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Nonexistence of global solutions of fractional diffusion equation with time-space nonlocal source

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Abstract

We prove the nonexistence of solutions of the fractional diffusion equation with time-space nonlocal source

$$u_t + (-\Delta)^{\frac{\beta}{2}} u = (1 + |x|)^{\gamma} \int_0^t (t-s)^{\alpha-1} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r ds$$

for $(x, t) \in \mathbb{R}^N \times (0, \infty)$ with initial data $u(x, 0) = u_0(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$, where $p, q, r > 1$, $q(p+r) > q+r$, $0 < \gamma \leq 2$, $0 < \alpha < 1$, $0 < \beta \leq 2$, $(-\Delta)^{\frac{\beta}{2}}$ stands for the fractional Laplacian operator of order β , the weight function $v(x)$ is positive and singular at the origin, and $\|\cdot\|_q$ is the norm of L^q space.

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1 Introduction

In this paper, we consider the fractional diffusion problem with time-space nonlocal source term of the form

$$u_t + (-\Delta)^{\frac{\beta}{2}} u = (1 + |x|)^{\gamma} \int_0^t (t-s)^{\alpha-1} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r ds, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (1)$$

subject to the initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad (2)$$

where $u = u(x, t)$ is a real-valued unknown function of (x, t) , $p, q, r > 1$, $q(p+r) > q+r$, $0 < \gamma \leq 2$, $0 < \alpha < 1$, $0 < \beta \leq 2$, the weight function $v(x)$ is positive and singular at the origin, that is, there exist $c > 0$ and $s \geq 0$ such that $v(x) \geq c|x|^{-s}$, $x \in \mathbb{R}^N \setminus \{0\}$, and $\|\cdot\|_q$ is the norm of the space $L^q(\mathbb{R}^N)$.

Here we emphasize that equation (1) is a possible model of invasion of a population, and the fractional Laplacian represents the dispersion of the individuals, where $u(x, t)$ repre-

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sents the density of the species at position x and time t . A nonlocal term is a way to express that the evolution of the species at a point of space depends not only on nearby density but also on the mean value of the total amount of species (see [1]). Equation (1) also suggests the possibility of an interesting physical model in which a superdiffusive medium is coupled to a classically diffusive medium. The right-hand side of (1) can be interpreted as the effect of a classical diffusive medium that is nonlinearly linked to a superdiffusive medium. Such a link may come in the form of a porous material with reactive properties that is partially insulated by contact with a classical diffusive material.

Many researchers have shown a keen interest in the study of differential equations with fractional diffusion in the last few decades. In fact, it has developed into a hot topic of research nowadays. We can find its applications in probability theory, potential theory, fluid dynamics, conformal geometry, mathematical finance, and so on. In addition, fractional diffusion is also important in physics and biology (see [2]).

Let us now recall some known results on fractional diffusion equations. Since the literature on these equations is extensive, we only mention the papers related to our work. Fujita [6] studied the following Cauchy problem involving the semilinear heat equation with a local source term:

$$u_t - \Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, t > 0, \quad (3)$$

supplemented with initial data (2). He has shown that every solution of the problem blows up in a finite time if $u \geq 0$, $u_0(x) \not\equiv 0$, and $p < 1 + 2/N$. On the other hand, for $p > 1 + 2/N$ and the initial data bounded by sufficiently small Gaussian, the solution of problem (3)–(2) exists globally. The critical case $p = 1 + 2/N$ was proved later in the blowup range (see [8, 11]). In [15], it is proved that the solution of (3)–(2) exists globally if the initial value u_0 is small enough in $L^{q_c}(\mathbb{R}^N)$, $q_c = N(p-1)/2 > 1$.

Kirane, Laskri, and Tatar [9] discussed the evolution equation

$${}_0^C \mathbb{D}_t^\alpha u_t + (-\Delta)^{\frac{\beta}{2}} u = h(x, t)|u|^{p+1}, \quad x \in \mathbb{R}^N, t > 0, \quad (4)$$

where $u_0(x) \in C_0(\mathbb{R}^N)$, $0 < \alpha < 1$, $0 < \beta \leq 2$, $p > 0$, $h(x, t) \geq C|x|^\sigma t^\rho$ for $x \in \mathbb{R}^N$, $t > 0$, $C > 0$, and σ, ρ satisfy appropriate conditions. For $0 < p \leq (\alpha(\sigma + \beta) + \beta\rho)/(\alpha N + \beta(1 - \alpha))$, they showed that problem (4)–(2) has no global weak nonnegative solution.

Cazenave, Dickstein, and Weissler [3] have proved that all nontrivial nonnegative solutions of the equation

$$u_t - \Delta u = \int_0^t (t-s)^{-\gamma} |u(s)|^{p-1} u(s) ds, \quad x \in \mathbb{R}^N, t > 0, \quad (5)$$

with (2), $u_0 \in C_0(\mathbb{R}^N)$, and $0 < \gamma < 1$, blow up in a finite time when $u_0 \geq 0$, $u_0 \not\equiv 0$, and $p \leq \max\{1 + 2(2 - \gamma)/(N - 2 + 2\gamma)_+, 1/\gamma\}$, and the solution exists globally if $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$, $q_{sc} = N(p-1)/(4 - 2\gamma)$ with sufficiently small $\|u_0\|$, and $p > \max\{1 + 2(2 - \gamma)/(N - 2 + 2\gamma)_+, 1/\gamma\}$. In case of $\gamma = 0$, all nontrivial solutions blow up as established by Souplet [14].

Fino and Kirane [5] considered the equation

$$u_t + (-\Delta)^{\beta/2} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u(s)|^{p-1} u(s) ds, \quad x \in \mathbb{R}^N, t > 0, \quad (6)$$

with (2), where $u_0(x) \in C_0(\mathbb{R}^N)$, $0 < \beta \leq 2$, $0 < \gamma < 1$, $p > 1$. They obtained the blowup and global existence results by using the test function method. In precise terms, if $u_0 \geq 0$, $u_0 \not\equiv 0$, and $p \leq \max\{1 + \beta(2 - \gamma)/(N - \beta + \beta\gamma)_+, 1/\gamma\}$, then any solution blows up in finite time. On the other hand, if $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$, where $q_{sc} = N(p - 1)/\beta(2 - \gamma)$ with sufficiently small $\|u_0\|$, and $p > \max\{1 + \beta(2 - \gamma)/(N - \beta + \beta\gamma)_+, 1/\gamma\}$, then the solution u exists globally.

For the studies on a nonlocal problem, we refer the reader to the paper by Chen et al. [4], who studied some degenerated parabolic inequalities with local and nonlocal nonlinear terms

$$\frac{\partial u^m}{\partial t} \geq \Delta u + \|u\|_p^p + b(x, t)u^q(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (7)$$

$$\frac{\partial u^m}{\partial t} \geq \Delta u + u^q(x, t) \left(\int_{\mathbb{R}^N} \beta(y)u^p(y, t) dy \right)^{\frac{n}{p}}, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (8)$$

and proved the global nonexistence of nontrivial solutions by the test function method.

Furthermore, in [16] the authors considered the homogeneous and inhomogeneous inequalities with singular potential and weighted nonlocal source term

$$u_t \geq \Delta u^m - V(x) + |x|^\alpha u^p \left\| \beta^{\frac{1}{q}}(x)u \right\|_q^r, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (9)$$

$$u_t \geq \Delta u^m - V(x) + |x|^\alpha u^p \left\| \beta^{\frac{1}{q}}(x)u \right\|_q^r + \omega(x), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (10)$$

and showed the nonexistence of nontrivial global weak solutions for problems (9)–(2) and (10)–(2).

The purpose of this paper is establishing the nonexistence of solutions of problem (1)–(2) by the test function method (see [7, 12]). This paper is organized as follows. In Sect. 2, we introduce some preliminaries including some definitions and formulation of the main results. The proofs of the main results are presented in Sect. 3.

2 Preliminaries and main results

In this section, we present some preliminaries and announce the main results. Let us first recall some definitions and properties concerning fractional integrals and derivatives.

Let $AC([0, T])$ be the space of all absolutely continuous functions on $[0, T]$ with T finite. We denote by $\mathbb{D}_{0|t}^\alpha f(t)$ and $\mathbb{D}_{t|T}^\alpha f(t)$ the left- and right-sided Riemann–Liouville fractional derivatives of order $(0 < \alpha < 1)$ for a function $f(t) \in AC([0, T])$, $t > 0$ defined by

$$\begin{aligned} (\mathbb{D}_{0|t}^\alpha f)(t) &= \frac{d}{dt} \mathbb{I}_{0|t}^{1-\alpha} f(t), \\ (\mathbb{D}_{t|T}^\alpha f)(t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s-t)^{-\alpha} f(s) ds, \end{aligned}$$

where

$$(\mathbb{I}_{0|t}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (11)$$

is the Riemann–Liouville fractional integral of order $0 < \alpha < 1$ for all $f \in L^q(0, T)$, $1 \leq q \leq \infty$.

Lemma 2.1 (Formula of integration by parts) *Let $f, g, \mathbb{D}_{0|t}^\alpha f, \mathbb{D}_{t|T}^\alpha g \in C([0, T])$. Then the formula of integration by parts (see (2.64) p. 46 in [13]) is*

$$\int_0^T (\mathbb{D}_{0|t}^\alpha f)(t)g(t) dt = \int_0^T f(t)(\mathbb{D}_{t|T}^\alpha g)(t) dt. \quad (12)$$

Lemma 2.2 *Let $f \in AC^2([0, T]) := \{f : [0, T] \rightarrow \mathbb{R} \text{ be such that } f' \in AC([0, T])\}$. Then*

$$-\frac{d}{dt} \mathbb{D}_{t|T}^\alpha f(t) = \mathbb{D}_{t|T}^{\alpha+1} f(t). \quad (13)$$

Moreover, for all $1 \leq q \leq \infty$, we have the following equality almost everywhere on $[0, T]$:

$$\mathbb{D}_{0|t}^\alpha \mathbb{I}_{0|t}^\alpha = \text{Id}_{L^q}(0, T). \quad (14)$$

Lemma 2.3 (See [5]) *Let*

$$f(t) = \left(1 - \frac{t}{T}\right)_+^\sigma$$

with $t \geq 0$, $T > 0$, and $\sigma \gg 1$. Then for all $\alpha \in (0, 1)$, we have

$$\mathbb{D}_{t|T}^\alpha f(t) = C_1 T^{-\alpha} \left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha}, \quad (15)$$

$$\mathbb{D}_{t|T}^{\alpha+1} f(t) = C_2 T^{-\alpha-1} \left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha-1}, \quad (16)$$

and

$$(\mathbb{D}_{t|T}^\alpha f)(T) = 0, \quad (\mathbb{D}_{t|T}^\alpha f)(0) = C_1 T^{-\alpha}, \quad (17)$$

where $C_1 = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}$ and $C_2 = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}$.

Definition 2.4 (See [10]) *Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$. Then the fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}$ in $\mathbb{R}^N$, $0 < \beta \leq 2$ is the nonlocal operator given by*

$$(-\Delta)^{\frac{\beta}{2}} : \nu \in \mathcal{S} \longrightarrow (-\Delta)^{\frac{\beta}{2}} \nu(x) = C_{N,\nu} p.v. \int_{\mathbb{R}^N} \frac{\nu(x) - \nu(y)}{|x - y|^{N+\beta}} dy, \quad (18)$$

as long as the right-hand side exists, where $p.v.$ stands for Cauchy's principal value, $C_{N,\beta} = \frac{4^{\frac{\beta}{2}} \Gamma(\frac{N}{2} + \frac{\beta}{2})}{\pi^{\frac{N}{2}} \Gamma(-\frac{\beta}{2})}$ is a normalization constant, and Γ denotes the gamma function.

Lemma 2.5 (See [10]) *Let $0 < \beta \leq 2$, $\theta_1, \theta'_1, \theta_2 > 1$, $\frac{1}{\theta_1} + \frac{1}{\theta'_1} = 1$. Consider the function $\chi : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by*

$$\chi(x) = \begin{cases} 1 & (|x| \leq 1), \\ (2 - |x|)^\mu & (1 < |x| < 2), \\ 0 & (|x| \geq 2), \end{cases} \quad (19)$$

with $\mu > 2\theta'_1$. Then

$$0 < \int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} |(-\Delta)^{\frac{\beta}{2}} \chi|^{\theta_2} \chi^{-\frac{\theta_2}{\theta_1}} dx < \infty. \quad (20)$$

Lemma 2.6 Let $0 < \beta \leq 2$ and $T > 0$, and let φ be defined by

$$\varphi(x) = \chi\left(\frac{x}{T^{\frac{1}{\beta}}}\right) \quad \text{for all } x \in \mathbb{R}^N,$$

where χ is given in (19). Then $(-\Delta)^{\frac{\beta}{2}} \varphi$ satisfies

$$(-\Delta)^{\frac{\beta}{2}} \varphi(x) = T^{-1} (-\Delta)^{\frac{\beta}{2}} \chi\left(\frac{x}{T^{\frac{1}{\beta}}}\right). \quad (21)$$

Next, we give some definitions.

Definition 2.7 Let $\Omega = \mathbb{R}^N \times (0, T)$ and $u_0(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$. The function $u(x, t)$ is called a weak solution of problem (1)–(2) if $u \in C([0, T]; H^\beta(\mathbb{R}^N))$, $(1 + |x|)^\gamma \|u\|^{\frac{1}{q}} v^{\frac{1}{q}}(x) u\|_q^r \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ and satisfies

$$\begin{aligned} & \int_{\Omega} (1 + |x|)^\gamma \int_0^t (t-s)^{\alpha-1} \|u\|^p v^{\frac{1}{q}}(x) u\|_q^r ds \zeta(x, t) dx dt + \int_{\mathbb{R}^N} u(x, 0) \zeta(x, 0) dx \\ &= \int_{\Omega} u (-\Delta)^{\frac{\beta}{2}} \zeta(x, t) dx dt - \int_{\Omega} u \zeta_t(x, t) dx dt \end{aligned} \quad (22)$$

for any nonnegative function $\zeta \in C([0, T]; H^\beta(\mathbb{R}^N)) \cap C^1([0, T]; L^2(\mathbb{R}^N))$ such that $\zeta(\cdot, T) = 0$. If $T = \infty$, then we say that u is a global weak solution of problem (1)–(2).

Definition 2.8 Suppose that a function $u(x, t)$ is a weak solution of problem (1)–(2) in $\mathcal{S} = \mathbb{R}^N \times (0, \infty)$. We say that $u(x, t)$ is a solution with positive lower bound in $\mathcal{S}$ if there exists a positive constant $c > 0$ such that $u(x, t) \geq c$ in $\mathcal{S}$.

Finally, we formulate our main results as follows.

Theorem 2.9 Let $u_0 \geq 0, u_0 \not\equiv 0$. Problem (1)–(2) has no nontrivial global solutions for $\gamma > \frac{rs}{q}$ and

$$N \leq \frac{(1 + \alpha(p+r))\beta q + \gamma q - sr}{(p+r-1)q - r}. \quad (23)$$

Theorem 2.10 Let $u_0 \geq 0, u_0 \not\equiv 0$. Problem (1)–(2) has no solution with positive lower bound in $\mathcal{S}$ for

$$s < N + \frac{q}{r}(\alpha\beta + \beta + \gamma). \quad (24)$$

3 Proof of main results

Proof of Theorem 2.9 We assume that u exists globally. In the definition of a weak solution of problem (1)–(2), we take $\zeta(x, t) = \mathbb{D}_{t|T}^\alpha \phi(x, t) = \varphi(x) \mathbb{D}_{t|T}^\alpha \lambda(t)$, $0 < \alpha < 1$, where

$$\varphi(x) = \chi\left(\frac{x}{T^{\frac{1}{\beta}}}\right), \quad \lambda(t) = \left(1 - \frac{t}{T}\right)_+^\sigma$$

with $\sigma > (1 + \alpha)\theta'_1$, $\theta'_1 > 1$, $\alpha\theta'_1 > 1$, $T > 0$.

Then from (11), using the integration-by-parts formula and the fact that $\mathbb{D}_{0|t}^\alpha(\mathbb{I}_{0|t}^\alpha \phi)(x, t) = \phi(x, t)$ in the first term on the left-hand side of (22), we obtain

$$\begin{aligned} & \int_{\Omega} (1 + |x|)^\gamma \int_0^t (t-s)^{\alpha-1} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r ds \mathbb{D}_{t|T}^\alpha \phi \, dx \, dt \\ &= \Gamma(\alpha) \int_{\Omega} (1 + |x|)^\gamma \mathbb{I}_{0|t}^\alpha (|u|^p \|v(x)^{\frac{1}{q}}u\|_q^r) \mathbb{D}_{t|T}^\alpha \phi \, dx \, dt \\ &= \Gamma(\alpha) \int_{\Omega} (1 + |x|)^\sigma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r \phi \, dx \, dt. \end{aligned}$$

Therefore expression (22) can be written as

$$\begin{aligned} & \Gamma(\alpha) \int_{\Omega} (1 + |x|)^\gamma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r \phi \, dx \, dt + \int_{\Omega} u_0(x) \mathbb{D}_{t|T}^\alpha \phi(x, 0) \, dx \, dt \\ &= \int_{\Omega} u \mathbb{D}_{t|T}^{\alpha+1} \phi \, dx \, dt + \int_{\Omega} u(-\Delta)^{\frac{\beta}{2}} \mathbb{D}_{t|T}^\alpha \phi \, dx \, dt. \end{aligned}$$

Thus

$$\begin{aligned} & \Gamma(\alpha) \int_{\Omega} (1 + |x|)^\sigma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r \phi \, dx \, dt + \int_{\Omega} u_0(x) \varphi \mathbb{D}_{t|T}^\alpha \lambda(0) \, dx \, dt \\ &\leq \int_{\Omega} |u| |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)| \varphi \, dx \, dt + \int_{\Omega} |u| |(-\Delta)^{\frac{\beta}{2}} \varphi| |\mathbb{D}_{t|T}^\alpha \lambda(t)| \, dx \, dt. \end{aligned} \quad (25)$$

Now we estimate the right-hand side of inequality (25). First, applying the Hölder inequality to the first term in the right-hand side of (25), we have

$$\begin{aligned} & \int_{\Omega} |u| |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)| \varphi \, dx \, dt \\ &\leq \int_0^T \left(\int_{\mathbb{R}^N} (1 + |x|)^\gamma |u|^p \phi \, dx \right)^{\frac{1}{\theta_1}} \left(\int_{\mathbb{R}^N} v(x) |u|^q \, dx \right)^{\frac{r}{q\theta_1}} \\ &\quad \times \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} \varphi^{\theta_2} |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{1}{\theta_2}} dt \\ &\leq \left(\int_{\Omega} (1 + |x|)^\gamma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r \phi \, dx \, dt \right)^{\frac{1}{\theta_1}} \\ &\quad \times \left(\int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} \varphi^{\theta_2} |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt \right)^{\frac{1}{\theta'_1}}. \end{aligned}$$

Applying ε -Young's inequality, we obtain

$$\begin{aligned} & \int_{\Omega} |u| |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)| \varphi \, dx \, dt \\ & \leq \varepsilon \int_{\Omega} (1 + |x|)^{\gamma} \|u^p\|_{v^{\frac{1}{q}}(x)u}^r \phi \, dx \, dt \\ & \quad + C \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} \varphi^{\theta_2} |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt, \end{aligned}$$

where

$$\begin{aligned} \theta_1 &= p + r, & \theta'_1 &= \frac{p + r}{p + r - 1}, & \theta_2 &= \frac{q(p + r)}{q(p + r) - (q + r)}, \\ \frac{1}{\theta_1} + \frac{r}{q\theta_1} + \frac{1}{\theta_2} &= 1, & \frac{1}{\theta_1} + \frac{1}{\theta'_1} &= 1. \end{aligned}$$

Similarly, we get the following estimate to the second term:

$$\begin{aligned} & \int_{\Omega} |u| |(-\Delta)^{\frac{\beta}{2}} \varphi| |\mathbb{D}_{t|T}^{\alpha} \lambda(t)| \, dx \, dt \\ & \leq \varepsilon \int_{\Omega} (1 + |x|)^{\gamma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r \phi \, dx \, dt \\ & \quad + C \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} |(-\Delta)^{\frac{\beta}{2}} \varphi|^{\theta_2} |\mathbb{D}_{t|T}^{\alpha} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt. \end{aligned}$$

Therefore inequality (25) takes the form

$$\begin{aligned} & \int_{\Omega} (1 + |x|)^{\gamma} \|u^p\|_{v^{\frac{1}{q}}(x)u}^r \phi \, dx \, dt + CT^{1-\alpha} \int_{\Omega} u_0(x) \varphi \, dx \, dt \\ & \leq C \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} \varphi^{\theta_2} |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt \\ & \quad + C \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} |(-\Delta)^{\frac{\beta}{2}} \varphi|^{\theta_2} |\mathbb{D}_{t|T}^{\alpha} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt. \quad (26) \end{aligned}$$

Now, to estimate the right-hand side of (26), we consider the change of variables $\tau = t/T$ and $\xi = x/T^{\frac{1}{\beta}}$. Then we get

$$\begin{aligned} & \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} \varphi^{\theta_2} |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt \\ & \leq CT^{\ell} \left(\int_0^1 (1 - \tau)^{\sigma - \theta'_1(1+\alpha)} \, d\tau \right) \left(\int_{|\xi| < 2} |\xi|^{\frac{sr\theta_2}{q\theta_1}} (1 + |\xi|)^{-\frac{\gamma\theta_2}{\theta_1}} \chi^{\frac{\theta_2}{\theta'_1}} \, d\xi \right)^{\frac{\theta'_1}{\theta_2}}. \quad (27) \end{aligned}$$

Using the fact that $(-\Delta)^{\frac{\beta}{2}} \varphi(x) = T^{-1}(-\Delta)^{\frac{\beta}{2}} \chi(\xi)$, we obtain

$$\int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1 + |x|)^{-\frac{\gamma\theta_2}{\theta_1}} |(-\Delta)^{\frac{\beta}{2}} \varphi|^{\theta_2} |\mathbb{D}_{t|T}^{\alpha} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} \, dx \right)^{\frac{\theta'_1}{\theta_2}} dt$$

$$\begin{aligned} &\leq CT^\ell \left(\int_0^1 (1-\tau)^{\sigma-\theta'_1\alpha} d\tau \right) \\ &\quad \times \left(\int_{|\xi|<2} |\xi|^{\frac{sr\theta_2}{q\theta_1}} (1+|\xi|)^{-\frac{\gamma\theta_2}{\theta_1}} |(-\Delta)^{\frac{\beta}{2}} \chi|^{\theta_2} \chi^{-\frac{\theta_2}{\theta_1}} d\xi \right)^{\frac{\theta'_1}{\theta_2}}. \end{aligned} \quad (28)$$

Combining (27) and (28) together with $u_0(x) \geq 0$, we arrive at

$$\int_{\Omega} (1+|x|)^{\sigma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \leq CT^\ell, \quad (29)$$

where $\ell = \theta'_1(\frac{N}{\beta\theta_2} + \frac{sr}{q\beta\theta_1} - \frac{\gamma}{\beta\theta_1} - \alpha - 1) + 1$. Note that inequality (23) is equivalent to $\ell \leq 0$. Thus we consider two cases, $\ell < 0$ and $\ell = 0$.

Case $\ell < 0$. Taking $T \rightarrow \infty$ in (29), we get a contradiction

$$\int_0^\infty \int_{\mathbb{R}^N} (1+|x|)^{\sigma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt = 0 \quad \Rightarrow \quad u = 0,$$

which implies that u cannot exist.

Case $\ell = 0$. Passing to the limit as $T \rightarrow \infty$ in (29), we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (1+|x|)^{\sigma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt < \infty.$$

On the other hand, repeating the same calculation as before by letting

$$\varphi(x) = \chi\left(\frac{|x|}{R^{-1}T^{\frac{1}{\beta}}}\right),$$

where $1 \leq R < T$ is large enough, we do not have $R \rightarrow \infty$ as $T \rightarrow \infty$. Now we estimate the right-hand side of inequality (25) using again the Hölder and ε -Young inequalities as follows:

$$\begin{aligned} &\Gamma(\alpha) \int_{\Omega} (1+|x|)^{\gamma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \\ &\leq \varepsilon \int_{\Omega} (1+|x|)^{\gamma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \\ &\quad + C \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1+|x|)^{-\frac{\gamma\theta_2}{\theta_1}} \varphi^{\theta_2} |\mathbb{D}_{t|T}^{\alpha+1} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} dx \right)^{\frac{\theta'_1}{\theta_2}} dt. \\ &\leq \varepsilon \int_{\Omega} (1+|x|)^{\gamma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \\ &\quad + C \int_0^T \left(\int_{\mathbb{R}^N} v^{-\frac{r\theta_2}{q\theta_1}}(x) (1+|x|)^{-\frac{\gamma\theta_2}{\theta_1}} |(-\Delta)^{\frac{\beta}{2}} \varphi|^{\theta_2} |\mathbb{D}_{t|T}^{\alpha} \lambda(t)|^{\theta_2} \phi^{-\frac{\theta_2}{\theta_1}} dx \right)^{\frac{\theta'_1}{\theta_2}} dt, \end{aligned}$$

which, on introducing the change of variables $\tau = t/T$ and $\xi = x/R^{-1}T^{\frac{1}{\beta}}$, takes the form

$$\begin{aligned} &\int_{\Omega} (1+|x|)^{\gamma} |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \\ &\leq CT^\ell R^{\theta'_1(-\frac{rs}{q\theta_1} + \frac{\gamma}{\theta_1} - \frac{N}{\theta_2})} + CT^\ell R^{\theta'_1(-\frac{rs}{q\theta_1} + \frac{\gamma}{\theta_1} - \frac{N}{\theta_2} + \beta)}. \end{aligned} \quad (30)$$

Taking the limit in (30) as $T \rightarrow \infty$, we get

$$\int_0^\infty \int_{\mathbb{R}^N} (1 + |x|)^\gamma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \leq CR^{\ell'} + CR^{\ell''}, \quad (31)$$

where $\ell' = \beta(1 - \theta'_1(\alpha + 1)) - \beta\ell$ and $\ell'' = \beta(1 - \theta'_1\alpha) - \beta\ell$. Finally, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (1 + |x|)^\gamma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \leq 0$$

by taking the limit as $R \rightarrow \infty$, which leads to $u \equiv 0$, and hence u cannot exist. The proof of Theorem 2.9 is completed. $\square$

Proof of Theorem 2.10 Assume that $u(x, t) \geq c > 0$ is a solution of problem (1)–(2). Then from (29) we have

$$\int_\Omega (1 + |x|)^\gamma |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r \phi dx dt \leq CT^\ell, \quad (32)$$

where $\ell = \theta'_1(\frac{N}{\beta\theta_2} + \frac{sr}{q\beta\theta_1} - \frac{\gamma}{\beta\theta_1} - \alpha - 1) + 1$.

Now, introducing the change of variables $\tau = t/T$ and $\xi = x/T^{\frac{1}{\beta}}$ in the right-hand side of (32) and using the fact that $u(x, t) \geq c > 0$, we get

$$\begin{aligned} & \int_\Omega (1 + |x|)^\gamma \phi |u|^p \|v^{\frac{1}{q}}(x)u\|_q^r dx dt \\ & \geq CR^{\frac{\gamma}{\beta} + 1 + \frac{N}{\beta} + \frac{r}{\beta q}(N-s)} \int_\Sigma (1 + |\xi|)^\gamma \phi d\xi \left(\int_{\mathbb{R}^N} |\xi|^{-s} d\xi \right)^{\frac{r}{q}} d\tau \\ & \geq CR^{\frac{\gamma}{\beta} + 1 + \frac{N}{\beta} + \frac{r}{\beta q}(N-s)}. \end{aligned} \quad (33)$$

Next, combining (32) and (33), we have

$$\ell \geq \frac{1}{\beta q} ((\gamma + \beta + N)q + r(N - s)).$$

Hence

$$-(\alpha + 1)\beta q \geq \gamma q + r(N - s),$$

which implies that

$$s < N + \frac{q}{r}(\alpha\beta + \beta + \gamma).$$

This completes the proof. $\square$

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Authors' contributions

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