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A note on Nakano generalized difference sequence space

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Abstract

In this paper, we investigate the necessary conditions on any s-type sequence space to form an operator ideal. As a result, we show that the s-type Nakano generalized difference sequence space X fails to generate an operator ideal. We investigate the sufficient conditions on X to be premodular Banach special space of sequences and the constructed prequasi-operator ideal becomes a small, simple, and closed Banach space and has eigenvalues identical with its s-numbers. Finally, we introduce necessary and sufficient conditions on X explaining some topological and geometrical structures of the multiplication operator defined on X.

Keywords: Premodular; Generalized difference; Simple Banach space; Multiplication operator; Approximable operator; Fredholm operator

1 Introduction

By $\mathbb{C}^{\mathbb{N}}$, c, ℓ_∞, ℓ_r, and c_0, we denote the spaces of all, convergent, bounded, r-absolutely summable, and convergent to zero sequences of complex numbers, and $\mathbb{N}$ is the set of nonnegative integers. Tripathy et al. [14] introduced and studied the forward and backward generalized difference sequence spaces $U(\Delta_n^{(m)}) = \{(w_k) \in \mathbb{C}^{\mathbb{N}} : (\Delta_n^{(m)} w_k) \in U\}$ and $U(\Delta_n^m) = \{(w_k) \in \mathbb{C}^{\mathbb{N}} : (\Delta_n^m w_k) \in U\}$, where $m, n \in \mathbb{N}$, $U = \ell_\infty$, c or c_0, with $\Delta_n^{(m)} w_k = \sum_{\nu=0}^m (-1)^\nu C_\nu^m w_{k+\nu n}$, and $\Delta_n^m w_k = \sum_{\nu=0}^m (-1)^\nu C_\nu^m w_{k-\nu n}$, respectively. When $n = 1$, the generalized difference sequence spaces reduced to $U(\Delta^{(m)})$ were defined and investigated by Et and Çolak [3]. For $m = 1$, the generalized difference sequence spaces reduced to $U(\Delta_n)$ were defined and investigated by Tripathy and Esi [13]. For $n = 1$ and $m = 1$, the generalized difference sequence spaces reduced to $U(\Delta)$ were defined and studied by Kizmaz [6]. Summability is very important in mathematical models and has numerous implementations, such as normal series theory, approximation theory, ideal transformations, fixed point theory, and so forth. Let $r = (r_j) \in \mathbb{R}^{+\mathbb{N}}$, where $\mathbb{R}^{+\mathbb{N}}$ is the space of sequences with positive reals. We define the Nakano backward generalized difference sequence space as follows: $(\ell(r, \Delta_{n+1}^m))_\tau = \{w = (w_j) \in \mathbb{C}^{\mathbb{N}} : \exists \sigma > 0 \text{ with } \tau(\sigma w) < \infty\}$, where $\tau(w) = \sum_{j=0}^\infty |\Delta_{n+1}^m |w_j||^{r_j}$, $w_j = 0$ for $j < 0$, $\Delta_{n+1}^m |w_j| = \Delta_{n+1}^{m-1}|w_j| - \Delta_{n+1}^{m-1}|w_{j-1}|$ and $\Delta^0 w_j = w_j$ for all $j, n, m \in \mathbb{N}$. It is a Banach space with norm $\|w\| = \inf\{\sigma > 0 : \tau(\frac{w}{\sigma}) \le 1\}$. If $(r_j) \in \ell_\infty$, then $\ell(r, \Delta_{n+1}^m) = \{w = (w_j) \in \mathbb{C}^{\mathbb{N}} : \sum_{j=0}^\infty |\Delta_{n+1}^m |w_j||^{r_j} < \infty\}$. Several geometric and topological characteristics of $\ell(r, \Delta_{n+1}^m)$ have been studied (see [5, 16]). By $\mathfrak{B}(W, Z)$ we denote the set of all linear bounded operators between Banach spaces W and Z, and if $W = Z$, then we write $\mathfrak{B}(W)$. The multiplication operators and operator ideals have a wide field of mathematics in functional analysis, for instance, in eigenvalue distributions theorem, geometric structure of Banach spaces, theory of fixed point, and so forth. An s-number function [12] is a map defined on $\mathfrak{B}(W,Z)$ that associates with each operator $T \in \mathfrak{B}(W,Z)$ a nonnegative scaler sequence $(s_n(T))_{n=0}^{\infty}$ satisfying the following conditions:

(a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for $T \in \mathfrak{B}(W,Z)$,
(b) $s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in \mathfrak{B}(W,Z)$ and $m, n \in \mathbb{N}$,
(c) ideal property: $s_n(RVT) \leq \|R\| s_n(V) \|T\|$ for all $T \in \mathfrak{B}(W_0, W)$, $V \in \mathfrak{B}(W,Z)$, and $R \in \mathfrak{B}(Z, Z_0)$, where W_0 and Z_0 are arbitrary Banach spaces,
(d) if $G \in \mathfrak{B}(W,Z)$ and $\lambda \in \mathbb{C}$, then $s_n(\lambda G) = |\lambda| s_n(G)$.
(e) rank property: If $\operatorname{rank}(T) \leq n$, then $s_n(T) = 0$ for each $T \in \mathfrak{B}(W,Z)$,
(f) norming property: $s_{r \geq n}(I_n) = 0$ or $s_{r<n}(I_n) = 1$, where I_n is the unit operator on the n-dimensional Hilbert space ℓ_2^n.

The s-numbers have many examples such as the rth approximation number

$$\alpha_r(V) = \inf\bigl\{\|V - B\| : B \in \mathfrak{B}(W,Z) \text{ and } \operatorname{rank}(B) \leq r\bigr\}$$

and the rth Kolmogorov number

$$d_r(V) = \inf_{\dim W \leq r} \sup_{\|w\| \leq 1} \inf_{v \in W} \|Vw - v\|.$$

The following notations will be further used:

$$\begin{aligned}
&X^{\mathcal{S}} := \bigl\{X^{\mathcal{S}}(W,Z)\bigr\}, \quad \text{where } X^{\mathcal{S}}(W,Z) := \bigl\{V \in \mathfrak{B}(W,Z) : ((s_j(V))_{j=0}^{\infty} \in X\bigr\};\\
&X^{\mathrm{app}} := \bigl\{X^{\mathrm{app}}(W,Z)\bigr\}, \quad \text{where } X^{\mathrm{app}}(W,Z) := \bigl\{V \in \mathfrak{B}(W,Z) : ((\alpha_j(V))_{j=0}^{\infty} \in X\bigr\};\\
&X^{\mathrm{Kol}} := \bigl\{X^{\mathrm{Kol}}(W,Z)\bigr\}, \quad \text{where } X^{\mathrm{Kol}}(W,Z) := \bigl\{V \in \mathfrak{B}(W,Z) : ((d_j(V))_{j=0}^{\infty} \in X\bigr\};\\
&X^{\nu} := \bigl\{X^{\nu}(W,Z)\bigr\}, \quad \text{where}\\
&X^{\nu}(W,Z) := \bigl\{V \in \mathfrak{B}(W,Z) : ((\nu_j(V))_{j=0}^{\infty} \in X \text{ and } \bigl\|V - \nu_j(V)I\bigr\| = 0 \text{ for all} j \in \mathbb{N}\bigr\}.
\end{aligned}$$

The s-type Nakano generalized difference sequence space under $\tau : \ell(r, \Delta_{n+1}^m)) \to [0,\infty)$ is defined as

$$\begin{aligned}
&s\text{-type } \bigl(\ell\bigl(r, \Delta_{n+1}^m\bigr)\bigr)_{\tau}\\
&\quad := \bigl\{\bigl(s_j(V)\bigr)_{j=0}^{\infty} \in \mathbb{C}^{\mathbb{N}} : V \in \mathfrak{B}(W,Z) \text{ and } \tau\bigl(\lambda\bigl(s_j(V)\bigr)\bigr)_{j=0}^{\infty} < \infty \text{ for some } \lambda > 0\bigr\}.
\end{aligned}$$

If $(r_j) \in \ell_{\infty}$, then

$$s\text{-type } \bigl(\ell\bigl(r, \Delta_{n+1}^m\bigr)\bigr)_{\tau} = \left\{\bigl(s_j(V)\bigr)_{j=0}^{\infty} \in \mathbb{C}^{\mathbb{N}} : V \in \mathfrak{B}(W,Z) \text{ and } \sum_{j=0}^{\infty} \bigl|\Delta_{n+1}^m s_j(V)\bigr|^{r_j} < \infty\right\}.$$

Some examples of s-type Nakano generalized difference sequence spaces are

$$s\text{-type}\left(\ell\left(\left(\frac{j}{j+1}\right),\Delta_2^3\right)\right)_\tau = \left\{\left(s_j(V)\right)_{j=0}^{\infty}\in\mathbb{C}^{\mathbb{N}}: V\in\mathfrak{B}(W,Z)\text{ and }\sum_{j=0}^{\infty}\left|\Delta_2^3 s_j(V)\right|^{\frac{j}{j+1}}<\infty\right\}$$

and

$$s\text{-type}\left(\ell_r(\Delta)\right)_\tau = \left\{\left(s_j(V)\right)_{j=0}^{\infty}\in\mathbb{C}^{\mathbb{N}}: V\in\mathfrak{B}(W,Z)\text{ and }\left(\sum_{j=0}^{\infty}\left|\Delta s_j(V)\right|^{r}\right)^{\frac{1}{r}}<\infty\right\}.$$

A few operator ideals in the class of Hilbert or Banach spaces are defined by distinct scalar sequence spaces such as the ideal of compact operators $\mathfrak{B}_c$ formed by $(d_r(V))$ and c_0. Pietsch [12] studied the smallness of the quasi-ideals $(\ell_r)^{\mathrm{app}}$ for $r\in(0,\infty)$, the ideals of Hilbert–Schmidt operators between Hilbert spaces constructed by ℓ_2, and the ideals of nuclear operators generated by ℓ_1. He explained that $\overline{\mathfrak{F}} = (\ell_r)^{\mathrm{app}}$ for $r\in[1,\infty)$, where $\overline{\mathfrak{F}}$ is the closed class of all finite rank operators, and the class $(\ell_r)^{\mathrm{app}}$ became simple Banach [11]. The strict inclusions $(\ell_r)^{\mathrm{app}}(W,Z)\subsetneqq(\ell_j)^{\mathrm{app}}(W,Z)\subsetneqq\mathfrak{B}(W,Z)$ for $j>r>0$, where W and Z are infinite-dimensional Banach spaces, were investigated by Makarov and Faried [7]. Faried and Bakery [4] gave a generalization of the class of quasi-operator ideal, which is the prequasi-operator ideal and examined several geometric and topological structures of $(\ell_M)^{\mathcal{S}}$ and $(\mathrm{ces}(r))^{\mathcal{S}}$. On sequence spaces, Mursaleen and Noman [10] investigated the compact operators on some difference sequence spaces. Kiliçman and Raj [5] studied the matrix transformations of Norlund–Orlicz difference sequence spaces of nonabsolute type. Yaying et al. [15] examined the operator ideal of type sequence space whose q-Cesáro matrix in ℓ_p for all $q\in(0,1]$ and $1<p<\infty$. The point of this paper is explaining some results of $(\ell(p,\Delta_{n+1}^m))_\tau$ equipped with a prequasi-norm τ. Firstly, we give necessary conditions on any s-type sequence space to give an operator ideal. Secondly, we study some geometric and topological structures of $(\ell(p,\Delta_{n+1}^m))_\tau^{\mathcal{S}}$ such as closed, small, and simple Banach and $(\ell(p,\Delta_{n+1}^m))^{\mathcal{S}} = (\ell(p,\Delta_{n+1}^m))^{\nu}$. We determine a strict inclusion relation of $(\ell(p,\Delta_{n+1}^m))^{\mathcal{S}}$ for different p and Δ_{n+1}^m. Finally, we investigate the multiplication operator defined on $(\ell(p,\Delta_{n+1}^m))_\tau$.

2 Preliminaries and definitions

Definition 2.1 ([12]) An operator $V\in\mathfrak{B}(W)$ is called approximable if there are $D_r\in\mathfrak{F}(W)$ for every $r\in\mathbb{N}$ and $\lim_{r\to\infty}\|V-D_r\|=0$.

By $\Upsilon(W,Z)$ we denote the space of all approximable operators from W to Z.

Lemma 2.2 ([12]) *Let $V\in\mathfrak{B}(W,Z)$. If $V\notin\Upsilon(W,Z)$, then there are $G\in\mathfrak{B}(W)$ and $B\in\mathfrak{B}(Z)$ such that $BVGe_r=e_r$ for all $r\in\mathbb{N}$.*

Definition 2.3 ([12]) A Banach space W is called simple if $\mathfrak{B}(W)$ includes a unique nontrivial closed ideal.

Theorem 2.4 ([12]) *If W is Banach space with* $\dim(W) = \infty$, *then*

$$\mathfrak{F}(W) \subsetneqq \Upsilon(W) \subsetneqq \mathfrak{B}_c(W) \subsetneqq \mathfrak{B}(W).$$

Definition 2.5 ([9]) An operator $V \in \mathfrak{B}(W)$ is called Fredholm if $\dim(R(V))^c < \infty$, $\dim(\ker V) < \infty$, and $R(V)$ is closed, where $(R(V))^c$ denotes the complement of range V.

We will further use the sequence $e_j = (0, 0, \dots, 1, 0, 0, \dots)$ with 1 in the jth coordinate for all $j \in \mathbb{N}$.

Definition 2.6 ([4]) The space of linear sequence spaces $\mathbb{Y}$ is called a special space of sequences (sss) if

(1) $e_r \in \mathbb{Y}$ with $r \in \mathbb{N}$,
(2) if $u = (u_r) \in \mathbb{C}^{\mathbb{N}}$, $v = (v_r) \in \mathbb{Y}$, and $|u_r| \le |v_r|$ for every $r \in \mathbb{N}$, then $u \in \mathbb{Y}$. This means that $\mathbb{Y}$ is "solid",
(3) if $(u_r)_{r=0}^{\infty} \in \mathbb{Y}$, then $(u_{[\frac{r}{2}]})_{r=0}^{\infty} \in \mathbb{Y}$, where $[\frac{r}{2}]$ means the integral part of $\frac{r}{2}$.

Definition 2.7 ([2]) A subspace of the (sss) $\mathbb{Y}_\tau$ is called a premodular (sss) if there is a function $\tau : \mathbb{Y} \to [0, \infty)$ satisfying the following conditions:

(i) $\tau(y) \ge 0$ for each $y \in \mathbb{Y}$ and $\tau(y) = 0 \Leftrightarrow y = \theta$, where θ is the zero element of $\mathbb{Y}$,
(ii) there exists $a \ge 1$ such that $\tau(\eta y) \le a|\eta|\tau(y)$ for all $y \in \mathbb{Y}$ and $\eta \in \mathbb{C}$,
(iii) for some $b \ge 1$, $\tau(y + z) \le b(\tau(y) + \tau(z))$ for all $y, z \in \mathbb{Y}$,
(iv) $|y_r| \le |z_r|$ with $r \in \mathbb{N}$, implies $\tau((y_r)) \le \tau((z_r))$,
(v) for some $b_0 \ge 1$, $\tau((y_r)) \le \tau((y_{[\frac{r}{2}]})) \le b_0\tau((y_i))$,
(vi) if $y = (y_r)_{r=o}^{\infty} \in \mathbb{Y}$ and $d > 0$, then there is $r_0 \in \mathbb{N}$ with $\tau((y_r)_{r=r_0}^{\infty}) < d$,
(vii) there is $t > 0$ with $\tau(v, 0, 0, 0, \dots) \ge t|v|\tau(1, 0, 0, 0, \dots)$ for all $v \in \mathbb{C}$.

The (sss) $\mathbb{Y}_\tau$ is called prequasi-normed (sss) if τ satisfies parts (i)–(iii) of Definition 2.7, and when the space $\mathbb{Y}$ is complete under τ, then $\mathbb{Y}_\tau$ is called a prequasi-Banach (sss).

Theorem 2.8 ([2]) *A prequasi-norm* (*sss*) $\mathbb{Y}_\tau$, *whenever it is premodular* (*sss*).

By $\mathfrak{B}$ we denote the class of all bounded linear operators between any pair of Banach spaces.

Definition 2.9 ([2]) A class $\mathfrak{G} \subseteq \mathfrak{B}$ is called an operator ideal if every component $\mathfrak{G}(W, Z) = \mathfrak{G} \cap \mathfrak{B}(W, Z)$, where W and Z are Banach spaces, satisfies the following conditions:

(i) $\mathfrak{G} \supseteq \mathfrak{F}$, that is, the class $\mathfrak{G}$ contains the class of all finite-rank Banach space operators $\mathfrak{F}$.
(ii) The space $\mathfrak{G}(W, Z)$ is linear over $\mathbb{C}$.
(iii) If $V \in \mathfrak{B}(W_0, W)$, $G \in \mathfrak{G}(W, Z)$, and $Q \in \mathfrak{B}(Z, Z_0)$, then $QGV \in \mathfrak{G}(W_0, Z_0)$, where W_0 and Z_0 are Banach spaces.

Definition 2.10 ([2]) A prequasi-norm on the ideal B is a function $\zeta : B \to [0, \infty)$ that satisfies the following conditions:

(1) For all $V \in B(W,Z)$, $\zeta(V) \geq 0$ and $\zeta(V) = 0$ if and only if $V = 0$,
(2) there is $H \geq 1$ such that $\zeta(\eta V) \leq H|\eta|\zeta(V)$ for all $V \in B(W,Z)$ and $\eta \in \mathbb{C}$,
(3) there is $b \geq 1$ such that $\zeta(V_1 + V_2) \leq b[\zeta(V_1) + \zeta(V_2)]$ for all $V_1, V_2 \in B(W,Z)$,
(4) there is $D \geq 1$ such that if $U \in \mathfrak{B}(W_0, W)$, $T \in B(W,Z)$, and $V \in \mathfrak{B}(Z,Z_0)$, then $\zeta(VTU) \leq D\|V\|\zeta(T)\|U\|$.

Theorem 2.11 ([4]) *The function $\zeta(V) = \tau(s_r(V))_{r=0}^{\infty}$ forms a prequasi-norm on $X_\tau^{\mathcal{S}}$ whenever X_τ is a premodular (sss).*

We will further use the inequality $|a_i + b_i|^{q_i} \leq H(|a_i|^{q_i} + |b_i|^{q_i})$, where $q_i \geq 0$ for all $i \in \mathbb{N}$, $H = \max\{1, 2^{h-1}\}$, and $h = \sup_i q_i$ (see [1]).

3 Main results

Pietsch [12] investigated the quasi-ideals $(\ell_r)^{\mathrm{app}}$ for $r \in (0,\infty)$. Faried and Bakery [4] introduced sufficient conditions on any linear sequence space X such that the class $X^{\mathcal{S}}$ of all bounded linear operators between arbitrary Banach spaces with its sequence of s-numbers belongs to X generates an operator ideal. In this section, we give necessary conditions on s-type X under $\tau : X \to [0,\infty)$ such that $X_\tau^{\mathcal{S}}$ forms an operator ideal. Consequently, any none solid s-type sequence space does not form an operator ideal. We explain sufficient conditions on Nakano backward generalized difference sequence space to be premodular Banach (sss).

Theorem 3.1 *For s-type $X_\tau := \{x = (s_n(V)) \in \mathbb{C}^{\mathbb{N}} : V \in \mathfrak{B}(W,Z)$ and $\tau(x) < \infty\}$, if $X_\tau^{\mathcal{S}}$ is an operator ideal, then the following conditions are satisfied*:
1. *The set X_τ contains F, the space of all sequences with finite nonzero numbers.*
2. *If $(s_r(V_1))_{r=0}^{\infty} \in X_\tau$ and $(s_r(V_2))_{r=0}^{\infty} \in X_\tau$, then $(s_r(V_1 + V_2))_{r=0}^{\infty} \in X_\tau$.*
3. *For all $\lambda \in \mathbb{C}$ and $(s_r(V))_{r=0}^{\infty} \in X_\tau$, we have $|\lambda|(s_r(V))_{r=0}^{\infty} \in X_\tau$.*
4. *The sequence space X_τ is solid. This means that if $(s_r(V))_{r=0}^{\infty} \in \mathbb{C}^{\mathbb{N}}$, $(s_r(T))_{r=0}^{\infty} \in X_\tau$ and $s_r(V) \leq s_r(T)$ for every $r \in \mathbb{N}$ and $T, V \in \mathfrak{B}(W,Z)$, then $(s_r(V))_{r=0}^{\infty} \in X_\tau$.*

Proof Let $X_\tau^{\mathcal{S}}$ be an operator ideal.
(i) We have $\mathfrak{F}(W,Z) \subset X_\tau^{\mathcal{S}}(W,Z)$. Hence for all $T \in \mathfrak{F}(W,Z)$, we have $(s_r(V))_{r=0}^{\infty} \in F$. This gives $(s_r(V))_{r=0}^{\infty} \in X_\tau$. Hence $F \subset X_\tau$.
(ii) The space $X_\tau^{\mathcal{S}}(W,Z)$ is linear over $\mathbb{C}$. Hence for all $\lambda \in \mathbb{C}$ and $V_1, V_2 \in X_\tau^{\mathcal{S}}(W,Z)$, we have $V_1 + V_2 \in X_\tau^{\mathcal{S}}(W,Z)$ and $\lambda V_1 \in X_\tau^{\mathcal{S}}(W,Z)$. This implies

$$\big(s_r(V_1)\big)_{r=0}^{\infty} \in X_\tau \quad \text{and} \quad \big(s_r(V_2)\big)_{r=0}^{\infty} \in X_\tau \quad \Rightarrow \quad \big(s_r(V_1 + V_2)\big)_{r=0}^{\infty} \in X_\tau$$

and

$$\lambda \in \mathbb{C} \quad \text{and} \quad \big(s_r(V_1)\big)_{r=0}^{\infty} \in X_\tau \quad \Rightarrow \quad |\lambda|\big(s_r(V_1)\big)_{r=0}^{\infty} \in X_\tau.$$

(iii) If $A \in \mathfrak{B}(W_0, W)$, $B \in X_\tau^{\mathcal{S}}(W,Z)$, and $D \in \mathfrak{B}(Z,Z_0)$, then $DBA \in X_\tau^{\mathcal{S}}(W_0,Z_0)$, where W_0 and Z_0 are arbitrary Banach spaces. Therefore, if $A \in \mathfrak{B}(W_0, W)$, $(s_r(B))_{r=0}^{\infty} \in X_\tau$, and $D \in \mathfrak{B}(Z,Z_0)$, then $(s_r(DBA))_{r=0}^{\infty} \in X_\tau$ since $s_r(DBA) \leq \|D\|s_r(B)\|A\|$. By using condition 3, if $(\|D\|\|A\|s_r(B))_{r=0}^{\infty} \in X_\tau$, then we have $(s_r(DBA))_{r=0}^{\infty} \in X_\tau$. This means that X_τ is solid. $\square$

Corollary 3.2 *The s-type q-Cesáro sequence space of nonabsolute type* χ_p^q *is solid for all* $q \in (0,1]$ *and* $1 < p < \infty$.

Proof From Theorem 5.6 in [15], since the class of all bounded linear operators between any two Banach spaces such that its s-numbers belong to q-Cesáro sequence space of nonabsolute type forms an operator ideal if $q \in (0,1]$ and $1 < p < \infty$. Then by Theorem 3.1 the s-type q-Cesáro sequence space of nonabsolute type is solid for all $q \in (0,1]$ and $1 < p < \infty$. □

Theorem 3.3 *The space* $(\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau$ *is not operator ideal, where* (p_i) *satisfies* $0 < p_i < \infty$ *for all* $i \in \mathbb{N}$ *and* $\tau(w) = \sum_{i=0}^\infty |\Delta^m_{n+1}|w_i||^{p_i}$ *for all* $w \in \ell(p, \Delta^m_{n+1})$.

Proof We choose $m = 2$, $n = 1$, $w_k = 1$, $v_k = w_k$ for $k = 3s$ and, otherwise, $v_k = 0$ for all $s, k \in \mathbb{N}$. We have $|v_k| \leq |w_k|$ for all $k \in \mathbb{N}$, $w \in (\ell(p, \Delta^2_2))_\tau$, and $v \notin (\ell(p, \Delta^2_2))_\tau$. Hence the space $(\ell(p, \Delta^m_{n+1}))_\tau$ is not solid. This finishes the proof. □

According to Theorem 3.3, we correct Theorem 4.2 in [8], that is, the class of all bounded linear operators constructed by Musielak–Lorentz forward difference sequence spaces equipped with the Luxemburg norm and s-numbers fails to form a quasi-operator ideal, since it is not solid.

Definition 3.4 The backward generalized difference Δ^m_{n+1} is called absolutely nondecreasing if from $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ it follows that $|\Delta^m_{n+1}|x_i|| \leq |\Delta^m_{n+1}|y_i||$.

Theorem 3.5 *If* $(p_i) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ *is an increasing and* Δ^m_{n+1} *is absolutely nondecreasing, then the space* $(\ell(p, \Delta^m_{n+1}))_\tau$ *is a premodular Banach* (sss), *where*

$$\tau(w) = \sum_{i=0}^\infty \big|\Delta^m_{n+1}|w_i|\big|^{p_i} \quad \text{for all } w \in \ell\big(p, \Delta^m_{n+1}\big).$$

Proof

(1-i) Suppose $v, w \in \ell(p, \Delta^m_{n+1})$. Since $(p_i) \in \ell_\infty$ and Δ^m_{n+1} is absolutely nondecreasing, we have

$$\begin{aligned}
\tau(v + w) &= \sum_{i=0}^\infty \big|\Delta^m_{n+1}|v_i + w_i|\big|^{p_i} \\
&\leq H\left(\sum_{i=0}^\infty \big|\Delta^m_{n+1}|v_i|\big|^{p_i} + \sum_{i=0}^\infty \big|\Delta^m_{n+1}|w_i|\big|^{p_i}\right) \\
&= H\big(\tau(v) + \tau(w)\big) < \infty,
\end{aligned}$$

where $H = \max\{1, 2^{\sup_i p_i - 1}\}$. Then $v + w \in \ell(p, \Delta^m_{n+1})$.

(1-ii) Let $\lambda \in \mathbb{C}$ and $v \in \ell(p, \Delta^m_{n+1})$. Since (p_i) is bounded, we have

$$\tau(\lambda v) = \sum_{r=0}^\infty \big|\Delta^m_{n+1}|\lambda v_r|\big|^{p_r} \leq \sup_r |\lambda|^{p_r} \sum_{r=0}^\infty \big|\Delta^m_{n+1}|v_r|\big|^{p_r} = \sup_r |\lambda|^{p_r} \tau(v) < \infty.$$

Then $\lambda v \in \ell(p, \Delta^m_{n+1})$. Hence from parts (1-i) and (1-ii) the space $\ell(p, \Delta^m_{n+1})$ is linear. Since $e_r \in \ell(p) \subseteq \ell(p, \Delta^m_{n+1})$ for all $r \in \mathbb{N}$, we have $e_r \in \ell(p, \Delta^m_{n+1})$ for all $r \in \mathbb{N}$.

(2) Suppose $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ and $y \in \ell(p, \Delta^m_{n+1})$. Since Δ^m_{n+1} is absolutely nondecreasing. Hence we have

$$\tau(x) = \sum_{i=0}^{\infty} \big|\Delta^m_{n+1}|x_i|\big|^{p_i} \leq \sum_{i=0}^{\infty} \big|\Delta^m_{n+1}|y_i|\big|^{p_i} = \tau(y) < \infty,$$

so that $x \in \ell(p, \Delta^m_{n+1})$.

(3) Let $(v_r) \in \ell(p, \Delta^m_{n+1})$. Since (p_r) is an increasing and Δ^m_{n+1} is linear, we have

$$\begin{aligned}
\tau\big((v_{[\frac{r}{2}]})\big) &= \sum_{r=0}^{\infty} \big|\Delta^m_{n+1}|v_{[\frac{r}{2}]}|\big|^{p_r} \\
&= \sum_{r=0}^{\infty} \big|\Delta^m_{n+1}|v_r|\big|^{p_{2r}} + \sum_{r=0}^{\infty} \big|\Delta^m_{n+1}|v_r|\big|^{p_{2r+1}} \\
&\leq 2\sum_{r=0}^{\infty} \big|\Delta^m_{n+1}|v_r|\big|^{p_r} = 2\tau(v),
\end{aligned}$$

and then $(v_{[\frac{r}{2}]}) \in \ell(p, \Delta^m_{n+1})$.

(i) Obviously, $\tau(w) \geq 0$ and $\tau(w) = 0 \Leftrightarrow w = \theta$.

(ii) $a = \max\{1, \sup_r |\eta|^{p_r - 1}\} \geq 1$, where $\tau(\eta w) \leq a|\eta|\tau(w)$ for all $w \in \ell(p, \Delta^m_{n+1})$ and $\eta \in \mathbb{C}$.

(iii) The inequality $\tau(v + w) \leq H(\tau(v) + \tau(w))$ for all $v, w \in \ell(p, \Delta^m_{n+1})$ is satisfied.

(iv) Clearly from (2).

(v) From (3) we have that $b_0 = 2 \geq 1$.

(vi) It is obvious that $\overline{F} = \ell(p, \Delta^m_{n+1})$.

(vii) There is ζ with $0 < \zeta \leq |\eta|^{p_0 - 1}$ such that $\tau(\eta, 0, 0, 0, \ldots) \geq \zeta|\eta|\tau(1, 0, 0, 0, \ldots)$ for all $\eta \neq 0$ and $\zeta > 0$ if $\eta = 0$.

Hence the space $(\ell(p, \Delta^m_{n+1}))_\tau$ is premodular (sss). To explain that $(\ell(p, \Delta^m_{n+1}))_\tau$ is a premodular Banach (sss). Let $x^i = (x^i_k)_{k=0}^{\infty}$ be a Cauchy sequence in $(\ell(p, \Delta^m_{n+1}))_\tau$. Then for each $\varepsilon \in (0, 1)$, there is $i_0 \in \mathbb{N}$ such that for all $i, j \geq i_0$, we have

$$\tau\big(x^i - x^j\big) = \sum_{k=0}^{\infty} \big|\Delta^m_{n+1}\big|x^i_k - x^j_k\big|\big|^{p_k} < \varepsilon^{\sup_k p_k}.$$

Hence, for $i, j \geq i_0$ and $k \in \mathbb{N}$, we conclude

$$\big|\Delta^m_{n+1}\big|x^i_k\big| - \Delta^m_{n+1}\big|x^j_k\big|\big| < \varepsilon.$$

Therefore $(\Delta^m_{n+1}|x^j_k|)$ is a Cauchy sequence in $\mathbb{C}$ for fixed $k \in \mathbb{N}$, so $\lim_{j\to\infty} \Delta^m_{n+1} x^j_k = \Delta^m_{n+1} x^0_k$ for fixed $k \in \mathbb{N}$. Hence $\tau(x^i - x^0) < \varepsilon^{\sup_i p_i}$ for all $i \geq i_0$. Finally, to show that $x^0 \in \ell(p, \Delta^m_{n+1})$, we have

$$\tau\big(x^0\big) = \tau\big(x^0 - x^n + x^n\big) \leq H\big(\tau\big(x^n - x^0\big) + \tau\big(x^n\big)\big) < \infty.$$

Therefore $x^0 \in \ell(p, \Delta^m_{n+1})$. This gives that $(\ell(p, \Delta^m_{n+1}))_\tau$ is a premodular Banach (sss). $\square$

In view of Theorem 2.8, we get the following theorem.

Theorem 3.6 *If $(p_i) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing and Δ^m_{n+1} is absolutely nondecreasing, then the space $(\ell(p, \Delta^m_{n+1}))_\tau$ is prequasi-Banach (sss), where*

$$\tau(x) = \sum_{i=0}^{\infty} \big|\Delta^m_{n+1}|x_i|\big|^{p_i} \quad \text{for all } x \in \ell\big(p, \Delta^m_{n+1}\big).$$

Corollary 3.7 *If $0 < p < \infty$ and Δ^m_{n+1} is absolutely nondecreasing, then $(\ell_p(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss), where $\tau(x) = \sum_{i=0}^{\infty} |\Delta^m_{n+1}|x_i||^p$ for all $x \in \ell_p(\Delta^m_{n+1})$.*

4 Prequasi-Banach closed ideal

Pietsch [12] examined the Banach quasi-ideals $(\ell_r)^{\text{app}}$ for $r \in (0,\infty)$ and the Banach quasi-ideals of Hilbert–Schmidt and nuclear operators between Hilbert spaces formed by ℓ_2 and ℓ_1, respectively. Yaying et al. [15] made current the Banach quasi-operator ideal of type sequence space whose q-Cesáro matrix is in ℓ_p for all $q \in (0,1]$ and $1 < p < \infty$. Bakery and Mohammed [2] introduced the concept of prequasi-ideal, which is more general than the class of quasi-ideals. In this section, we introduce sufficient conditions on $\ell(p, \Delta^m_{n+1})$ such that the class $(\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau$ is a prequasi-Banach and closed ideal.

Theorem 4.1 *If $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing and Δ^m_{n+1} is absolutely nondecreasing, then $((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau, \zeta)$ is a prequasi-Banach operator ideal with $\tau(w) = \sum_{i=0}^{\infty} |\Delta^m_{n+1}|w_i||^{p_i}$ for all $w \in \ell(p, \Delta^m_{n+1})$ and $\zeta(V) = \tau((s_n(V))_{n=0}^{\infty})$.*

Proof By Theorems 3.5 and 2.11 the function ζ is a prequasi-norm on $(\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau$. Let (V_j) be a Cauchy sequence in $(\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau(W,Z)$. Since $\mathfrak{B}(W,Z) \supseteq (\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau(W,Z)$, we have

$$\zeta(V_i - V_j) = \sum_{k=0}^{\infty} \big|\Delta^m_{n+1} s_k(V_i - V_j)\big|^{p_k} \geq \big|\Delta^m_{n+1} \|V_i - V_j\|\big)\big|^{p_0}.$$

Therefore $(V_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{B}(W,Z)$. Since $\mathfrak{B}(W,Z)$ is a Banach space, $T \in \mathfrak{B}(W,Z)$ with $\lim_{j\to\infty} \|V_j - V\| = 0$ and $(s_n(V_i))_{n=0}^{\infty} \in (\ell(p, \Delta^m_{n+1}))_\tau$ for each $i \in \mathbb{N}$. From parts (ii), (iii), and (iv) of Definition 2.7 we have

$$\begin{aligned}
\zeta(V) &= \sum_{r=0}^{\infty} \big|\Delta^m_{n+1} s_r(V - V_j + V_j)\big|^{p_r} \\
&\leq H\left(\sum_{r=0}^{\infty} \big|\Delta^m_{n+1} s_{[\frac{r}{2}]}(V - V_j)\big|^{p_r} + \sum_{r=0}^{\infty} \big|\Delta^m_{n+1} s_{[\frac{r}{2}]}(V_j)\big|^{p_r}\right) \\
&\leq H \sum_{r=0}^{\infty} \big|\Delta^m_{n+1} \|V - V_j\|\big|^{p_0} + H b_0 \sum_{r=0}^{\infty} \big|\Delta^m_{n+1} s_r(V_j)\big|^{p_r} < \varepsilon.
\end{aligned}$$

Therefore $(s_r(V))_{r=0}^{\infty} \in (\ell(p, \Delta^m_{n+1}))_\tau$. Hence $V \in (\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau(W,Z)$. □

Theorem 4.2 *If $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing and Δ^m_{n+1} is absolutely nondecreasing, then $((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau, \zeta)$ is a prequasi-closed operator ideal with $\tau(w) = \sum_{i=0}^{\infty} |\Delta^m_{n+1}|w_i||^{p_i}$ for all $w \in \ell(p, \Delta^m_{n+1})$ and $\zeta(V) = \tau((s_n(V))_{n=0}^{\infty})$.*

Proof By Theorems 3.5 and 2.11 the function ζ is a prequasi-norm on $(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}_{\tau}$. Assume that $V_j \in (\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}_{\tau}(W,Z)$ for all $j\in\mathbb{N}$ and $\lim_{j\to\infty}\zeta(V_j-V)=0$. Since $\mathfrak{B}(W,Z)\supseteq(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}_{\tau}(W,Z)$, we have

$$\zeta(V-V_j)=\sum_{k=0}^{\infty}\bigl|\Delta^m_{n+1}s_k(V-V_j)\bigr|^{p_k}\geq\bigl|\Delta^m_{n+1}\|V-V_j\|\bigr|^{p_0}.$$

Hence $(V_j)_{j\in\mathbb{N}}$ is a convergent sequence in $\mathfrak{B}(W,Z)$. Since $(s_n(V_j))_{n=0}^{\infty}\in(\ell(p,\Delta^m_{n+1})_{\tau}$ for each $j\in\mathbb{N}$, from parts (ii), (iii), and (iv) of Definition 2.7 we get

$$\begin{aligned}\zeta(V)&=\sum_{r=0}^{\infty}\bigl|\Delta^m_{n+1}s_r(V-V_j+V_j)\bigr|^{p_r}\\&\leq H\Biggl(\sum_{r=0}^{\infty}\bigl|\Delta^m_{n+1}s_{[\frac{r}{2}]}(V-V_j)\bigr|^{p_r}+\sum_{r=0}^{\infty}\bigl|\Delta^m_{n+1}s_{[\frac{r}{2}]}(V_j)\bigr|^{p_r}\Biggr)\\&\leq H\sum_{r=0}^{\infty}\bigl|\Delta^m_{n+1}\|V-V_j\|\bigr|^{p_0}+Hb_0\sum_{r=0}^{\infty}\bigl|\Delta^m_{n+1}s_r(V_j)\bigr|^{p_r}<\varepsilon.\end{aligned}$$

Therefore $(s_r(V))_{r=0}^{\infty}\in(\ell(p,\Delta^m_{n+1})_{\tau}$. This gives $V\in(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}_{\tau}(W,Z)$. □

Corollary 4.3 *$((\ell_p(\Delta^m_{n+1}))^{\mathcal{S}}_{\tau},\zeta)$ is prequasi-closed and Banach with $\tau(w)=\sum_{i=0}^{\infty}|\Delta^m_{n+1}|w_i||^p$ for all $w\in\ell_p(\Delta^m_{n+1})$ and $\zeta(V)=\tau((s_n(V))_{n=0}^{\infty})$ if $0<p<\infty$ and Δ^m_{n+1} is absolutely nondecreasing.*

5 Small and simple of $(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}$

Makarov and Faried [7] explained the strict inclusion $(\ell_r)^{\mathrm{app}}(W,Z)\subsetneqq(\ell_j)^{\mathrm{app}}(W,Z)\subsetneqq\mathfrak{B}(W,Z)$ for $j>r>0$. Pietsch [11] proved that the class $(\ell_r)^{\mathrm{app}}$ became simple and small Banach space for $r\in[1,\infty)$ and $r\in(0,\infty)$, respectively. In this section, we explain sufficient conditions on $\ell(p,\Delta^m_{n+1})$ for the strict inclusion relation of $(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}$ for different p and Δ^m_{n+1}. We study the conditions such that the class $(\ell(p,\Delta^m_{n+1}))^{\mathrm{app}}$ is small. We also investigate sufficient conditions on $\ell(p,\Delta^m_{n+1})$ such that $(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}$ equals $(\ell(p,\Delta^m_{n+1}))^{\nu}$. Finally, we give an answer of the following question: For which $\ell(p,\Delta^m_{n+1})$, $(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}$ is simple?

Theorem 5.1 *Let W and Z be infinite-dimensional Banach spaces, $0<p_i\leq q_i$ for all $i\in\mathbb{N}$, and let Δ^m_n be absolutely nondecreasing for all $n,m\in\mathbb{N}$. Then*

$$\bigl(\ell\bigl(p,\Delta^m_{n+2}\bigr)\bigr)^{\mathcal{S}}(W,Z)\subsetneqq\bigl(\ell\bigl(q,\Delta^{m+1}_{n+1}\bigr)\bigr)^{\mathcal{S}}(W,Z)\subsetneqq\mathfrak{B}(W,Z).$$

Proof If $V\in(\ell(p,\Delta^m_{n+2}))^{\mathcal{S}}(W,Z)$, then we have $(s_i(V))\in\ell(p,\Delta^m_{n+2})$. We can see that

$$\sum_{j=0}^{\infty}\bigl|\Delta^{m+1}_{n+1}s_j(V)\bigr|^{q_j}<\sum_{j=0}^{\infty}\bigl|\Delta^m_{n+2}s_j(V)\bigr|^{p_j}<\infty.$$

Therefore $V\in(\ell(q,\Delta^{m+1}_{n+1}))^{\mathcal{S}}(W,Z)$. Next, if we choose $(s_j(V))_{j=0}^{\infty}$ such that $\Delta^m_{n+2}s_j(V)=(j+1)^{-\frac{1}{p_j}}$ for $n,m\in\mathbb{N}$, then we can find $V\in\mathfrak{B}(W,Z)$ with $\sum_{j=0}^{\infty}|\Delta^m_{n+2}s_j(V)|^{p_j}=\sum_{j=0}^{\infty}\frac{1}{j+1}=\infty$

and

$$\sum_{j=0}^{\infty}\left(\left|\Delta_{n+2}^{m}s_j(V)\right|\right)^{q_j}=\sum_{j=0}^{\infty}\left(\frac{1}{j+1}\right)^{\frac{q_j}{p_j}}<\infty.$$

Since $\ell(q,\Delta_{n+2}^{m})\subseteq\ell(q,\Delta_{n+1}^{m+1})$, $V\notin(\ell(p,\Delta_{n+2}^{m}))^{\mathcal{S}}(W,Z)$ and $V\in(\ell(q,\Delta_{n+1}^{m+1}))^{\mathcal{S}}(W,Z)$. Clearly, $(\ell(q,\Delta_{n+1}^{m+1}))^{\mathcal{S}}(W,Z)\subset\mathfrak{B}(W,Z)$. By choosing $(s_j(V))_{j=0}^{\infty}$ such that $\Delta_{n+1}^{m+1}s_j(V)=(j+1)^{-\frac{1}{q_j}}$ for $n,m\in\mathbb{N}$, we have $V\in\mathfrak{B}(W,Z)$ such that $V\notin(\ell(q,\Delta_{n+1}^{m+1}))^{\mathcal{S}}(W,Z)$. □

Corollary 5.2 *For any infinite-dimensional Banach spaces W and Z, $j\geq r>0$, and absolutely nondecreasing Δ_n^m for all $n,m\in\mathbb{N}$, we have*

$$\left(\ell_r\left(\Delta_{n+2}^{m}\right)\right)^{\mathcal{S}}(W,Z)\subsetneqq\left(\ell_j\left(\Delta_{n+1}^{m+1}\right)\right)^{\mathcal{S}}(W,Z)\subsetneqq\mathfrak{B}(W,Z).$$

Theorem 5.3 *For any Banach spaces W and Z with $\dim(W)=\dim(Z)=\infty$, let $(p_r)\in\mathbb{R}^{+\mathbb{N}}\cap\ell_\infty$ be increasing, and let Δ_{n+1}^{m} be absolutely nondecreasing. Then the class $(\ell(p,\Delta_{n+1}^{m}))^{\mathrm{app}}$ is small.*

Proof $((\ell(p,\Delta_{n+1}^{m}))^{\mathrm{app}},\zeta)$ is a prequasi-Banach operator ideal, where $\zeta(V)=(\sum_{k=0}^{\infty}|\Delta_{n+1}^{m}\alpha_k(V)|^{p_k})^{\frac{1}{h}}$. Let $(\ell(p,\Delta_{n+1}^{m}))^{\mathrm{app}}(W,Z)=\mathfrak{B}(W,Z)$. Then there is $\delta>0$ with $\zeta(V)\leq\delta\|V\|$ for all $V\in\mathfrak{B}(W,Z)$. By Dvoretzky's theorem [12] for $j\in\mathbb{N}$, there are subspaces M_j and quotient spaces W/N_j of Z. By isomorphisms, A_j and H_j will be mapped Z onto ℓ_2^j with $\|H_j\|\|H_j^{-1}\|\leq 2$ and $\|A_j\|\|A_j^{-1}\|\leq 2$. Let J_j be the natural embedding map from M_j into Z, and let Q_j be the quotient map from W onto W/N_j. Denoting the Bernstein numbers [12] by u_j, we have

$$\begin{aligned}1&=u_k(I_j)=u_k\left(A_jA_j^{-1}I_jH_jH_j^{-1}\right)\\&\leq\|A_j\|u_k\left(A_j^{-1}I_jH_j\right)\left\|H_j^{-1}\right\|\\&=\|A_j\|u_k\left(J_jA_j^{-1}I_jH_j\right)\left\|H_j^{-1}\right\|\\&\leq\|A_j\|d_k\left(J_jA_j^{-1}I_jH_j\right)\left\|H_j^{-1}\right\|\\&=\|A_j\|d_k\left(J_jA_j^{-1}I_jH_jQ_j\right)\left\|H_j^{-1}\right\|\\&\leq\|A_j\|\alpha_k\left(J_jA_j^{-1}I_jH_jQ_j\right)\left\|H_j^{-1}\right\|\end{aligned}$$

for $0\leq k\leq i$. Therefore

$$\begin{aligned}&1\leq\|A_j\|\left|\Delta_{n+1}^{m}\alpha_k\left(J_jA_j^{-1}I_jH_jQ_j\right)\right|\left\|H_j^{-1}\right\|\\&\quad\Rightarrow\quad(i+1)\leq\left(\|A_j\|\left\|H_j^{-1}\right\|\right)^{p_i}\sum_{k=0}^{i}\left|\Delta_{n+1}^{m}\alpha_k\left(J_jA_j^{-1}I_jH_jQ_j\right)\right|^{p_k}.\end{aligned}$$

Hence

$$\begin{aligned}&(i+1)^{\frac{1}{h}}\leq a\|A_m\|\left\|H_m^{-1}\right\|\left[\sum_{k=0}^{i}\left|\Delta_{n+1}^{m}\alpha_k\left(J_jA_j^{-1}I_jH_jQ_j\right)\right|^{p_k}\right]^{\frac{1}{h}}\\&\quad\Rightarrow\quad(i+1)^{\frac{1}{h}}\leq a\|A_j\|\left\|H_j^{-1}\right\|g\left(J_jA_j^{-1}I_jH_jQ_j\right)\end{aligned}$$

$$\Rightarrow \quad (i+1)^{\frac{1}{h}} \leq a\delta \|A_j\| \left\|H_j^{-1}\right\| \left\|J_jA_j^{-1}I_jH_jQ_j\right\|$$
$$\Rightarrow \quad (i+1)^{\frac{1}{h}} \leq a\delta \|A_j\| \left\|H_j^{-1}\right\| \left\|J_jA_j^{-1}\right\| \|I_j\| \|H_jQ_j\| = L\delta \|A_j\| \left\|H_j^{-1}\right\| \left\|A_j^{-1}\right\| \|I_j\| \|H_j\|$$
$$\Rightarrow \quad (i+1)^{\frac{1}{h}} \leq 4a\delta$$

for some $a \geq 1$. Since i is arbitrary, we have a contradiction. So, W and Z cannot be infinite-dimensional while $(\ell(p, \Delta^m_{n+1}))^{\mathrm{app}}(W,Z) = \mathfrak{B}(W,Z)$.

In the same manner we can prove that the class $(\ell(p, \Delta^m_{n+1}))^{\mathrm{Kol}}$ is small. □

Theorem 5.4 *Let W and Z be any Banach spaces with* $\dim(W) = \dim(Z) = \infty$. *Let $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ be increasing, and let Δ^m_{n+1} be absolutely nondecreasing. Then the class $(\ell(p, \Delta^m_{n+1}))^{\mathrm{Kol}}$ is small.*

Theorem 5.5 *Pick any Banach spaces W and Z with* $\dim(W) = \dim(Z) = \infty$. *If $(p_r), (q_r) \in \ell_\infty$ are increasing with $1 \leq p_i < q_i$ for all $i \in \mathbb{N}$ and Δ^m_n is absolutely nondecreasing, then*

$$\mathfrak{B}\big(\big(\ell\big(q, \Delta^{m+1}_{n+1}\big)\big)^{\mathcal{S}}, \big(\ell\big(p, \Delta^m_{n+2}\big)\big)^{\mathcal{S}}\big) = \Upsilon\big(\big(\ell\big(q, \Delta^{m+1}_{n+1}\big)\big)^{\mathcal{S}}, \big(\ell\big(p, \Delta^m_{n+2}\big)\big)^{\mathcal{S}}\big).$$

Proof Assume that there is $V \in \mathfrak{B}((\ell(q, \Delta^{m+1}_{n+1}))^{\mathcal{S}}, (\ell(p, \Delta^m_{n+2}))^{\mathcal{S}})$ that is not approximable. By Lemma 2.2 we have $G \in \mathfrak{B}((\ell(q, \Delta^{m+1}_{n+1}))^{\mathcal{S}})$ and $B \in \mathfrak{B}((\ell(p, \Delta^m_{n+2}))^{\mathcal{S}})$ with $BVGI_k = I_k$. Therefore for all $k \in \mathbb{N}$, we get

$$\|I_k\|_{(\ell(p,\Delta^m_{n+2}))^{\mathcal{S}}} = \sum_{n=0}^{\infty} \big|\Delta^m_{n+2} s_n(I_k)\big|^{p_k} \leq \|BVG\| \|I_k\|_{(\ell(q,\Delta^{m+1}_{n+1}))^{\mathcal{S}}} \leq \sum_{n=0}^{\infty} \big|\Delta^{m+1}_{n+1} s_n(I_k)\big|^{q_k}.$$

From Theorem 5.1 we obtain a contradiction. Hence $V \in \Upsilon((\ell(q, \Delta^{m+1}_{n+1}))^{\mathcal{S}}, (\ell(p, \Delta^m_{n+2}))^{\mathcal{S}})$. □

Corollary 5.6 *Let W and Z be any Banach spaces with* $\dim(W) = \dim(Z) = \infty$. *If $(p_r), (q_r) \in \ell_\infty$ are increasing with $1 \leq p_i < q_i$ for all $i \in \mathbb{N}$ and Δ^m_n is absolutely nondecreasing, then*

$$\mathfrak{B}\big(\big(\ell\big(q, \Delta^{m+1}_{n+1}\big)\big)^{\mathcal{S}}, \big(\ell\big(p, \Delta^m_{n+2}\big)\big)^{\mathcal{S}}\big) = \mathfrak{B}_c\big(\big(\ell\big(q, \Delta^{m+1}_{n+1}\big)\big)^{\mathcal{S}}, \big(\ell\big(p, \Delta^m_{n+2}\big)\big)^{\mathcal{S}}\big).$$

Proof Since each approximable operator is compact, the result follows. □

Theorem 5.7 *Let W and Z be any Banach spaces with* $\dim(W) = \dim(Z) = \infty$. *If $(p_r) \in \ell_\infty$ is increasing with $p_0 \geq 1$ for all $i \in \mathbb{N}$ and Δ^m_n is absolutely nondecreasing, then the class $(\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau$ is simple.*

Proof Suppose that there is $V \in \mathfrak{B}_c((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$ such that $V \notin \Upsilon((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$. Therefore by Lemma 2.2 one find $A, B \in \mathfrak{B}((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$ with $BVAI_k = I_k$. This means that $I_{(\ell(p,\Delta^m_{n+1}))^{\mathcal{S}}_\tau} \in \mathfrak{B}_c((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$. Consequently, $\mathfrak{B}((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau) = \mathfrak{B}_c((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$. Therefore $\mathfrak{B}((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$ includes one and only one nontrivial closed ideal $\Upsilon((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}_\tau)$. □

5.1 Eigenvalues of s-type $\ell(p, \Delta^m_{n+1})$

Theorem 5.8 *Let W and Z be Banach spaces with* $\dim(W) = \dim(Z) = \infty$. *If* $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ *is increasing and* Δ^m_{n+1} *is absolutely nondecreasing, then*

$$\left(\ell\left(p, \Delta^m_{n+1}\right)\right)^{\mathcal{S}}(W, Z) = \left(\ell\left(p, \Delta^m_{n+1}\right)\right)^{\nu}(W, Z).$$

Proof Suppose $V \in (\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}(W, Z)$. Then $(s_r(V))_{r=0}^\infty \in \ell(p, \Delta^m_{n+1})$, and we have $\sum_{r=0}^\infty (|\Delta^m_{n+1} s_r(V)|)^{p_r} < \infty$. Since Δ^m_{n+1} is continuous, $\lim_{r\to\infty} s_r(V) = 0$. Let $\|V - s_r(V)I\|$ be invertible for all $r \in \mathbb{N}$. Then $\|V - s_r(V)I\|^{-1}$ exists and is bounded for each $r \in \mathbb{N}$. Therefore $\lim_{r\to\infty} \|V - s_r(V)I\|^{-1} = \|V\|^{-1}$ with $V^{-1} \in \mathfrak{B}(Z, W)$. From the prequasi-operator ideal of $((\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}, \zeta)$ we have

$$I = VV^{-1} \in \left(\ell\left(p, \Delta^m_{n+1}\right)\right)^{\mathcal{S}}(Z) \quad \Rightarrow \quad \left(s_r(I)\right)_{r=0}^\infty \in \ell\left(p, \Delta^m_{n+1}\right) \quad \Rightarrow \quad \lim_{r\to\infty} s_r(I) = 0.$$

Since $\lim_{r\to\infty} s_r(I) = 1$, we have a contradiction. Then $\|V - s_r(V)I\|$ is not invertible for all $r \in \mathbb{N}$. Hence $(s_r(V))_{r=0}^\infty$ represents the eigenvalues of V. Conversely, if $V \in (\ell(p, \Delta^m_{n+1}))^{\nu}(W, Z)$, then $(\nu_r(V))_{r=0}^\infty \in \ell(p, \Delta^m_{n+1})$ and $\|V - \nu_r(V)I\| = 0$ for all $n \in \mathbb{N}$. This gives $V = \nu_r(V)I$ for all $r \in \mathbb{N}$. Then $s_r(V) = s_r(\nu_r(V)I) = |\nu_r(V)|$ for all $r \in \mathbb{N}$. Therefore $(s_r(V))_{r=0}^\infty \in \ell(p, \Delta^m_{n+1})$, and so $V \in (\ell(p, \Delta^m_{n+1}))^{\mathcal{S}}(W, Z)$. This completes the proof. □

6 Multiplication operator on $\ell(p, \Delta^m_{n+1})$

Mursaleen and Noman [10] examined compact operators on some difference sequence spaces. Kiliçman and Raj [5] introduced the matrix transformations of Norlund–Orlicz difference sequence spaces of nonabsolute type. Yaying et al. [15] investigated the matrix transformations on q-Cesáro sequence spaces of nonabsolute type. In this section, we introduce some topological and geometric structures of the multiplication operator acting on $\ell(p, \Delta^m_{n+1})$ such as bounded, invertible, approximable, closed range, and Fredholm operator.

Definition 6.1 Let $\kappa \in \mathbb{C}^{\mathbb{N}} \cap \ell_\infty$, and let W_τ be a prequasi-normed (sss). An operator $V_\kappa : W_\tau \to W_\tau$ is called a multiplication operator if $V_\kappa w = \kappa w = (\kappa_r w_r)_{r=0}^\infty \in W$ for all $w \in W$. If $V_\kappa \in \mathfrak{B}(W)$, then we call it a multiplication operator generated by κ.

Theorem 6.2 *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$, $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ be increasing, and let Δ^m_{n+1} be absolutely nondecreasing. Then $\kappa \in \ell_\infty$ if and only if, $V_\kappa \in \mathfrak{B}(\ell(p, \Delta^m_{n+1})_\tau)$, where $\tau(x) = \sum_{r=0}^\infty |\Delta^m_{n+1}|x_r||^{p_r}$ for all $x \in \ell(p, \Delta^m_{n+1})$.*

Proof Let $\kappa \in \ell_\infty$. Then there is $\varepsilon > 0$ with $|\kappa_r| \le \varepsilon$ for every $r \in \mathbb{N}$. For $x \in (\ell(p, \Delta^m_{n+1})_\tau$, since Δ^m_{n+1} is absolutely nondecreasing and (p_r) is bounded from above with $p_r > 0$ for all $r \in \mathbb{N}$, we have

$$\begin{aligned}
\tau(V_\kappa x) &= \tau(\kappa x) = \tau\left((\kappa_r x_r)_{r=0}^\infty\right) \\
&= \sum_{r=0}^\infty \left|\Delta^m_{n+1}\left(|\kappa_r||x_r|\right)\right|^{p_r} \\
&\le \sum_{r=0}^\infty \left|\Delta^m_{n+1}\left(\varepsilon|x_r|\right)\right|^{p_r}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_r \varepsilon^{p_r} \sum_{r=0}^{\infty} \big|\Delta_{n+1}^m |x_r|\big|^{p_r} \\
&= D\tau(x).
\end{aligned}$$

This gives $V_\kappa \in \mathfrak{B}(\ell(p,\Delta_{n+1}^m)_\tau)$. Conversely, let $V_\kappa \in \mathfrak{B}(\ell(p,\Delta_{n+1}^m)_\tau)$. Suppose $\kappa \notin \ell_\infty$. Then for each $j \in \mathbb{N}$, there is $i_j \in \mathbb{N}$ such that $\kappa_{i_j} > j$. Since Δ_{n+1}^m is absolutely nondecreasing, we have

$$\begin{aligned}
\tau(V_\kappa e_{i_j}) = \tau(\kappa e_{i_j}) &= \tau\big((\kappa_r (e_{i_j})_r)_{r=0}^{\infty}\big) \\
&= \sum_{r=0}^{\infty} \big|\Delta_{n+1}^m \big(|\kappa_r|\big|(e_{i_j})_r\big|\big)\big|^{p_r} \\
&= \big|\Delta_{n+1}^m |\kappa_{i_j}|\big|^{p_{i_j}} > \big|\Delta_{n+1}^m |j|\big|^{p_{i_j}} \\
&= \big|\Delta_{n+1}^m |j|\big|^{p_{i_j}} \tau(e_{i_j}).
\end{aligned}$$

This shows that $V_\kappa \notin \mathfrak{B}(\ell(p,\Delta_{n+1}^m)_\tau)$. Therefore $\kappa \in \ell_\infty$. □

Theorem 6.3 *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$, and let $(\ell(p,\Delta_{n+1}^m))_\tau$ be a prequasi-normed (sss) with $\tau(x) = \sum_{r=0}^{\infty} |\Delta_{n+1}^m |x_r||^{p_r}$ for all $x \in \ell(p,\Delta_{n+1}^m)$. Then $|\kappa_r| = 1$ for all $r \in \mathbb{N}$ if and only if V_κ is an isometry.*

Proof Suppose $|\kappa_r| = 1$ for all $r \in \mathbb{N}$. Then

$$\begin{aligned}
\tau(V_\kappa x) = \tau(\kappa x) &= \tau\big((\kappa_r x_r)_{r=0}^{\infty}\big) \\
&= \sum_{r=0}^{\infty} \big|\Delta_{n+1}^m \big(|\kappa_r||x_r|\big)\big|^{p_r} = \sum_{r=0}^{\infty} \big|\Delta_{n+1}^m |x_r|\big|^{p_r} = \tau(x)
\end{aligned}$$

for all $x \in (\ell(p,\Delta_{n+1}^m))_\tau$. Therefore V_κ is an isometry. Conversely, assume that $|\kappa_i| < 1$ for some $i = i_0$. Since Δ_{n+1}^m is absolutely nondecreasing, we obtain

$$\begin{aligned}
\tau(V_\kappa e_{i_0}) = \tau(\kappa e_{i_0}) &= \tau\big((\kappa_r (e_{i_0})_r)_{r=0}^{\infty}\big) \\
&= \sum_{r=0}^{\infty} \big|\Delta_{n+1}^m \big(|\kappa_r|\big|(e_{i_0})_r\big|\big)\big|^{p_r} \\
&< \sum_{r=0}^{\infty} \big|\Delta_{n+1}^m \big|(e_{i_0})_r\big|\big|^{p_r} = \tau(e_{i_0}).
\end{aligned}$$

When $|\kappa_{i_0}| > 1$, we can prove that $\tau(V_\kappa e_{i_0}) > \tau(e_{i_0})$. Therefore, in both cases, we have a contradiction. So $|\kappa_r| = 1$ for every $r \in \mathbb{N}$.

By card(A) we denote the cardinality of a set A. □

Theorem 6.4 *If $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell(p,\Delta_{n+1}^m))_\tau$ is a prequasi-normed (sss), where $\tau(x) = \sum_{r=0}^{\infty} |\Delta_{n+1}^m |x_r||^{p_r}$ for all $x \in \ell(p,\Delta_{n+1}^m)$. Then $V_\kappa \in \Upsilon((\ell(p,\Delta_{n+1}^m))_\tau)$ if and only if $(\kappa_r)_{r=0}^{\infty} \in c_0$.*

Proof Let $V_\kappa \in \Upsilon((\ell(p,\Delta_{n+1}^m))_\tau)$. Therefore $V_\kappa \in \mathfrak{B}_c((\ell(p,\Delta_{n+1}^m))_\tau)$. To prove that the sequence $(\kappa_r)_{r=0}^{\infty}$ belongs to c_0, suppose $(\kappa_r)_{r=0}^{\infty} \notin c_0$. Then there is $\delta > 0$ such that the set $A_\delta = \{r \in \mathbb{N} : |\kappa_r| \geq \delta\}$ has $\operatorname{card}(A_\delta) = \infty$. Assume that $a_i \in A_\delta$ for all $i \in \mathbb{N}$. Hence $\{e_{a_i} : a_i \in A_\delta\}$ is an infinite bounded set in $(\ell(p, \Delta^m_{n+1}))_\tau$. Let

$$\begin{aligned}\tau(V_\kappa e_{a_i} - V_\kappa e_{a_j}) &= \tau(\kappa e_{a_i} - \kappa e_{a_j})\\ &= \tau\Big(\big(\kappa_r\big((e_{a_i})_r - (e_{a_j})_r\big)\big)_{r=0}^\infty\Big) = \sum_{r=0}^\infty \big|\Delta^m_{n+1}\big|\kappa_r\big((e_{a_i})_r - (e_{a_j})_r\big)\big|\big|^{p_r}\\ &\geq \sum_{r=0}^\infty \big|\Delta^m_{n+1}\big|\delta\big((e_{a_i})_r - (e_{a_j})_r\big)\big|\big|^{p_r} = \tau(\delta e_{a_i} - \delta e_{a_j})\end{aligned}$$

for all $a_i, a_j \in A_\delta$. This shows that $\{e_{a_i} : a_i \in B_\delta\} \in \ell_\infty$, which cannot have a convergent subsequence under V_κ. This proves that $V_\kappa \notin \mathfrak{B}_c((\ell(p, \Delta^m_{n+1}))_\tau)$. Then $V_\kappa \notin \Upsilon((\ell(p, \Delta^m_{n+1}))_\tau)$, a contradiction. So, $\lim_{i\to\infty} \kappa_i = 0$. Conversely, let $\lim_{i\to\infty} \kappa_i = 0$. Then for each $\delta > 0$, the set $A_\delta = \{i \in \mathbb{N} : |\kappa_i| \geq \delta\}$ has $\operatorname{card}(A_\delta) < \infty$. Hence, for every $\delta > 0$, the space

$$\Big(\big(\ell\big(p, \Delta^m_{n+1}\big)\big)_\tau\Big)_{A_\delta} = \Big\{x = (x_i) \in \big(\ell\big(p, \Delta^m_{n+1}\big)\big)_\tau : i \in A_\delta\Big\}$$

is finite-dimensional. Then $V_\kappa|((\ell(p, \Delta^m_{n+1}))_\tau)_{A_\delta}$ is a finite rank operator. For every $i \in \mathbb{N}$, define $\kappa_i \in \mathbb{C}^\mathbb{N}$ by

$$(\kappa_i)_j = \begin{cases}\kappa_j, & j \in A_{\frac{1}{i}},\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that V_{κ_i} has $\operatorname{rank}(V_{\kappa_i}) < \infty$ as $\dim((\ell(p, \Delta^m_{n+1}))_\tau)_{A_{\frac{1}{i}}} < \infty$ for $i \in \mathbb{N}$. Therefore, since Δ^m_{n+1} is absolutely nondecreasing, we get

$$\begin{aligned}\tau\big((V_\kappa - V_{\kappa_i})x\big) &= \tau\Big(\big(\big(\kappa_j - (\kappa_i)_j\big)x_j\big)_{j=0}^\infty\Big)\\ &= \sum_{j=0}^\infty \big|\Delta^m_{n+1}\big(\big|\big(\kappa_j - (\kappa_i)_j\big)x_j\big|\big)\big|^{p_j}\\ &= \sum_{j=0, j\in A_{\frac{1}{i}}}^\infty \big|\Delta^m_{n+1}\big(\big|\big(\kappa_j - (\kappa_i)_j\big)x_j\big|\big)\big|^{p_j} + \sum_{j=0, j\notin A_{\frac{1}{i}}}^\infty \big|\Delta^m_{n+1}\big(\big|\big(\kappa_j - (\kappa_i)_j\big)x_j\big|\big)\big|^{p_j}\\ &= \sum_{j=0, j\notin A_{\frac{1}{i}}}^\infty \big|\Delta^m_{n+1}|\kappa_j x_j|\big|^{p_j}\\ &\leq \frac{1}{i}\sum_{j=0, j\notin A_{\frac{1}{i}}}^\infty \big|\Delta^m_{n+1}|x_j|\big|^{p_j} < \frac{1}{i}\sum_{j=0}^\infty \big|\Delta^m_{n+1}|x_j|\big|^{p_j} = \frac{1}{i}\tau(x).\end{aligned}$$

This implies that $\|V_\kappa - V_{\kappa_i}\| \leq \frac{1}{i}$ and that V_κ is a limit of finite rank operators. Therefore V_κ is an approximable operator. □

Theorem 6.5 *Let $\kappa \in \mathbb{C}^\mathbb{N}$, and let $(\ell(p, \Delta^m_{n+1}))_\tau$ be a prequasi-normed (sss), where $\tau(x) = \sum_{r=0}^\infty |\Delta^m_{n+1}|x_r||^{p_r}$ for $x \in \ell(p, \Delta^m_{n+1})$. Then $V_\kappa \in \mathfrak{B}_c((\ell(p, \Delta^m_{n+1}))_\tau)$ if and only if $(\kappa_i)_{i=0}^\infty \in c_0$.*

Proof It is simple and so overlooked. □

Corollary 6.6 *If $\kappa \in \mathbb{C}^{\mathbb{N}}$, $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing, and Δ_{n+1}^m is absolutely nondecreasing, then $\mathfrak{B}_c((\ell(p,\Delta_{n+1}^m))_\tau) \subsetneqq \mathfrak{B}((\ell(p,\Delta_{n+1}^m))_\tau)$, where $\tau(x) = \sum_{r=0}^{\infty} |\Delta_{n+1}^m |x_r||^{p_r}$ for all $x \in \ell(p,\Delta_{n+1}^m)$.*

Proof Since I is a multiplication operator on $(\ell(p,\Delta_{n+1}^m))_\tau$ generated by $\kappa = (1,1,\dots)$, $I \notin \mathfrak{B}_c((\ell(p,\Delta_{n+1}^m))_\tau)$ and $I \in \mathfrak{B}((\ell(p,\Delta_{n+1}^m))_\tau)$. □

Theorem 6.7 *If $\kappa \in \mathbb{C}^{\mathbb{N}}$, then $(\ell(p,\Delta_{n+1}^m))_\tau$ is prequasi-Banach (sss), where $\tau(x) = \sum_{r=0}^{\infty} |\Delta_{n+1}^m |x_r||^{p_r}$ for all $x \in \ell(p,\Delta_{n+1}^m)$, and $V_\kappa \in \mathfrak{B}((\ell(p,\Delta_{n+1}^m))_\tau)$. Then κ is bounded away from zero on $(\ker(\kappa))^c$ if and only if $R(V_\kappa)$ is closed.*

Proof Let the sufficient condition be satisfied. Then there is $\epsilon > 0$ with $|\kappa_i| \geq \epsilon$ for all $i \in (\ker(\kappa))^c$. To show that $R(V_\kappa)$ is closed, let d be a limit point of $R(V_\kappa)$. Therefore there is $V_\kappa x_i$ in $(\ell(p,\Delta_{n+1}^m))_\tau$ for all $i \in \mathbb{N}$ such that $\lim_{i\to\infty} V_\kappa x_i = d$. Obviously, $(V_\kappa x_i)$ is a Cauchy sequence. Since Δ_{n+1}^m is absolutely nondecreasing, we have

$$\begin{aligned}
&\tau(V_\kappa x_i - V_\kappa x_j)\\
&\quad= \sum_{r=0}^{\infty} \bigl|\Delta_{n+1}^m \bigl|\kappa_r(x_i)_r - \kappa_r(x_j)_r\bigr|\bigr|^{p_r}\\
&\quad= \sum_{r=0, r\in(\ker(\kappa))^c}^{\infty} \bigl|\Delta_{n+1}^m \bigl|\kappa_r(x_i)_r - \kappa_r(x_j)_r\bigr|\bigr|^{p_r} + \sum_{r=0, r\notin(\ker(\kappa))^c}^{\infty} \bigl|\Delta_{n+1}^m \bigl|\kappa_r(x_i)_r - \kappa_r(x_j)_r\bigr|\bigr|^{p_r}\\
&\quad\geq \sum_{r=0, r\in(\ker(\kappa))^c}^{\infty} \bigl|\Delta_{n+1}^m \bigl(|\kappa_r|\bigl|(x_i)_r - (x_j)_r\bigr|\bigr)\bigr|^{p_r} = \sum_{r=0}^{\infty} \bigl|\Delta_{n+1}^m \bigl(|\kappa_r|\bigl|(y_i)_r - (y_j)_r\bigr|\bigr)\bigr|^{p_r}\\
&\quad> \epsilon \sum_{r=0}^{\infty} \bigl|\Delta_{n+1}^m \bigl|(y_i)_r - (y_j)_r\bigr|\bigr|^{p_r} = \epsilon\tau(y_n - y_m),
\end{aligned}$$

where

$$(y_i)_r = \begin{cases} (x_i)_r, & r \in (\ker(\kappa))^c,\\ 0, & r \notin (\ker(\kappa))^c. \end{cases}$$

This shows that (y_i) is a Cauchy sequence in $(\ell(p,\Delta_{n+1}^m))_\tau$. Since $(\ell(p,\Delta_{n+1}^m))_\tau$ is complete, there is $x \in (\ell(p,\Delta_{n+1}^m))_\tau$ such that $\lim_{i\to\infty} y_i = x$. Since V_κ is continuous, $\lim_{i\to\infty} V_\kappa y_i = V_\kappa x$. But $\lim_{i\to\infty} V_\kappa x_i = \lim_{i\to\infty} V_\kappa y_i = d$. Hence $V_\kappa x = d$. Therefore $d \in R(V_\kappa)$. This shows that $R(V_\kappa)$ is closed. Conversely, let $R(V_\kappa)$ be closed. Then V_κ is bounded away from zero on $((\ell(p,\Delta_{n+1}^m))_\tau)_{(\ker(\kappa))^c}$. Hence there exists $\epsilon > 0$ such that $\tau(V_\kappa x) \geq \epsilon\tau(x)$ for all $x \in ((\ell(p,\Delta_{n+1}^m))_\tau)_{(\ker(\kappa))^c}$.

Let $B = \{r \in (\ker(\kappa))^c : |\kappa_r| < \epsilon\}$. If $B \neq \phi$, then for $i_0 \in B$, we obtain

$$\tau(V_\kappa e_{i_0}) = \tau\bigl(\bigl(\kappa_r(e_{i_0})_r\bigr)_{r=0}^{\infty}\bigr) = \sum_{r=0}^{\infty} \bigl|\Delta_{n+1}^m \bigl|\kappa_r(e_{n_0})_r\bigr|\bigr|^{p_r} < \sum_{r=0}^{\infty} \bigl|\Delta_{n+1}^m \bigl|\epsilon(e_{n_0})_r\bigr|\bigr|^{p_r} = \epsilon\tau(e_{n_0}),$$

which gives a contradiction. So, $B = \phi$ such that $|\kappa_r| \geq \epsilon$ for all $r \in (\ker(\kappa))^c$. This completes the proof of the theorem. □

Theorem 6.8 *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$, and let $(\ell(p, \Delta^m_{n+1}))_\tau$ be a prequasi-Banach (sss) with $\tau(w) = \sum_{r=0}^{\infty} |\Delta^m_{n+1}|w_r||^{p_r}$ for all $w \in \ell(p, \Delta^m_{n+1})$. Then there are $b > 0$ and $B > 0$ such that $b < \kappa_r < B$ for all $r \in \mathbb{N}$ if and only if $V_\kappa \in \mathfrak{B}((\ell(p, \Delta^m_{n+1}))_\tau)$ is invertible.*

Proof Define $\gamma \in \mathbb{C}^{\mathbb{N}}$ by $\gamma_r = \frac{1}{\kappa_r}$. From Theorem 6.2 we have $V_\kappa, V_\gamma \in \mathfrak{B}((\ell(p, \Delta^m_{n+1}))_\tau)$ and $V_\kappa.V_\gamma = V_\gamma.V_\kappa = I$. Then V_γ is the inverse of V_κ. Conversely, let V_κ be invertible. Then $R(V_\kappa) = ((\ell(p, \Delta^m_{n+1}))_\tau)_{\mathbb{N}}$. This implies that $R(V_\kappa)$ is closed. By Theorem 6.7 there is $b > 0$ such that $|\kappa_r| \geq b$ for all $r \in (\ker(\kappa))^c$. Now $\ker(\kappa) = \phi$, else $\kappa_{r_0} = 0$ for several $r_0 \in \mathbb{N}$, and we get $e_{r_0} \in \ker(V_\kappa)$. This gives a contradiction, since $\ker(V_\kappa)$ is trivial. So, $|\kappa_r| \geq a$ for all $r \in \mathbb{N}$. Since V_κ is bounded, by Theorem 6.2 there is $B > 0$ such that $|\kappa_r| \leq B$ for all $r \in \mathbb{N}$. Therefore we have shown that $b \leq |\kappa_r| \leq B$ for all $r \in \mathbb{N}$. □

Theorem 6.9 *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$, and let $(\ell(p, \Delta^m_{n+1}))_\tau$ be a prequasi-Banach (sss), where $\tau(w) = \sum_{r=0}^{\infty} |\Delta^m_{n+1}|w_r||^{p_r}$ for all $w \in \ell(p, \Delta^m_{n+1})$. Then $V_\kappa \in \mathfrak{B}((\ell(p, \Delta^m_{n+1}))_\tau)$ is a Fredholm operator if and only if* (i) $\mathrm{card}(\ker(\kappa)) < \infty$ *and* (ii) $|\kappa_r| \geq \epsilon$ *for all* $r \in (\ker(\kappa))^c$.

Proof Let V_κ be Fredholm. If $\mathrm{card}(\ker(\kappa)) = \infty$, then $e_n \in \ker(V_\kappa)$ for all $n \in \ker(\kappa)$. Since e_n are linearly independent, this gives $\mathrm{card}(\ker(V_\kappa) = \infty$, a contradiction. Therefore $\mathrm{card}(\ker(\kappa)) < \infty$. By Theorem 6.7 condition (ii) is satisfied. Next, if the necessary conditions are satisfied, then V_κ is Fredholm. Indeed, by Theorem 6.7 condition (ii) gives that $R(V_\kappa)$ is closed. Condition (i) indicates that $\dim(\ker(V_\kappa)) < \infty$ and $\dim((R(V_\kappa))^c) < \infty$, and therefore V_κ is Fredholm. □

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