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$L^p - L^q$ -Maximal regularity of the Van Wijngaarden–Eringen equation in a cylindrical domain

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Abstract

We consider the maximal regularity problem for a PDE of linear acoustics, named the Van Wijngaarden–Eringen equation, that models the propagation of linear acoustic waves in isothermal bubbly liquids, wherein the bubbles are of uniform radius. If the dimensionless bubble radius is greater than one, we prove that the inhomogeneous version of the Van Wijngaarden–Eringen equation, in a cylindrical domain, admits maximal regularity in Lebesgue spaces. Our methods are based on the theory of operator-valued Fourier multipliers.

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1 Introduction

In this paper, we study the following model arising in acoustics propagation in viscous, isothermal bubbly liquids known as the Van Wijngaarden–Eringen (VWE) equation [14, p. 1121]:

$$\begin{aligned} \partial_{tt}u(x, t) - \Delta u(x, t) - (\text{Re}_d)^{-1} \Delta \partial_t u(x, t) - a_0^2 \Delta \partial_{tt}u(x, t) \\ = 0, \quad t \in \mathbb{T} := [0, 2\pi], x \in \Omega \subset \mathbb{R}^N, \end{aligned} \quad (1.1)$$

where Δ denotes the Laplacian operator defined in a domain $\Omega \subset \mathbb{R}^N$ and subject to appropriate boundary conditions. The parameter $\text{Re}_d = c_e L / \delta$ is a *Reynolds number*, where $c_e (> 0)$ is the adiabatic sound speed, δ is the diffusivity of sound [25], and L is a characteristic (macroscopic length). The constant $a_0 > 0$ is a *Knudsen number* that corresponds to the dimensionless bubble radius.

In the case $N = 1$, equation (1.1) was obtained by Van Wijngaarden [26] to describe the propagation of linear acoustic waves in isothermal bubbly liquids. In the case $N = 3$, Eringen [10] re-derived equation (1.1) based on a microcontinuum theory. Later, Rubin et al. [23] found that equation (1.1) also describes acoustic waves in a thermoelastic compress-

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ible Newtonian viscous fluid, and Hayes and Saccomandi [13] showed that it also governs the propagation of transverse plane waves in a particular class of viscoelastic media.

When the Knudsen number a_0 is less than 1, it was proved in [7] that model (1.1) can exhibit chaotic behavior. However, the analysis of mathematical behavior of the model for the case $a_0 > 1$ was left open.

In the present paper we are concerned with the $L^p - L^q$ -maximal regularity problem in a cylindrical domain $\Omega = U \times V \subset \mathbb{R}^{n+d}$ for the following inhomogeneous version of the VWE equation subject to Dirichlet boundary conditions:

$$\begin{cases} (I - a_0^2 \Delta) \partial_{tt} u(x, y, t) - \Delta u(x, y, t) - (\text{Re}_d)^{-1} \Delta \partial_t u(x, y, t) \\ = f(x, y, t), \quad \text{for } (x, y, t) \in \Omega \times (0, 2\pi); \\ \mathcal{B}_U u(x, y, t) = 0, \quad \text{for } (x, y, t) \in \partial U \times V \times (0, 2\pi); \\ \mathcal{B}_V u(x, y, t) = 0, \quad \text{for } (x, y, t) \in U \times \partial V \times (0, 2\pi); \\ u(x, y, 0) = u(x, y, 2\pi), \quad \partial_t u(x, y, 0) = \partial_t u(x, y, 2\pi), \\ \partial_{tt} u(x, y, 0) = \partial_{tt} u(x, y, 2\pi), \quad (x, y) \in \Omega, \end{cases} \tag{1.2}$$

where $U = \mathbb{R}_+^n$, $n \in \mathbb{N}$ and $V \subset \mathbb{R}^d$, $d \in \mathbb{N}_0$ is bounded, open, and connected, Δ denotes a cylindrical decomposition of the Dirichlet Laplacian operator on $L^q(\Omega)$ with respect to the two cross-sections, i.e., $\Delta = \Delta_1 + \Delta_2$ and each Δ_i acts on the according component of Ω . It is well known that many situations in applied sciences naturally lead to problems in cylindrical domains Ω . We refer, e.g., to the textbooks [5] and [6] and the references [9, 20, 22] for a demonstration of the significance of problems on such Ω .

Suppose that we know something about the behavior of the forcing function f in (1.2). For example, f could be bounded or asymptotically periodic, or f might satisfy $f \in L^p(\mathbb{T}; L^q(\Omega))$, where $1 < p, q < \infty$. In the last case, the $L^p - L^q$ -maximal regularity problem consists of obtaining conditions on the parameters $a_0^2, (\text{Re}_d)^{-1}$ in order to conclude that the solution u of (1.2) has the same behavior as f and the following estimate

$$\begin{aligned} & \|u\|_{L^p(\mathbb{T}; L^q(\Omega))} + \|u'\|_{W^{1,p}(\mathbb{T}; L^q(\Omega))} + \|u''\|_{W^{2,p}_{per}(\mathbb{T}; L^q(\Omega))} + \|\Delta u\|_{L^p(\mathbb{T}; [D(\Delta)])} \\ & + \|\Delta u'\|_{W^{1,p}_{per}(\mathbb{T}; [D(\Delta)])} + \|\Delta u''\|_{W^{2,p}_{per}(\mathbb{T}; [D(\Delta)])} \leq C \|f\|_{L^p(\mathbb{T}; L^q(\Omega))} \end{aligned} \tag{1.3}$$

holds.

One of the main tools to address the maximal regularity problem for equation (1.2) is the theory of discrete operator-valued Fourier multipliers. Taking the Fourier series, we are faced with the question under which conditions an operator-valued Fourier series defines a bounded operator in $L^p(\mathbb{T}; X)$ where X is a Banach space. This question was answered by Arendt and Bu in [2], where a discrete operator-valued Fourier multiplier result for UMD spaces X and applications to Cauchy problems of first and second order in Lebesgue spaces can be found. A generalization of the results in [2] to first order integro-differential equations in Lebesgue, Besov, and Hölder spaces is given in [16]. In [17] one finds a comprehensive treatment of second order differential equations in Lebesgue and Hölder spaces. In particular, the special case of the linearized Kuznetsov equation, i.e., $a_0 = 0$ is investigated. More references concerning abstract degenerate Volterra integro-differential equations can be found in [4, 11, 18, 24].

Replacing in (1.2) the negative Laplacian operator $-\Delta$ by a closed linear operator A with domain $D(A)$ defined on a Banach space X , one of the main difficulties we are faced with in order to analyze maximal regularity for (1.2) relies in the unbounded operator $M := I + a_0^2 A$ in front of the second order term ∂_{tt} which, for general A , produces a kind of degenerate second order problem. When M is bounded, this problem was studied by Anufrieva [1]. Very recently, Bu and Cai [3] treated the case of M unbounded.

On the other hand, the usage of operator-valued Fourier multipliers to treat cylindrical in space boundary value problems was first carried out in [12] in a Besov space setting. In that paper the author constructs semiclassical fundamental solutions for a class of elliptic operators on infinite cylindrical domains $\mathbb{R}^n \times V$. This proves to be a strong tool for the treatment of related elliptic and parabolic, as well as hyperbolic problems. Operators in cylindrical domains with a similar splitting property as in the present paper were, in the case of an infinite cylinder, also considered by Nau et al. in [9, 19–22].

In this paper, we directly apply general results of [3] and [20] to our case of the VWE equation and obtain a $L^p - L^q$ -maximal regularity result. The main difficulty relies in the verification of the so-called R -boundedness property that must be satisfied by certain sets of operators. To overcome this difficulty, we will employ the criteria established by Denk, Hieber, and Prüss in the reference [8] that reduce the problem to the localization of the spectrum of the Laplacian. We highlight that our method is sufficiently general to admit a wider class of operators than the Laplacian in (1.2) allowing also the possibility of the fractional Laplacian, the bi-Laplacian Δ^2 , or other operators of practical interest. Therefore, we first establish our main result in an abstract setting that roughly states that under certain conditions of sectoriality of the operator A , and for all $\eta > 0$, the equation

$$(I + a_0^2 A^\eta)u''(t) + A^\eta u(t) + (\operatorname{Re} a)^{-1} A^\eta u'(t) = f(t), \quad t \in \mathbb{T},$$

has $L^p - L^q$ -maximal regularity. Then, using the results of [20], we establish our main findings concerning (1.2), namely: for any given $f \in L^p(\mathbb{T}, L^q(\Omega))$ and under the condition

$$a_0 > 1,$$

the solution u of problem (1.2) exists, is unique, and belongs to the space $W_{per}^{2,p}(\mathbb{T}, [D(\Delta_q)]) \cap W_{per}^{2,p}(\mathbb{T}, X)$. Moreover, for any $1 < p, q < \infty$, the a priori estimate (1.3) holds.

2 Preliminaries

We will use recent results obtained in [3] where L^p -maximal regularity was obtained for an abstract degenerate model of second order given by

$$(Mu)'(t) - aBu(t) - \alpha Bu'(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{2.1}$$

where a, α are real numbers and B and M are linear operators with domains $D(B)$ and $D(M)$ defined on a Banach space X such that $D(B) \subset D(M)$.

We recall the notion of the M -resolvent set of B as follows:

$$\begin{aligned} \rho_M(B) := \{s \in \mathbb{R} : -k^2 M - (a + \alpha ik)B : D(B) \rightarrow X \\ \text{is invertible and } [-k^2 M - (a + \alpha ik)B]^{-1} \in \mathcal{B}(X)\}. \end{aligned} \tag{2.2}$$

Observe that $D(B)$ and $D(M)$ are Banach spaces when endowed with the graph norm.

For any $n \in \mathbb{N}$ and $1 \leq p < \infty$, we define the vector-valued function spaces [3, Definition 2.4]:

$$W_{per}^{n,p}(\mathbb{T}, X) := \{u \in L^p(\mathbb{T}, X) : \text{there exists } v \in L^p(\mathbb{T}, X), \hat{v}(k) = (ik)^n \hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Let $u \in L^p(\mathbb{T}; X)$, then $u \in W_{per}^{n,p}(\mathbb{T}; X)$ if and only if u is n -times differentiable a.e. on \mathbb{T} and $u^{(n)} \in L^p(\mathbb{T}, X)$, in this case $u^{(k)}(0) = u^{(k)}(2\pi)$, $0 \leq k \leq n - 1$ [2, Lemma 2.1]. We refer to [2] and [3] for more information about these spaces.

Let $1 < p \leq \infty$, we define the solution space of (2.1) by

$$S_p(B, M) := \{u \in L^p(\mathbb{T}; D(B)) \cap W_{per}^{1,p}(\mathbb{T}; X) \mid u' \in L^p(\mathbb{T}; D(M)), Mu' \in W_{per}^{1,p}(\mathbb{T}; X)\}.$$

We have that $S_p(B, M)$ is a Banach space with the norm

$$\|u\|_{S_p(B,M)} := \|u\|_{L^p} + \|u'\|_{L^p} + \|Bu\|_{L^p} + \|Bu'\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu)'\|_{L^p}.$$

The notion of L^p -maximal regularity is given as follows.

Definition 2.1 Let $1 < p < \infty$ and $f \in L^p(\mathbb{T}, X)$ be given. We say that (2.1) has L^p -maximal regularity if there exists a unique $u \in S_p(B, M)$ that solves equation (2.1) on \mathbb{T} and there exists a constant $C > 0$ such that the estimate

$$\|u\|_{S_p(B,M)} \leq C \|f\|_{L^p}$$

holds.

In particular, for $X = L^q(\Omega)$, $1 < q < \infty$, we say that (2.1) has $L^p - L^q$ -maximal regularity. Since the characterization given by Bu and Cai in the reference [3] is provided in terms of the R -boundedness of certain sets of operators, we first recall this definition.

Definition 2.2 Let X and Y be Banach spaces. A set $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R -bounded if there is a constant $c \geq 0$ such that

$$\|(T_1 x_1, \dots, T_n x_n)\|_R \leq c \|(x_1, \dots, x_n)\|_R \tag{2.3}$$

for all $T_1, \dots, T_n \in \mathcal{T}$, $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$, where

$$\|(x_1, \dots, x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1, 1\}^n} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|.$$

The least c such that (2.3) is satisfied is called the R -bound of \mathcal{T} and is denoted by $R(\mathcal{T})$.

The property of R -boundedness is preserved under sum or product by a constant. Moreover, if X and Y are Hilbert spaces, R -boundedness is equivalent to uniform boundedness. More information about these properties is summarized in [8].

The class of Banach spaces X such that the Hilbert transform defined by

$$(Hf)(t) = \lim_{\epsilon, R \rightarrow \infty} \frac{1}{\pi} \int_{\epsilon \leq |s| \leq R} \frac{f(t-s)}{s} ds, \quad t \in \mathbb{R},$$

is bounded in $L^p(\mathbb{R}; X)$ for some $p \in (1, \infty)$ is denoted by \mathcal{HT} (or UMD).

The UMD spaces include Hilbert spaces, Sobolev spaces $H_p^s(\Omega)$, $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, ℓ_p , $1 < p < \infty$, and vector-valued Lebesgue spaces $L^p(\Omega, \mu; X)$, where X is a UMD space. On the other hand, the space of continuous functions $C(K)$ does not have the UMD property.

We next recall the result obtained in [3].

Theorem 2.3 *Let $1 < p < \infty$ and $\alpha, a \in \mathbb{R}$. Assume that B and M are closed linear operators defined on a UMD space X such that $D(B) \subset D(M)$. The following assertions are equivalent:*

- (i) Equation (2.1) has L^p -maximal regularity;
- (ii) $\mathbb{Z} \subset \rho_M(B)$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R -bounded where

$$N_k := -[k^2M + (a + i\alpha k)B]^{-1}, \quad k \in \mathbb{Z}. \tag{2.4}$$

We also need to recall some preliminaries on sectorial operators. Let $\Sigma_\phi \subset \mathbb{C}$ denote the open sector $\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$. We define the spaces of functions as follows: $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \rightarrow \mathbb{C} \text{ holomorphic}\}$, and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \rightarrow \mathbb{C} \text{ holomorphic and bounded}\}$$

which is endowed with the norm $\|f\|_\infty^\phi = \sup_{|\arg \lambda| < \phi} |f(\lambda)|$. We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows:

$$\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha, \beta < 0} \{f \in \mathcal{H}(\Sigma_\phi) : \|f\|_{\alpha, \beta}^\phi < \infty\},$$

with $\|f\|_{\alpha, \beta}^\phi = \sup_{|\lambda| \leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda| \geq 1} |\lambda^{-\beta} f(\lambda)|$.

Definition 2.4 Given a closed linear operator A in X , we say that A is sectorial if the following conditions hold:

- (i) $\overline{D(A)} = X$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$;
 - (ii) $\|t(t + A)^{-1}\| \leq M$ for all $t > 0$ and some $M > 0$.
- A is called R -sectorial if the set $\{t(t + A)^{-1}\}_{t > 0}$ is R -bounded.

If A is sectorial, then $\Sigma_\phi \subset \rho(-A)$ for some $\phi > 0$ and

$$\sup_{|\arg \lambda| < \phi} \|\lambda(\lambda + A)^{-1}\| < \infty.$$

We denote the *spectral angle* of a sectorial operator A by

$$\phi_A = \inf \left\{ \phi : \Sigma_{\pi-\phi} \subset \rho(-A), \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty \right\}.$$

Definition 2.5 Given a sectorial operator A , we say that it admits a bounded \mathcal{H}^∞ -calculus if there exist $\phi > \phi_A$ and a constant $K_\phi > 0$ such that

$$\|f(A)\| \leq K_\phi \|f\|_\infty^\phi \quad \text{for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{2.5}$$

The class of sectorial operators A which admit a bounded \mathcal{H}^∞ -calculus is denoted by $\mathcal{H}^\infty(X)$. Moreover, the \mathcal{H}^∞ -angle is defined by $\phi_A^\infty = \inf\{\phi > \phi_A : (2.5) \text{ holds}\}$. When $A \in \mathcal{H}^\infty(X)$, we say that A admits an R -bounded \mathcal{H}^∞ -calculus if the set

$$\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta), \|h\|_\infty^\theta \leq 1\}$$

is R -bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$. The corresponding angle is defined in an obvious way and is denoted by $\theta_A^{R\infty}$.

Remark 2.6 If A is a sectorial operator on a Hilbert space, Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$, Sobolev spaces $W^{s,p}(\Omega)$, $1 < p < \infty$, $s \in \mathbb{R}$, or Besov spaces $B_{p,q}^s(\Omega)$, $1 < p, q < \infty$, $s \in \mathbb{R}$ and A admits a bounded \mathcal{H}^∞ -calculus of angle β , then A admits a \mathcal{RH}^∞ -calculus on the same angle β on each of the above described spaces (see Kalton and Weis [15]). More generally, this property is true whenever X is a UMD space with the so-called property (α) (see [15]).

There exist well-known examples for general classes of closed linear operators with bounded \mathcal{H}^∞ such as: normal sectorial operators in a Hilbert space; m -accretive operators in a Hilbert space; generators of bounded C_0 -groups on L^p -spaces, and negative generators of positive contraction semigroups on L^p -spaces.

We also recall the following result [8, Proposition 4.10], which will be needed for our characterization, which shows under suitable conditions of uniform boundedness the R -boundedness of certain sets of operators.

Proposition 2.7 *Let $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_A^{R\infty}$, where Λ is an arbitrary index set. Then the set $\{h_\lambda(A)\}_{\lambda \in \Lambda}$ is R -bounded.*

3 Main results

Let $1 \leq p < \infty$, $\eta > 0$ and X be a Banach space. In this section, we want to give necessary conditions on a given sectorial operator A with domain $D(A)$ defined on X that describe the $L^p - L^q$ -maximal regularity of the VWE equation given in an abstract form as follows:

$$(I + a_0^2 A^\eta)u''(t) + A^\eta u(t) + (\operatorname{Re}_d)^{-1} A^\eta u'(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{3.1}$$

where $a_0 > 0$, $\operatorname{Re}_d > 0$, and $f \in L^p(\mathbb{T}; X)$. We state the main abstract result of this paper.

Theorem 3.1 *Assume that X is a UMD -space, $1 < p < \infty$, $a_0 > 1$, and suppose that $A \in \mathcal{RH}^\infty(X)$ with angle $\theta_A^{R\infty} \in (0, \frac{\pi}{2\eta})$ and $0 \in \rho(A)$. Then, for all $\eta > 0$, equation (3.1) admits $L^p - L^q$ -maximal regularity.*

Proof We first point out that our equation (3.1) labels into (2.1) for $M = (I + a_0^2 A^\eta)$, $a = -1$, $\alpha = -(\operatorname{Re}_d)^{-1}$, and $B = A^\eta$. Moreover, it is clear that $D(A^\eta) = D(I + a_0^2 A^\eta)$. In order to prove well-posedness for (3.1), we only need to show that condition (ii) in Theorem 2.3 holds, that is, we have to prove that the sets $\{k^2 MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R -bounded. Indeed, we have

$$N_k = [-k^2 + ((1 - a_0^2 k^2) + ik(\operatorname{Re}_d)^{-1})A^\eta]^{-1}.$$

It follows that

$$\begin{aligned} N_k &= \frac{1}{(1 - a_0^2 k^2) + ik(\operatorname{Re}_d)^{-1}} \left[\frac{-k^2}{(1 - a_0^2 k^2) + ik(\operatorname{Re}_d)^{-1}} + A^\eta \right]^{-1} \\ &= \frac{-1}{k^2} d_k (d_k + A^\eta)^{-1}, \end{aligned}$$

where $d_k := \frac{-k^2}{(1 - a_0^2 k^2) + ik(\operatorname{Re}_d)^{-1}}$. A computation shows that

$$\Re(d_k) = \frac{k^2(k^2 a_0^2 - 1)}{(1 - a_0^2 k^2)^2 + (k(\operatorname{Re}_d)^{-1})^2}$$

and

$$\Im(d_k) = \frac{k^3(\operatorname{Re}_d)^{-1}}{(1 - a_0^2 k^2)^2 + (k(\operatorname{Re}_d)^{-1})^2}.$$

Since $a_0 > 1$, by hypothesis, we obtain $\theta^* := \sup_{k \in \mathbb{Z}} |\arg(d_k)| < \pi/2$.

On the other hand, we have $0 < \theta_A^{R_\infty} < \frac{\pi}{2\eta}$, and hence there exists $s > \theta_A^{R_\infty}$ such that $s < \frac{\pi}{2\eta}$. For every $z \in \Sigma_s$ and $k \in \mathbb{Z}$, $k \neq 0$, we can define

$$F(k, z) = d_k (d_k + z^\eta)^{-1}.$$

Observe that $\frac{z^\eta}{d_k}$ belongs to the sector $\Sigma_{s\eta + \pi/2}$. We immediately get that the distance from the sector $\Sigma_{s\eta + \pi/2}$ to -1 is always positive. Consequently, there exists a constant $M > 0$ independent of $k \in \mathbb{Z}$ and $z \in \Sigma_s$ such that

$$|F(k, z)| = \left| \frac{1}{1 + \frac{z^\eta}{d_k}} \right| \leq M.$$

Now, from Proposition 2.7 it follows that

$$\{F(k, A)\}_{k \in \mathbb{Z} \setminus \{0\}}$$

is R -bounded. In particular, since A is invertible, the operators $H(k) := (d_k + A^\eta)^{-1}$ exist for all $k \in \mathbb{Z}$, then $H(k)$ belongs to $\mathcal{B}(X)$ for all $k \in \mathbb{Z}$ and the sequence

$$\{d_k (d_k + A^\eta)^{-1}\}_{k \in \mathbb{Z}}$$

is R -bounded. Since the following identity holds $A^\eta(d_k + A^\eta)^{-1} = I - d_k(d_k + A^\eta)^{-1}$, we obtain

$$\begin{aligned} k^2MN_k &= -(I + a_0^2A^\eta)d_k(d_k + A^\eta)^{-1} \\ &= a_0^2d_kd_k(d_k + A^\eta)^{-1} - a_0^2d_kI - d_k(d_k + A^\eta)^{-1}. \end{aligned}$$

Due to the R -boundedness of the sets $\{d_k\}_{k \in \mathbb{Z}}$ and $\{d_kd_k(d_k + A^\eta)^{-1}\}$, we can state that the set $\{k^2MN_k : k \in \mathbb{Z}\}$ is also R -bounded. On the other hand, we have

$$kN_k = \frac{-1}{k}d_k(d_k + A^\eta)^{-1},$$

which implies that the set $\{kN_k\}_{k \in \mathbb{Z}}$ is R -bounded, too. From Theorem 2.3 we conclude that equation (3.1) admits $L^p - L^q$ -maximal regularity. \square

Taking into account Remark 2.6, we obtain the following corollary.

Corollary 3.2 *Let $1 < p, q < \infty$ be given. Suppose that $a_0 > 1$ and that A is a sectorial operator that admits a bounded \mathcal{H}^∞ -calculus of angle $\theta_A^{\mathcal{R}^\infty} \in (0, \frac{\pi}{2\eta})$ and $0 \in \rho(A)$. Then, for all $\eta > 0$, equation (3.1) admits $L^p - L^q$ -maximal regularity.*

Finally, we consider the Van Wijngaarden–Eringen equation in a cylindrical domain $\Omega = U \times V \subset \mathbb{R}^{n+d}$, where $U = \mathbb{R}_+^n, n \in \mathbb{N}$ and $V \subset \mathbb{R}^d, d \in \mathbb{N}_0$ is bounded, open, and connected

$$\begin{cases} (I - a_0^2\Delta)\partial_{tt}u(x, y, t) - \Delta u(x, y, t) - (\text{Re}_d)^{-1}\Delta\partial_tu(x, y, t) \\ \quad = f(x, y, t), \quad \text{for } (x, y, t) \in \Omega \times (0, 2\pi), \\ \mathcal{B}_Uu(x, y, t) = 0, \quad \text{for } (x, y, t) \in \partial U \times V \times (0, 2\pi), \\ \mathcal{B}_Vu(x, y, t) = 0, \quad \text{for } (x, y, t) \in U \times \partial V \times (0, 2\pi), \\ u(x, y, 0) = u(x, y, 2\pi), \quad \partial_tu(x, y, 0) = \partial_tu(x, y, 2\pi), \\ \partial_{tt}u(x, y, 0) = \partial_{tt}u(x, y, 2\pi), \quad (x, y) \in \Omega, \end{cases} \tag{3.2}$$

where Δ denotes a cylindrical decomposition of the Dirichlet Laplacian operator on $L^q(\Omega)$ with respect to the two cross-sections, i.e., $\Delta = \Delta_1 + \Delta_2$, where Δ_i acts on the according component of Ω . Following [20] we introduce L^q -realizations $\Delta_{q,i} = \Delta_i$ as follows:

$$\begin{aligned} D(\Delta_{q,1}) &:= \{u \in W^{2,q}(\mathbb{R}_+^n, L^q(V)) : \mathcal{B}_U = 0\}; \\ D(\Delta_{q,2}) &:= W^{2,q}(V) \cap W_0^{1,q}(V), \end{aligned}$$

see also [27] for the description of $\Delta_{q,2}$. We define the Laplacian Δ_q in $L^q(\Omega)$ subject to the Dirichlet boundary conditions \mathcal{B}_U and \mathcal{B}_V to be

$$\begin{aligned} D(\Delta_q) &:= D(\Delta_{q,1}) \cap D(\Delta_{q,2}) \\ \Delta_q u &:= \Delta_{q,1}u + \Delta_{q,2}u = \Delta u, \quad u \in D(\Delta_q). \end{aligned}$$

Suppose now that V is a C^2 -standard domain (see [20, Definition 3.1] for the precise definition). Then, applying [20, Theorem 4.2], we have that $-\Delta_q \in \mathcal{RH}^\infty(L^q(\Omega))$ and $0 \in \rho(\Delta_q)$. Moreover, by [20, Proposition 5.1(i)], we have $\theta_{-\Delta_q}^{\mathcal{R}^\infty} < \frac{\pi}{2}$.

From Corollary 3.2 with $\eta = 1$ and $A = -\Delta_q$, we deduce the following result.

Theorem 3.3 *Let $1 < p, q < \infty$ and assume the condition*

$$a_0 > 1.$$

Then, for any given $f \in L^p(\mathbb{T}, L^q(\Omega))$, the solution u of problem (3.2) exists, is unique, and belongs to the space $W_{per}^{2,p}(\mathbb{T}, [D(\Delta_q)]) \cap W_{per}^{2,p}(\mathbb{T}, X)$. Moreover, for any $1 < p, q < \infty$, the estimate

$$\begin{aligned} & \|u\|_{L^p(\mathbb{T}, L^q(\Omega))} + \|u'\|_{W^{1,p}(\mathbb{T}, L^q(\Omega))} + \|u''\|_{W_{per}^{2,p}(\mathbb{T}, L^q(\Omega))} + \|\Delta u\|_{L^p(\mathbb{T}, [D(\Delta_q)])} \\ & + \|\Delta u'\|_{W_{per}^{1,p}(\mathbb{T}, [D(\Delta_q)])} + \|\Delta u''\|_{W_{per}^{2,p}(\mathbb{T}, [D(\Delta_q)])} \leq C \|f\|_{L^p(\mathbb{T}, L^q(\Omega))} \end{aligned}$$

holds.

We remark that an analogous result holds when we replace the Laplacian by the fractional Laplacian $(-\Delta_q)^\eta$, $0 < \eta < 1$.

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Authors' contributions

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