) = 0, \qquad \eta^{*\mathcal{C}}\mathfrak{D}_q\varpi(1) + \lambda^{*\mathcal{C}}\mathfrak{D}_q^{2}\varpi(1) = 0, \end{cases}$$

such that $0 < q < 1$, $2 < \sigma < 3$, $\eta^*, \lambda^* \geq 0$, and $\hat{h}_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ is supposed to be continuous.

In the usual fractional calculus setting, Niyom et al. [50] designed the following multi-order boundary problem in a new framework including Riemann–Liouville derivatives:

$$\begin{cases} (\eta^{*\mathcal{RL}}\mathfrak{D}^{\sigma_1} + (1-\eta^*)^{\mathcal{RL}}\mathfrak{D}^{\sigma_2})\varpi(z) = \hat{h}_*(z,\varpi(z)), \quad z \in [0,T], \\ \varpi(0) = 0, \qquad \mu^{*\mathcal{RL}}\mathfrak{D}^{\sigma_3}\varpi(T) + (1-\mu^*)^{\mathcal{RL}}\mathfrak{D}^{\sigma_4}\varpi(T) = \tilde{s}, \end{cases}$$

where $\sigma_1, \sigma_2 \in (1,2)$, $0 < \sigma_3$, $\sigma_4 < \sigma_1 - \sigma_2$, and ${}^{\mathcal{RL}}\mathfrak{D}^{\gamma}$ stands for the standard Riemann–Liouville derivative of order $\gamma \in \{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$, and also $\eta^*, \mu^* \in (0,1]$, $\tilde{s} \in \mathbb{R}$, and $\hat{h}_* \in \mathcal{C}_{\mathbb{R}}([0,T] \times \mathbb{R})$ for $T > 0$. Recently in 2019, Etemad, Ntouyas, and Ahmad [51] formulated a novel framework of the nonlinear fractional quantum integro-differential equation equipped with quantum integral conditions as follows;

$$\begin{cases} (\eta^{*\mathcal{RL}}\mathfrak{D}_q^{\sigma_1} + (1-\eta^*)^{\mathcal{RL}}\mathfrak{D}_q^{\sigma_2})\varpi(z) = a\hat{h}_*(z,\varpi(z)) + b^{\mathcal{RL}}\mathcal{I}_q^{\delta}\hat{f}_*(z,\varpi(z)), \\ \varpi(0) = 0, \qquad \mu^* \int_0^1 \frac{(1-qr)^{(\theta_1^*-1)}}{\Gamma_q(\theta_1^*)}\varpi(r)\,\mathrm{d}_q r + (1-\mu^*)\int_0^1 \frac{(1-qr)^{(\theta_2^*-1)}}{\Gamma_q(\theta_2^*)}\varpi(r)\,\mathrm{d}_q r = 0, \end{cases}$$

where $z \in [0,1]$, $q \in (0,1)$, $\sigma_1, \sigma_2 \in (1,2)$ with $\sigma_1 - \sigma_2 > 1$, $\eta^*, \mu^* \in (0,1)$, $\theta_1^*, \theta_2^* > 0$, $\delta \in (0,1)$, $a, b \in \mathbb{R}^+$, and ${}^{\mathcal{RL}}\mathfrak{D}_q^{\sigma}$ stands for the Riemann–Liouville quantum derivative of order σ while $\hat{h}_*, \hat{f}_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ are supposed to be continuous functions.

Inspired by the aforementioned ideas given in the above-cited papers, we formulate a new generalized structure of the nonlinear Caputo fractional quantum multi-integro-differential equation furnished with fractional multi-order quantum integrals conditions:

$$\begin{cases} (\eta^{*\mathcal{C}}\mathfrak{D}_q^{\sigma} - (\eta^*+1)^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*} - (\eta^*+2)^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*})\varpi(z) \\ \quad = \lambda_1^{*\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*}\hat{h}_*(z,\varpi(z)) + \lambda_2^{*\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*}\hat{f}_*(z,\varpi(z)), \\ \varpi(0) = 0, \qquad \mu^{*\mathcal{RL}}\mathcal{I}_q^{\theta_1^*}\varpi(1) + (\mu^*+1)^{\mathcal{RL}}\mathcal{I}_q^{\theta_2^*}\varpi(1) + (\mu^*+2)^{\mathcal{RL}}\mathcal{I}_q^{\theta_3^*}\varpi(1) = 0, \end{cases} \tag{1}$$

such that $z \in [0,1]$, $\sigma \in (1,2)$, $q \in (0,1)$, $\delta_1^*, \delta_2^*, \gamma_1^*, \gamma_2^* \in (0,1)$, $\theta_1^*, \theta_2^*, \theta_3^* > 0$, and η^*, μ^* are nonzero real positive constants and $\lambda_1^*, \lambda_2^* \in \mathbb{R}^{\geq 0}$. Moreover, two operators, ${}^{\mathcal{C}}\mathfrak{D}_q^{(\cdot)}$ and ${}^{\mathcal{RL}}\mathcal{I}_q^{(\cdot)}$, stand for the Caputo quantum derivative and the Riemann–Liouville quantum integral of given fractional orders, respectively. Also, both real-valued functions $\hat{h}_*, \hat{f}_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ are supposed to be continuous. It is necessary that all researchers pay attention to that the proposed multi-order Caputo quantum multi-integro-differential equation has a novel and unique structure. In other words, the formulated structure for given fractional multi-integro-differential problem (1) includes one quantum derivative in the Caputo sense and also seven quantum integrals of the Riemann–Liouville type. This combined boundary problem covers many different special cases of various nonlinear integro-differential equations. Therefore, we emphasize that this kind of the Caputo quantum multi-integro-differential problem has not been investigated in the literature so far. In this direction, we apply well-known analytical techniques to derive desired criteria which guarantee the existence of solutions for the proposed Caputo quantum multi-integro-differential boundary problem (1).

The organization of the contents of the current manuscript is as follows. In the next section, some required notions in the context of the quantum calculus are assembled. Section 3 is devoted to establishing the main theorems in which the existence criteria can be obtained under some necessary conditions. In Sect. 4, numerical examples are considered to examine our theoretical and analytical findings by using the proposed methods.

2 Preliminaries

In this part of the present research manuscript, some required notions in the context of the quantum calculus are assembled. Let us assume that $q \in (0,1)$. For the given power function $(m_1 - m_2)^n$ with $n \in \mathbb{N}_0$, its q-analogue is defined by $(m_1 - m_2)^{(0)} = 1$ and

$$(m_1 - m_2)^{(n)} = \prod_{k=0}^{n-1} \left(m_1 - m_2 q^k\right),$$

such that $m_1, m_2 \in \mathbb{R}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ [52]. Here, the constant $n = \sigma$ is supposed to be an arbitrary real number. In this case, one can define the q-analogue of mentioned power function $(m_1 - m_2)^n$ in the q-fractional setting as follows:

$$(m_1 - m_2)^{(\sigma)} = m_1^\sigma \prod_{n=0}^{\infty} \frac{1 - (\frac{m_2}{m_1}) q^n}{1 - (\frac{m_2}{m_1}) q^{\sigma + n}} \tag{2}$$

for $m_1 \neq 0$. Notice that if we take $m_2 = 0$, then we reach an equality $m_1^{(\sigma)} = m_1^\sigma$ immediately [52]. For the given real number $m_1 \in \mathbb{R}$, a q-number $[m_1]_q$ is considered as

$$[m_1]_q = \frac{1 - q^{m_1}}{1 - q} = q^{m_1 - 1} + \cdots + q + 1.$$

The quantum Gamma function, or simply the q-Gamma function, is provided by the following rule:

$$\Gamma_q(z) = \frac{(1-q)^{(z-1)}}{(1-q)^{z-1}}, \tag{3}$$

Algorithm 1 The pseudo-code to compute different values of $\Gamma_q(\sigma)$

Require: $n, q \in (0,1)$, $\sigma \in \mathbb{R}\backslash\{0,-1,-2,-3,\ldots\}$

```
1: a ← 1
2: for k = 0 to n do
3:     a ← a((1 − q^(k+1))/(1 − q^(σ+k)))
4: end for
5: Γ_q(σ) ← a/(1 − q)^(σ−1)
```

Ensure: $\Gamma_q(\sigma)$

Algorithm 2 The pseudo-code to compute different values of $(\mathfrak{D}_q\varpi)(z)$

Require: $q \in (0,1)$, $\varpi(z)$, z

```
1: syms s
2: if z = 0 then
3:     h ← lim((ϖ(s) − ϖ(q ∗ s))/((1 − q)s), s, 0)
4: else
5:     h ← (ϖ(z) − ϖ(q ∗ z))/((1 − q) ∗ z)
6: end if
```

Ensure: $(\mathfrak{D}_q\varpi)(z)$

such that $z \in \mathbb{R}\setminus\{0,-1,-2,\ldots\}$ [19, 52]. It is notable that $\Gamma_q(z+1) = [z]_q\Gamma_q(z)$ is true [19]. In Algorithm 1, we provide a pseudo-code based on relations (2) and (3) to compute different values of the Gamma function in the quantum setting.

In the following, the quantum derivative of a real-valued continuous function ϖ is defined by

$$(\mathfrak{D}_q\varpi)(z) = \frac{\varpi(z) - \varpi(qz)}{(1-q)z} \tag{4}$$

and also $(\mathfrak{D}_q\varpi)(0) = \lim_{z\to 0}(\mathfrak{D}_q\varpi)(z)$ [22]. One can simply extend the quantum derivative of a function ϖ to arbitrary higher order by $(\mathfrak{D}_q^n\varpi)(z) = \mathfrak{D}_q(\mathfrak{D}_q^{n-1}\varpi)(z)$ for any $n \in \mathbb{N}$ [22]. It is obvious that $(\mathfrak{D}_q^0\varpi)(z) = \varpi(z)$. Similar to above, a pseudo-code based on (4) is provided to compute the quantum derivative of a function ϖ in Algorithm 2.

The quantum integral of a real-valued continuous function ϖ defined on $[0, m_2]$ is formulated by

$$(\mathcal{I}_q\varpi)(z) = \int_0^z \varpi(r)\,\mathrm{d}_q r = z(1-q)\sum_{k=0}^{\infty}\varpi\left(zq^k\right)q^k, \quad z \in [0, m_2], \tag{5}$$

provided that the series is absolutely convergent [22]. Similar to a quantum derivative, we can extend the quantum integral of a function ϖ to arbitrary higher order by iterative rule $(\mathcal{I}_q^n\varpi)(z) = \mathcal{I}_q(\mathcal{I}_q^{n-1}\varpi)(z)$ for all $n \geq 1$ [22]. In addition, it is evident that $(\mathcal{I}_q^0\varpi)(z) = \varpi(z)$. Note that a pseudo-code based on (5) is provided to compute the quantum integral of a function ϖ in Algorithm 3. At this moment, let us assume that $m_1 \in [0, m_2]$. In this case,

Algorithm 3 The pseudo-code to compute different values of $(\mathcal{I}_q^{\sigma}\varpi)(z)$

Require: $q \in (0,1)$, σ, n, $\varpi(z)$, z

$sum \leftarrow 0$
for $k = 0$ to n **do**
  $pf \leftarrow (1-q^{k+1})^{\sigma-1}$
  $sum \leftarrow sum + pf * q^k * \varpi(z * q^k)$
end for
$h \leftarrow (z^{\sigma} * (1-q) * sum)/(\Gamma_q(z))$

Ensure: $(\mathcal{I}_q^{\sigma}\varpi)(z)$

Algorithm 4 The pseudo-code to compute different values of $\int_{m_1}^{m_2}\varpi(r)\,\mathrm{d}_q r$

Require: $q \in (0,1)$, σ, n, $\varpi(z)$, $m1$, $m2$

$sum \leftarrow 0$
for $k = 0 : n$ **do**
  $sum \leftarrow sum + q^k * (m2 * \varpi(m2 * q^k) - m1 * \varpi(m1 * q^k))$
end for
$h \leftarrow (1-q) * sum$

Ensure: $\int_{m_1}^{m_2}\varpi(r)\,\mathrm{d}_q r$

the quantum integral of the function ϖ from m_1 to m_2 is defined as

$$\begin{aligned}\int_{m_1}^{m_2}\varpi(r)\,\mathrm{d}_q r &= \mathcal{I}_q\varpi(m_2) - \mathcal{I}_q\varpi(m_1)\\ &= \int_0^{m_2}\varpi(r)\,\mathrm{d}_q r - \int_0^{m_1}\varpi(r)\,\mathrm{d}_q r\\ &= (1-q)\sum_{k=0}^{\infty}\bigl[m_2\varpi\bigl(m_2q^k\bigr) - m_1\varpi\bigl(m_1q^k\bigr)\bigr]q^k \qquad (6)\end{aligned}$$

if the right-hand side series has a finite value [22]. A pseudo-code based on (6) is provided to compute the quantum integral of a function ϖ from m_1 to m_2 in Algorithm 4.

Notice that if the function ϖ is supposed to be continuous at the point $z = 0$, then we have $(\mathcal{I}_q\mathfrak{D}_q\varpi)(z) = \varpi(z) - \varpi(0)$ [22]. Moreover, the equality $(\mathfrak{D}_q\mathcal{I}_q\varpi)(z) = \varpi(z)$ holds for each z. At this point, consider the real number $\sigma \geq 0$ so that $n-1 < \sigma < n$, i.e., $n = [\sigma] + 1$. The Riemann–Liouville quantum integral for the given function $\varpi \in \mathcal{C}_{\mathbb{R}}([0,+\infty))$ is introduced by

$$^{\mathcal{RL}}\mathcal{I}_q^{\sigma}\varpi(z) = \frac{1}{\Gamma_q(\sigma)}\int_0^z (z-qr)^{(\sigma-1)}\varpi(r)\,\mathrm{d}_q r, \quad \sigma > 0,$$

whenever the existing integral has finite value and $^{\mathcal{RL}}\mathcal{I}_q^{0}\varpi(z) = \varpi(z)$ [53, 54]. Further, the semigroup property for this q-operator is valid, and so we have $^{\mathcal{RL}}\mathcal{I}_q^{\sigma_1}(^{\mathcal{RL}}\mathcal{I}_q^{\sigma_2}\varpi)(z) = {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma_1+\sigma_2}\varpi(z)$ for $\sigma_1, \sigma_2 \geq 0$ [55]. For $\theta \in (-1,\infty)$, the following property is valid:

$$^{\mathcal{RL}}\mathcal{I}_q^{\sigma}z^{\theta} = \frac{\Gamma_q(\theta+1)}{\Gamma_q(\theta+\sigma+1)}z^{\theta+\sigma}, \quad z > 0.$$

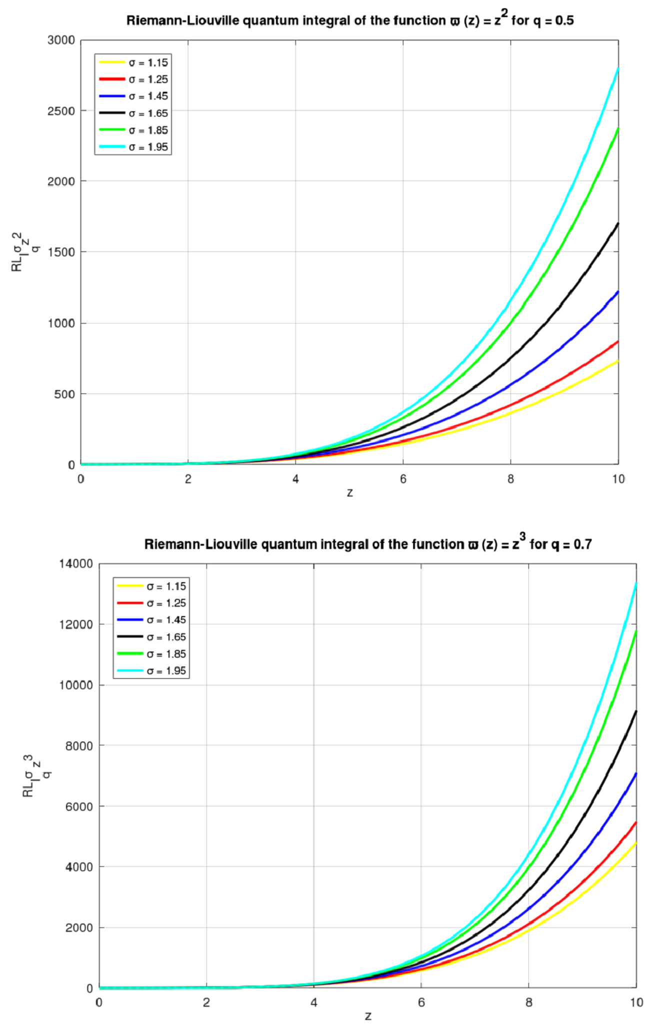

Figure 1 The Riemann–Liouville quantum integral of $\varpi(z) = z^2, z^3$ for $q = 0.5, 0.7$

It is evident that if we take $\theta = 0$, then ${}^{\mathcal{RL}}\mathcal{I}_q^\sigma 1(z) = \frac{1}{\Gamma_q(\sigma+1)} z^\sigma$ for any $z > 0$. In the sequel, the Caputo quantum derivative for the given function $\varpi \in \mathcal{C}_{\mathbb{R}}^{(n)}([0,+\infty))$ is provided by

$$ {}^{\mathcal{C}}\mathfrak{D}_q^\sigma \varpi(z) = \frac{1}{\Gamma_q(n-\sigma)} \int_0^z (z-qr)^{(n-\sigma-1)} \mathfrak{D}_q^n \varpi(r)\, \mathrm{d}_q r $$

whenever the integral is finite-valued [53, 54]. Notice that the following property is valid:

$$ {}^{\mathcal{C}}\mathfrak{D}_q^\sigma z^\theta = \frac{\Gamma_q(\theta+1)}{\Gamma_q(\theta-\sigma+1)} z^{\theta-\sigma}, \quad z > 0. $$

It is evident that ${}^{\mathcal{C}}\mathfrak{D}_q^\sigma 1(z) = 0$ for any $z > 0$. In Figs. 1 and 2, the dynamical behavior of the Riemann–Liouville fractional quantum integral and the Caputo fractional quantum derivative can be observed on two given functions $\varpi(z) = z^2$ and $\varpi(z) = z^3$ for $q = 0.5$ and $q = 0.7$, respectively.

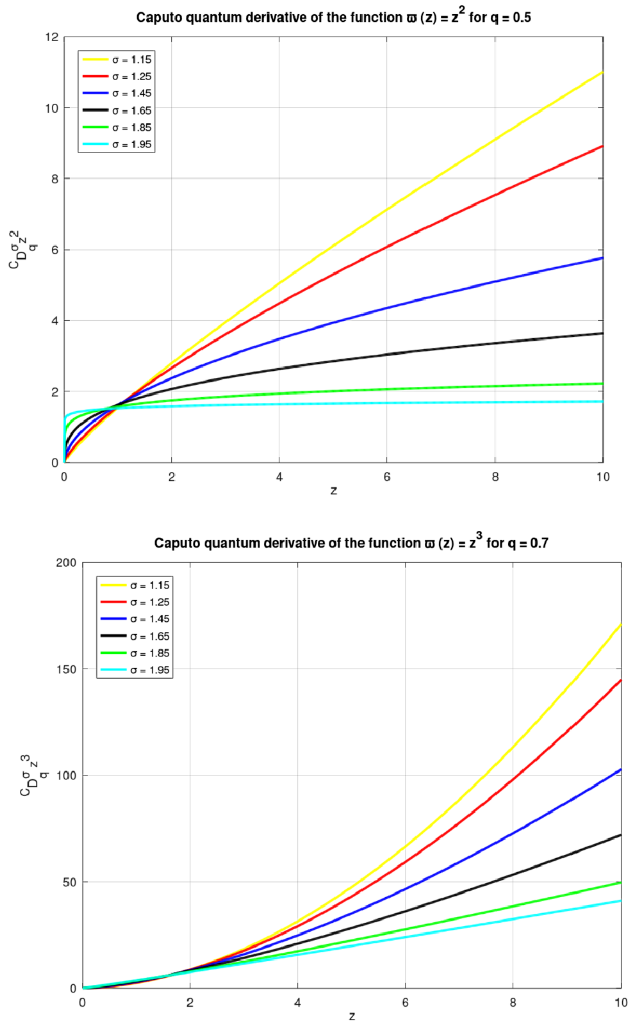

Figure 2 The Caputo quantum derivative of $\varpi(z) = z^2, z^3$ for $q = 0.5, 0.7$

Lemma 2.1 ([56]) *Let* $n-1<\sigma<n$. *Then,*

$$\big({}^{\mathcal{RL}}\mathcal{I}_q^{\sigma}\,{}^{\mathcal{C}}\mathfrak{D}_q^{\sigma}\varpi\big)(z)=\varpi(z)-\sum_{k=0}^{n-1}\frac{z^k}{\Gamma_q(k+1)}\big(\mathfrak{D}_q^k\varpi\big)(0).$$

Due to the latter lemma, the general solution for the given fractional quantum differential equation ${}^{\mathcal{C}}\mathfrak{D}_q^{\sigma}\varpi(z)=0$ is obtained by $\varpi(z)=\tilde{\alpha}_0+\tilde{\alpha}_1 z+\tilde{\alpha}_2 z^2+\cdots+\tilde{\alpha}_{n-1}z^{n-1}$ where $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_{n-1}$ are arbitrary real numbers and $n=[\sigma]+1$ [56]. Note that for every continuous function ϖ, by Lemma 2.1, we have

$$\big({}^{\mathcal{RL}}\mathcal{I}_q^{\sigma}\,{}^{\mathcal{C}}\mathfrak{D}_q^{\sigma}\varpi\big)(z)=\varpi(z)+\tilde{\alpha}_0+\tilde{\alpha}_1 z+\tilde{\alpha}_2 z^2+\cdots+\tilde{\alpha}_{n-1}z^{n-1},$$

where $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_{n-1}$ are constants which belong to $\mathbb{R}$ and $n=[\sigma]+1$ [56]. In what follows, some required fixed point theorems related to the proposed boundary problem are recalled.

Theorem 2.2 (Krasnoselskii's fixed point theorem, [57]) *Let $\mathbb{E}$ be a closed, convex, bounded, and nonempty subset of a Banach space $\mathfrak{W}$. Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two operators mapping $\mathbb{E}$ into $\mathfrak{W}$ so that the following statements are valid*:

(i1) $\mathbb{A}_1\varpi_1 + \mathbb{A}_2\varpi_2 \in \mathbb{E}$, *where* $\varpi_1, \varpi_2 \in \mathbb{E}$;

(i2) $\mathbb{A}_1$ *is a contraction*;

(i3) $\mathbb{A}_2$ *is a continuous and compact operator.*

Then there is an element $\varpi^* \in \mathbb{E}$ such that $\varpi^* = \mathbb{A}_1\varpi^* + \mathbb{A}_2\varpi^*$.

Theorem 2.3 (Nonlinear alternative for single-valued maps, [58]) *Let $\mathfrak{W}$ be a Banach space, $\mathbb{M}$ a convex and closed subset of $\mathfrak{W}$, and $\mathbb{O}$ an open subset of $\mathbb{M}$ and $0 \in \mathbb{O}$. Moreover, let $\mathbb{A} : \overline{\mathbb{O}} \to \mathbb{M}$ be a continuous and compact operator* (*that is, $\mathbb{A}(\overline{\mathbb{O}})$ is a relatively compact subset of $\mathbb{M}$*). *Then either*

(ii1) $\mathbb{A}$ *has a fixed point in* $\overline{\mathbb{O}}$; *or*

(ii2) *there exists an element $\varpi^* \in \partial\mathbb{O}$* (*as the boundary of the set $\mathbb{O}$ in $\mathbb{M}$*) *and $\hat{c} \in (0,1)$ with $\varpi^* = \hat{c}\mathbb{A}(\varpi^*)$.*

Theorem 2.4 (Banach fixed point theorem, [59]) *Let $\mathfrak{W}$ be a Banach space. Assume that $\mathbb{E} \subset \mathfrak{W}$ is closed and $\mathbb{A} : \mathbb{E} \to \mathbb{E}$ is a contraction. Then $\mathbb{A}$ is an operator having a fixed point in $\mathbb{E}$.*

3 Main results

Let $\mathfrak{W} = \mathcal{C}_{\mathbb{R}}([0,1])$ be the space of all real-valued continuous functions on $[0,1]$. One can simply verify that the set $\mathfrak{W}$ will be a Banach space if we define the sup norm $\|\varpi\|_{\mathfrak{W}} = \sup_{z\in[0,1]} |\varpi(z)|$ for all members $\varpi \in \mathfrak{W}$. At this point, we first provide the following structural lemma which characterizes the construction of solutions for the equivalent quantum integral equation related to the proposed quantum multi-integro-differential problem (1).

Lemma 3.1 *Let $\Phi_* \in \mathfrak{W}$, $\sigma \in (1,2)$, $\delta_j^* \in (0,1)$, $\theta_i^* > 0$ for $j = 1,2$ and $i = 1,2,3$. Also, let η^*, μ^* be nonzero real positive constants and consider the following nonzero positive constant*:

$$\tilde{\Xi}_* := \frac{\mu^*}{\Gamma_q(\theta_1^* + 2)} + \frac{\mu^* + 1}{\Gamma_q(\theta_2^* + 2)} + \frac{\mu^* + 2}{\Gamma_q(\theta_3^* + 2)} \neq 0. \tag{7}$$

Then the function ϖ^* is a solution to the nonlinear Caputo quantum fractional problem

$$\begin{cases} (\eta^{*\mathcal{C}}\mathfrak{D}_q^{\sigma} - (\eta^* + 1)^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*} - (\eta^* + 2)^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*})\varpi(z) = \Phi_*(z), \\ \varpi(0) = 0, \\ \mu^{*\mathcal{RL}}\mathcal{I}_q^{\theta_1^*}\varpi(1) + (\mu^* + 1)^{\mathcal{RL}}\mathcal{I}_q^{\theta_2^*}\varpi(1) + (\mu^* + 2)^{\mathcal{RL}}\mathcal{I}_q^{\theta_3^*}\varpi(1) = 0 \end{cases} \tag{8}$$

if and only if ϖ^* is a solution to the fractional quantum integral equation

$$\begin{aligned} \varpi(z) &= \frac{\eta^* + 1}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma}\varpi(z) + \frac{\eta^* + 2}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma}\varpi(z) + \frac{1}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\sigma}\Phi_*(z) \\ &\quad + \frac{z}{\tilde{\Xi}_*}\Bigg[-\frac{\mu^*(\eta^* + 1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*}\varpi(1) - \frac{\mu^*(\eta^* + 2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*}\varpi(1) \\ &\quad - \frac{\mu^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\theta_1^*}\Phi_*(1) - \frac{(\mu^* + 1)(\eta^* + 1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*}\varpi(1) \end{aligned}$$

$$
\begin{aligned}
&- \frac{(\mu^*+1)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*} \varpi(1) \\
&- \frac{\mu^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\theta_2^*} \Phi_*(1) - \frac{(\mu^*+2)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*} \varpi(1) \\
&- \frac{(\mu^*+2)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*} \varpi(1) - \frac{\mu^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\theta_3^*} \Phi_*(1)\Bigg]. \qquad (9)
\end{aligned}
$$

Proof At first, we regard the given function ϖ^* as a solution for the Caputo quantum fractional problem (8). Then we get

$$
{}^{\mathcal{C}}\mathfrak{D}_q^{\sigma} \varpi^*(z) = \frac{\eta^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*} \varpi^*(z) + \frac{\eta^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*} \varpi^*(z) + \frac{1}{\eta^*} \Phi_*(z).
$$

Taking fractional quantum integral in the Riemann–Liouville sense of order σ on both sides of the latter equation, we reach

$$
\begin{aligned}
\varpi^*(z) = {} & \frac{\eta^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma} \varpi^*(z) + \frac{\eta^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma} \varpi^*(z) + \frac{1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma} \Phi_*(z) \\
& + \tilde{\alpha}_0 + \tilde{\alpha}_1 z, \qquad (10)
\end{aligned}
$$

where $\tilde{\alpha}_0, \tilde{\alpha}_1 \in \mathbb{R}$ are some constants that we need to find. It is immediately deduced that $\tilde{\alpha}_0 = 0$ by the first boundary condition and (10). On the other hand, by considering the properties of the Riemann–Liouville quantum integral, we have

$$
\begin{aligned}
{}^{\mathcal{RL}}\mathcal{I}_q^{\upsilon} \varpi^*(z) = {} & \frac{\eta^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\upsilon} \varpi^*(z) + \frac{\eta^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\upsilon} \varpi^*(z) \\
& + \frac{1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\upsilon} \Phi_*(z) + \tilde{\alpha}_0 \frac{z^{\upsilon}}{\Gamma_q(\upsilon+1)} + \tilde{\alpha}_1 \frac{z^{\upsilon+1}}{\Gamma_q(\upsilon+2)}
\end{aligned}
$$

for $\upsilon \in \{\theta_1^*, \theta_2^*, \theta_3^*\}$. Then the second boundary condition (8) implies

$$
\begin{aligned}
\tilde{\alpha}_1 = {} & \frac{1}{\tilde{\Xi}_*}\Bigg[- \frac{\mu^*(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*} \varpi^*(1) \\
& - \frac{\mu^*(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*} \varpi^*(1) - \frac{\mu^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\theta_1^*} \Phi_*(1) \\
& - \frac{(\mu^*+1)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*} \varpi^*(1) - \frac{(\mu^*+1)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*} \varpi^*(1) \\
& - \frac{\mu^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\theta_2^*} \Phi_*(1) - \frac{(\mu^*+2)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*} \varpi^*(1) \\
& - \frac{(\mu^*+2)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*} \varpi^*(1) - \frac{\mu^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\sigma+\theta_3^*} \Phi_*(1)\Bigg],
\end{aligned}
$$

where $\tilde{\Xi}_* \neq 0$ is provided by (7). Eventually, we substitute both obtained values of $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ into (10). In this case, we observe that the function ϖ^* satisfies the quantum integral equation (9), and so ϖ^* is a solution for the mentioned integral equation. In the opposite direction, it is simple to confirm that ϖ^* is a solution for the given nonlinear Caputo quantum fractional boundary problem (8) whenever ϖ^* is regarded as a solution for the quantum integral equation (9). This completes the proof. □

Notation 3.2 From here onwards, for the sake of convenience in writing and computation, we consider the compact notations $\hat{h}_*(z,\varpi(z)) = \hat{h}_*(z)$ and $\hat{f}_*(z,\varpi(z)) = \hat{f}_*(z)$.

In the light of Lemma 3.1 and in relation to the proposed nonlinear Caputo fractional quantum multi-integro-differential equation (1), we construct an operator $\mathbb{A} : \mathfrak{W} \to \mathfrak{W}$ as follows:

$$\begin{aligned}
(\mathbb{A}\varpi)(z) &= \frac{\eta^*+1}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma}\varpi(z) + \frac{\eta^*+2}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma}\varpi(z) \\
&\quad + \frac{\lambda_1^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma}\hat{h}_*(z) + \frac{\lambda_2^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma}\hat{f}_*(z) \\
&\quad + \frac{z}{\tilde{\Xi}_*}\bigg[-\frac{\mu^*(\eta^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*}\varpi(1) - \frac{\mu^*(\eta^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*}\varpi(1) \\
&\quad - \frac{\lambda_1^*\mu^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*}\hat{h}_*(1) - \frac{\lambda_2^*\mu^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_1^*}\hat{f}_*(1) \\
&\quad - \frac{(\mu^*+1)(\eta^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*}\varpi(1) \\
&\quad - \frac{(\mu^*+1)(\eta^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*}\varpi(1) - \frac{\lambda_1^*(\mu^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*}\hat{h}_*(1) \\
&\quad - \frac{\lambda_2^*(\mu^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_2^*}\hat{f}_*(1) - \frac{(\mu^*+2)(\eta^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*}\varpi(1) \\
&\quad - \frac{(\mu^*+2)(\eta^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*}\varpi(1) - \frac{\lambda_1^*(\mu^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*}\hat{h}_*(1) \\
&\quad - \frac{\lambda_2^*(\mu^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_3^*}\hat{f}_*(1)\bigg]
\end{aligned} \tag{11}$$

for each $\varpi \in \mathfrak{W}$ and $z \in [0,1]$. Consider the following constants which we will utilize these nonzero constants later:

$$\begin{aligned}
\tilde{\Omega}_*^{(1)} &:= \frac{\eta^*+1}{\eta^*\Gamma_q(\delta_1^*+\sigma+1)} + \frac{\eta^*+2}{\eta^*\Gamma_q(\delta_2^*+\sigma+1)} + \frac{\mu^*(\eta^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\delta_1^*+\sigma+\theta_1^*+1)} \\
&\quad + \frac{\mu^*(\eta^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\delta_2^*+\sigma+\theta_1^*+1)} + \frac{(\mu^*+1)(\eta^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\delta_1^*+\sigma+\theta_2^*+1)} \\
&\quad + \frac{(\mu^*+1)(\eta^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\delta_2^*+\sigma+\theta_2^*+1)} \\
&\quad + \frac{(\mu^*+2)(\eta^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\delta_1^*+\sigma+\theta_3^*+1)} + \frac{(\mu^*+2)(\eta^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\delta_2^*+\sigma+\theta_3^*+1)}, \\
\tilde{\Omega}_*^{(2)} &:= \frac{\lambda_1^*}{\eta^*\Gamma_q(\gamma_1^*+\sigma+1)} + \frac{\lambda_1^*\mu^*}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_1^*+\sigma+\theta_1^*+1)} \\
&\quad + \frac{\lambda_1^*(\mu^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_1^*+\sigma+\theta_2^*+1)} + \frac{\lambda_1^*(\mu^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_1^*+\sigma+\theta_3^*+1)}, \\
\tilde{\Omega}_*^{(3)} &:= \frac{\lambda_2^*}{\eta^*\Gamma_q(\gamma_2^*+\sigma+1)} + \frac{\lambda_2^*\mu^*}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_1^*+1)} \\
&\quad + \frac{\lambda_2^*(\mu^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_2^*+1)} + \frac{\lambda_2^*(\mu^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_3^*+1)}.
\end{aligned} \tag{12}$$

Now we are in the position to derive required existence criteria for the given nonlinear Caputo fractional quantum multi-integro-differential problem (1). To begin this process, we first invoke the well-known Krasnoselskii's fixed point theorem.

Theorem 3.3 *Assume that two single-valued operators $\hat{h}_* : [0,1] \times \mathfrak{W} \to \mathfrak{W}$ and $\hat{f}_* : [0,1] \times \mathfrak{W} \to \mathfrak{W}$ are continuous and also satisfy the following hypotheses:*

(HK1) *there exists a constant $\hat{b}^* > 0$ such that for each $\varpi_1, \varpi_2 \in \mathfrak{W}$ and for any $z \in [0,1]$, the inequality $|\hat{h}_*(z,\varpi_1) - \hat{h}_*(z,\varpi_2)| \le \hat{b}^* |\varpi_1 - \varpi_2|$ holds;*

(HK2) *there is a continuous function Υ on $[0,1]$ such that the inequality $|\hat{f}_*(z,\varpi)| \le \Upsilon(z)$ is valid for any $z \in [0,1]$ and for every $\varpi \in \mathfrak{W}$.*

Then the given nonlinear Caputo fractional quantum multi-integro-differential problem (1) has at least one solution on $[0,1]$ whenever $\tilde{\Omega}_*^{(1)} + \hat{b}^* \tilde{\Omega}_*^{(2)} < 1$, where $\tilde{\Omega}_*^{(1)}$ and $\tilde{\Omega}_*^{(2)}$ are introduced by (12).

Proof We take $\|\Upsilon\| = \sup_{z\in[0,1]} |\Upsilon(z)|$ and construct $\mathbb{B}_{\tilde{r}} := \{\varpi \in \mathfrak{W} : \|\varpi\| \le \tilde{r}\}$ with

$$\tilde{r} \ge \frac{\|\Upsilon\| \tilde{\Omega}_*^{(3)} + \hat{H}_* \tilde{\Omega}_*^{(2)}}{1 - (\tilde{\Omega}_*^{(1)} + \hat{b}^* \tilde{\Omega}_*^{(2)})},$$

where $\hat{H}_* := \sup_{z\in[0,1]} |\hat{h}_*(z,0)|$ and $\tilde{\Omega}_*^{(1)}$, $\tilde{\Omega}_*^{(2)}$, and $\tilde{\Omega}_*^{(3)}$ are introduced by (12). As we know, the so-defined ball $\mathbb{B}_{\tilde{r}}$ is a bounded, convex, closed, and nonempty subset of the Banach space $\mathfrak{W}$. In addition, we consider an operator $\mathbb{A} : \mathfrak{W} \to \mathfrak{W}$ as in (11). In the light of Lemma 3.1, it is natural that the fixed point of $\mathbb{A}$ is considered as a solution for the nonlinear Caputo fractional quantum multi-integro-differential problem (1). To begin the proof, for any $z \in [0,1]$, we construct two operators $\mathbb{A}_1$ and $\mathbb{A}_2$ from $\mathbb{B}_{\tilde{r}}$ to $\mathfrak{W}$ as follows:

$$\begin{aligned}
\mathbb{A}_1\varpi(z) = {}& \frac{\eta^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma} \varpi(z) + \frac{\eta^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma} \varpi(z) + \frac{\lambda_1^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma} \hat{h}_*(z) \\
& + \frac{z}{\tilde{\Xi}_*} \biggl[-\frac{\mu^*(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*} \varpi(1) - \frac{\mu^*(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*} \varpi(1) \\
& - \frac{\lambda_1^*\mu^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*} \hat{h}_*(1) - \frac{(\mu^*+1)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*} \varpi(1) \\
& - \frac{(\mu^*+1)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*} \varpi(1) - \frac{\lambda_1^*(\mu^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*} \hat{h}_*(1) \\
& - \frac{(\mu^*+2)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*} \varpi(1) - \frac{(\mu^*+2)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*} \varpi(1) \\
& - \frac{\lambda_1^*(\mu^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*} \hat{h}_*(1) \biggr]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{A}_2\varpi(z) = {}& \frac{\lambda_2^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma} \hat{f}_*(z) + \frac{z}{\tilde{\Xi}_*} \biggl[-\frac{\lambda_2^*\mu^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_1^*} \hat{f}_*(1) \\
& - \frac{\lambda_2^*(\mu^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_2^*} \hat{f}_*(1) - \frac{\lambda_2^*(\mu^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_3^*} \hat{f}_*(1) \biggr].
\end{aligned}$$

At first, due to hypothesis (HK1), we know that for any $z \in [0,1]$,

$$\left|\hat{h}_*(z)\right| = \left|\hat{h}_*\big(z,\varpi(z)\big)\right| \le \left(\left|\hat{h}_*\big(z,\varpi(z)\big) - \hat{h}_*(z,0)\right| + \left|\hat{h}_*(z,0)\right|\right) \le \hat{b}^*\left|\varpi(z)\right| + \hat{H}_*.$$

Also, hypothesis (HK2) implies that $|\hat{f}_*(z)| = |\hat{f}_*(z,\varpi)| \le \Upsilon(z)$ for $z \in [0,1]$. Then for any elements $\varpi_1, \varpi_2 \in \mathbb{B}_{\tilde{r}}$, one can write

$$\begin{aligned}
&\left|\mathbb{A}_1\varpi_1(z) + \mathbb{A}_2\varpi_2(z)\right| \\
&\le \frac{\eta^*+1}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma}\|\varpi_1\| + \frac{\eta^*+2}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma}\|\varpi_1\| + \frac{\lambda_1^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma}\big(\hat{b}^*\|\varpi_1\| + \hat{H}_*\big) \\
&\quad + \frac{1}{\tilde{\Xi}_*}\Bigg[\frac{\mu^*(\eta^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*}\|\varpi_1\| + \frac{\mu^*(\eta^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*}\|\varpi_1\| \\
&\quad + \frac{\lambda_1^*\mu^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*}\big(\hat{b}^*\|\varpi_1\| + \hat{H}_*\big) \\
&\quad + \frac{(\mu^*+1)(\eta^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*}\|\varpi_1\| \\
&\quad + \frac{(\mu^*+1)(\eta^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*}\|\varpi_1\| \\
&\quad + \frac{\lambda_1^*(\mu^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*}\big(\hat{b}^*\|\varpi_1\| + \hat{H}_*\big) \\
&\quad + \frac{(\mu^*+2)(\eta^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*}\|\varpi_1\| + \frac{(\mu^*+2)(\eta^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*}\|\varpi_1\| \\
&\quad + \frac{\lambda_1^*(\mu^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*}\big(\hat{b}^*\|\varpi_1\| + \hat{H}_*\big)\Bigg] \\
&\quad + \frac{\lambda_2^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma}\|\Upsilon\| + \frac{1}{\tilde{\Xi}_*}\Bigg[\frac{\lambda_2^*\mu^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_1^*}\|\Upsilon\| \\
&\quad + \frac{\lambda_2^*(\mu^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_2^*}\|\Upsilon\| + \frac{\lambda_2^*(\mu^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_3^*}\|\Upsilon\|\Bigg] \\
&= \big(\tilde{\Omega}_*^{(1)} + \hat{b}^*\tilde{\Omega}_*^{(2)}\big)\|\varpi_1\| + \tilde{\Omega}_*^{(3)}\|\Upsilon\| + \tilde{\Omega}_*^{(2)}\hat{H}_* \\
&\le \big(\tilde{\Omega}_*^{(1)} + \hat{b}^*\tilde{\Omega}_*^{(2)}\big)\tilde{r} + \tilde{\Omega}_*^{(3)}\|\Upsilon\| + \tilde{\Omega}_*^{(2)}\hat{H}_* \le \tilde{r}.
\end{aligned}$$

The latter inequality demonstrates that $\|\mathbb{A}_1\varpi_1 + \mathbb{A}_2\varpi_2\| \le \tilde{r}$ and thus $\mathbb{A}_1\varpi_1 + \mathbb{A}_2\varpi_2 \in \mathbb{B}_{\tilde{r}}$ for each $\varpi_1, \varpi_2 \in \mathbb{B}_{\tilde{r}}$. This also means that condition (i1) of Theorem 2.2 holds for both operators $\mathbb{A}_1$ and $\mathbb{A}_2$. At this point, we proceed to verify that $\mathbb{A}_1$ is a contraction. For arbitrary elements $\varpi_1, \varpi_2 \in \mathbb{B}_{\tilde{r}}$ and $z \in [0,1]$, and in view of hypothesis (HK1), we have

$$\begin{aligned}
&\left|\mathbb{A}_1\varpi_1(z) - \mathbb{A}_1\varpi_2(z)\right| \\
&\le \frac{\eta^*+1}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma}\left|\varpi_1(z) - \varpi_2(z)\right| \\
&\quad + \frac{\eta^*+2}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma}\left|\varpi_1(z) - \varpi_2(z)\right| + \frac{\lambda_1^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma}\hat{b}^*\left|\varpi_1(z) - \varpi_2(z)\right| \\
&\quad + \frac{1}{\tilde{\Xi}_*}\Bigg[\frac{\mu^*(\eta^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*}\left|\varpi_1(z) - \varpi_2(z)\right| + \frac{\mu^*(\eta^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*}
\end{aligned}$$

$$
\begin{aligned}
&\times\big|\varpi_1(z)-\varpi_2(z)\big| \\
&+\frac{\lambda_1^*\mu^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*}\hat{b}^*\big|\varpi_1(z)-\varpi_2(z)\big|+\frac{(\mu^*+1)(\eta^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*} \\
&\times\big|\varpi_1(z)-\varpi_2(z)\big| \\
&+\frac{(\mu^*+1)(\eta^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*}\big|\varpi_1(z)-\varpi_2(z)\big| \\
&+\frac{\lambda_1^*(\mu^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*}\hat{b}^*\big|\varpi_1(z)-\varpi_2(z)\big| \\
&+\frac{(\mu^*+2)(\eta^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*}\big|\varpi_1(z)-\varpi_2(z)\big| \\
&+\frac{(\mu^*+2)(\eta^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*}\big|\varpi_1(z)-\varpi_2(z)\big| \\
&+\frac{\lambda_1^*(\mu^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*}\hat{b}^*\big|\varpi_1(z)-\varpi_2(z)\big|\bigg] \\
=&\big(\tilde{\Omega}_*^{(1)}+\hat{b}^*\tilde{\Omega}_*^{(2)}\big)\|\varpi_1-\varpi_2\|.
\end{aligned}
$$

In the light of the given hypothesis, we know that $\tilde{\Omega}_*^{(1)}+\hat{b}^*\tilde{\Omega}_*^{(2)}<1$. Thus we conclude that $\mathbb{A}_1$ is a contraction and so condition (i2) of Theorem 2.2 is valid for the operator $\mathbb{A}_1$.

In the sequel, we intend to verify the continuity of $\mathbb{A}_2$. To reach this goal, let us assume that $\{\varpi_n\}_{n\ge1}$ is a convergent sequence belonging to the given ball $\mathbb{B}_{\tilde{r}}$ such that ϖ_n approaches ϖ. Then for any $z\in[0,1]$, we obtain

$$
\begin{aligned}
\big|\mathbb{A}_2\varpi_n(z)-\mathbb{A}_2\varpi(z)\big| \le{}& \frac{\lambda_2^*}{\eta^*\Gamma_q(\gamma_2^*+\sigma+1)}\big|\hat{f}_*\big(z,\varpi_n(z)\big)-\hat{f}_*\big(z,\varpi(z)\big)\big| \\
&+\frac{\lambda_2^*\mu^*}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_1^*+1)}\big|\hat{f}_*\big(z,\varpi_n(z)\big)-\hat{f}_*\big(z,\varpi(z)\big)\big| \\
&+\frac{\lambda_2^*(\mu^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_2^*+1)}\big|\hat{f}_*\big(z,\varpi_n(z)\big)-\hat{f}_*\big(z,\varpi(z)\big)\big| \\
&+\frac{\lambda_2^*(\mu^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_3^*+1)}\big|\hat{f}_*\big(z,\varpi_n(z)\big)-\hat{f}_*\big(z,\varpi(z)\big)\big|.
\end{aligned}
$$

But by the hypothesis, we know that the function $\hat{f}_*$ is continuous on $[0,1]\times\mathfrak{W}$, thus we find that $\|\mathbb{A}_2\varpi_n-\mathbb{A}_2\varpi\|$ approaches zero whenever $\varpi_n\to\varpi$. Therefore, we conclude that $\mathbb{A}_2$ is a continuous operator defined on $\mathbb{B}_{\tilde{r}}$. In the subsequent stage, we claim that the operator $\mathbb{A}_2$ is compact. To confirm this claim, we first check the uniform boundedness of $\mathbb{A}_2$. For given member $\varpi\in\mathbb{B}_{\tilde{r}}$ and $z\in[0,1]$, we may write

$$
\begin{aligned}
\big|\mathbb{A}_2\varpi(z)\big| \le{}& \frac{\lambda_2^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma}\big|\hat{f}_*(z)\big|+\frac{z}{\tilde{\Xi}_*}\bigg[\frac{\lambda_2^*\mu^*}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_1^*}\big|\hat{f}_*(1)\big| \\
&+\frac{\lambda_2^*(\mu^*+1)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_2^*}\big|\hat{f}_*(1)\big|+\frac{\lambda_2^*(\mu^*+2)}{\eta^*}\,{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_3^*}\big|\hat{f}_*(1)\big|\bigg] \\
\le{}& \|\Upsilon\|\bigg[\frac{\lambda_2^*}{\eta^*\Gamma_q(\gamma_2^*+\sigma+1)}+\frac{\lambda_2^*\mu^*}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_1^*+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda_2^*(\mu^*+1)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_2^*+1)} \\
&+ \frac{\lambda_2^*(\mu^*+2)}{\eta^*\tilde{\Xi}_*\Gamma_q(\gamma_2^*+\sigma+\theta_3^*+1)}\Bigg] \\
= {}& \tilde{\Omega}_*^{(3)}\|\Upsilon\|,
\end{aligned}
$$

which illustrates that $\|\mathbb{A}_2\varpi\| \le \tilde{\Omega}_*^{(3)}\|\Upsilon\|$ and $\mathbb{A}_2$ is uniformly bounded. Besides, we establish that $\mathbb{A}_2$ is an equicontinuous operator. To establish this result, we consider two elements $z, x \in [0,1]$ such that $z < x$. In fact, we shall verify that bounded sets are mapped to equicontinuous sets by the operator $\mathbb{A}_2$. Hence for every $\varpi \in \mathbb{B}_{\tilde{r}}$, we get

$$
\begin{aligned}
&\big|\mathbb{A}_2\varpi(x) - \mathbb{A}_2\varpi(z)\big| \\
&\quad\le \frac{\lambda_2^*}{\eta^*\Gamma_q(\gamma_2^*+\sigma)}\int_0^z \big[(x-qr)^{(\gamma_2^*+\sigma-1)} - (z-qr)^{(\gamma_2^*+\sigma-1)}\big]\big|\hat{f}_*\big(r,\varpi(r)\big)\big|\,\mathrm{d}_q r \\
&\qquad + \frac{\lambda_2^*}{\eta^*\Gamma_q(\gamma_2^*+\sigma)}\int_z^x (x-qr)^{(\gamma_2^*+\sigma-1)}\big|\hat{f}_*\big(r,\varpi(r)\big)\big|\,\mathrm{d}_q r \\
&\qquad + \frac{(x-z)}{\tilde{\Xi}_*}\Bigg[\frac{\lambda_2^*\mu^*}{\eta^*\Gamma_q(\gamma_2^*+\sigma+\theta_1^*)}\int_0^1 (1-qr)^{(\gamma_2^*+\sigma+\theta_1^*-1)}\big|\hat{f}_*\big(r,\varpi(r)\big)\big|\,\mathrm{d}_q r \\
&\qquad + \frac{\lambda_2^*(\mu^*+1)}{\eta^*\Gamma_q(\gamma_2^*+\sigma+\theta_2^*)}\int_0^1 (1-qr)^{(\gamma_2^*+\sigma+\theta_2^*-1)}\big|\hat{f}_*\big(r,\varpi(r)\big)\big|\,\mathrm{d}_q r \\
&\qquad + \frac{\lambda_2^*(\mu^*+2)}{\eta^*\Gamma_q(\gamma_2^*+\sigma+\theta_3^*)}\int_0^1 (1-qr)^{(\gamma_2^*+\sigma+\theta_3^*-1)}\big|\hat{f}_*\big(r,\varpi(r)\big)\big|\,\mathrm{d}_q r\Bigg].
\end{aligned}
$$

We find that the right-hand side of the obtained inequality is not dependent on $\varpi \in \mathbb{B}_{\tilde{r}}$ and also approaches 0 when z tends to x. In consequence, we realize that $\mathbb{A}_2$ is equicontinuous. Hence, it is concluded that $\mathbb{A}_2$ is a relatively compact operator on $\varpi \in \mathbb{B}_{\tilde{r}}$ and thus the Arzelá–Ascoli theorem implies that $\mathbb{A}_2$ is completely continuous, and eventually $\mathbb{A}_2$ is a compact operator on the given ball $\varpi \in \mathbb{B}_{\tilde{r}}$. Therefore condition (i3) of Theorem 2.2 is valid for the operator $\mathbb{A}_2$. In consequence, all three hypotheses of Theorem 2.2 are valid for both single-valued operators $\mathbb{A}_1$ and $\mathbb{A}_2$. Therefore Theorem 2.2 implies that the given nonlinear Caputo fractional quantum multi-integro-differential problem (1) has at least one solution on the interval $[0,1]$, and so this completes the proof. □

Leray–Schauder nonlinear alternative theorem is another analytical tool by which we will be able to derive our desired existence criteria for the mentioned nonlinear Caputo fractional quantum multi-integro-differential problem (1).

Theorem 3.4 *Let the functions $\hat{h}_* : [0,1]\times\mathfrak{W} \to \mathfrak{W}$ and $\hat{f}_* : [0,1]\times\mathfrak{W} \to \mathfrak{W}$ be continuous and satisfy the following assumptions*:

(HK3) *there exist two functions $\Phi_1, \Phi_2 \in \mathcal{C}_{\mathbb{R}^+}([0,1])$ along with two continuous nondecreasing functions $\Psi_1, \Psi_2 : [0,\infty) \to (0,\infty)$ such that for any $(z,\varpi) \in [0,1]\times\mathfrak{W}$, we have*

$$
\big|\hat{h}_*(z,\varpi)\big| \le \Phi_1(z)\Psi_1\big(|\varpi|\big) \quad \textit{and} \quad \big|\hat{f}_*(z,\varpi)\big| \le \Phi_2(z)\Psi_2\big(|\varpi|\big);
$$

(HK4) *there exists a real constant* $\mathfrak{N} > 0$ *such that* $\tilde{\Omega}_*^{(1)} < 1$ *and*

$$\frac{(1 - \tilde{\Omega}_*^{(1)})\mathfrak{N}}{\tilde{\Omega}_*^{(2)} \|\Phi_1\| \Psi_1(\mathfrak{N}) + \tilde{\Omega}_*^{(3)} \|\Phi_2\| \Psi_2(\mathfrak{N})} > 1,$$

where $\tilde{\Omega}_*^{(1)}$, $\tilde{\Omega}_*^{(2)}$, *and* $\tilde{\Omega}_*^{(3)}$ *are given by* (12).

Then the nonlinear Caputo fractional quantum multi-integro-differential problem (1) *has at least one solution on* $[0,1]$.

Proof To reach the desired conclusion, we check all the hypotheses of Leray–Schauder nonlinear alternative (Theorem 2.3) in the subsequent steps. At first, we are going to show that the operator $\mathbb{A}$ defined by (11) maps bounded sets (i.e., balls) into bounded sets in $\mathfrak{W}$. For a positive real number $\tilde{R}$, construct a bounded ball $\mathbb{B}_{\tilde{R}} = \{\varpi \in \mathfrak{W} : \|\varpi\| \leq \tilde{R}\}$ in $\mathfrak{W}$. Then for any $z \in [0,1]$ and in view of hypothesis (HK3), we can write

$$\begin{aligned}
&\left|\mathbb{A}\varpi(z)\right| \\
&\quad\leq \frac{\eta^*+1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma} \|\varpi\| + \frac{\eta^*+2}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma} \|\varpi\| + \frac{\lambda_1^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma} \big(\|\Phi_1\| \Psi_1\big(\|\varpi\|\big)\big) \\
&\qquad + \frac{1}{\tilde{\Xi}_*} \biggl[\frac{\mu^*(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*} \|\varpi\| + \frac{\mu^*(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*} \|\varpi\| \\
&\qquad + \frac{\lambda_1^*\mu^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*} \big(\|\Phi_1\| \Psi_1\big(\|\varpi\|\big)\big) + \frac{(\mu^*+1)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*} \|\varpi\| \\
&\qquad + \frac{(\mu^*+1)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*} \|\varpi\| \\
&\qquad + \frac{\lambda_1^*(\mu^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*} \big(\|\Phi_1\| \Psi_1\big(\|\varpi\|\big)\big) \\
&\qquad + \frac{(\mu^*+2)(\eta^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*} \|\varpi\| + \frac{(\mu^*+2)(\eta^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*} \|\varpi\| \\
&\qquad + \frac{\lambda_1^*(\mu^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*} \big(\|\Phi_1\| \Psi_1\big(\|\varpi\|\big)\big) \biggr] \\
&\qquad + \frac{\lambda_2^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma} \big(\|\Phi_2\| \Psi_2\big(\|\varpi\|\big)\big) + \frac{1}{\tilde{\Xi}_*} \biggl[\frac{\lambda_2^*\mu^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_1^*} \big(\|\Phi_2\| \Psi_2\big(\|\varpi\|\big) \\
&\qquad + \frac{\lambda_2^*(\mu^*+1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_2^*} \big(\|\Phi_2\| \Psi_2\big(\|\varpi\|\big) \\
&\qquad + \frac{\lambda_2^*(\mu^*+2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_3^*} \big(\|\Phi_2\| \Psi_2\big(\|\varpi\|\big) \biggr] \\
&\quad= \tilde{\Omega}_*^{(1)} \|\varpi\| + \tilde{\Omega}_*^{(2)} \|\Phi_1\| \Psi_1\big(\|\varpi\|\big) + \tilde{\Omega}_*^{(3)} \|\Phi_2\| \Psi_2\big(\|\varpi\|\big).
\end{aligned}$$

Hence, the above inequality yields

$$\|\mathbb{A}\varpi\| \leq \tilde{\Omega}_*^{(1)} \tilde{R} + \tilde{\Omega}_*^{(2)} \|\Phi_1\| \Psi_1(\tilde{R}) + \tilde{\Omega}_*^{(3)} \|\Phi_2\| \Psi_2(\tilde{R}).$$

This indicates that the operator $\mathbb{A}$ is uniformly bounded. In the second stage, we proceed to verify that $\mathbb{A}$ maps bounded sets (i.e., balls) into equicontinuous subsets of $\mathfrak{W}$. To see this, take $z, x \in [0,1]$ with $z < x$ and $\varpi \in \mathbb{B}_{\tilde{R}}$. In this case, we get

$$\begin{aligned}
&\left|\mathbb{A}\varpi(x) - \mathbb{A}\varpi(z)\right| \\
&\quad \leq \frac{(\eta^* + 1)\tilde{R}}{\eta^*\Gamma_q(\delta_1^* + \sigma)} \\
&\qquad \times \left(\int_0^z \left[(x - qr)^{(\delta_1^*+\sigma-1)} - (z - qr)^{(\delta_1^*+\sigma-1)}\right] \mathrm{d}_q r + \int_z^x (x - qr)^{(\delta_1^*+\sigma-1)} \, \mathrm{d}_q r\right) \\
&\qquad + \frac{(\eta^* + 2)\tilde{R}}{\eta^*\Gamma_q(\delta_2^* + \sigma)} \\
&\qquad \times \left(\int_0^z \left[(x - qr)^{(\delta_2^*+\sigma-1)} - (z - qr)^{(\delta_2^*+\sigma-1)}\right] \mathrm{d}_q r + \int_z^x (x - qr)^{(\delta_2^*+\sigma-1)} \, \mathrm{d}_q r\right) \\
&\qquad + \frac{\lambda_1^* \|\Phi_1\| \Psi_1(\tilde{R})}{\eta^*\Gamma_q(\gamma_1^* + \sigma)} \\
&\qquad \times \left(\int_0^z \left[(x - qr)^{(\gamma_1^*+\sigma-1)} - (z - qr)^{(\gamma_1^*+\sigma-1)}\right] \mathrm{d}_q r + \int_z^x (x - qr)^{(\gamma_1^*+\sigma-1)} \, \mathrm{d}_q r\right) \\
&\qquad + \frac{(x - z)}{\tilde{\Xi}_*}\left[\frac{\mu^*(\eta^* + 1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*} \|\varpi(1)\| + \frac{\mu^*(\eta^* + 2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*} \|\varpi(1)\|\right. \\
&\qquad + \frac{\lambda_1^*\mu^*}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*} \|\hat{h}_*(1)\| + \frac{(\mu^* + 1)(\eta^* + 1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*} \|\varpi(1)\| \\
&\qquad + \frac{(\mu^* + 1)(\eta^* + 2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*} \|\varpi(1)\| + \frac{\lambda_1^*(\mu^* + 1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*} \|\hat{h}_*(1)\| \\
&\qquad + \frac{(\mu^* + 2)(\eta^* + 1)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*} \|\varpi(1)\| + \frac{(\mu^* + 2)(\eta^* + 2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*} \|\varpi(1)\| \\
&\qquad \left. + \frac{\lambda_1^*(\mu^* + 2)}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*} \|\hat{h}_*(1)\|\right] \\
&\qquad + \frac{\lambda_2^* \|\Phi_2\| \Psi_2(\tilde{R})}{\eta^*\Gamma_q(\gamma_2^* + \sigma)} \\
&\qquad \times \left(\int_0^z \left[(x - qr)^{(\gamma_2^*+\sigma-1)} - (z - qr)^{(\gamma_2^*+\sigma-1)}\right] \mathrm{d}_q r + \int_z^x (x - qr)^{(\gamma_2^*+\sigma-1)} \, \mathrm{d}_q r\right) \\
&\qquad + \frac{(x - z)\|\Phi_2\| \Psi_2(\tilde{r})}{\tilde{\Xi}_*}\left[\frac{\lambda_2^*\mu^*}{\eta^*\Gamma_q(\gamma_2^* + \sigma + \theta_1^*)} \int_0^1 (1 - qr)^{(\gamma_2^*+\sigma+\theta_1^*-1)} \, \mathrm{d}_q r\right. \\
&\qquad + \frac{\lambda_2^*(\mu^* + 1)}{\eta^*\Gamma_q(\gamma_2^* + \sigma + \theta_2^*)} \int_0^1 (1 - qr)^{(\gamma_2^*+\sigma+\theta_2^*-1)} \, \mathrm{d}_q r \\
&\qquad \left. + \frac{\lambda_2^*(\mu^* + 2)}{\eta^*\Gamma_q(\gamma_2^* + \sigma + \theta_3^*)} \int_0^1 (1 - qr)^{(\gamma_2^*+\sigma+\theta_3^*-1)} \, \mathrm{d}_q r\right].
\end{aligned}$$

We find that the right-hand side of the obtained inequality is not dependent on $\varpi \in \mathbb{B}_{\tilde{R}}$ and also approaches 0 when z tends to x. In consequence, $\mathbb{A}$ is equicontinuous, and hence we have confirmed the complete continuity of $\mathbb{A} : \mathfrak{W} \to \mathfrak{W}$ by the Arzelá–Ascoli theorem. Consequently, $\mathbb{A}$ is a compact operator.

Eventually, in order to finish checking all the assumptions of the Leray–Schauder nonlinear alternative (Theorem 2.3), it will be verified that the set of all obtained solutions of an operator equation $\varpi = \hat{c}(\mathbb{A}\varpi)$ is bounded for $\hat{c} \in [0,1]$. For this purpose, assume that ϖ^* is a solution of equation $\varpi^* = \hat{c}\mathbb{A}\varpi^*$ for $\hat{c} \in [0,1]$. Then by utilizing the strategy applied in the first stage, for any $z \in [0,1]$, we have

$$\|\varpi^*\| \le \tilde{\Omega}_*^{(1)}\|\varpi^*\| + \tilde{\Omega}_*^{(2)}\|\Phi_1\|\Psi_1(\|\varpi^*\|) + \tilde{\Omega}_*^{(3)}\|\Phi_2\|\Psi_2(\|\varpi^*\|).$$

In this case, we get

$$\frac{(1-\tilde{\Omega}_*^{(1)})\|\varpi^*\|}{\tilde{\Omega}_*^{(2)}\|\Phi_1\|\Psi_1(\|\varpi^*\|) + \tilde{\Omega}_*^{(3)}\|\Phi_2\|\Psi_2(\|\varpi^*\|)} \le 1.$$

In the light of hypothesis (HK4), we can find a real number $\mathfrak{N} > 0$ so that $\|\varpi^*\| \neq \mathfrak{N}$. Now, we construct a set

$$\mathbb{O} = \{\varpi^* \in \mathfrak{W} : \|\varpi^*\| < \mathfrak{N}\}.$$

We simply see that $\mathbb{A} : \overline{\mathbb{O}} \to \mathfrak{W}$ is an operator which is continuous and completely continuous. In view of this choice of $\mathbb{O}$, we cannot find $\varpi^* \in \partial\mathbb{O}$ which satisfies an equation $\varpi^* = \hat{c}(\mathbb{A}\varpi^*)$ for some $\hat{c} \in (0,1)$. Finally, by the nonlinear alternative of Leray–Schauder type, we realize that the operator $\mathbb{A}$ has a fixed point belonging to $\overline{\mathbb{O}}$. In consequence, there is at least one solution on $[0,1]$ for the nonlinear Caputo fractional quantum multi-integro-differential problem (1). □

In the following part of the present section, the uniqueness criterion for solutions of the given nonlinear Caputo fractional quantum multi-integro-differential problem (1) is checked with the aid of Banach contraction principle (Theorem 2.4).

Theorem 3.5 *Suppose that $\hat{h}_* : [0,1] \times \mathfrak{W} \to \mathfrak{W}$ is a function which satisfies hypothesis* (HK1). *Moreover, let the following assumption be valid for the function $\hat{f}_* : [0,1] \times \mathfrak{W} \to \mathfrak{W}$:*

(HK5) *there is a real constant $\mathfrak{K} > 0$ so that for any $\varpi_1, \varpi_2 \in \mathfrak{W}$, we have*

$$|\hat{f}_*(z,\varpi_1) - \hat{f}_*(z,\varpi_2)| \le \mathfrak{K}|\varpi_1 - \varpi_2|, \quad z \in [0,1].$$

Then there exists a unique solution on $[0,1]$ *for the nonlinear Caputo fractional quantum multi-integro-differential problem* (1) *such that $\tilde{\Omega}_*^{(1)} + \hat{b}^*\tilde{\Omega}_*^{(2)} + \mathfrak{K}\tilde{\Omega}_*^{(3)} < 1$, where $\tilde{\Omega}_*^{(1)}$, $\tilde{\Omega}_*^{(2)}$, and $\tilde{\Omega}_*^{(3)}$ are given by* (12).

Proof By utilizing Theorem 2.4, we shall verify that $\mathbb{A} : \mathfrak{W} \to \mathfrak{W}$ defined by (11) is an operator having a unique fixed point which corresponds to a unique solution of the mentioned nonlinear Caputo fractional quantum multi-integro-differential problem (1). By taking $\sup_{z\in[0,1]} |\hat{h}_*(z,0)| = \hat{H}_* < \infty$, $\sup_{z\in[0,1]} |\hat{f}_*(z,0)| = \hat{F}_* < \infty$, choosing $\tilde{\varepsilon} > 0$ so that

$$\tilde{\varepsilon} \ge \frac{\hat{F}_*\tilde{\Omega}_*^{(3)} + \hat{H}_*\tilde{\Omega}_*^{(2)}}{1 - (\tilde{\Omega}_*^{(1)} + \hat{b}^*\tilde{\Omega}_*^{(2)} + \mathfrak{K}\tilde{\Omega}_*^{(3)})},$$

and by constructing a bounded ball $\mathbb{B}_{\tilde{\varepsilon}} = \{\varpi \in \mathfrak{W} : \|\varpi\| \le \tilde{\varepsilon}\}$, we claim that $\mathbb{A}\mathbb{B}_{\tilde{\varepsilon}} \subset \mathbb{B}_{\tilde{\varepsilon}}$. For an arbitrary element $\varpi \in \mathbb{B}_{\tilde{\varepsilon}}$ and due to hypotheses (HK1) and (HK5), we get

$$\|\mathbb{A}\varpi\| \le \frac{\eta^* + 1}{\eta^*} {}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^* + \sigma}\|\varpi\|$$

$$
\begin{aligned}
&+\frac{\eta^*+2}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma}\|\varpi\|+\frac{\lambda_1^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma}\big(\hat{b}^*\|\varpi\|+\hat{H}_*\big)\\
&+\frac{1}{\tilde{\Xi}_*}\bigg[\frac{\mu^*(\eta^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_1^*}\|\varpi\|+\frac{\mu^*(\eta^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_1^*}\|\varpi\|\\
&+\frac{\lambda_1^*\mu^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_1^*}\big(\hat{b}^*\|\varpi\|+\hat{H}_*\big)+\frac{(\mu^*+1)(\eta^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_2^*}\|\varpi\|\\
&+\frac{(\mu^*+1)(\eta^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_2^*}\|\varpi\|+\frac{\lambda_1^*(\mu^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_2^*}\big(\hat{b}^*\|\varpi\|+\hat{H}_*\big)\\
&+\frac{(\mu^*+2)(\eta^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_1^*+\sigma+\theta_3^*}\|\varpi\|+\frac{(\mu^*+2)(\eta^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\delta_2^*+\sigma+\theta_3^*}\|\varpi\|\\
&+\frac{\lambda_1^*(\mu^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_1^*+\sigma+\theta_3^*}\big(\hat{b}^*\|\varpi\|+\hat{H}_*\big)\bigg]\\
&+\frac{\lambda_2^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma}\big(\mathfrak{K}\|\varpi\|+\hat{F}_*\big)+\frac{1}{\tilde{\Xi}_*}\bigg[\frac{\lambda_2^*\mu^*}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_1^*}\big(\mathfrak{K}\|\varpi\|+\hat{F}_*\big)\\
&+\frac{\lambda_2^*(\mu^*+1)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_2^*}\big(\mathfrak{K}\|\varpi\|+\hat{F}_*\big)\\
&+\frac{\lambda_2^*(\mu^*+2)}{\eta^*}{}^{\mathcal{RL}}\mathcal{I}_q^{\gamma_2^*+\sigma+\theta_3^*}\big(\mathfrak{K}\|\varpi\|+\hat{F}_*\big)\bigg]\\
\le{}&\big(\tilde{\Omega}_*^{(1)}+\hat{b}^*\tilde{\Omega}_*^{(2)}+\mathfrak{K}\tilde{\Omega}_*^{(3)}\big)\tilde{\varepsilon}+\tilde{\Omega}_*^{(3)}\hat{F}_*+\tilde{\Omega}_*^{(2)}\hat{H}_*<\tilde{\varepsilon}.
\end{aligned}
$$

In view of the above result, it is seen that the claim is valid, and so we have $\mathbb{A}\mathbb{B}_{\tilde{\varepsilon}}\subset\mathbb{B}_{\tilde{\varepsilon}}$. To confirm that the operator $\mathbb{A}:\mathfrak{W}\to\mathfrak{W}$ given by (11) is a contraction, let us assume that $z\in[0,1]$ and $\varpi_1,\varpi_2\in\mathfrak{W}$ are arbitrary. Now, by some straightforward computations, we can simply observe that

$$
\big\|(\mathbb{A}\varpi_1)-(\mathbb{A}\varpi_2)\big\|\le\big(\tilde{\Omega}_*^{(1)}+\hat{b}^*\tilde{\Omega}_*^{(2)}+\mathfrak{K}\tilde{\Omega}_*^{(3)}\big)\|\varpi_1-\varpi_2\|.
$$

By invoking the hypothesis $\tilde{\Omega}_*^{(1)}+\hat{b}^*\tilde{\Omega}_*^{(2)}+\mathfrak{K}\tilde{\Omega}_*^{(3)}<1$, we conclude that $\mathbb{A}$ is a contraction. Therefore, as a conclusion of Theorem 2.4, $\mathbb{A}$ has a unique fixed point. In consequence, there exists a unique solution for the nonlinear Caputo fractional quantum multi-integro-differential problem (1), and this ends the argument. □

4 Examples

In the current section of this manuscript, three illustrative numerical examples are considered to examine our theoretical and analytical findings by using the proposed methods.

Example 4.1 (Illustration of Theorem 3.3) With due attention to the defined structure for the proposed quantum multi-integro-differential problem (1), we here design the following multi-order Caputo fractional quantum multi-integro-differential boundary value problem:

$$
\begin{cases}
[0.1{}^{\mathcal{C}}\mathfrak{D}_{0.5}^{1.46}-1.1{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.72}-2.1{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.56}]\varpi(z)=0.002{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.12}\frac{0.008|\arctan\varpi(z)|}{|\arctan\varpi(z)|+1}\\
\qquad\qquad+0.003{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.18}\frac{\cos\varpi(z)}{(4+z)^2},\\
\varpi(0)=0,\qquad 0.001{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.27}\varpi(1)+1.001{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.16}\varpi(1)+2.001{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.31}\varpi(1)=0.
\end{cases}
\tag{13}
$$

Here, we consider some constants as follows: $z = [0,1]$, $\eta^* = 0.1$, $\mu^* = 0.001$, $\lambda_1^* = 0.002$, $\lambda_2^* = 0.003$, $\sigma = 1.46$, $q = 0.5$, $\delta_1^* = 0.72$, $\delta_2^* = 0.56$, $\gamma_1^* = 0.12$, $\gamma_2^* = 0.18$, $\theta_1^* = 0.27$, $\theta_2^* = 0.16$, and $\theta_3^* = 0.31$. Further, two continuous functions $\hat{h}_*, \hat{f}_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ are formulated by follows:

$$\hat{h}_*\big(z, \varpi(z)\big) = \frac{0.008|\arctan \varpi(z)|}{|\arctan \varpi(z)| + 1}, \qquad \hat{f}_*\big(z, \varpi(z)\big) = \frac{\cos \varpi(z)}{(4+z)^2}.$$

Notice that for each $\varpi_1, \varpi_2 \in \mathbb{R}$, we get

$$\begin{aligned}\big|\hat{h}_*(z,\varpi_1) - \hat{h}_*(z,\varpi_2)\big| &\le \left|\frac{0.008|\arctan \varpi_1(z)|}{|\arctan \varpi_1(z)| + 1} - \frac{0.008|\arctan \varpi_2(z)|}{|\arctan \varpi_2(z)| + 1}\right| \\ &\le \frac{8}{1000}\big|\arctan \varpi_1(z) - \arctan \varpi_2(z)\big| \\ &\le \frac{8}{1000}\big|\varpi_1(z) - \varpi_2(z)\big|.\end{aligned}$$

Thus we get $|\hat{h}_*(z,\varpi_1) - \hat{h}_*(z,\varpi_2)| \le 0.008|\varpi_1(z) - \varpi_2(z)|$ so that $\hat{b}^* = 0.008 > 0$. Furthermore, there is a continuous function $\Upsilon(z) = \frac{1}{(4+z)^2}$ on the interval $[0,1]$ so that an inequality $|\hat{f}_*(z,\varpi(z))| \le |\frac{\cos \varpi(z)}{(4+z)^2}| \le \Upsilon(z)$ holds for any $\varpi \in \mathbb{R}$. In this case, we have $\|\Upsilon\| = \sup_{z\in[0,1]} \Upsilon(z) = 0.0625$. By utilizing the above-given values, it is immediately obtained that $\tilde{\Xi}_* = 0.0538$, $\tilde{\Omega}_*^{(1)} = 0.095734$, $\tilde{\Omega}_*^{(2)} = 0.000000292$, and $\tilde{\Omega}_*^{(3)} = 0.00000042$. Hence we reach required value $\tilde{\Omega}_*^{(1)} + \hat{b}^* \tilde{\Omega}_*^{(2)} = 0.0957340023 < 1$. It is observed that all the hypotheses of Theorem 3.3 are valid for this problem. In consequence, the conclusion of Theorem 3.3 yields that the nonlinear multi-order Caputo fractional quantum multi-integro-differential boundary problem (13) has at least one solution on $[0,1]$.

Example 4.2 (Illustration of Theorem 3.4) With due attention to the defined structure for the proposed quantum multi-integro-differential problem (1), we here consider the following nonlinear multi-order Caputo fractional quantum multi-integro-differential boundary value problem:

$$\begin{cases} [0.1{}^{\mathcal{C}}\mathfrak{D}_{0.5}^{1.46} - 1.1{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.72} - 2.1{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.56}]\varpi(z) \\ \quad = 0.002{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.12}\frac{1}{\sqrt{64+z^3}} \\ \qquad \times (\cos \varpi(z) + \frac{|\sin \varpi(z)|}{1+|\sin \varpi(z)|}) + 0.003{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.18}\frac{1}{z+5}(\frac{1}{4} + \frac{|\arcsin \varpi(z)|}{1+|\arcsin \varpi(z)|}), \\ \varpi(0) = 0, \qquad 0.001{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.27}\varpi(1) + 1.001{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.16}\varpi(1) + 2.001{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.31}\varpi(1) = 0, \end{cases} \tag{14}$$

such that $z = [0,1]$, $\eta^* = 0.1$, $\mu^* = 0.001$, $\lambda_1^* = 0.002$, $\lambda_2^* = 0.003$, $\sigma = 1.46$, $q = 0.5$, $\delta_1^* = 0.72$, $\delta_2^* = 0.56$, $\gamma_1^* = 0.12$, $\gamma_2^* = 0.18$, $\theta_1^* = 0.27$, $\theta_2^* = 0.16$, and $\theta_3^* = 0.31$. Moreover, the functions $\hat{h}_*, \hat{f}_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{h}_*\big(z, \varpi(z)\big) = \frac{1}{\sqrt{64+z^3}}\left(\cos \varpi(z) + \frac{|\sin \varpi(z)|}{1 + |\sin \varpi(z)|}\right)$$

and

$$\hat{f}_*\big(z, \varpi(z)\big) = \frac{1}{z+5}\left(\frac{1}{4} + \frac{|\arcsin \varpi(z)|}{1 + |\arcsin \varpi(z)|}\right)$$

are continuous. Evidently, we have the following inequalities:

$$\left|\hat{h}_*\big(z,\varpi(z)\big)\right| \le \frac{1}{\sqrt{64+z^3}}\big(1+\|\varpi\|\big), \qquad \left|\hat{f}_*\big(z,\varpi(z)\big)\right| \le \frac{1}{z+5}\big(1+\|\varpi\|\big).$$

Thus, we take $\Phi_1(z) = \frac{1}{\sqrt{64+z^3}}$ and $\Phi_2(z) = \frac{1}{z+5}$ and $\Psi_1(\|\varpi\|) = \Psi_2(\|\varpi\|) = 1 + \|\varpi\|$. Notice that $\|\Phi_1\| = \frac{1}{8} = 0.125$, $\|\Phi_2\| = \frac{1}{5} = 0.2$, and $\Psi_1(\mathfrak{N}) = \Psi_2(\mathfrak{N}) = 1 + \mathfrak{N}$. In view of the above data, we find that $\tilde{\Xi}_* = 0.0538$, $\tilde{\Omega}_*^{(1)} = 0.095734 < 1$, $\tilde{\Omega}_*^{(2)} = 0.000000292$, and $\tilde{\Omega}_*^{(3)} = 0.00000042$. Therefore, by taking into account hypothesis (HK4), we find that $\mathfrak{N} > 0.00000013325 = 1.3325 \times 10^{-7}$. At this point, we see that all the hypotheses of Theorem 3.4 hold for this problem. Therefore, by Theorem 3.4, the nonlinear multi-order Caputo fractional quantum multi-integro-differential boundary problem (14) has at least one solution on $[0,1]$.

Example 4.3 (Illustration of Theorem 3.5) With due attention to the defined structure for the proposed quantum multi-integro-differential problem (1), we here design the following multi-order Caputo fractional quantum multi-integro-differential boundary value problem:

$$\begin{cases} [0.1\,{}^{\mathcal{C}}\mathfrak{D}_{0.5}^{1.46} - 1.1\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.72} - 2.1\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.56}]\varpi(z) \\ \quad = 0.002\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.12}\frac{0.005|\cos(\pi z))||\varpi(z)|}{|\varpi(z)|+1} + 0.003\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.18}\frac{2|\varpi(z)|}{125+125|\varpi(z)|)}, \\ \varpi(0) = 0, \qquad 0.001\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.27}\varpi(1) + 1.001\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.16}\varpi(1) + 2.001\,{}^{\mathcal{RL}}\mathcal{I}_{0.5}^{0.31}\varpi(1) = 0, \end{cases} \tag{15}$$

such that $z = [0,1]$, $\eta^* = 0.1$, $\mu^* = 0.001$, $\lambda_1^* = 0.002$, $\lambda_2^* = 0.003$, $\sigma = 1.46$, $q = 0.5$, $\delta_1^* = 0.72$, $\delta_2^* = 0.56$, $\gamma_1^* = 0.12$, $\gamma_2^* = 0.18$, $\theta_1^* = 0.27$, $\theta_2^* = 0.16$, and $\theta_3^* = 0.31$. Besides, two functions $\hat{h}_*, \hat{f}_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{h}_*\big(z,\varpi(z)\big) = \frac{0.005|\cos(\pi z))||\varpi(z)|}{|\varpi(z)|+1}, \qquad \hat{f}_*\big(z,\varpi(z)\big) = \frac{2|\varpi(z)|}{125+125|\varpi(z)|)}$$

are supposed to be continuous on the relevant domain. Then we get $\hat{b}^* = 0.005$ and $\mathfrak{K} = 0.016$, since one can simply see that

$$\left|\hat{h}_*\big(z,\varpi_1(z)\big) - \hat{h}_*\big(z,\varpi_2(z)\big)\right| \le 0.005\big(\left|\varpi_1(z) - \varpi_2(z)\right|\big)$$

and

$$\left|\hat{f}_*\big(z,\varpi_1(z)\big) - \hat{f}_*\big(z,\varpi_2(z)\big)\right| \le 0.016\big(\left|\varpi_1(z) - \varpi_2(z)\right|\big).$$

Eventually, in the light of the above assumptions, we find that $\tilde{\Xi}_* = 0.0538$ and

$$\tilde{\Omega}_*^{(1)} + \hat{b}^*\tilde{\Omega}_*^{(2)} + \mathfrak{K}\tilde{\Omega}_*^{(3)} = 0.095734009 < 1.$$

In consequence, we conclude that all the hypotheses of Theorem 3.5 hold for this problem. Therefore, by Theorem 3.5, the nonlinear multi-order Caputo fractional quantum multi-integro-differential boundary problem (15) has at least one solution on $[0,1]$.

Table 1 Numerical values of $\tilde{\Omega}_*^{(1)}$, $\tilde{\Omega}_*^{(2)}$, and $\tilde{\Omega}_*^{(3)}$ for $q = 0.3, 0.5, 0.7$

	$q = 0.3$	$q = 0.5$	$q = 0.7$
$\tilde{\Omega}_*^{(1)}$	0.075999	0.095734	0.012214
$\tilde{\Omega}_*^{(2)}$	0.0000011	0.000000292	0.00000016
$\tilde{\Omega}_*^{(3)}$	0.0000017	0.00000042	0.00000023

Remark 4.4 Notice that one can find other values of three nonzero constants $\tilde{\Omega}_*^{(1)}$, $\tilde{\Omega}_*^{(2)}$, and $\tilde{\Omega}_*^{(3)}$ for different values of $q = 0.3, 0.5, 0.7$ in Table 1. Indeed, we only calculated required numerical data of above examples for $q = 0.5$.

5 Conclusion

As years and even decades go by, the human beings need to be acquainted with different natural phenomena more and more. One possible way to achieve this purpose is to study the mathematical structures of these processes by means of the logical techniques and tools available in mathematics. In the present framework of this research manuscript, we formulate a new generalized structure of the nonlinear Caputo fractional quantum multi-integro-differential equation in which such multi-order structure of quantum integrals are considered for the first time. In fact, in the light of this type of boundary value problem equipped with the multi-integro-differential setting, one can simply study different cases of the existing usual integro-differential problems in the literature. In this direction, we utilize well-known analytical techniques to derive desired criteria which guarantee the existence of solutions for the proposed multi-order quantum multi-integro-differential problem. Further, some numerical examples are provided to examine our theoretical and analytical findings based on the proposed methods.

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