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Monotone iterative method for fractional p-Laplacian differential equations with four-point boundary conditions

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Abstract

A four-point boundary problem for a fractional p-Laplacian differential equation is studied. The existence of two positive solutions is established by means of the monotone iterative method. An example supporting the abstract result is given.

Keywords: Four-point boundary value problem; Monotone iterative; Fractional differential equation

1 Introduction

Fractional differential equations (FrDEs) are widely used in many fields: physical chemistry, financial mathematics, diffusion theory, transportation theory, chaos and turbulence, viscoelastic mechanics, non-newtonian fluid mechanics, seismic analysis. Therefore, many scholars have studied fractional differential equations, to mention a few (see, for example, [1–15]). The standard approach to study boundary value problems (BVPs) for FrDEs is based on the passage to equivalent integral equations and further application of the methods and techniques of modern nonlinear analysis. In particular, to study (multiple) positive solutions, one can combine the classical Green function methods with fixed point theorems in cones (see, for example, [1–3, 5, 7–9, 11]).

On the other hand, BVPs involving p-Laplacian have attracted a lot of attention during the last decades (see, for example, [16–19]). Also, we refer to [20–28], where BVPs for FrDE involving the p-Laplacian were considered. References [22, 25, 28], where the four-point BVPs were considered, are of our special interest. In [25], Wang et al. considered the BVP of the form

$$\begin{cases} D^{\alpha}_{0+}(\phi_p(D^{\beta}_{0+}x))(t) + h(t,x(t)) = 0, \quad 0 < t < 1, \\ x(0) = D^{\beta}_{0+}x(0) = 0, \qquad x(1) = ax(\xi), \qquad D^{\beta}_{0+}x(1) = bD^{\beta}_{0+}x(\eta), \end{cases} \tag{1.1}$$

with $\alpha, \beta \in R$; $1 < \alpha, \beta \leq 2$; $0 \leq a, b \leq 1$; $0 < \xi, \eta < 1$. The authors imposed certain monotonicity conditions and applied the upper and lower solutions method.

In [22] and [28], Tian et al. studied the differential system

$$D^{\alpha}_{0+}\big(\phi_p\big(D^{\beta}_{0+}x\big)\big)(t) = h\big(t,x(t)\big), \quad 0 < t < 1,$$

with the boundary conditions

$$x(0) = D^{\beta}_{0+}x(0) = 0, \qquad D^{\gamma}_{0+}x(1) = ax(\xi), \qquad D^{\beta}_{0+}x(1) = bD^{\beta}_{0+}x(\eta), \tag{1.2}$$

and

$$x(0) = D^{\beta}_{0+}x(0) = 0, \qquad D^{\gamma}_{0+}x(1) = aD^{\gamma}_{0+}x(\xi), \qquad D^{\beta}_{0+}x(1) = bD^{\beta}_{0+}x(\eta), \tag{1.3}$$

respectively (here $1 < \alpha$, $\beta \le 2$; $\gamma > 0$; $1 + \gamma \le \beta$; $a, b > 0$; $\xi, \eta \in (0, 1)$). To be more specific, in [22], the existence of multiple positive solutions was established by means of the Leggett–Williams fixed-point theorem, while in [28], some existence results were obtained using a monotone iterative method.

In this paper, we study the BVP

$$\begin{cases} D^{\alpha}_{0+}(\phi_p(D^{\beta}_{0+}x(t))) = h(t, x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad x(1) = aD^{\gamma}_{0+}x(\xi), \qquad D^{\beta}_{0+}x(0) = 0, \qquad D^{\beta}_{0+}x(1) = bD^{\beta}_{0+}x(\eta), \end{cases} \tag{1.4}$$

where $h \in C([0, 1] \times [0, +\infty), [0, +\infty))$, D^{α}_{0+}, D^{β}_{0+}, and D^{γ}_{0+} stand for the standard Riemann–Liouville differentiations, $\phi_p(z) = |z|^{p-2}z$, $p > 1$; $1 < \alpha, \beta \le 2$, and $\gamma = \frac{\beta-1}{2}$; $0 < \xi \le \frac{1}{2}$; $0 < \eta < 1$; $a, b \in [0, +\infty)$ and $a\Gamma(\beta)\xi^{\frac{\beta-1}{2}} < \Gamma(\frac{\beta+1}{2})$; $b^{p-1}\eta^{\alpha-1} < 1$. By applying a monotone iterative method, we establish the existence of two positive solutions of (1.4) (see Theorem 3.1) and support the general result by an example (see Sect. 4).

Our paper is distinguished from [22, 25, 28] in the following three aspects. Firstly, the boundary condition $x(1) = aD^{\gamma}_{0+}x(\xi)$ in (1.4) is different from the condition $x(1) = ax(\xi)$ in (1.1). Next, the condition $D^{\gamma}_{0+}x(1) = aD^{\gamma}_{0+}x(\xi)$ in (1.3) links the values of derivatives of the same order. At the same time, condition $x(1) = aD^{\gamma}_{0+}x(\xi)$ in (1.4) links the derivatives of different order (as usual, we regard $x(1)$ as the derivative of order 0 of x at $t = 1$). Finally, in (1.2), the authors imposed the boundary condition $D^{\gamma}_{0+}x(1) = ax(\xi)$, where $1 + \gamma \le \beta$, $\xi \in (0, 1)$, and applied the Leggett–Williams fixed-point theorem. This is in a sharp contrast with the boundary condition $x(1) = aD^{\gamma}_{0+}x(\xi)$, $\gamma = \frac{\beta-1}{2}$, $0 < \xi \le \frac{1}{2}$, in (1.4), which allows us to use a monotone iterative method. Summing up: although the methodology that we use rests on the one developed in [28], our setting is different from the ones considered in [22, 25, 28].

2 Preliminaries

In this section, we present some preliminary results including estimates for Green functions and solvability of non-homogeneous fractional p-Laplacian BVPs. These results constitute key ingredients of the proof of our main result (Theorem 3.1). All the function spaces considered below consist of scalar functions. The two lemmas following below are well known.

Lemma 2.1 ([29])

(1) *If* $F \in L(0,1)$ *and* $\mu > \nu > 0$, *then*

$$D_{0+}^{\nu} I_{0+}^{\mu} F(x) = I_{0+}^{\mu-\nu} F(x), \qquad D_{0+}^{\nu} I_{0+}^{\nu} F(x) = F(x).$$

(2) *If* $\mu, \nu > 0$, *then*

$$D_{0+}^{\mu} x^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} x^{\nu-\mu-1}.$$

Lemma 2.2 ([11]) *Let* $A_i \in R$, $i = 1,2,\ldots,N$, *and* $N = [\alpha] + 1$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} F(x) = F(x) + A_1 x^{\alpha-1} + A_2 x^{\alpha-2} + \cdots + A_N x^{\alpha-N},$$

where $\alpha > 0$, $F \in C(0,1) \cap L(0,1)$, $D_{0+}^{\alpha} F \in C(0,1) \cap L(0,1)$.

Let $B_1 = b^{p-1}\eta^{\alpha-1} \neq 1$, $B_2 = a\Gamma(\beta)\xi^{\frac{\beta-1}{2}} \neq \Gamma(\frac{\beta+1}{2})$, and denote

$$G(t,z) = \begin{cases} \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1} - (\Gamma(\frac{\beta+1}{2})-B_2)(t-z)^{\beta-1} - a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}, & 0 \le z \le t \le 1, z \le \xi, \\ \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1} - (\Gamma(\frac{\beta+1}{2})-B_2)(t-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}, & 0 < \xi \le z \le t \le 1, \\ \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1} - a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}, & 0 \le t \le z \le \xi \le 1, \\ \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}, & 0 \le t \le z \le 1, \xi \le z, \end{cases} \tag{2.1}$$

$$M(z,r) = \begin{cases} \frac{[z(1-r)]^{\alpha-1} - b^{p-1}z^{\alpha-1}(\eta-r)^{\alpha-1} - (1-B_1)(z-r)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)}, & 0 \le r \le z \le 1, r \le \eta, \\ \frac{[z(1-r)]^{\alpha-1} - (1-B_1)(z-r)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)}, & 0 \le \eta \le r \le z \le 1, \\ \frac{[z(1-r)]^{\alpha-1} - b^{p-1}z^{\alpha-1}(\eta-r)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)}, & 0 \le z \le r \le \eta \le 1, \\ \frac{[z(1-r)]^{\alpha-1}}{\Gamma(\alpha)(1-B_1)}, & 0 \le z \le r \le 1, \eta \le r. \end{cases} \tag{2.2}$$

The following technical statement plays an important role in studying Green functions relevant to our considerations.

Lemma 2.3 *Let* G *and* M *be defined by* (2.1) *and* (2.2), *respectively. If* $a\Gamma(\beta)\xi^{\frac{\beta-1}{2}} < \Gamma(\frac{\beta+1}{2})$ *and* $b^{p-1}\eta^{\alpha-1} < 1$, *then*:

(a) $G, M \in C([0,1] \times [0,1])$;

(b) $G(t,z) > 0$, $M(t,z) > 0$ *for all* $t, z \in (0,1)$;

(c) *there exist two positive functions* $\mu, \nu \in C((0,1),(0,+\infty))$ *such that, for all* $z \in (0,1)$, *one has*

$$\mu(z) \ge \max_{0 \le t \le 1} G(t,z), \qquad \nu(z) \ge \max_{0 \le t \le 1} M(t,z).$$

Proof (i) This statement follows immediately from (2.1) and (2.2).

(ii) In order to prove that $G(t,z) > 0$ for all $t, z \in (0,1)$, consider, first, the case

$$0 \le z \le t \le 1, \quad z \le \xi.$$

Put

$$g(t,z)=\frac{t^{\beta-1}(1-z)^{\beta-1}-(t-z)^{\beta-1}}{\Gamma(\beta)}. \tag{2.3}$$

Obviously, in the considered case, $g(t,z)>0$.

On the other hand, $\forall\xi\in(0,\frac{1}{2}]$, $z\in[0,\xi]$, we have

$$1-\frac{z}{\xi}\leq(1-z)^2.$$

Then

$$\left(1-\frac{z}{\xi}\right)^{\frac{\beta-1}{2}}\leq(1-z)^{\beta-1},$$

which implies that

$$(\xi-z)^{\frac{\beta-1}{2}}\leq\xi^{\frac{\beta-1}{2}}(1-z)^{\beta-1}.$$

Obviously,

$$g^*(\xi,z)=\xi^{\frac{\beta-1}{2}}(1-z)^{\beta-1}-(\xi-z)^{\frac{\beta-1}{2}}\geq 0.$$

Therefore,

$$\begin{aligned}
G(t,z)&=\frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1}-(\Gamma(\frac{\beta+1}{2})-B_2)(t-z)^{\beta-1}-a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}\\
&=\frac{1}{\Gamma(\beta)}\left(1+\frac{B_2}{\Gamma(\frac{\beta+1}{2})-B_2}\right)\bigl[t(1-z)\bigr]^{\beta-1}-\frac{(t-z)^{\beta-1}}{\Gamma(\beta)}-\frac{a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}\\
&=\frac{t^{\beta-1}(1-z)^{\beta-1}-(t-z)^{\beta-1}}{\Gamma(\beta)}+\frac{B_2t^{\beta-1}(1-z)^{\beta-1}-a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2})-B_2)}\\
&=g(t,z)+\frac{at^{\beta-1}}{(\Gamma(\frac{\beta+1}{2})-B_2)}g^*(\xi,z)\\
&>0.
\end{aligned}$$

The remaining three cases $0<\xi\leq z\leq t<1$ or $0<t\leq z\leq\xi<1$ or $0\leq t\leq z<1$, $\xi\leq z$, can be treated using the similar method, so that we omit the obvious modifications. Thus, $G(t,z)>0$ for all $t,z\in(0,1)$.

Similarly, to prove that $M(t,z)>0$, for all $t,z\in(0,1)$, consider, first, the case

$$0\leq z\leq t\leq 1,\quad z\leq\eta.$$

Put

$$m(t,z)=\frac{(1-z)^{\alpha-1}t^{\alpha-1}-(t-z)^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.4}$$

Obviously, $m(t,z) > 0$ for $0 \leq z \leq t \leq 1$ in the considered case. So

$$\begin{aligned}
M(t,z) &= \frac{[t(1-z)]^{\alpha-1} - b^{p-1}t^{\alpha-1}(\eta-z)^{\alpha-1} - (1-B_1)(t-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \\
&= \frac{1}{\Gamma(\alpha)}\left(1+\frac{B_1}{1-B_1}\right)\bigl[t(1-z)\bigr]^{\alpha-1} - \frac{(t-z)^{\alpha-1}}{\Gamma(\alpha)} - \frac{b^{p-1}t^{\alpha-1}(\eta-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \\
&= \frac{t^{\alpha-1}(1-z)^{\alpha-1} - (t-z)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b^{p-1}t^{\alpha-1}[\eta^{\alpha-1}(1-z)^{\alpha-1} - (\eta-z)^{\alpha-1}]}{\Gamma(\alpha)(1-B_1)} \\
&= m(t,z) + \frac{b^{p-1}t^{\alpha-1}}{1-B_1}m(\eta,z) \\
&> 0.
\end{aligned}$$

One can apply a similar argument in order to treat the remaining three cases $0 < \eta \leq z \leq t < 1$ or $0 < t \leq z \leq \eta < 1$ or $0 \leq t \leq z < 1$, $\eta \leq z$. Thus, $M(t,z) > 0$ for $t,z \in (0,1)$.

(iii) Obviously, for a fixed z, the functions g and m, given by (2.3) and (2.4), respectively, are increasing in t for $t \leq z$ and decreasing in t for $t \geq z$. Therefore,

$$\begin{aligned}
\max_{0\leq t\leq 1} g(t,z) &= g(z,z) = \frac{z^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)}, \quad z \in (0,1); \\
\max_{0\leq t\leq 1} m(t,z) &= m(z,z) = \frac{z^{\alpha-1}(1-z)^{\alpha-1}}{\Gamma(\alpha)}, \quad z \in (0,1).
\end{aligned}$$

Put

$$\begin{aligned}
\mu(z) &= g(z,z) + \frac{B_2(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)}, \quad z \in (0,1); \\
\nu(z) &= m(z,z) + \frac{B_2(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)}, \quad z \in (0,1).
\end{aligned}$$

It is clear that $\mu, \nu \in C((0,1),(0,+\infty))$.

Consider four cases.

If $0 \leq z \leq t \leq 1$, $z \leq \xi$, then

$$\begin{aligned}
\max_{0\leq t\leq 1} G(t,z) &= \max_{0\leq t\leq 1}\left(g(t,z) + \frac{B_2 t^{\beta-1}(1-z)^{\beta-1} - a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)}\right) \\
&\leq g(z,z) + \frac{B_2(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \mu(z).
\end{aligned}$$

If $0 < \xi \leq z \leq t \leq 1$, then

$$\begin{aligned}
\max_{0\leq t\leq 1} G(t,z) &= \max_{0\leq t\leq 1} \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1} - (\Gamma(\frac{\beta+1}{2}) - B_2)(t-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \max_{0\leq t\leq 1}\left(\frac{t^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)} + \frac{B_2 t^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} - \frac{(t-z)^{\beta-1}}{\Gamma(\beta)}\right)
\end{aligned}$$

$$\begin{aligned}
&\le g(z,z) + \frac{B_2(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \mu(z).
\end{aligned}$$

If $0 \le t \le z \le \xi < 1$, then

$$\begin{aligned}
\max_{0\le t\le 1} G(t,z) &= \max_{0\le t\le 1} \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1} - a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \max_{0\le t\le 1} \left(\frac{t^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)} + \frac{B_2 t^{\beta-1}(1-z)^{\beta-1} - a\Gamma(\beta)t^{\beta-1}(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \right) \\
&\le g(z,z) + \frac{B_2(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \mu(z).
\end{aligned}$$

If $0 \le t \le z \le 1$, $\xi \le z$, then

$$\begin{aligned}
\max_{0\le t\le 1} G(t,z) &= \max_{0\le t\le 1} \frac{\Gamma(\frac{\beta+1}{2})[t(1-z)]^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \max_{0\le t\le 1} \left(\frac{t^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)} + \frac{B_2 t^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \right) \\
&\le g(z,z) + \frac{B_2(1-z)^{\beta-1}}{\Gamma(\beta)(\Gamma(\frac{\beta+1}{2}) - B_2)} \\
&= \mu(z).
\end{aligned}$$

Thus,

$$\max_{0\le t\le 1} G(t,z) \le \mu(z), z \in (0,1).$$

Similarly, consider four cases for the function ν.

If $0 \le z \le t \le 1$, $z \le \eta$, then

$$\begin{aligned}
\max_{0\le t\le 1} M(t,z) &= \max_{0\le t\le 1} \left(m(t,z) + \frac{b^{p-1}t^{\alpha-1}[\eta^{\alpha-1}(1-z)^{\alpha-1} - (\eta-z)^{\alpha-1}]}{\Gamma(\alpha)(1-B_1)} \right) \\
&\le m(z,z) + \frac{B_1(1-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \\
&= \nu(z).
\end{aligned}$$

If $0 < \eta \le z \le t \le 1$, then

$$\begin{aligned}
\max_{0\le t\le 1} M(t,z) &= \max_{0\le t\le 1} \left(\frac{[t(1-z)]^{\alpha-1} - (1-B_1)(t-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \right) \\
&= \max_{0\le t\le 1} \left(h(t,z) + \frac{B_1 t^{\alpha-1}(1-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \right) \\
&\le m(z,z) + \frac{B_1(1-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} = \nu(z).
\end{aligned}$$

If $0 \leq t \leq z \leq \eta < 1$, then

$$\begin{aligned}
\max_{0\leq t\leq 1} M(t,z) &= \max_{0\leq t\leq 1}\left(\frac{[t(1-z)]^{\alpha-1} - b^{p-1}z^{\alpha-1}(\eta-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)}\right) \\
&= \max_{0\leq t\leq 1}\left(\frac{t^{\alpha-1}(1-z)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b^{p-1}t^{\alpha-1}[\eta^{\alpha-1}(1-z)^{\alpha-1} - (\eta-z)^{\alpha-1}]}{\Gamma(\alpha)(1-B_1)}\right) \\
&\leq m(z,z) + \frac{B_1(1-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \\
&= \nu(z).
\end{aligned}$$

If $0 \leq t \leq z \leq 1$, $\eta \leq z$, then

$$\begin{aligned}
\max_{0\leq t\leq 1} M(t,z) &= \max_{0\leq t\leq 1}\frac{[t(1-z)]^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \\
&= \max_{0\leq t\leq 1}\left(\frac{t^{\alpha-1}(1-z)^{\alpha-1}}{\Gamma(\alpha)} + \frac{B_1[t(1-z)]^{\alpha-1}}{\Gamma(\alpha)(1-B_1)}\right) \\
&\leq m(z,z) + \frac{B_1(1-z)^{\alpha-1}}{\Gamma(\alpha)(1-B_1)} \\
&= \nu(z).
\end{aligned}$$

Thus,

$$\max_{0\leq t\leq 1} M(t,z) \leq \nu(z), \quad z \in (0,1).$$

The proof of Lemma 2.3 is complete. □

The next statement provides the existence and uniqueness result for the non- homogeneous problems of our interest.

Lemma 2.4 *Assume that*

(i) $\phi_p(z) = |z|^{p-2}z$, $p > 1$;
(ii) $\phi_q = (\phi_p)^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$;
(iii) $1 < \alpha, \beta \leq 2$ *and* $\gamma = \frac{\beta-1}{2}$;
(iv) $0 < \xi \leq \frac{1}{2}$, $0 < \eta < 1$, $a, b \in [0, +\infty)$.

Then, for any $y \in C[0,1]$, *the problem*

$$\begin{cases} D_{0+}^{\alpha}(\phi_p(D_{0+}^{\beta}x(t))) = y(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x(1) = aD_{0+}^{\gamma}x(\xi), \qquad D_{0+}^{\beta}x(0) = 0, \qquad D_{0+}^{\beta}x(1) = bD_{0+}^{\beta}x(\eta), \end{cases} \tag{2.5}$$

admits the unique solution

$$x(t) = \int_0^1 G(t,z)\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz. \tag{2.6}$$

Proof By Lemma 2.2, one has

$$\phi_p\big(D_{0+}^{\beta}x(t)\big) = I_{0+}^{\alpha}y(t) + A_1t^{\alpha-1} + A_2t^{\alpha-2}, \tag{2.7}$$

where $A_1, A_2 \in R$. Combining (2.7) with $D^{\beta}_{0+}x(0) = 0$ (cf. (2.5)), we have $A_2 = 0$. Then

$$\phi_p\bigl(D^{\beta}_{0+}x(t)\bigr) = I^{\alpha}_{0+}y(t) + A_1 t^{\alpha-1}, \tag{2.8}$$

from which it follows that

$$\phi_p\bigl(D^{\beta}_{0+}x(t)\bigr) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}y(r)\,dr + A_1 t^{\alpha-1}. \tag{2.9}$$

Hence,

$$\phi_p\bigl(D^{\beta}_{0+}x(1)\bigr) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-r)^{\alpha-1}y(r)\,dr + A_1 \tag{2.10}$$

and

$$\phi_p\bigl(D^{\beta}_{0+}x(\eta)\bigr) = \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-r)^{\alpha-1}y(r)\,dr + A_1 \eta^{\alpha-1}. \tag{2.11}$$

Next, combining (2.10) and (2.11) with $D^{\beta}_{0+}x(1) = bD^{\beta}_{0+}x(\eta)$ (cf. once again (2.5)), we obtain

$$A_1 = -\int_0^1 \frac{(1-r)^{\alpha-1}}{\Gamma(\alpha)(1-b^{p-1}\eta^{\alpha-1})}y(r)\,dr + \int_0^\eta \frac{b^{p-1}(\eta-r)^{\alpha-1}}{\Gamma(\alpha)(1-b^{p-1}\eta^{\alpha-1})}y(r)\,dr.$$

So

$$\begin{aligned}\phi_p\bigl(D^{\beta}_{0+}x(t)\bigr) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}y(r)\,dr - \int_0^1 \frac{t^{\alpha-1}(1-r)^{\alpha-1}}{\Gamma(\alpha)(1-b^{p-1}\eta^{\alpha-1})}y(r)\,dr \\ &\quad + \int_0^\eta \frac{b^{p-1}t^{\alpha-1}(\eta-r)^{\alpha-1}}{\Gamma(\alpha)(1-b^{p-1}\eta^{\alpha-1})}y(r)\,dr \\ &= -\int_0^1 M(t,r)y(r)\,dr.\end{aligned}$$

Then

$$D^{\beta}_{0+}x(t) = -\phi_q\left(\int_0^1 M(t,r)y(r)\,dr\right). \tag{2.12}$$

Applying now Lemma 2.2 to (2.12), we have

$$x(t) = -I^{\beta}_{0+}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right) + C_1 t^{\beta-1} + C_2 t^{\beta-2}. \tag{2.13}$$

where $C_1, C_2 \in R$. Since $x(0) = 0$ (see (2.5), we have $C_2 = 0$. Therefore, (2.13) reduces to

$$x(t) = -I^{\beta}_{0+}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right) + C_1 t^{\beta-1}. \tag{2.14}$$

Applying D_{0+}^{γ} to both sides of (2.14), and by Lemma 2.1, we have

$$\begin{aligned} D_{0+}^{\gamma}x(t) &= -D_{0+}^{\gamma}I_{0+}^{\beta}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right) + C_1 D_{0+}^{\gamma}t^{\beta-1} \\ &= -I_{0+}^{\beta-\gamma}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right) + C_1\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}t^{\beta-\gamma-1}. \\ &= -\int_0^t \frac{(t-z)^{\frac{\beta-1}{2}}}{\Gamma(\frac{\beta+1}{2})}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz + C_1\frac{\Gamma(\beta)}{\Gamma(\frac{\beta+1}{2})}t^{\frac{\beta-1}{2}}. \end{aligned}$$

So

$$x(1) = -\int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz + C_1, \tag{2.15}$$

$$D_{0+}^{\gamma}x(\xi) = -\int_0^{\xi} \frac{(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\frac{\beta+1}{2})}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz + C_1\frac{\Gamma(\beta)}{\Gamma(\frac{\beta+1}{2})}\xi^{\frac{\beta-1}{2}}. \tag{2.16}$$

Combining (2.15) and (2.16) with $x(1) = aD_{0+}^{\gamma}x(\xi)$ (see again (2.5)), we have

$$\begin{aligned} C_1 = {}& \frac{\Gamma(\frac{\beta+1}{2})}{\Gamma(\frac{\beta+1}{2}) - B_2}\left\{\int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz\right. \\ & \left. - a\int_0^{\xi} \frac{(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\frac{\beta+1}{2})}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz\right\}. \end{aligned}$$

Thus, we obtain the unique solution of problem (2.5):

$$\begin{aligned} x(t) &= -\int_0^t \frac{(t-z)^{\beta-1}}{\Gamma(\beta)}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz \\ &\quad + \frac{t^{\beta-1}\Gamma(\frac{\beta+1}{2})}{\Gamma(\frac{\beta+1}{2}) - B_2}\left\{\int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz\right. \\ &\quad \left. - a\int_0^{\xi} \frac{(\xi-z)^{\frac{\beta-1}{2}}}{\Gamma(\frac{\beta+1}{2})}\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz\right\} \\ &= \int_0^1 G(t,z)\phi_q\left(\int_0^1 M(z,r)y(r)\,dr\right)dz. \end{aligned}$$

The proof of Lemma 2.4 is complete. □

We complete this section with the following simple observation.

Lemma 2.5 *Let $E = C[0,1]$ be the space of continuous functions equipped with the standard sup-norm $\|x\| = \max_{0\leq t\leq 1}|x(t)|$ and denote by $P = \{x \in E \mid x(t) \geq 0, 0 \leq t \leq 1\}$ the corresponding cone. Let $T : P \to E$ be given by*

$$Tx(t) = \int_0^1 G(t,z)\phi_q\left(\int_0^1 M(z,r)h\bigl(r,x(r)\bigr)\,dr\right)dz,$$

where $h \in C([0,1] \times [0,+\infty), [0,+\infty))$ and G and M are defined by (2.1) and (2.2), respectively. Then T takes P into itself, and as such is completely continuous.

Proof Since G, M and h are nonnegative and continuous, one has $T(P) \subset P$ and T is continuous. To prove the complete continuity of T, one needs to use the standard argument based on the Arzela–Ascoli theorem and Lebesgue dominated convergence theorem (see, for example, [23]). □

3 Main result

We are now in a position to formulate our main result. To this end, denote

$$J^{-1} = \int_0^1 \mu(z)\phi_q\left(\int_0^1 \nu(r)\,dr\right)dz,$$

where μ and ν are provided by Lemma 2.3(iii).

Theorem 3.1 *Let $h \in C([0,1] \times [0,+\infty), [0,+\infty))$ and assume that there exists a positive constant k satisfying the following conditions*:

(S_1) *if* $0 \le t \le 1$ *and* $0 \le s_1 \le s_2 \le k$, *then* $h(t,s_1) \le h(t,s_2)$;

(S_2) $\max_{0\le t\le 1} h(t,k) \le \phi_p(kJ)$;

(S_3) $h(t,0) \ne 0$ *for all* $0 \le t \le 1$.

Then problem (1.4) *admits two positive solutions* x^* *and* y^* *such that*:

(i) $0 < \|x^*\| \le k$ *and* $\lim_{n\to\infty} T^n x_0 = x^*$, *where* $x_0(t) = k$ *for all* $0 \le t \le 1$;

(ii) $0 < \|y^*\| \le k$ *and* $\lim_{n\to\infty} T^n y_0 = y^*$, *where* $y_0(t) = 0$ *for all* $0 \le t \le 1$.

Proof Let $\Omega = \{x \in P \mid \|x\| \le k\}$. Assume $x \in \Omega$. Obviously, $0 \le x(t) \le \|x\| \le k$. From ($S_1$) and ($S_2$) it follows immediately that

$$0 \le h\bigl(t,x(t)\bigr) \le h(t,k) \le \max_{0\le t\le 1} h(t,k) \le \phi_p(kJ).$$

We claim that $T(\Omega) \subseteq \Omega$. In fact, for any $x \in \Omega$, we have $Tx \in P$, and by Lemma 2.3, one has

$$\begin{aligned}
\|Tx\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,z)\phi_q\left(\int_0^1 M(s,r)h\bigl(r,x(r)\bigr)\,dr\right)dz\right| \\
&\le \int_0^1 \mu(z)\phi_q\left(\int_0^1 \nu(r)\phi_p(kJ)\,dr\right)dz \\
&= kJ\int_0^1 \mu(z)\phi_q\left(\int_0^1 \nu(r)\,dr\right)dz \\
&= k.
\end{aligned}$$

Hence, $Tx \in \Omega$ and the claim follows.

Let us show the existence of the required x^*. Take the function x_0 equal to k identically on $0 \le t \le 1$. Clearly, $\|x_0\| = k$ (in particular, $x_0 \in \Omega$). Also, $x_1(t) = Tx_0(t) \in \Omega$. Define

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad (n = 0,1,2,\ldots).$$

Then, for all $n = 0,1,2,\ldots$, one has $x_n \in \Omega$.

Also, using (S_2) and the formula for T, and Lemma 2.3, one obtains for any $t \in [0,1]$:

$$\begin{aligned} x_1(t) = Tx_0(t) &= \int_0^1 G(t,z)\phi_q\left(\int_0^1 M(z,r)h\bigl(r,x_0(r)\bigr)\,dr\right)dz \\ &\le \int_0^1 \mu(z)\phi_q\left(\int_0^1 \nu(r)\phi_p(kJ)\,dr\right)dz \\ &\le kJ\int_0^1 \mu(z)\phi_q\left(\int_0^1 \nu(r)\,dr\right)dz \\ &= k = x_0(t). \end{aligned}$$

Hence,

$$x_2(t) = Tx_1(t) \le Tx_0(t) = x_1(t), \quad 0 \le t \le 1.$$

By induction, one has

$$x_{n+1}(t) \le x_n(t), \quad 0 \le t \le 1, n = 0,1,2,\ldots$$

Moreover, by Lemma 2.5, T is completely continuous, we know that $\overline{T(\Omega)}$ is a compact set.

Hence, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ convergent to $x^* \in \Omega$. Since $\{x_n\}_{n=1}^{\infty}$ is monotone, one has $x_n \to x^*$. Combining the continuity of T with $Tx_n = x_{n+1} \to x^*$ yields $Tx^* = x^*$.

Below, using a similar approach, we prove $Ty^* = y^*$. Take the function y_0 equal to 0 identically on $0 \le t \le 1$. Clearly, $\|y_0\| = 0$, and $y_0 \in \Omega$. Also, $y_1 = Ty_0 \in \Omega$. Define

$$y_{n+1} = Ty_n = T^{n+1}y_0, \quad n = 0,1,2,\ldots$$

Then, for all $n = 0,1,2,\ldots$, one has $y_n \in \Omega$. By the same computation as above,

$$y_{n+1}(t) \ge y_n(t), \quad 0 \le t \le 1, n = 0,1,2,\ldots$$

Hence, there exists a subsequence $\{y_{n_i}\}_{i=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ convergent to $y^* \in \Omega$. Since $\{y_n\}_{n=1}^{\infty}$ is monotone, one has $y_n \to x^*$. Combining the continuity of T with $Ty_n = y_{n+1} \to y^*$ yields $Ty^* = y^*$. It remains to observe that, by assumption (S_3), the zero function is not a solution of problem (1.4). So $\|x^*\| > 0$, and $\|y^*\| > 0$. The proof is completed. □

4 Example

Consider the following BVP:

$$\begin{cases} D_{0+}^{\frac{3}{2}}(\phi_{\frac{3}{2}}(D_{0+}^{\frac{3}{2}}x(t))) = \frac{x^2}{15} + \frac{tx}{12} + \frac{1}{20}, & 0 < t < 1, \\ x(0) = 0, \quad x(1) = \frac{1}{4}D_{0+}^{\frac{1}{4}}x(\frac{1}{2}), & D_{0+}^{\frac{3}{2}}x(0) = 0, \quad D_{0+}^{\frac{3}{2}}x(1) = \frac{1}{2}D_{0+}^{\frac{3}{2}}x(\frac{1}{2}). \end{cases} \tag{4.1}$$

A simple computation gives

$$J \approx 3.2218,$$

$$\Gamma\left(\frac{\beta+1}{2}\right) - a\Gamma(\beta)\xi^{\frac{\beta-1}{2}} = \Gamma\left(\frac{5}{4}\right) - \left(\frac{1}{4}\right)\Gamma\left(\left(\frac{3}{2}\right)\right)\left(\frac{1}{2}\right)^{\frac{1}{4}} > 0,$$

$$1 - b^{p-1}\eta^{\alpha-1} = 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}} = 0.5 > 0.$$

Take $k = 8$. Then:

(1) For any $0 \le t \le 1$, $0 \le s_1 \le s_2 \le 8$, $h(t,s_1) \le h(t,s_2)$;

(2) $\max_{0\le t\le 1} h(t,k) = h(1,8) \approx 4.9834 < \phi_p(kJ) \approx 5.0768$;

(3) $h(t,0) = 0.05 \ne 0$, for $0 \le t \le 1$.

Then problem (1.4) has two positive solutions, x^* and y^*, such that

$0 < \|x^*\| \le 8$ and $\lim_{n\to\infty} T^n x_0 = x^*$, where $x_0(t) = 8$,

$0 < \|y^*\| \le 8$ and $\lim_{n\to\infty} T^n y_0 = y^*$, where $y_0(t) = 0$.

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