Advances in Difference Equations
a SpringerOpen Journal

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Stability of deficiency indices for discrete Hamiltonian systems under bounded perturbations

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Abstract

This paper is concerned with stability of deficiency indices for discrete Hamiltonian systems under perturbations. By applying the perturbation theory of Hermitian linear relations we establish the invariance of deficiency indices for discrete Hamiltonian systems under bounded perturbations. As a consequence, we obtain the invariance of limit types for the systems under bounded perturbations. In particular, we build several criteria of the invariance of the limit circle and limit point cases for the systems. Some of these results improve and extend some previous results.

MSC: 39A70; 47A06; 47A55; 47B15

Keywords: Discrete Hamiltonian systems; Deficiency index; Perturbation; Stability; Hermitian relation

1 Introduction

In this paper, we consider the following discrete linear Hamiltonian systems with one singular endpoint:

$$J\triangle y(t) - P(t)R(y)(t) = \lambda W(t)R(y)(t), \quad t \in I, \tag{1.1_λ}$$

where I is the integer set $\{t\}_{t=0}^{+\infty}$, J is the canonical symplectic matrix, that is,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

I_n is the $n \times n$ unit matrix, $\triangle$ is the forward difference operator, that is, $\triangle y(t) = y(t + 1) - y(t)$; the weight function $W(t) = \text{diag}\{W_1(t), W_2(t)\}$, where $W_1(t)$ and $W_2(t)$ are $n \times n$ nonnegative Hermitian matrices, the matrix $P(t)$ can be written as

$$P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix},$$

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where $A(t)$, $B(t)$, and $C(t)$ are $n \times n$ complex-valued matrices, $B(t)$ and $C(t)$ are Hermitian matrices, $A^*(t)$ is the complex conjugate transpose of $A(t)$, the partial right shift operator $R(y)(t) = (u^T(t+1), v^T(t))^T$ with $y(t) = (u^T(t), v^T(t))^T$ and $u(t), v(t) \in \mathbb{C}^n$, and λ is a complex spectral parameter.

To ensure the existence and uniqueness of the solution of any initial value problem for (1.1_λ), we always assume that

(A_1) $I_n - A(t)$ is invertible in I.

For any $\lambda \in \mathbb{C}$, by (A_1), (1.1_λ) can be rewritten as a discrete symplectic system

$$y(t+1) = S(t, \lambda)y(t), \quad t \in I, \tag{1.2_λ}$$

in the sense of [1, 2], where $E(t) = (I_n - A(t))^{-1}$, and

$$S(t, \lambda) = \begin{pmatrix} E(t) & E(t)(B(t) + \lambda W_2(t)) \\ (C(t) - \lambda W_1(t))E(t) & I_n - A^*(t) + (C(t) - \lambda W_1(t))E(t)(B(t) + \lambda W_2(t)) \end{pmatrix}$$

satisfies

$$S^*(t, \bar{\lambda})JS(t, \lambda) = J, \quad \forall t \in I.$$

Some interesting issues related to discrete symplectic system (1.2_λ), such as associated maximal and minimal linear relations, Weyl–Titchmarsh theory, and nonhomogeneous problems, were studied in [1–3].

Discrete Hamiltonian systems are of growing interest in recent years because of their wide applications (see [3–12] and references therein). Although discrete Hamiltonian systems originate from the discretization of continuous Hamiltonian systems, there is an important difference between them. It is well known that under certain condition, the minimal and maximal operators generated by continuous Hamiltonian systems are densely defined and single-valued, respectively [13, 14]. However, the minimal and maximal operators generated by the general discrete Hamiltonian systems may be neither densely defined nor single-valued in general even though the definiteness condition is satisfied [5–7, 15, 16]. This fact was ignored in some existing literature including [3, 17]. This is an essential difficulty that we would encounter in the study of the stability of deficiency indices for discrete Hamiltonian systems under perturbations because the classical operator theory is not applicable in this case.

To overcome this difficulty, we will apply the theory of linear relations to study system (1.1_λ). In 1961, Arens [18] initiated the study of linear relations, and his work was followed by many scholars [19–30]. In particular, perturbation theory of linear relations has received lots of attention, and some excellent results have been obtained, including stability of closedness, boundedness, self-adjointness, and spectra of linear relations (see [25, 27–30]). Recently, we studied the stability of deficiency indices of Hermitian relations and obtained several criteria of invariance of deficiency indices of Hermitian relations under relatively bounded perturbations [31]. Then, using our perturbation results, we obtained the invariance of deficiency indices of second-order symmetric linear difference equations under perturbations [32], which can be seen as the simplest example of system (1.1_λ). In this paper, we apply the results given in [31] to study the stability of deficiency indices for (1.1_λ).

It is well known that the deficiency indices of symmetric operators or Hermitian relations play a decisive role in their self-adjoint extensions. By the generalized von Neumann theory [19] and the GKN theory [26] a symmetric operator or Hermitian relation has a self-adjoint extension if and only if its positive and negative deficiency indices are equal; moreover, the numbers and types of boundary conditions of its self-adjoint extensions are determined by its deficiency indices. So it is necessary to pay attention to the stability of their deficiency indices under perturbations.

To the best of our knowledge, there seems to be a few results about stability of deficiency indices for discrete Hamiltonian systems under perturbations. In 2013, by using the generalized von Neumann theory Zheng [33] obtained the invariance of the minimal and maximal deficiency indices for (1.1_λ) with $P(t)$ under bounded perturbations. In the present paper, we apply the perturbation theory of Hermitian relations obtained in [31] to establish several criteria of stability of deficiency indices for (1.1_λ) with both of $P(t)$ and $W(t)$ under bounded perturbations. Our technique is obviously different from that in [33]. By using it we could obtain the invariance of any deficiency index for (1.1_λ) with $P(t)$ under bounded perturbations. These results not only cover the results obtained in [33], but also some of them improve or weaken the conditions of the existing results. In addition, we note that almost all criteria for limit types of (1.1_λ) were established only for the limit point and limit circle cases. However, there are seldom criteria of the intermediate cases for (1.1_λ). We remark that the results given in the present paper provide an alternate way to determine the limit types of system (1.1_λ).

The rest of this paper is organized as follows. In Sect. 2, we introduce some notations, basic concepts, and useful fundamental results about linear relations and recall some fundamental results about system (1.1_λ). In Sect. 3, we establish several criteria of stability of deficiency indices for system (1.1_λ) under bounded perturbations by using the perturbation theory of Hermitian relations obtained in [31]. As a consequence, we obtain the invariance of limit types for the systems under bounded perturbations. In particular, we build several criteria of the invariance of the limit circle and limit point cases for the systems. Finally, we present an example to illustrate the perturbation results obtained in Sect. 4.

2 Preliminaries

This section is divided into two parts. In Sect. 2.1, we introduce some notations, basic concepts, and fundamental results about linear relations. In Sect. 2.2, we first recall the maximal, preminimal, and minimal relations corresponding to system (1.1_λ). Then we list some useful results about (1.1_λ), which will be used in the sequent sections.

2.1 Some notations, concepts, and results about linear relations

By $\mathbb{C}$ and $\mathbb{R}$ we denote the sets of complex numbers and real numbers, respectively. Let X be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let T be a linear relation in the product space X^2 with the following induced inner product, still denoted by $\langle \cdot, \cdot \rangle$ without any confusion:

$$\langle (x,f),(y,g) \rangle = \langle x,y \rangle + \langle f,g \rangle, \quad (x,f),(y,g) \in X^2.$$

The domain $D(T)$ and range $R(T)$ of T are respectively defined by

$$D(T) := \{x \in X : (x,f) \in T \text{ for some } f \in X\},$$
$$R(T) := \{f \in X : (x,f) \in T \text{ for some } x \in X\}.$$

Its adjoint is defined by

$$T^* = \{(y,g) \in X^2 : \langle g,x \rangle = \langle y,f \rangle \text{ for all } (x,f) \in T\}.$$

Further, denote

$$T(x) := \{f \in X : (x,f) \in T\}.$$

It is evident that $T(0) = \{0\}$ if and only if T can uniquely determine a linear operator from $D(T)$ into X whose graph is T.

A linear relation T is called closed if it is a closed subspace in X^2, Hermitian if $T \subset T^*$, and self-adjoint if $T = T^*$.

Let T and S be two linear relations in X^2, and let $\lambda \in \mathbb{C}$. Define

$$\lambda T := \{(x, \lambda f) : (x,f) \in T\},$$
$$T + S := \{(x, f + g) : (x,f) \in T, (x,g) \in S\}.$$

The subspace $R(T - \lambda I)^\perp$ and the number $d_\lambda(T) := \dim(R(T - \lambda I))^\perp$ are called the deficiency space and deficiency index of T and λ, respectively [26, Definition 2.3], where $I := \{(x,x) : x \in X\}$. It can be easily verified that the deficiency indices of T and $\overline{T}$ with the same λ are equal. Further, if T is Hermitian, then $d_\lambda(T)$ is constant in the upper and lower half-planes according to [26, Theorem 2.3]. Denote $d_\pm(T) := d_{\pm i}(T)$. We say that $(d_+(T), d_-(T))$ are the deficiency indices of T and that $d_\pm(T)$ are the positive and negative deficiency indices of T, respectively.

In the following, we recall concepts of the norm of a linear relation and relatively boundedness of two linear relations.

Let T be a linear relation in X^2. The quotient space $X/\overline{T(0)}$ is a Hilbert space [34] with the inner product

$$\langle [x], [y] \rangle = \langle x^\perp, y^\perp \rangle, \quad [x], [y] \in X/\overline{T(0)},$$

where $x = x_0 + x^\perp$ and $y = y_0 + y^\perp$ with $x_0, y_0 \in \overline{T(0)}$ and $x^\perp, y^\perp \in T(0)^\perp$.

Now define the natural quotient map

$$Q_T : X \to X/\overline{T(0)}, \quad x \mapsto [x].$$

Further, define

$$\tilde{T}_s = G(Q_T)T,$$

where $G(Q_T)$ is the graph of Q_T. Then $\tilde{T}_s$ is a linear operator with domain $D(T)$ [25, Proposition II.1.2]. The norm of T at $x \in D(T)$ and the norm of T are defined by, respectively (see [25, II.1]),

$$\|T(x)\| := \|\tilde{T}_s(x)\|,$$

$$\|T\| := \|\tilde{T}_s\| = \sup\{\|\tilde{T}_s(x)\| : x \in D(T) \text{ with } \|x\| \le 1\}.$$

If $\|T\| < \infty$, then T is said to be bounded [30].

Definition 2.1 ([25, Definition VII.2.1]) Let S and T be two linear relations in X^2.

(1) S is said to be T-bounded if $D(T) \subset D(S)$ and there exists a constant $c \ge 0$ such that

$$\|S(x)\| \le c(\|x\| + \|T(x)\|), \quad x \in D(T).$$

(2) If S is T-bounded, then the infimum of all numbers $b \ge 0$ for which there exists a constant $a \ge 0$ such that

$$\|S(x)\| \le a\|x\| + b\|T(x)\|, \quad x \in D(T),$$

is called the T-bound of S.

Next, we recall a criterion of stability of defect indices of Hermitian relations under relatively bounded perturbations, which will take a key role in the study of stability of deficiency indices for (1.1_λ) under perturbations.

Lemma 2.1 ([31, Corollary 3.1]) *Let T and S be Hermitian relations in X^2 with $D(T) \subset D(S)$ and $S(0) \subset T(0)$. If S is T-bounded with T-bound less than 1, then $d_\pm(T + S) = d_\pm(T)$.*

Lemma 2.2 ([30, Proposition 2.1]) *Let T and S be two linear relations in X^2. Then $T = (T - S) + S$ if and only if $D(T) \subset D(S)$ and $S(0) \subset T(0)$.*

2.2 Some fundamental results about system (1.1_λ)

In this subsection, we first introduce the concepts of maximal, preminimal, and minimal relations and then list some useful results about system (1.1_λ).

We denote

$$\mathcal{L}_W^2(I) := \left\{ y = \{y(t)\}_{t=0}^{+\infty} \subset \mathbb{C}^{2n} : \sum_{t \in I} R(y)^*(t) W(t) R(y)(t) < +\infty \right\}$$

with the semiscalar product

$$\langle y, z \rangle := \sum_{t \in I} R^*(z)(t) W(t) R(y)(t).$$

Further, we define $\|y\| := (\langle y, y \rangle)^{1/2}$ for $y \in \mathcal{L}_W^2(I)$. Since the weight function $W(t)$ may be singular in I, $\|\cdot\|$ is a seminorm. We denote

$$L_W^2(I) := \mathcal{L}_W^2(I)/\{y \in \mathcal{L}_W^2(I) : \|y\| = 0\}.$$

Then $L^2_W(I)$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ (see [3, Lemma 2.5]). For a function $y \in \mathcal{L}^2_W(I)$, we denote by $[y]$ the corresponding class in $L^2_W(I)$. Set

$$\mathcal{L}^2_{W,0}(I) := \big\{y \in \mathcal{L}^2_W(I): \text{ there exist two integer } s, k \in I \text{ with } s \le k$$
$$\text{such that } y(t) = 0 \text{ for } t \le s \text{ and } t \ge k + 1\big\}.$$

The natural difference operator corresponding to system (1.1_λ) is

$$\mathfrak{L}(y)(t) := J\triangle y(t) - P(t)R(y)(t).$$

Set

$$H := \big\{\big([y],[g]\big) \in L^2_W(I) \times L^2_W(I): \text{ there exists } y \in [y] \text{ such that}$$
$$\mathfrak{L}(y)(t) = W(t)R(g)(t), t \in I\big\},$$
$$H_{00} := \big\{\big([y],[g]\big) \in H: \text{ there exists } y \in [y] \text{ such that } y \in \mathcal{L}^2_{W,0}(I)$$
$$\text{and } \mathfrak{L}(y)(t) = W(t)R(g)(t), t \in I\big\},$$

where H is called the maximal relation, and H_{00} is called the preminimal relation corresponding to $\mathfrak{L}$ or system (1.1_λ); $H_0 := \overline{H}_{00}$ is called the minimal relation corresponding to $\mathfrak{L}$ or to system (1.1_λ).

A classification of $\mathfrak{L}$ or (1.1_λ) at $t = +\infty$ is given in terms of $d_\pm(H_0)$ in [3, Definition 5.1]. In particular, $\mathfrak{L}$ is said to be in the limit point case (l.p.c.) at $t = +\infty$ if $d_+(H_0) = d_-(H_0) = n$, and in the limit circle case (l.c.c.) at $t = +\infty$ if $d_+(H_0) = d_-(H_0) = 2n$. We refer to the cases $n < d_\pm(H_0) < 2n$ as $\mathfrak{L}$ in the intermediate cases at $t = +\infty$.

By $n_\lambda(H_0)$ we denote the number of linearly independent solutions of (1.1_λ) in $\mathcal{L}^2_W(I)$. By [5, Corollary 5.1] we know that $n_\lambda(H_0) = d_\lambda(H_0)$ if and only if the following condition is satisfied:

(A$_2$) There exists a finite subset $I_0 := [s_0, t_0] \subset I$ such that for some $\lambda \in \mathbb{C}$ and any nontrivial solution $y(t)$ of (1.1_λ),

$$\sum_{t\in I_0} R(y)^*(t)W(t)R(y)(t) > 0. \tag{2.1}$$

Remark 2.1 By [5, Theorem 3.1] H_0 and H_{00} are both Hermitian relations in $L^2_W(I) \times L^2_W(I)$. Condition (A$_2$) is called the definiteness condition for (1.1_λ). It was shown in [5] that if (2.1) holds for some $\lambda \in \mathbb{C}$, then it holds for any $\lambda \in \mathbb{C}$. As pointed out in Sect. 1, the definiteness condition (A$_2$) cannot guarantee H_0 to be densely defined or H to be single-valued. In fact, if there exists $t_0 \in I$ such that $W(t_0) \ne 0$, then H_0 is Hermitian and nondensely defined in $L^2_W(I) \times L^2_W(I)$, and H is multivalued in $L^2_W(I) \times L^2_W(I)$ by [5, Theorem 3.1] and [7, Theorems 3.1 and 3.2]. So the classical perturbation theory of symmetric operators is not available in the study of stability of deficiency indices of H_0 under perturbations. We will apply the result about the perturbation of Hermitian relations, that is, Lemma 2.1, to study this problem in the present paper.

Finally, we introduce a criterion of limit point case (see [5, Theorem 6.1]) and the largest index theorem (see [3, Theorem 5.5]) for (1.1_λ).

Lemma 2.3 (1.1_λ) *is in l.p.c. at* $t = +\infty$ *if* (A_1) *and*

$$\sum_{t=1}^{+\infty} \sqrt{w_1(t-1)w_2(t)} = \infty, \tag{2.2}$$

where $w_j(t)$ *is the minimal eigenvalue of* $W_j(t)$ *for* $j = 1, 2$.

Theorem 2.1 *Assume that* (A_1) *holds. If there exists* $\lambda_0 \in \mathbb{C}$ *such that all the solutions of* (1.1_{λ_0}) *are in* $\mathcal{L}^2_W(I)$, *then this is true for all* $\lambda \in \mathbb{C}$.

Remark 2.2 Theorem 2.1 was significantly generalized in [35]. In the smallest deficiency index case, that is, when (1.1_λ) is in l.p.c. at $t = +\infty$, it follows from [5, Theorem 5.2] that $d_\lambda(H_0) \equiv n$ for all $\lambda \in \mathbb{C}$ if (A_1) holds. In addition, if (A_2) holds, then $n_\lambda(H_0) \equiv n$ for all $\lambda \in \mathbb{C}$.

3 Main results

In this section, we study the stability of deficiency indices for system (1.1_λ) with coefficient matrices under bounded perturbations with respect to the weight functions. We establish several criteria of stability of deficiency indices for (1.1_λ) under bounded perturbations by using the perturbation theory of Hermitian relations obtained in [31]. As a consequence, we obtain the invariance of limit types of (1.1_λ) under bounded perturbations. In particular, we build several criteria of the invariance of the limit circle and limit point cases for the systems.

Consider the perturbed discrete Hamiltonian system with $P(t)$ and $W(t)$ perturbed by $\tilde{P}(t)$ and $\tilde{W}(t)$, respectively, that is,

$$\tilde{\mathfrak{L}}(y)(t) := J \triangle y(t) - \tilde{P}(t)R(y)(t) = \lambda \tilde{W}(t)R(y)(t), \quad t \in I, \tag{3.1_\lambda}$$

where the weight function $\tilde{W}(t) = \mathrm{diag}\{\tilde{W}_1(t), \tilde{W}_2(t)\}$ with $n \times n$ matrices $\tilde{W}_j(t) \geq 0$, $j = 1, 2$; $\tilde{P}(t)$ can be written as

$$\tilde{P}(t) = \begin{pmatrix} -\tilde{C}(t) & \tilde{A}^*(t) \\ \tilde{A}(t) & \tilde{B}(t) \end{pmatrix},$$

where $\tilde{A}(t)$, $\tilde{B}(t)$, and $\tilde{C}(t)$ are $n \times n$ complex-valued matrices, $\tilde{B}(t)$ and $\tilde{C}(t)$ are Hermitian matrices, and $\tilde{A}^*(t)$ is the complex conjugate transpose of $\tilde{A}(t)$. Similarly, $(\tilde{A}_1)$, $(\tilde{A}_2)$, $\mathcal{L}^2_{\tilde{W}}(I)$, $\mathcal{L}^2_{\tilde{W},0}(I)$, $L^2_{\tilde{W}}(I)$, $\| \cdot \|_{\tilde{W}}$, $\langle \cdot, \cdot \rangle_{\tilde{W}}$, $\tilde{H}$, $\tilde{H}_{00}$, $\tilde{H}_0$, $d_\lambda(\tilde{H}_0)$, and $n_\lambda(\tilde{H}_0)$ are defined as in Sects. 1 and 2.2.

The rest of this section is divided into two parts based on whether the weight matrix is perturbed.

3.1 Stability of deficient indices for (1.1_λ) in the case of $\tilde{W}(t) = W(t)$

In this subsection, we pay our attention on stability of deficiency indices for system (1.1_λ) in the case of $\tilde{W}(t) = W(t)$ for $t \in I$. By applying the perturbation theory of Hermitian relations, we establish several criteria of stability of deficiency indices for (1.1_λ) under bounded perturbations. As a consequence, we obtain the invariance of limit types of (1.1_λ) under bounded perturbations.

Theorem 3.1 *Assume that* (A_1) *holds. Let* $\tilde{W}(t) = W(t)$ *for* $t \in I$. *If there exist nonnegative constants* c_j *and real-valued functions* $c_j(t)$ *with* $|c_j(t)| \leq c_j$ *for* $j = 1, 2$ *such that*

$$\tilde{P}(t) - P(t) = \text{diag}\{c_1(t)W_1(t), c_2(t)W_2(t)\}, \quad t \in I, \tag{3.2}$$

then $d_\pm(H_0) = d_\pm(\tilde{H}_0)$.

Proof It follows from (3.2) that $\tilde{A}(t) = A(t)$ for $t \in I$. So $(\tilde{A}_1)$ holds since (A_1) holds. Since the deficient indices of H_0 and $\tilde{H}_0$ are equal to those of H_{00} and $\tilde{H}_{00}$, respectively, it suffices to show that

$$d_\pm(H_{00}) = d_\pm(\tilde{H}_{00}). \tag{3.3}$$

By Remark 2.1 and the assumption that $\tilde{W}(t) = W(t)$ for $t \in I$ it follows that H_{00} and $\tilde{H}_{00}$ are both Hermitian relations in $L_W^2(I) \times L_W^2(I)$. Next, we will prove (3.3) by Lemma 2.1. The proof is divided into three steps.

Step 1. We prove that $D(H_{00}) = D(\tilde{H}_{00})$.

It is evident that $\mathcal{L}_{\tilde{W},0}^2(I) = \mathcal{L}_{W,0}^2(I)$ since $\tilde{W}(t) = W(t)$ for $t \in I$. For any $[y] \in D(H_{00})$, there exist $y \in [y]$ and $g \in \mathcal{L}_W^2(I)$ such that $y \in \mathcal{L}_{W,0}^2(I)$ and

$$\mathfrak{L}(y)(t) = J\triangle y(t) - P(t)R(y)(t) = W(t)R(g)(t), \quad t \in I. \tag{3.4}$$

This, together with (3.2), yields that

$$\begin{aligned}
\tilde{\mathfrak{L}}(y)(t) &= J\triangle y(t) - \tilde{P}(t)R(y)(t) \\
&= J\triangle y(t) - P(t)R(y)(t) + \big(P(t) - \tilde{P}(t)\big)R(y)(t) \\
&= W(t)R(g)(t) - \text{diag}\{c_1(t)W_1(t), c_2(t)W_2(t)\}R(y)(t) \quad t \in I. \tag{3.5}
\end{aligned}$$

Take $\tilde{g}(t) = (\tilde{g}_1^T(t), \tilde{g}_2^T(t))^T$ with $\tilde{g}_j(t) \in \mathbb{C}^n, j = 1, 2$, such that

$$\tilde{g}_1(t + 1) = g_1(t + 1) - c_1(t)y_1(t + 1), \qquad \tilde{g}_2(t) = g_2(t) - c_2(t)y_2(t), \quad t \in I,$$

where $y(t) = (y_1^T(t), y_2^T(t))^T$ and $g(t) = (g_1^T(t), g_2^T(t))^T$ with $y_j(t), g_j(t) \in \mathbb{C}^n, j = 1, 2$. Then it follows from (3.5) that $\tilde{\mathfrak{L}}(y)(t) = W(t)R(\tilde{g})(t)$ for $t \in I$. In addition, noting that $y \in \mathcal{L}_{W,0}^2(I)$, $g \in \mathcal{L}_W^2(I)$, and $|c_j(t)| \leq c_j$, we get that $\tilde{g} \in \mathcal{L}_W^2(I)$, and so $([y], [\tilde{g}]) \in \tilde{H}_{00}$. Thus $[y] \in D(\tilde{H}_{00})$. Hence $D(H_{00}) \subset D(\tilde{H}_{00})$. With a similar argument, we can show that $D(\tilde{H}_{00}) \subset D(H_{00})$. Consequently, $D(H_{00}) = D(\tilde{H}_{00})$.

Step 2. We show that $H_{00}(0) = \tilde{H}_{00}(0)$.

For any $[g] \in H_{00}(0)$, there exist $g \in [g]$ and $y \in \mathcal{L}_{W,0}^2(I)$ with $[y] = 0$ such that (3.4) and (3.5) hold. Note that $[y] = 0$ implies that $W(t)R(y)(t) = 0$ for $t \in I$. Hence it follows from (3.5) that $\tilde{\mathfrak{L}}(y)(t) = W(t)R(g)(t)$ for $t \in I$. This implies that $[g] \in \tilde{H}_{00}(0)$. Thus $H_{00}(0) \subset \tilde{H}_{00}(0)$. Similarly, we can show that $\tilde{H}_{00}(0) \subset H_{00}(0)$, and thus $H_{00}(0) = \tilde{H}_{00}(0)$.

Step 3. We show that $\tilde{H}_{00} - H_{00}$ is H_{00}-bounded with H_{00}-bound 0, that is, $\tilde{H}_{00} - H_{00}$ is bounded.

It is evident that $\tilde{H}_{00} - H_{00}$ is a Hermitian relation in $L_W^2(I) \times L_W^2(I)$ since H_{00} and $\tilde{H}_{00}$ are Hermitian relations in $L_W^2(I) \times L_W^2(I)$. In addition, since $D(H_{00}) = D(\tilde{H}_{00})$ and $H_{00}(0) = \tilde{H}_{00}(0)$, $\tilde{H}_{00} = \tilde{H}_{00} - H_{00} + H_{00}$ according to Lemma 2.2.

Based on the discussions in Step 1, for any $[y] \in D(H_{00}) = D(\tilde{H}_{00}) = D(\tilde{H}_{00} - H_{00})$, there exist $g, \tilde{g} \in \mathcal{L}_W^2(I)$ and $y \in [y]$ with $y \in \mathcal{L}_{W,0}^2(I)$ such that

$$R(\tilde{g} - g)(t) = -\operatorname{diag}\{c_1(t)I_n, c_2(t)I_n\}R(y)(t)$$

for $t \in I$ and $([y], [\tilde{g} - g]) \in \tilde{H}_{00} - H_{00}$, that is, $[\tilde{g} - g] \in (\tilde{H}_{00} - H_{00})([y])$. Therefore

$$\left\| (\tilde{H}_{00} - H_{00})([y]) \right\| \leq \left\| [\tilde{g} - g] \right\| \leq \max\{|c_1|, |c_2|\}\|y\|.$$

So $\tilde{H}_{00} - H_{00}$ is bounded, that is, $\tilde{H}_{00} - H_{00}$ is H_{00}-bounded with H_{00}-bound 0.

Based on these three statements, it follows that H_{00} and $\tilde{H}_{00} - H_{00}$ satisfy the conditions in Lemma 2.1. Therefore $d_\pm(\tilde{H}_{00} - H_{00} + H_{00}) = d_\pm(\tilde{H}_{00}) = d_\pm(H_{00})$ by Lemma 2.1. So (3.3) holds. This completes the proof. □

Remark 3.1

(1) By using the generalized von Neumann theory, Zheng [33] showed the invariance of the minimal and maximal deficiency indices of (1.1_λ) under assumptions (A_1) and (A_2), $\tilde{W}(t) = W(t)$ for $t \in I$, and the condition that

$$\tilde{P}(t) - P(t) = \operatorname{diag}\{C_0 W_1(t), B_0 W_2(t)\}, \tag{3.6}$$

where B_0 and C_0 are constants. Theorem 3.1 extends the results in [33]. By applying the perturbation theory of Hermitian relations we do not need that condition (A_2) holds in Theorem 3.1. Furthermore, we obtain the invariance for any deficient index of (1.1_λ) under assumption (A_1), $\tilde{W}(t) = W(t)$ for $t \in I$, and condition (3.2) in Theorem 3.1.

(2) Note that (3.2) implies that $c_1(t)$ and $c_2(t)$ are both real-valued functions. In fact, $\tilde{P}(t) - P(t)$ is Hermitian since $\tilde{P}(t)$ and $P(t)$ are Hermitian matrices. This, together with $W(t) \geq 0$ and (3.2), yields that $c_1(t)$ and $c_2(t)$ are real-valued. With a similar argument, B_0 and C_0 in (3.6) also must be real numbers. This fact was ignored in the proof of Corollaries 3.1 and 3.2 in [33], where the author regarded the perturbation as $(\lambda - \lambda_0)W(t)$ for any $\lambda, \lambda_0 \in \mathbb{C}$.

Let E and F be two Hermitian matrices. In this paper, we write $E \geq F$ if $E - F \geq 0$.

By comparing with Theorem 3.1 the following result imposes a weaker restriction on $\tilde{P}(t) - P(t)$ when $W(t) = \tilde{W}(t) > 0$.

Theorem 3.2 *Assume that (A_1) and $(\tilde{A}_1)$ hold. Let $\tilde{W}(t) = W(t) > 0$ for $t \in I$. If there exist two constants c_1 and c_2 such that*

$$c_1 W(t) \leq \tilde{P}(t) - P(t) \leq c_2 W(t), \quad t \in I, \tag{3.7}$$

then $d_\pm(H_0) = d_\pm(\tilde{H}_0)$.

Proof The main idea of the proof is similar to that of Theorem 3.1. It suffices to show that (3.3) holds by Lemma 2.1. The proof is divided into three steps.

Step 1. We prove that $D(H_{00}) = D(\tilde{H}_{00})$.

It is evident that $\mathcal{L}^2_{\tilde{W},0}(I) = \mathcal{L}^2_{W,0}(I)$ since $\tilde{W}(t) = W(t) > 0$ for $t \in I$. For any $[y] \in D(H_{00})$, there exist $y \in [y]$ and $g \in \mathcal{L}^2_W(I)$ such that $y \in \mathcal{L}^2_{W,0}(I)$ and (3.4) holds. Take $\tilde{g}(t) := (\tilde{g}_1^T(t), \tilde{g}_2^T(t))^T$ with $\tilde{g}_j(t) \in \mathbb{C}^n, j = 1, 2$, such that

$$\left(\tilde{g}_1^T(t+1), \tilde{g}_2^T(t)\right)^T = W^{-1}(t)\left[J\triangle y(t) - \tilde{P}(t)R(y)(t)\right], \quad t \in I.$$

Then the corresponding class $[\tilde{g}] \in L^2_W(I)$ since $y \in \mathcal{L}^2_{W,0}(I)$ and

$$\begin{aligned}
\tilde{\mathfrak{L}}(y)(t) &= J\triangle y(t) - \tilde{P}(t)R(y)(t) \\
&= W(t)R(g)(t) - \left(\tilde{P}(t) - P(t)\right)R(y)(t) \\
&= W(t)R(\tilde{g})(t), \quad t \in I.
\end{aligned} \tag{3.8}$$

Thus $([y], [\tilde{g}]) \in \tilde{H}_{00}$. Hence, $[y] \in D(\tilde{H}_{00})$. Therefore, $D(H_{00}) \subset D(\tilde{H}_{00})$. With a similar argument, we can show that $D(\tilde{H}_{00}) \subset D(H_{00})$. Consequently, $D(H_{00}) = D(\tilde{H}_{00})$.

Step 2. We show that $H_{00}(0) = \tilde{H}_{00}(0)$.

For any $[g] \in H_{00}(0)$, there exist $g \in [g]$ and $y \in \mathcal{L}^2_{W,0}(I)$ with $[y] = 0$ such that (3.4) holds. Then, based on the discussions in Step 1, there exists $\tilde{g} \in \mathcal{L}^2_W(I)$ such that (3.8) holds. It follows from (3.7) that

$$c_1 R(y)^*(t)W(t)R(y)(t) \le R(y)^*(t)\left(\tilde{P}(t) - P(t)\right)R(y)(t) \le c_2 R(y)^*(t)W(t)R(y)(t), \quad t \in I,$$

which, together with $[y] = 0$, yields that $(\tilde{P}(t) - P(t))R(y)(t) = 0$ for $t \in I$ since $W(t)R(y)(t) = 0$ for $t \in I$. Hence it follows from (3.8) that $\tilde{\mathfrak{L}}(y)(t) = W(t)R(g)(t)$ for $t \in I$. This implies that $[g] \in \tilde{H}_{00}(0)$. Thus $H_{00}(0) \subset \tilde{H}_{00}(0)$. Similarly, we can show that $\tilde{H}_{00}(0) \subset H_{00}(0)$, and thus $H_{00}(0) = \tilde{H}_{00}(0)$.

Step 3. We show that $\tilde{H}_{00} - H_{00}$ is H_{00}-bounded with H_{00}-bound 0, that is, $\tilde{H}_{00} - H_{00}$ is bounded.

It is evident that $\tilde{H}_{00} - H_{00}$ is a Hermitian relation in $L^2_W(I) \times L^2_W(I)$ and $\tilde{H}_{00} = \tilde{H}_{00} - H_{00} + H_{00}$ according to Lemma 2.2.

Based on the discussions in Step 1, for any $[y] \in D(H_{00}) = D(\tilde{H}_{00}) = D(\tilde{H}_{00} - H_{00})$, there exist $y \in [y]$ with $y \in \mathcal{L}^2_{W,0}(I)$ and $g, \tilde{g} \in \mathcal{L}^2_W(I)$ such that (3.4) and (3.8) hold. This means that $([y], [g]) \in H_{00}$ and $([y], [\tilde{g}]) \in \tilde{H}_{00}$. Then $([y], [\tilde{g} - g]) \in \tilde{H}_{00} - H_{00}$, that is, $[\tilde{g} - g] \in (\tilde{H}_{00} - H_{00})([y])$. This, together with (3.8), yields that

$$\begin{aligned}
\left\| (\tilde{H}_{00} - H_{00})([y]) \right\|^2 &\le \left\| [\tilde{g} - g] \right\|^2 = \sum_{t \in I} R(\tilde{g} - g)^*(t)W(t)R(\tilde{g} - g)(t) \\
&= \sum_{t \in I} R(\tilde{g} - g)^*(t)\left(P(t) - \tilde{P}(t)\right)R(y)(t) \\
&= \sum_{t \in I} R(\tilde{g} - g)^*(t)\left(P(t) - \tilde{P}(t) + c_2 W(t)\right)R(y)(t) \\
&\quad - c_2 \sum_{t \in I} R(\tilde{g} - g)^*(t)W(t)R(y)(t).
\end{aligned} \tag{3.9}$$

It follows from (3.7) that $0 \leq P(t) - \tilde{P}(t) + c_2 W(t) \leq (c_2 - c_1) W(t)$ for $t \in I$. So $P(t) - \tilde{P}(t) + c_2 W(t)$ is positive semidefinite. Thus by using Cauchy's inequality from (3.9) it follows that

$$
\begin{aligned}
\left\| \left(\tilde{H}_{00} - H_{00} \right) \left([y] \right) \right\|^2 \\
\leq \left\| [\tilde{g} - g] \right\|^2 \\
\leq \left(\sum_{t \in I} R(\tilde{g} - g)^*(t) \left(P(t) - \tilde{P}(t) + c_2 W(t) \right) R(\tilde{g} - g)(t) \right)^{1/2} \\
\times \left(\sum_{t \in I} R(y)^*(t) \left(P(t) - \tilde{P}(t) + c_2 W(t) \right) R(y)(t) \right)^{1/2} \\
+ |c_2| \left(\sum_{t \in I} R(\tilde{g} - g)^*(t) W(t) R(\tilde{g} - g)(t) \right)^{1/2} \left(\sum_{t \in I} R(y)^*(t) W(t) R(y)(t) \right)^{1/2} \\
\leq (c_2 - c_1) \left(\sum_{t \in I} R(\tilde{g} - g)^*(t) W(t) R(\tilde{g} - g)(t) \right)^{1/2} \left(\sum_{t \in I} R(y)^*(t) W(t) R(y)(t) \right)^{1/2} \\
+ |c_2| \|\tilde{g} - g\| \|y\| \\
\leq (c_2 - c_1) \|\tilde{g} - g\| \|y\| + |c_2| \|\tilde{g} - g\| \|y\|,
\end{aligned}
$$

which yields that $\|(\tilde{H}_{00} - H_{00})([y])\| \leq (c_2 - c_1 + |c_2|)\|y\|$. This implies that $\tilde{H}_{00} - H_{00}$ is H_{00}-bounded with H_{00}-bound 0, that is, $\tilde{H}_{00} - H_{00}$ is bounded.

Based on these three statements, it follows that H_{00} and $\tilde{H}_{00} - H_{00}$ satisfy the conditions in Lemma 2.1. Therefore $d_\pm(\tilde{H}_{00}) = d_\pm(H_{00})$ by Lemma 2.1. So (3.3) holds. This completes the proof.   □

In the particular case that $W_1(t) > 0$ and $W_2(t) = 0$, we obtain the following result.

Theorem 3.3 *Assume that* (A_1) *holds. Let* $W_1(t) = \tilde{W}_1(t) > 0$ *and* $W_2(t) = \tilde{W}_2(t) \equiv 0$ *for* $t \in I$. *If there exist two constants* c_1 *and* c_2 *such that* (3.7) *holds, then* $d_\pm(H_0) = d_\pm(\tilde{H}_0)$.

Proof The main idea of the proof is similar to that of Theorem 3.2 with only Step 1 being replaced by the following:

Step 1. We prove that $D(H_{00}) = D(\tilde{H}_{00})$.

It is evident that $\mathcal{L}^2_{\tilde{W},0}(I) = \mathcal{L}^2_{W,0}(I)$ since $\tilde{W}(t) = W(t)$ for $t \in I$. For any $[y] \in D(H_{00})$, there exist $y \in [y]$ and $g \in \mathcal{L}^2_W(I)$ such that $y \in \mathcal{L}^2_{W,0}(I)$ and (3.4) holds. Next, we will show that $[y] \in D(\tilde{H}_{00})$. To prove $[y] \in D(\tilde{H}_{00})$, it suffices to prove that there exists $\tilde{g} \in \mathcal{L}^2_W(I)$ such that (3.8) holds. By (3.7) and the assumptions that $W_1(t) = \tilde{W}_1(t) > 0$ and $W_2(t) = \tilde{W}_2(t) \equiv 0$ for $t \in I$, we have that

$$
\begin{aligned}
\tilde{P}(t) - P(t) - c_1 W(t) = \begin{pmatrix} C(t) - \tilde{C}(t) - c_1 W_1(t) & \tilde{A}^*(t) - A^*(t) \\ \tilde{A}(t) - A(t) & \tilde{B}(t) - B(t) \end{pmatrix} \geq 0, \\
c_2 W(t) - \left(\tilde{P}(t) - P(t) \right) = \begin{pmatrix} c_2 W_1(t) - C(t) + \tilde{C}(t) & A^*(t) - \tilde{A}^*(t) \\ A(t) - \tilde{A}(t) & B(t) - \tilde{B}(t) \end{pmatrix} \geq 0.
\end{aligned}
\tag{3.10}
$$

According to [36, Observation 7.1.2.], we get that $\tilde{B}(t) - B(t) \geq 0$ and $B(t) - \tilde{B}(t) \geq 0$ for $t \in I$. Thus $B(t) = \tilde{B}(t)$ for $t \in I$. This, together with (3.10), yields that $A(t) = \tilde{A}(t)$ for $t \in I$ by

[36, Observation 7.1.10.]. Consequently, $(\tilde{A}_1)$ holds, and $\tilde{P}(t) - P(t) = \text{diag}\{C(t) - \tilde{C}(t), 0\}$. Inserting it into (3.8), we get that

$$\tilde{g}_1(t+1) = g_1(t+1) - W_1^{-1}(t)\big(C(t) - \tilde{C}(t)\big)y_1(t+1)$$

for $t \in I$, where $y(t) = (y_1^T(t), y_2^T(t))^T$, $g(t) = (g_1^T(t), g_2^T(t))^T$, and $\tilde{g}(t) = (\tilde{g}_1^T(t), \tilde{g}_2^T(t))^T$ with $y_j(t), g_j(t), \tilde{g}_j(t) \in \mathbb{C}^n$, $j = 1, 2$. In addition, noting that $y \in \mathcal{L}_{W,0}^2(I)$ and $g \in \mathcal{L}_W^2(I)$, we have that $\sum_{t \in I} R(\tilde{g})^*(t)W(t)R(\tilde{g})(t) < +\infty$. So the corresponding class $[\tilde{g}] \in L_W^2(I)$. Thus $[y] \in D(\tilde{H}_{00})$. Hence $D(H_{00}) \subset D(\tilde{H}_{00})$. With a similar argument, we can show that $D(\tilde{H}_{00}) \subset D(H_{00})$. Consequently, $D(H_{00}) = D(\tilde{H}_{00})$. The proof is complete.   □

Remark 3.2 We remark that (3.7) implies that c_1 and c_2 are real numbers since $P(t)$, $\tilde{P}(t)$, and $W(t)$ are Hermitian matrices.

The following result is a direct consequence of Theorems 3.1–3.3.

Corollary 3.1 *If any of the conditions of Theorems 3.1, 3.2, and 3.3 hold, then $\mathfrak{L}$ is in the limit $(d_+(H_0), d_-(H_0))$ case at $+\infty$ if and only if $\tilde{\mathfrak{L}}$ is in the limit $(d_+(H_0), d_-(H_0))$ case at $+\infty$. In particular, $\mathfrak{L}$ is in l.c.c. at $+\infty$ if and only if $\tilde{\mathfrak{L}}$ is in l.c.c. at $+\infty$; $\mathfrak{L}$ is in l.p.c. at $+\infty$ if and only if $\tilde{\mathfrak{L}}$ is in l.p.c. at $+\infty$.*

3.2 Stability of deficient indices for (1.1_λ) in the case of $\tilde{W}(t) \neq W(t)$

In this subsection, we study the stability of deficiency indices for system (1.1_λ) when $\tilde{W}(t) \neq W(t)$ for $t \in I$. Note that H_0 and $\tilde{H}_0$ are defined in $(L_W^2(I))^2$ and $(L_{\tilde{W}}^2(I))^2$, respectively, and it is difficult to study the stability of deficiency indices of H_0 and $\tilde{H}_0$ since $(L_W^2(I))^2$ and $(L_{\tilde{W}}^2(I))^2$ are different spaces. So we turn to study the invariance of the limit circle and limit point cases.

Theorem 3.4 *Assume that (A_1) holds. Let $\tilde{P}(t) = P(t)$ for $t \in I$. If there exist two positive constants c_1 and c_2 such that*

$$c_1 W(t) \le \tilde{W}(t) \le c_2 W(t), \quad t \in I, \tag{3.11}$$

then $\mathfrak{L}$ is in l.c.c. at $+\infty$ if and only if $\tilde{\mathfrak{L}}$ is in l.c.c. at $+\infty$.

Proof First, consider the necessity. Suppose that $\mathfrak{L}$ is in l.c.c. at $+\infty$. Then $d_\pm(H_0) = 2n$. By (2) of [5, Corollary 5.1] we get that $n_\pm(H_0) = 2n$. It follows from Theorem 2.1 that $n_\lambda(H_0) = 2n$ for all $\lambda \in \mathbb{C}$. Again by (2) of [5, Corollary 5.1], $n_0(H_0) - d_0(H_0) = n_\pm(H_0) - d_\pm(H_0)$, that is, $2n - d_0(H_0) = 2n - 2n$. Hence $d_0(H_0) = 2n$.

Next, we will show that $d_0(\tilde{H}_0) = d_0(H_0)$. It suffices to prove that $\tilde{H}_0$ and H_0 with $\lambda = 0$ have the same deficiency space, that is, $R(\tilde{H}_0)^\perp = R(H_0)^\perp$. By [26, Lemma 2.4] and [5, Theorem 3.1],

$$\begin{aligned}
R(H_0)^\perp &= \big\{[y] \in L_W^2(I) : ([y], 0) \in H_0^*\big\} \\
&= \big\{[y] \in L_W^2(I) : ([y], 0) \in H\big\} \\
&= \big\{[y] \in L_W^2(I) : \text{there exists } y \in [y] \text{ such that } \mathfrak{L}(y)(t) = 0, t \in I\big\}.
\end{aligned}$$

With a similar argument, we can show that

$$R(\tilde{H}_0)^\perp = \left\{ [y] \in L^2_{\tilde{W}}(I) : \text{ there exists } y \in [y] \text{ such that } \tilde{\mathfrak{L}}(y)(t) = 0, t \in I \right\}.$$

Since $\tilde{P}(t) = P(t)$ for $t \in I$, $\tilde{\mathfrak{L}}(y)(t) = \mathfrak{L}(y)(t)$ for $t \in I$. In addition, it follows from (3.11) that $L^2_W(I) = L^2_{\tilde{W}}(I)$. Thus $R(\tilde{H}_0)^\perp = R(H_0)^\perp$. Consequently, $d_0(\tilde{H}_0) = d_0(H_0) = 2n$. This, together with [5, Theorem 5.2] and the fact that $n \le d_\lambda(\tilde{H}_0) \le 2n$, yields that $d_\pm(\tilde{H}_0) = 2n$. Therefore $\tilde{\mathfrak{L}}$ is in l.c.c. at $+\infty$.

With a similar argument, the sufficiency can be shown by noting that (3.11) implies $\frac{1}{c_2}\tilde{W}(t) \le W(t) \le \frac{1}{c_1}\tilde{W}(t)$ for $t \in I$. This completes the proof. $\square$

Introduce the following new system:

$$J\triangle y(t) - \tilde{P}(t)R(y)(t) = \lambda W(t)R(y)(t), \quad t \in I. \tag{3.12_λ}$$

If we regard (3.12_λ) as the perturbation of (1.1_λ) and (3.1_λ) as the perturbation of (3.12_λ), then the following two results can be directly derived by Theorem 3.4 and Corollary 3.1.

Theorem 3.5 *Assume that* (A_1) *holds. If* (3.2) *and* (3.11) *hold, then* $\mathfrak{L}$ *is in l.c.c. at* $+\infty$ *if and only if* $\tilde{\mathfrak{L}}$ *is in l.c.c. at* $+\infty$.

Theorem 3.6 *Assume that* (A_1) *holds. If* (3.7) *and* (3.11) *with* $W_1(t) > 0$ *and either* $W_2(t) > 0$ *or* $W_2(t) = 0$ *for* $t \in I$ *hold, then* $\mathfrak{L}$ *is in l.c.c. at* $+\infty$ *if and only if* $\tilde{\mathfrak{L}}$ *is in l.c.c. at* $+\infty$.

Theorem 3.7 *Assume that* (A_1) *and* (3.11) *hold. Then* (2.2) *holds if and only if*

$$\sum_{t=1}^{+\infty} \sqrt{\tilde{w}_1(t-1)\tilde{w}_2(t)} = \infty, \tag{3.13}$$

where $\tilde{w}_j(t)$ *is the minimal eigenvalue of* $\tilde{W}_j(t)$ *for* $j = 1, 2$. *Moreover,* $\mathfrak{L}$ *is in l.p.c. at* $+\infty$ *if and only if* $\tilde{\mathfrak{L}}$ *is in l.p.c. at* $+\infty$ *in this case.*

Proof It follows from (3.11) that

$$c_1 w_j(t) \le \tilde{w}_j(t) \le c_2 w_j(t), \quad j = 1, 2, t \in I.$$

Therefore (2.2) holds if and only if (3.13) holds. Consequently, in this case, $\mathfrak{L}$ is in l.p.c. at $+\infty$ if and only if $\tilde{\mathfrak{L}}$ is in l.p.c. at $+\infty$ according to Lemma 2.3. This completes the proof. $\square$

Remark 3.3 The results of Theorem 3.7 shows that if the weight functions of (1.1_λ) and (3.1_λ) satisfy quite strong conditions, then the deficient indices of (1.1_λ) are invariant under any perturbation of $P(t)$.

Remark 3.4 Note that the number of linearly independent square summable solutions of (1.1_λ) is invariant if its coefficient matrices $P(t)$ and $W(t)$ vary at finite points. Therefore we remark that the results obtained in the present paper still hold if there exists $t_0 \in I$ such that the assumptions of our results are satisfied for $t \ge t_0$.

4 Examples

In this section, we give an example illustrating the perturbation results obtained in this paper. It is well known that there are a lot of limit circle and limit point criteria for system (1.1_λ). The criteria of stability of deficiency indices of (1.1_λ) obtained in this paper provide an alternate way to determine the extreme limit type. However, to the best of our knowledge, there are seldom criteria of the intermediate cases for (1.1_λ). Therefore it is very difficult to determine if (1.1_λ) is in the intermediate cases. So next, we give an example in the intermediate case to illustrate the advantages of our conclusions obtained in this paper.

Consider system (1.1_λ) with $n = 2$,

$$
A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B(t) = \mathrm{diag}\{0, 3^{-t}/2\},
$$
$$
C(t) = \mathrm{diag}\{0, -3^t\}, \qquad W(t) = \mathrm{diag}\{1, 0, 0, 0\}
\tag{4.1}
$$

for $t \in I$. It is obvious that assumptions (A_1) and (A_2) hold. So, $d_\lambda(H_0) = n_\lambda(H_0)$. Next, we will show that (1.1_λ) with those matrix functions is in the limit-3 case at $t = +\infty$.

In fact, by Lemma 2.1 in [37], $n_\lambda(H_0)$ is equal to the number of linearly independent solutions of the following fourth-order difference equation:

$$
\triangle^2\big[2 \cdot 3^t \triangle^2 x(t-2)\big] - \triangle\big[-3^t \triangle x(t-1)\big] = \lambda x(t), \quad t \in I,
\tag{4.2$_\lambda$}
$$

in l^2 for any $\lambda \in \mathbb{C}$. The solutions of (4.2_λ) with $\lambda = 0$ are of the form α^t with α satisfying the equation

$$
\big(6\alpha^2 - 5\alpha + 1\big)\big(3\alpha^2 - 5\alpha + 2\big) = 0.
$$

By a simple calculation there are four distinct roots of this equation:

$$
\alpha_1 = 1/2, \qquad \alpha_2 = 1/3, \qquad \alpha_3 = 2/3, \qquad \alpha_4 = 1.
$$

Then $x_j(t) = \alpha_j^t$ for $j = 1, 2, 3, 4$ are four linearly independent solutions of (4.2_λ) with $\lambda = 0$. It is evident that $x_j \in l^2$ for $j = 1, 2, 3$, but $x_4 \notin l^2$. This implies that (1.1_λ) is neither in l.p.c. nor in l.c.c. at $t = +\infty$ by Theorem 2.1 and Remark 2.2. So (1.1_λ) is in the limit-3 case at $t = +\infty$.

Now we consider system (3.1_λ) with $n = 2$,

$$
\tilde{A}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{B}(t) = \mathrm{diag}\{0, 3^{-t}/2\},
$$
$$
\tilde{C}(t) = \mathrm{diag}\{\sin t + e^{-t} - 2, -3^t\}, \qquad \tilde{W}(t) = \mathrm{diag}\{1, 0, 0, 0\}
\tag{4.3}
$$

for $t \in I$. It is evident that assumptions $(\tilde{A}_1)$ and $(\tilde{A}_2)$ still hold. But it is difficult to determine the limit type of this system as discussed before. As pointed out in the first paragraph of this section, almost all criteria for limit types were established only for the limit point and limit circle cases. However, there are seldom criteria of the intermediate cases. In addition, we note that the existing criteria for limit types are not applicable to (3.1_λ) with

matrix functions satisfying (4.3). At this point, we find that we may regard (4.3) as the perturbations of (4.1). Moreover, it is easy to verify that the conditions of Theorem 3.1 are satisfied with $c_1(t) = 2 - \sin t - e^{-t}$, $c_2(t) = c_2 = 0$, and $c_1 = 4$. Therefore (3.1_λ) with (4.3) is also in the limit-3 case at $t = +\infty$ by Theorem 3.1.

Availability of data and materials
No applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this manuscript. Both authors read and approved the final manuscript.

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