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Dynamic properties of a discrete population model with diffusion

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Abstract

We study the dynamical properties of a discrete population model with diffusion. We survey the transcritical, pitchfork, and flip bifurcations of nonhyperbolic fixed points by using the center manifold theorem. For the degenerate fixed point with eigenvalues ± 1 of the model, we obtain the normal form of the mapping by using the coordinate transformation. Then we give an approximating system of the normal form via an approximation by a flow. We give the local behavior near a degenerate equilibrium of the vector field by the blowup technique. By the conjugacy between the reflection of time-one mapping of a vector field and the model we obtain the stability and qualitative structures near the degenerate fixed point of the model. Finally, we carry out a numerical simulation to illustrate the analytical results of the model.

Keywords: Discrete population models; Centre manifold theorem; Flip bifurcation; Degenerate fixed point; Stability

1 Introduction

In recent years, discrete dynamic systems developed rapidly and achieved fruitful results (see [1–9]). Particularly, the dynamic properties of discrete population models received extensive attention by many researchers [10–15]. In [10] the authors considered the following discrete population model with diffusion:

$$\mu_i^{t+1} = \frac{p\mu_i^t}{1 + q\mu_i^t} + d\Delta^2\mu_{i-1}^t, \quad (1)$$

with Dirichlet boundary condition $\mu_0^t = 0 = \mu_{n+1}^t$, where $d > 0$ is the diffusion coefficient, $t \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $p > 1$, $q > 0$, Δ^2 is second-order difference operator, and $\Delta^2\mu_{i-1}^t = \mu_{i+1}^t - 2\mu_i^t + \mu_{i-1}^t$, $i \in \{1, 2, \dots, n\}$.

When $n = 2$, $p = q = b$, and μ_1^t, μ_2^t are written as μ_t, v_t , respectively, and model (1) can be transformed into a particular two-patch discrete-time metapopulation model of the form

$$\begin{cases} \mu_{t+1} = \frac{b\mu_t}{1+b\mu_t} + d(-2\mu_t + v_t), \\ v_{t+1} = \frac{bv_t}{1+bv_t} + d(\mu_t - 2v_t). \end{cases} \quad (2)$$

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The properties of pitchfork bifurcation, one case (i.e., $b = \frac{(d+1)^2}{3d-1}, d > \frac{1}{3}$) of flip bifurcation at nontrivial fixed point $E_1(\mu^*, \nu^*)$ with $\mu^* = \nu^* = \frac{b-d-1}{b(d+1)}$, and Lyapunov exponential analysis are given in [10].

In this paper, we study the following problems. First, we focus on the stability and bifurcations of the nonhyperbolic fixed point $E_0(0, 0)$ and on the case (i.e., $b = \frac{(d+1)^2}{d-1}, d > 1$) of flip bifurcation at nontrivial fixed point E_1 . Second, we investigate the stability and qualitative structures near a degenerate fixed point $\tilde{E}_0(0, 0)$ formed by fixed points E_0 and E_1 , which collide and bond together when $(d, b) = (1, 2)$.

Compared to continuous models, discrete models can have different dynamic properties (see [1]), such as flip bifurcation (see [15]), Neimark–Sacker bifurcation (see [3, 14, 15]), invariant curve (see [15]), fold-flip bifurcation (see [8]), and strong resonance (see [3]).

The stability and qualitative structures near a degenerate fixed point of a discrete dynamic system are paid attention by many researchers; for example, Elaydi and Luís [7] proposed some open problems about stability of degenerate fixed point of discrete-time Guzowska–Luís–Elaydi competition model and discrete-time Ricker competition model. Due to the complexity of the degenerate fixed point, it is necessary to combine a variety of mathematical tools to study, for example, coordinate transformation [3], normal form theory (see [3, 16]), Picard iteration (see [3, 8]), approximation by a flow (see [3, 16]), qualitative theory of ordinary differential equations (see [17]), and blowup technique (see [17–19]). In this paper, we survey the local behavior near a degenerate equilibrium of a vector field by homogeneous polar blowup (see [18, 19]), which can avoid tedious calculations.

This paper is organized as follows. In Sect. 2, we give topological types of fixed point of model (2). In Sect. 3, we study transcritical bifurcation, pitchfork bifurcation, flip bifurcation, and the stability of the fixed point E_0 of model (2). In Sect. 4, a particular case ($b = \frac{(d+1)^2}{d-1}, d > 1$) of flip bifurcation and stability of fixed point E_1 of model (2) are surveyed. In Sect. 5, we apply the coordinate transformation to obtain the normal form of the mapping of model (2) near the degenerate fixed point \tilde{E}_0 . The approximate system (also called a vector field) of the normal form of the mapping is given by Takens’s theorem (see [16, pp. 142–148]) and Picard iteration. The local behavior near the degenerate equilibrium $(0, 0)$ of the vector field is obtained by using the blowup technique. The stability and qualitative structures near the degenerate fixed point \tilde{E}_0 of model (2) are obtained by the conjugacy between reflection of time-one mapping of the vector field and model (2). Finally, we perform numerical simulation analysis.

2 Topological types of the fixed point

In this section, we give topological types of fixed points E_0 and E_1 on the parameter (d, b) -plane. We write model (2) as the following planar mapping $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$:

$$T : \begin{pmatrix} \mu \\ \nu \end{pmatrix} \mapsto \begin{pmatrix} \frac{b\mu}{1+b\mu} + d(-2\mu + \nu) \\ \frac{b\nu}{1+b\nu} + d(\mu - 2\nu) \end{pmatrix}.$$

The mapping T expanded by Taylor series at the fixed point E_0 is

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} \mapsto \begin{pmatrix} b-2d & d \\ d & b-2d \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} + \begin{pmatrix} -b^2\mu^2 + b^3\mu^3 \\ -b^2\nu^2 + b^3\nu^3 \end{pmatrix} + O(3), \tag{3}$$

where $O(3)$ represents the terms of orders greater than or equal to 3. The Jacobian matrix of T at the point E_0 is

$$J(E_0) = \begin{pmatrix} b - 2d & d \\ d & b - 2d \end{pmatrix}.$$

The eigenvalues of matrix $J(E_0)$ are $\lambda_1 = b - d$ and $\lambda_2 = b - 3d$.

Translating the fixed point E_1 into the origin O by coordinate translation $y_1 = \mu - \mu^*$, $y_2 = \nu - \nu^*$, the Taylor expansion at the fixed point E_1 is

$$\tilde{T} : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{(d+1)^2}{b} - 2d & d \\ d & \frac{(d+1)^2}{b} - 2d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -\frac{(d+1)^3}{b}y_1^2 + \frac{(d+1)^4}{b}y_1^3 \\ -\frac{(d+1)^3}{b}y_2^2 + \frac{(d+1)^4}{b}y_2^3 \end{pmatrix} + O(3). \quad (4)$$

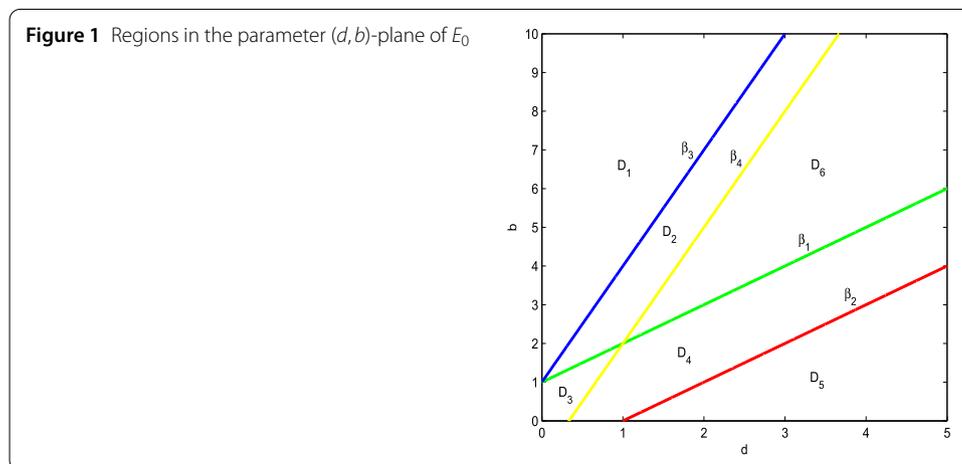
The Jacobian matrix of T at the fixed point E_1 (i.e., mapping \tilde{T} at the fixed point O) is as follows:

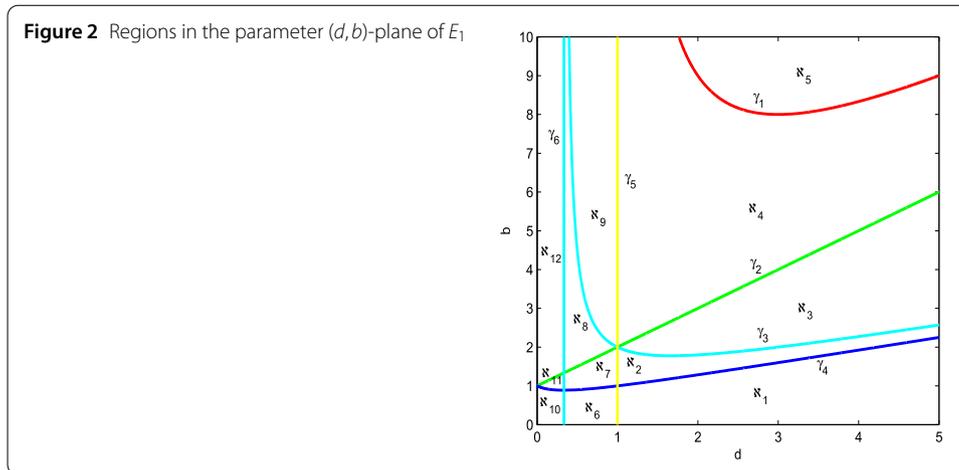
$$J(E_1) = \begin{pmatrix} \frac{(d+1)^2}{b} - 2d & d \\ d & \frac{(d+1)^2}{b} - 2d \end{pmatrix}.$$

The eigenvalues of matrix $J(E_1)$ are $\lambda_3 = \frac{(d+1)^2}{b} - d$ and $\lambda_4 = \frac{(d+1)^2}{b} - 3d$.

For convenience, we give some notations: D_i ($i = 1, 2, \dots, 6$) (see Fig. 1), and \mathfrak{R}_i ($i = 1, 2, \dots, 12$) (see Fig. 2) are the regions of the parameter (d, b) -plane of E_0 and E_1 , respectively, divided by the curves β_i ($i = 1, 2, 3, 4$) and γ_i ($i = 1, 2, \dots, 6$):

$$\begin{aligned} \beta_1 &\triangleq \{(d, b) \in \mathbb{R}^2 \mid b = d + 1, d > 0\}, & \beta_2 &\triangleq \{(d, b) \in \mathbb{R}^2 \mid b = d - 1, d > 1\}, \\ \beta_3 &\triangleq \{(d, b) \in \mathbb{R}^2 \mid b = 3d + 1, d > 0\}, & \beta_4 &\triangleq \left\{ (d, b) \in \mathbb{R}^2 \mid b = 3d - 1, d > \frac{1}{3} \right\}, \\ \gamma_1 &\triangleq \left\{ (d, b) \in \mathbb{R}^2 \mid b = \frac{(d+1)^2}{d-1}, d > 1, b > 0 \right\}, & \gamma_2 &\triangleq \beta_1, \\ \gamma_3 &\triangleq \left\{ (d, b) \in \mathbb{R}^2 \mid b = \frac{(d+1)^2}{3d-1}, d > \frac{1}{3}, b > 0 \right\}, \end{aligned}$$





$$\gamma_4 \triangleq \left\{ (d, b) \in \mathbb{R}^2 \mid b = \frac{(d+1)^2}{3d+1}, d > 0 \right\}, \quad \gamma_5 \triangleq \left\{ (d, b) \in \mathbb{R}^2 \mid b = 1, d > 0 \right\},$$

$$\gamma_6 \triangleq \left\{ (d, b) \in \mathbb{R}^2 \mid b = \frac{1}{3}, d > 0 \right\}.$$

Lemma 1 *The fixed point E_0 is nonhyperbolic if and only if (d, b) lies on the curves β_i ($i = 1, 2, 3, 4$). Otherwise, the fixed point E_0 has the following properties:*

- (1) *If $(d, b) \in D_i$ ($i = 3$), then it is a stable node;*
- (2) *If $(d, b) \in D_i$ ($i = 1, 5, 6$), then it is an unstable node;*
- (3) *If $(d, b) \in D_i$ ($i = 2, 4$), then it is a saddle.*

Lemma 2 *The fixed point E_1 is nonhyperbolic if and only if (d, b) lies on the curves γ_i ($i = 1, 2, 3, 4, 5, 6$). Otherwise, the fixed point E_1 has the following properties:*

- (1) *If $(d, b) \in \aleph_i$ ($i = 8, 12$), then it is a stable node;*
- (2) *If $(d, b) \in \aleph_i$ ($i = 1, 3, 5, 6, 10$), then it is a unstable node;*
- (3) *If $(d, b) \in \aleph_i$ ($i = 2, 4, 7, 9, 11$), then it is a saddle.*

3 Bifurcation at fixed point E_0

In this section, the parameter b is regarded as a bifurcation parameter. We survey stability and bifurcation properties of the mapping T at the fixed point E_0 .

Theorem 1 *As (d, b) passes through the curve β_1 , the mapping T undergoes a transcritical bifurcation at the fixed point E_0 . The details are as follows:*

- (1) *The fixed points E_0 and E_1 collide and bond together to form a fixed point when (d, b) passes through curve β_1 ;*
- (2) *The fixed point E_0 is nonhyperbolic and stable for $0 < d < 1$ when (d, b) lies on the curve β_1 ;*
- (3) *The fixed point E_0 is nonhyperbolic and unstable for $d > 1$ when (d, b) lies on the curve β_1 .*

Proof Mapping (3) can be diagonalized by the linear transformation $\mu = x + y, \nu = x - y$ when $b = d + 1$. We obtain the following mapping:

$$\begin{pmatrix} x \\ y \\ b \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ b \end{pmatrix} + \begin{pmatrix} -b^2(x^2 + y^2) + b^3(x^3 + 3xy^2) \\ -2b^2xy + b^3(y^3 + 3x^2y) \\ 0 \end{pmatrix} + O(3). \tag{5}$$

We choose b as a bifurcation parameter. According to the center manifold theorem [20], a center manifold of mapping (5) is

$$W^c = \{ (x, y, b) \mid y = h(x, b), h(0, b_0) = Dh(0, b_0) = 0, |x| < \varepsilon, |b - b_0| < \delta \},$$

where $b_0 := d + 1$, and ε and δ are sufficiently small positives. Therefore assume that the center manifold form is

$$y = h_1(x, b) = d_1x^2 + e_1(b - b_0)x + f_1(b - b_0)^2 + O(3),$$

which must satisfy

$$\begin{aligned} h_1(\lambda_1x - b^2(x^2 + h_1^2(x, b)) + b^3(x^3 + 3xh_1^2(x, b)) + O(3), b) \\ = \lambda_2h_1(x, b) - 2b^2xh_1(x, b) + b^3(h_1^3(x, b) + 3x^2(x, b)). \end{aligned}$$

By comparing the coefficients we obtain $d_1 = \lambda_2d_1, e_1 = \lambda_2e_1, f_1 = \lambda_2f_1$. We have $\lambda_1 = 1$ and $\lambda_2 = 1 - 2d$ when (d, b) lies on the curve β_1 . Therefore $d_1 = e_1 = f_1 = 0$. The center manifold is

$$y = h_1(x, b) = O(3).$$

By substituting the center manifold $y = h_1(x, b)$ into mapping (5) we obtain a one-dimensional mapping reduced to the center manifold as follows:

$$x \mapsto f_1(x) = x - b^2x^2 + b^3x^3 + O(3).$$

We can verify that the transversality and nondegeneracy conditions of transcritical bifurcation (see [4, pp. 504–507])

$$\left. \frac{\partial f_1}{\partial x} \right|_{(x,b)=(0,b_0)} = 1, \quad \left. \frac{\partial^2 f_1}{\partial x \partial b} \right|_{(x,b)=(0,b_0)} = 1 \neq 0, \quad \left. \frac{\partial^2 f_1}{\partial x^2} \right|_{(x,b)=(0,b_0)} = -2b^2 \neq 0$$

are established. Therefore the mapping T undergoes a transcritical bifurcation at the fixed point E_0 as (d, b) passes through the curve β_1 . We obtain by Theorem 1.15 in [5, p. 29] and $\lambda_1 = 1, \lambda_2 = 1 - 2d \neq 1$ that the fixed point E_0 is stable and nonhyperbolic for $0 < d < 1$ when (d, b) lies on the curve β_1 . The fixed point E_0 is unstable and nonhyperbolic for $d > 1$ when (d, b) lies on the curve β_1 . The proof of the theorem is completed. \square

Remark 1 The mapping

$$x \mapsto f_1(x) = x - b^2x^2 + b^3x^3 + O(3)$$

can be seen as

$$x \mapsto f_1(x) = (b - d)x - b^2x^2 + b^3x^3 + O(3)$$

when (b, d) lies on β_1 . Therefore

$$\frac{\partial^2 f_1}{\partial x \partial b} \Big|_{(x,b)=(0,b_0)} = 1.$$

Theorem 2 *As (d, b) passes through the curve β_2 , the mapping T undergoes a flip bifurcation at fixed point E_0 . The details are as follows:*

- (1) *The mapping T appears in a stable 2-period orbit near fixed point E_0 , and the stability of fixed point does not change when (d, b) passes through the curve β_2 ;*
- (2) *The fixed point E_0 is nonhyperbolic and unstable when (d, b) lies on the curve β_2 .*

Proof Similarly, the center manifold has the form

$$y = h_2(x, b) = d_2x^2 + e_2(b - b_1)x + f_2(b - b_1)^2 + O(3)$$

when $b = b_1 := d - 1$. The center manifold

$$y = \overline{h_2}(x, b) = O(3)$$

can be obtained by the same calculation method as in Theorem 1. By substituting the center manifold into mapping (5) we obtain the following one-dimensional mapping reduced to the center manifold:

$$x \mapsto f_2(x) = -x - b^2x^2 + b^3x^3 + O(3).$$

We can verify that the transversality and nondegeneracy conditions of flip bifurcation (see Theorem 4.3 in [3])

$$F_1 := \frac{\partial^2 f_2}{\partial x \partial b} \Big|_{(x,b)=(0,b_1)} = -1,$$

$$F_2 := \left[\frac{1}{2} \left(\frac{\partial^2 f_2}{\partial x^2} \right)^2 + \frac{1}{3} \left(\frac{\partial^3 f_2}{\partial x^3} \right) \right] \Big|_{(x,b)=(0,b_1)} = 2b^3(b + 1) > 0$$

are established. Note that the mapping T undergoes a flip bifurcation as (d, b) passes through the curve β_2 . By Theorem 3.5.1 in [6] and $F_2 > 0$ the mapping T appears in a stable 2-period orbit as (d, b) passes through the curve β_2 . We have $\lambda_1 = -1$ and $\lambda_2 < -1$ when (d, b) lies on the curve β_2 . So the nonhyperbolic fixed point E_0 is unstable. The proof of the theorem is completed. □

Theorem 3 *As (d, b) passes through the curve β_3 , the mapping T undergoes a pitchfork bifurcation at the fixed point E_0 . The details are as follows:*

- (1) *The stability of the fixed point E_0 does not change when (d, b) passes through the curve β_3 ;*

(2) The fixed point E_0 is nonhyperbolic and unstable when (d, b) lies on the curve β_3 .

Proof Applying the same calculation method as in Theorem 1, we obtain the center manifold

$$x = h_3(y, b) = \frac{b^2}{2d}y^2 + O(3)$$

when $b = b_2 := 3d + 1$. Substituting the center manifold $x = h_3(y, b)$ into mapping (2) yields a one-dimensional mapping reduced to the center manifold

$$y \mapsto f_3(y) = y - \frac{b^4}{d}y^3 + O(3).$$

We can check that the transversality and nondegeneracy conditions of pitchfork bifurcation (see [4, p. 511])

$$\begin{aligned} \frac{\partial f_3}{\partial y} \Big|_{(y,b)=(0,b_2)} &= 1, & \frac{\partial^2 f_3}{\partial y \partial b} \Big|_{(y,b)=(0,b_2)} &= 1, \\ \frac{\partial^2 f_3}{\partial y^2} \Big|_{(y,b)=(0,b_2)} &= 0, & \frac{\partial^3 f_3}{\partial y^3} \Big|_{(y,b)=(0,b_2)} &= \frac{18b^3(b+1)}{1-b} \end{aligned}$$

are established. Therefore the mapping T undergoes a pitchfork bifurcation at fixed point E_0 as (d, b) passes through the curve β_3 . We have $\lambda_1 = 3d + 1 > 1$ and $\lambda_2 = 1$ when (d, b) lies on the curve β_3 . Therefore the fixed point E_0 is nonhyperbolic and stable when (d, b) lies on the curve β_3 . The proof of the theorem is completed. \square

Theorem 4 As (d, b) passes through the curve β_4 , the mapping T undergoes a flip bifurcation at fixed point E_0 . The details are as follows:

- (1) For $d \in (\frac{1}{3}, 1)$, the mapping T appears in a stable 2-periodic orbit near fixed point E_0 , and the stability of fixed point E_0 has to change when (d, b) passes through the curve β_4 ;
- (2) For $d \in (1, +\infty)$, the mapping T appears in a repellent 2-period orbit near fixed point E_0 , and the stability of fixed point E_0 does not change when (d, b) passes through the curve β_4 ;
- (3) The fixed point E_0 is stable and nonhyperbolic for $d \in (\frac{1}{3}, 1)$ when (d, b) lies on the curve β_4 ;
- (4) The fixed point E_0 is unstable and nonhyperbolic for $d \in (1, +\infty)$ when (d, b) lies on the curve β_4 .

Proof Similarly, when $b = b_3 := 3d - 1$, we obtain the center manifold as follows:

$$x = h_4(y, b) = \frac{b^2}{2(d-1)}y^2 + O(3).$$

Substituting the center manifold $x = h_4(y, b)$ into mapping (5), we obtain a one-dimensional mapping reduced to the center manifold,

$$y \mapsto f_4(y) = -y - \frac{b^4}{d-1}y^3 + O(3).$$

We can check that the transversality and nondegeneracy conditions of flip bifurcation

$$F_3 := \frac{\partial^2 f_4}{\partial y \partial b} \Big|_{(y,b)=(0,b_3)} = -1, \quad F_4 := \left[\frac{1}{2} \left(\frac{\partial^2 f_4}{\partial y^2} \right)^2 + \frac{1}{3} \left(\frac{\partial^3 f_4}{\partial y^3} \right) \right] \Big|_{(y,b)=(0,b_3)} = \frac{2b^4}{1-d}$$

are established. The mapping T undergoes a flip bifurcation as (d, b) passes through the curve β_4 at fixed point E_0 . By Theorem 3.5.1 in [6] the mapping T appears in a stable 2-periodic orbit for $d \in (\frac{1}{3}, 1)$ and in a repellent 2-periodic orbit for $d \in (1, +\infty)$. We have $\lambda_1 = 2d - 1$ and $\lambda_2 = -1$ when (d, b) lies on the curve β_4 . By Theorem 1.61 in [5] the fixed point E_0 is unstable and nonhyperbolic for $d \in (1, +\infty)$ and stable and nonhyperbolic for $d \in (\frac{1}{3}, 1)$. The proof of the theorem is completed. \square

4 Bifurcation at fixed point E_1

In this section, we consider b as a bifurcation parameter and apply the center manifold theorem to survey the stability and flip bifurcation of the fixed point E_1 of the mapping T .

Theorem 5 *As (d, b) passes through the curve γ_1 , the mapping T undergoes a flip bifurcation at fixed point E_1 . The details are as follows:*

- (1) *The mapping T appears in a stable 2-periodic orbit near fixed point E_1 , and the stability of the fixed point E_1 does not change when (d, b) passes through the curve γ_1 ;*
- (2) *The fixed point E_1 is nonhyperbolic and unstable when (d, b) lies on the curve γ_1 .*

Proof The mapping \tilde{T} can be diagonalized into the following form by linear transformation $y_1 = x + y, y_2 = x - y$ when $b = b_4 := \frac{(d+1)^2}{d-1}$ ($d > 1$):

$$\begin{pmatrix} x \\ y \\ b \end{pmatrix} \mapsto \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ b \end{pmatrix} + \begin{pmatrix} -\frac{2(d+1)^2}{b}(x^2 + y^2) + \frac{6(d+1)^4}{b}(x^3 + 3xy^2) \\ -\frac{4(d+1)^3}{b}xy + \frac{6(d+1)^4}{b}(y^3 + 3x^2y) \\ 0 \end{pmatrix} + O(3). \quad (6)$$

Similarly to the previous proof method, we get that the center manifold of mapping (6) is

$$y = h_5(x, b) = O(3).$$

Substituting the center manifold into mapping (6), we obtain the one-dimensional mapping reduced to the center manifold

$$x \mapsto f_5(x) = -x - (d - 1)(d + 1)x^2 + (d - 1)(d + 1)^2x^3 + O(3).$$

We can verify that the transversality and nondegeneracy conditions of flip bifurcation

$$F_5 := \frac{\partial^2 f_5}{\partial x \partial b} \Big|_{(x,b)=(0,b_4)} = -1, \\ F_6 := \left[\frac{1}{2} \left(\frac{\partial^2 f_5}{\partial x^2} \right)^2 + \frac{1}{3} \left(\frac{\partial^3 f_5}{\partial x^3} \right) \right] \Big|_{(x,b)=(0,b_4)} = 2d(d - 1)(d + 1)^2$$

are established. The mapping \tilde{T} undergoes a flip bifurcation as (d, b) passes through the curve γ_1 at fixed point E_1 . By Theorem 3.5.1 in [6] and $F_6 > 0$ mapping (4) appears in a

stable 2-period orbit near fixed point E_1 for $d \in (1, +\infty)$ as (d, b) passes through the curve γ_1 . We have $\lambda_1 = 2d - 1 > 1$ and $\lambda_2 = -1$ when (d, b) lies on the curve γ_1 , so the fixed point E_1 is nonhyperbolic and unstable. The proof of the theorem is completed. \square

5 Qualitative structures and stability of degenerate fixed point

In this section, we apply Picard iteration, approximation by a flow, and near identity transformation (see [3, 8, 16]) to investigate the degenerate fixed point of the mapping T when $(d, b) = (1, 2)$. We obtain qualitative structures and the stability of the degenerate fixed point of model (2).

5.1 Normal form

To establish the normal form of the mapping T , we write it in the following form when $(d, b) = (1, 2)$:

$$x \mapsto f(x) = Ax + R(x), \quad x = (\mu, \nu)^T \in \mathbb{R}^2. \tag{7}$$

The matrix A has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. From

$$Aq_1 = \lambda_1 q_1, \quad Aq_2 = \lambda_2 q_2, \quad A^T p_1 = \lambda_1 p_1, \quad A^T p_2 = \lambda_2 p_2$$

we get the solution

$$p_1 = q_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T, \quad p_2 = q_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T.$$

Apply the reversible linear transformation

$$\Phi_1 : x = \xi_1 q_1 + \xi_2 q_2$$

to mapping (7). Since $\langle q_1, p_2 \rangle = \langle q_2, p_1 \rangle = 0$, the linear transformation Φ_1 transforms mapping (7) into the mapping

$$F : \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 \xi_1 + \sum_{i+j=2,3} \frac{1}{i!j!} g_{ij} \xi_1^i \xi_2^j \\ \lambda_2 \xi_2 + \sum_{i+j=2,3} \frac{1}{i!j!} h_{ij} \xi_1^i \xi_2^j \end{pmatrix} + O(\|\xi\|^4),$$

where

$$\begin{aligned} g_{20} = g_{02} = -4\sqrt{2}, \quad g_{11} = g_{21} = g_{03} = 0, \quad g_{30} = g_{12} = 24, \\ h_{11} = -4\sqrt{2}, \quad h_{20} = h_{02} = h_{30} = h_{12} = 0, \quad h_{21} = h_{03} = 24. \end{aligned}$$

Theorem 6 *The smooth mapping F can be transformed into the following normal form by reversible near identity transformation:*

$$\tilde{F} : \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \mapsto \begin{pmatrix} \eta_1 + a_0 \eta_1^2 + b_0 \eta_2^2 + c_0 \eta_1^3 + d_0 \eta_1 \eta_2^2 \\ -\eta_1 + \eta_1 \eta_2 \end{pmatrix} + O(\|\eta\|^4), \tag{8}$$

where

$$a_0 = \frac{1}{2}, \quad b_0 = 16, \quad c_0 = \frac{1}{8}, \quad d_0 = 12.$$

Proof First, we apply the near identity transformation

$$\Phi_2 : \xi_1 = \eta_1 - \frac{h_{21}}{2h_{11}}\eta_1^2 - \frac{h_{03}}{6h_{11}}\eta_2^2, \quad \xi_2 = \eta_2$$

and scale transformation

$$\eta_1 \mapsto \frac{\eta_1}{h_{11}}$$

to mapping F to eliminate quadratic and cubic items as much as possible. Then the resonance term is left, and the normal form is obtained. The proof is complete. \square

5.2 Approximation by a flow

For dealing with stability and qualitative structures near the degenerate fixed points of a planar mapping with eigenvalues ± 1 , the general idea is embedding the reflection of the normal form of a mapping into the flow of a vector field. Then we study the dynamic behavior of the mapping by the properties of the vector field. In [3, p. 424], it is stated that “any mapping sufficiently close to the identity mapping can be approximated up to any order by a flow shift.” That is, an approximating system (see [3]) can be constructed, in which the unit time shift φ^1 along orbits coincides with (or, at least, approximates) the near identity mapping up to order k . In this paper, we use the reflection transformation (see [3, 8]) to transform the truncated normal form of the mapping \tilde{F} into a near identity mapping and embed it into the flow of a vector field. We use the vector field to investigate the stability and qualitative structures near the degenerate fixed point of the mapping T when $(d, b) = (1, 2)$. The Taken theorem and the following lemma come from [21, p. 38].

Lemma 3 (Taken’s) *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the C^r -diffeomorphism ($r \geq 2$) defined by*

$$F(x) = Ax + F^2(x) + F^3(x) + \dots + F^r(x) + o(|x|^r) \quad \text{as } x \rightarrow 0,$$

where $A = S(I + N)$, S is semisimple, N is nilpotent, $SN = NS$, and I is the identity. Let $1 < l < r$ be any integer, and let $F^k \in H_n^k$, where H_n^k is the vector space of homogeneous polynomials of order k in n variables with values in C^n . Then there exist a diffeomorphism $\psi_l : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, where Ω is a neighborhood of the origin in \mathbb{R}^n , and a vector field $X(x)$ on \mathbb{R}^n such that

- (i) $j^l(\psi_l \circ F \circ \psi_l^{-1})$ is an A -normal form of diffeomorphism $F(x)$ up to order l ,
- (ii) $X(Sx) = SX(x)$ for all $x \in \mathbb{R}^n$,
- (iii) $j^l(\psi_l \circ F \circ \psi_l^{-1})(x) = j^l(\Phi_X(1, Sx))$,

where j^l is the truncation operator up to order l , and $\Phi_X(t, x)$ is the flow of $X(x)$. Furthermore, for such a vector field $X(x)$, $j^l X(x)$ is uniquely determined by $j^l F(x)$.

If there exist constants k, σ, η such that

$$\|Y(x, y)\| \geq \sigma \|(x, y)\|^k, \quad (x, y) \in \mathbb{R}^2, \|(x, y)\| \leq \eta,$$

then the C^∞ vector field is of Łojasiewicz type, where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^2 .

Lemma 4 *Let $g \in \text{Diff}(2)$ with associated formal normal form $\tilde{G} = R \circ \tilde{X}_1$ ($X \mapsto X_1$, i.e., time-one mapping), where \tilde{X} has in 0 a singularity of Łojasiewicz type with a characteristic orbit and $R^n = I$ for some $n > 0$. Suppose, moreover, that all singularities in some nice decomposition of \tilde{X} are of type I and, restricted to a fundamental conic domain Σ , we have $\alpha(g^n|_\Sigma) \in [\text{Diff}_{\text{rot}}^0(2)]^k$ (this especially is the case where \tilde{X} has no elliptic sectors). Then there exists a C^∞ -coordinate change H with $j_\infty(H - I)(0) = 0$ and an R -invariant representative X of \tilde{X} such that $H^{-1} \circ g \circ H = R \circ \tilde{X}_1$.*

Remark 2 Lemmas 3 and 4 deal with many concepts, the details of which can be found in [16] and [21], respectively.

For the truncated normal form of the mapping \tilde{F} ,

$$N : \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \mapsto \begin{pmatrix} \eta_1 + a_0\eta_1^2 + b_0\eta_2^2 + c_0\eta_1^3 + d_0\eta_1\eta_2^2 \\ -\eta_1 + \eta_1\eta_2 \end{pmatrix}, \tag{9}$$

we can see that

$$R \circ N : \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \mapsto \begin{pmatrix} \eta_1 + a_0\eta_1^2 + b_0\eta_2^2 + c_0\eta_1^3 + d_0\eta_1\eta_2^2 \\ \eta_1 - \eta_1\eta_2 \end{pmatrix}$$

and $RN(x) = N(Rx)$. For $R \circ N$ embedded in the flow of a vector field, we have the following results.

Theorem 7 *The mapping $R \circ N$ satisfies*

$$R \circ N(\eta) = \varphi^t(\eta) + O(\|\eta\|^4)$$

in a small neighborhood of the origin O , where φ^t is the flow generated by the planar vector field

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\eta_1^2 + 16\eta_2^2 - \frac{1}{8}\eta_1^3 + 20\eta_1\eta_2^2 \\ -\eta_1\eta_2 - \frac{1}{4}\eta_1^2\eta_2 + 8\eta_2^3 \end{pmatrix}. \tag{10}$$

The φ^1 is the unit-time shift along orbits of planar vector field (10), and $R = \text{diag}(1, -1)$, $R^2 = I_{2 \times 2}$.

Proof By Lemma 3 the reflection of the truncated normal form N can be embedded in the flow of a vector field, and φ^t is constructed as follows:

$$X \mapsto \phi^t(X) = \varphi^t, \quad \dot{X} = Y(X) = JX + Y_2(X) + Y_3(X) + \dots, \quad X = (\eta_1, \eta_2)^T \in R^2, \tag{11}$$

where $J = \mathbf{0}$ and $Y_k(X) = (Z_1^k(X), Z_2^k(X))^T$, where $Z_{1,2}^k(X)$ are homogeneous polynomial functions with unknown coefficients. To facilitate the expression of the Picard iteration

process, define $M(X) = N(\eta)$, which is introduced to solve the explicit expression of the vector field Y such that $RM(X) = \phi^t(X) + O(\|X\|^4)$. Because the highest order of $N(\eta)$ expressions is 3, it is only necessary to perform three Picard iterations on system (11) to achieve the approximate purpose. We start the iterative from $X^{(1)}(t) = e^t X$. The calculation shows that the linear part of $X^{(1)}(1)$ coincides with $RM(X)$. The expression of Y_2 can be assumed of the following form

$$Y_2 = \left(\frac{1}{2}A_{20}\eta_1^2 + \frac{1}{2}A_{02}\eta_2^2, B_{11}\eta_1\eta_2 \right)^T.$$

Therefore by Picard iteration we obtain

$$X^{(2)}(t) = e^t X + \int_0^t e^{J(t-\tau)} Y_2(X^{(1)}(\tau)) d\tau = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}A_{20}\eta_1^2 t + \frac{1}{2}A_{02}\eta_2^2 t \\ B_{11}\eta_1\eta_2 t \end{pmatrix} + O(\|\eta\|^2).$$

By comparing the quadratic terms of $RM(X)$ and $X^{(2)}(1)$ we obtain $A_{20} = 1, A_{02} = 32$, and $B_{11} = -1$. Then we put

$$Y_3 = \left(\sum_{i+j=3} A_{ij}\eta_1^i\eta_2^j, \sum_{i+j=3} B_{ij}\eta_1^i\eta_2^j \right)^T.$$

By iterating and letting $t = 1$ we get

$$\begin{aligned} X^{(3)}(1) &= e^J X + \int_0^1 e^{J(1-\tau)} [Y_2(X^{(2)}(\tau)) + Y_3(X^{(2)}(\tau))] d\tau \\ &= \begin{pmatrix} \eta_1 + \frac{1}{2}\eta_1^2 + 16\eta_2^2 \\ \eta_2 - \eta_1\eta_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} (\frac{1}{4} + \frac{1}{6}A_{30})\eta_1^3 + \frac{1}{2}A_{21}\eta_1^2\eta_2 + (-8 + \frac{1}{2}A_{12})\eta_1\eta_2^2 + \frac{1}{6}A_{03}\eta_2^3 \\ \frac{1}{6}B_{30}\eta_1^3 + (-\frac{1}{4} + \frac{1}{2}B_{21})\eta_1^2\eta_2 + \frac{1}{2}B_{12}\eta_1\eta_2^2 + (\frac{1}{6}B_{03} - 8)\eta_2^3 \end{pmatrix} + O(\|x\|^4). \end{aligned}$$

By comparing the cubic terms of $MR(X)$ and $X^{(3)}(1)$ we obtain

$$A_{30} = -\frac{3}{4}, \quad A_{21} = A_{03} = B_{30} = B_{12} = 0, \quad A_{12} = 40, \quad B_{21} = -\frac{1}{2}, \quad B_{03} = 48.$$

By substituting the expressions of Y_2, Y_3 into system (11) we obtain a planar vector field (10) and the following C^∞ vector field:

$$X : \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\eta_1^2 + 16\eta_2^2 - \frac{1}{8}\eta_1^3 + 20\eta_1\eta_2^2 \\ -\eta_1\eta_2 - \frac{1}{4}\eta_1^2\eta_2 + 8\eta_2^3 \end{pmatrix} + O(4).$$

We can see from this expression that the C^∞ vector field X is of Łojasiewicz type. The proof of the theorem is completed. \square

Theorem 8 *In the sufficiently small neighborhood of the origin O , the degenerate fixed point \tilde{E}_0 of model (2) is unstable. The fixed points E_0 and E_1 of model (2) collide and bond together to form a degenerate fixed point \tilde{E}_0 when $(d, b) = (1, 2)$. For a sufficiently small*

positive initial value $P(\mu_0, \nu_0) \in R^2_+$, the sequences $\{T^n(P)\}_0^{+\infty}$ enter the third quadrant from the first quadrant along both sides of straight line $\mu = \nu$. In this process the sequences

$$\{T^{2n}(P)\}_0^{+\infty} \quad \text{and} \quad \{T^{2n+1}(P)\}_0^{+\infty}$$

along both sides of straight line $\mu = \nu$ first “repel” and then “attract”. The phase portraits of model (2) near the degenerate fixed point \tilde{E}_0 are shown in Fig. 4.

Proof Vector field (10) is $\eta_2 \mapsto -\eta_2$ invariant, and equilibrium (0,0) is degenerate (i.e., equilibrium (0,0) without linear part), so the phase portraits of vector field (10) are symmetric about the η_1 -axis, so we consider only $\eta_2 \geq 0$. To study the local behavior near the degenerate equilibrium (0,0) of the vector field X , we will desingularize it by using the homogeneous polar blowup technique. Consider the mapping

$$\begin{aligned} \phi : S^1 \times R &\rightarrow R^2, \\ (\theta, r) &\mapsto (r \cos \theta, r \sin \theta) = (\eta_1, \eta_2). \end{aligned}$$

This mapping transforms $\{r = 0\}$ into (0,0), and inverse mapping ϕ^{-1} blows up the degenerate equilibrium (0,0) of the vector field X to a circle (see [18, pp. 91–92]).

By using coordinate transformation $\eta_1 = r \cos \theta, \eta_2 = r \sin \theta$ of vector field X , we obtain the system

$$\begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} r^2 \cos \theta (\frac{1}{2} \cos^2 \theta + 15 \sin^2 \theta) + O(r^3) \\ -r \sin \theta (\frac{3}{2} \cos^2 \theta + 15 \sin^2 \theta) + O(r^2) \end{pmatrix}. \tag{12}$$

We use the following time scale transformation $r dt = d\tau$ (see [3, 8, 19]) and convert system (12) into the equivalent system

$$\begin{pmatrix} \frac{dr}{d\tau} \\ \frac{d\theta}{d\tau} \end{pmatrix} = \begin{pmatrix} r \cos \theta (\frac{1}{2} \cos^2 \theta + 15 \sin^2 \theta) + O(r^2) \\ -\sin \theta (\frac{3}{2} \cos^2 \theta + 15 \sin^2 \theta) + O(r) \end{pmatrix}. \tag{13}$$

To investigate the phase portrait of the vector field X in a neighborhood Ω of the origin O , it clearly suffices to investigate the phase portrait of system (13) in the neighborhood $\phi^{-1}(\Omega)$ of the circle $S^1 \times \{0\}$. The phase portrait of the vector field X near the degenerate equilibrium (0,0) is easily obtained by shrinking the circle $S^1 \times \{0\}$ to a point.

Since

$$\left. \frac{d\theta}{dt} \right|_{\theta=0,\pi} > 0, \quad \left. \frac{dr}{dt} \right|_{\theta=0} > 0, \quad \left. \frac{dr}{dt} \right|_{\theta=\pi} < 0,$$

the trajectory of the vector field X on the η_1 -axis tends to $+\infty$ when $\theta = 0$ and $\eta_1 > 0$ and tends to the origin O when $\theta = \pi$ and $\eta_1 < 0$. The vector field X has an orbit connecting with origin O in a definite direction. If r is sufficiently small, then

$$\frac{dr}{dt} > 0, \quad \theta \in \left(0, \frac{\pi}{2}\right)$$

and

$$\frac{dr}{dt} < 0, \quad \theta \in \left(\frac{\pi}{2}, \pi\right).$$

Therefore the equilibrium $(0, 0)$ of the vector field X is unstable. Vector field (10) is $\eta_2 \mapsto -\eta_2$ invariant. The phase portraits of vector field (10) near $(0, 0)$ are shown in Fig. 3.

System (13) has equilibria $E^j = (\theta_j, 0)$, $j = 1, 2$, where $\theta_1 = 0$ and $\theta_2 = \pi$, on the circle $S^1 \times \{0\}$, and the Jacobian matrices at equilibria E^1 and E^2 are

$$J(E^1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}, \quad J(E^2) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

The equilibria E^1 and E^2 are hyperbolic saddles on the circle $S^1 \times \{0\}$. From the first equation of system (13) we have that

$$\frac{dr}{d\tau} > 0, \quad \theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$$

and

$$\frac{dr}{d\tau} < 0, \quad \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

From the second equation of system (13) we have that

$$\frac{d\theta}{d\tau} > 0, \quad \theta \in (0, \pi)$$

and

$$\frac{d\theta}{d\tau} < 0, \quad \theta \in (\pi, 2\pi).$$

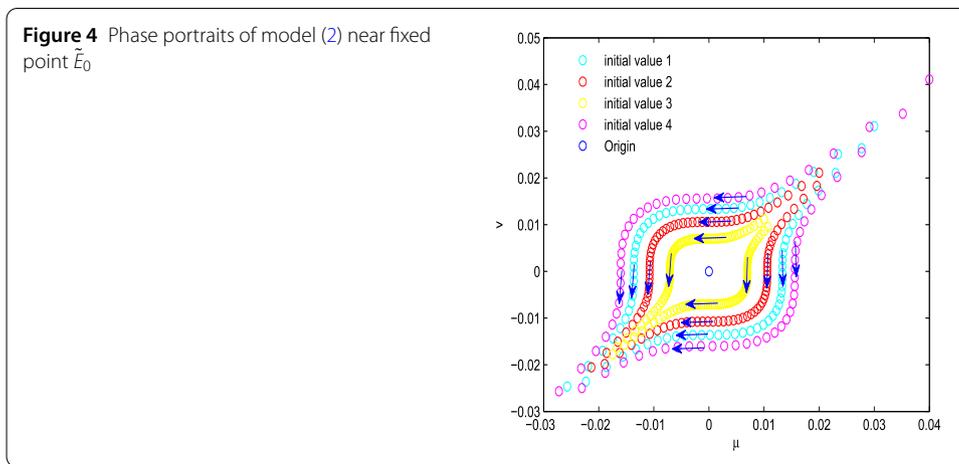
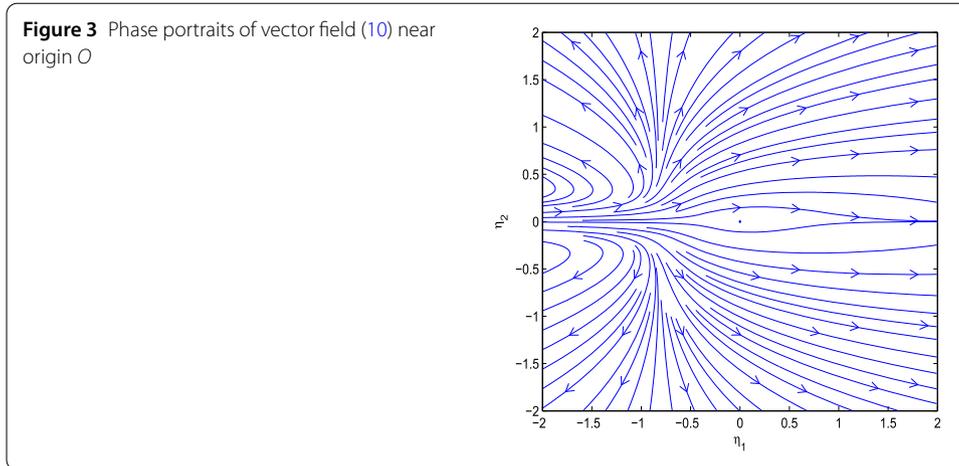
Therefore we can obtain the phase portrait of system (13) on the circle $S^1 \times \{0\}$. By shrinking the circle $S^1 \times \{0\}$ to a point we can obtain the local behavior of the vector field X near the degenerate equilibrium $(0, 0)$. Figure 3 also can be seen as the phase portrait of the vector field X near the degenerate equilibrium $(0, 0)$.

From the previous proof process we can see that the vector field X has no elliptic sectors and that the vector field X obtained by desingularization has only hyperbolic equilibrium. Therefore the vector field X satisfies the assumption of Lemma 4. From Lemma 4 we see that there is a C^∞ diffeomorphism Ψ satisfying $j_\infty(\Psi - I) = 0$ such that

$$\Phi_1 \circ T = \tilde{\Psi} \circ R \circ \varphi_X(1, \cdot) \circ \tilde{\Psi}^{-1}, \quad \tilde{\Psi} = \Phi_2 \circ \Psi$$

at the origin O . Thus the degenerate fixed point \tilde{E}_0 of model (2) is unstable.

We see from [3] that the orbit unit-time shift φ^1 along vector field (10) coincides with the near-identity mapping at least up to order 3. For a sufficiently small initial value $P(\mu_0, \nu_0) \in R_+^2$, by means of C^∞ diffeomorphism Ψ , linear transformation Φ_1 , and the local behavior of the vector field X near the origin O we can see that the sequence $\{T^n(P)\}_0^{+\infty}$ enters



the third quadrant from the first quadrant along both sides of straight line $\mu = \nu$. In this process the sequences

$$\{T^{2n}(P)\}_0^{+\infty} \quad \text{and} \quad \{T^{2n+1}(P)\}_0^{+\infty}$$

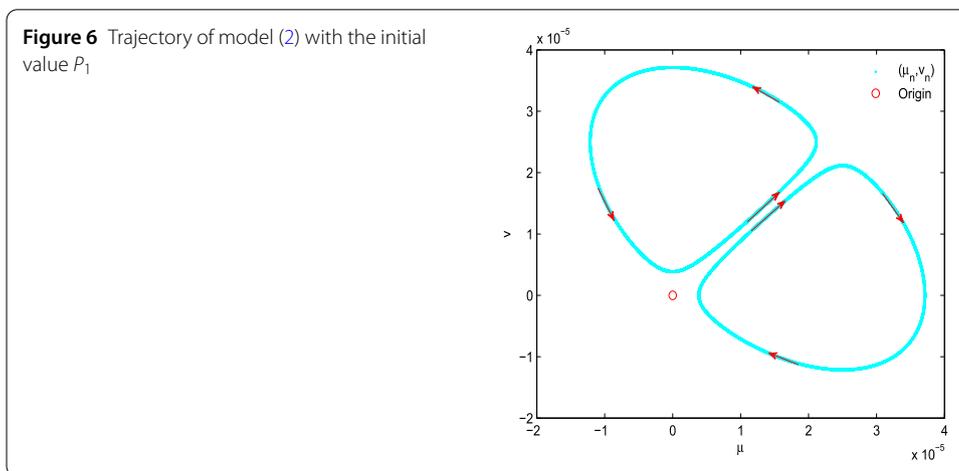
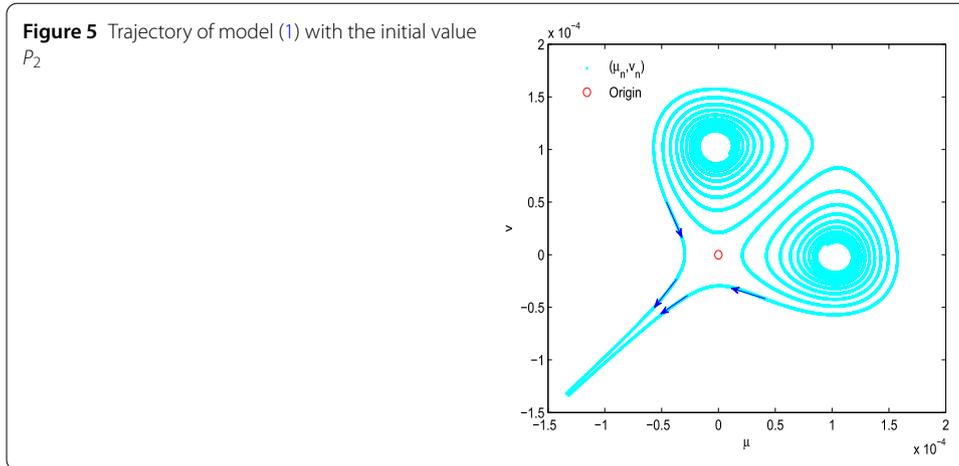
along both sides of straight line $\mu = \nu$ “repel” and then “attract”. The proof is completed. \square

Remark 3 For dynamic properties of the normal form N , the general description is given in [8] when $a_0 b_0 \neq 0$. In this paper, the normal form N satisfies $a_0 = \frac{1}{2}$, $b_0 = 16$, which is one of the situations given in [8].

Remark 4 In [17] the author investigates qualitative properties of the vector field near degenerate equilibrium by desingularization. This desingularization is obtained by homogeneous directional blowup. In this paper, we realize desingularization by homogeneous polar blowup. As can be seen from Remark 3.1 in [19], the homogeneous directional blowup and the homogeneous polar blowup are equivalent.

6 Numerical simulation

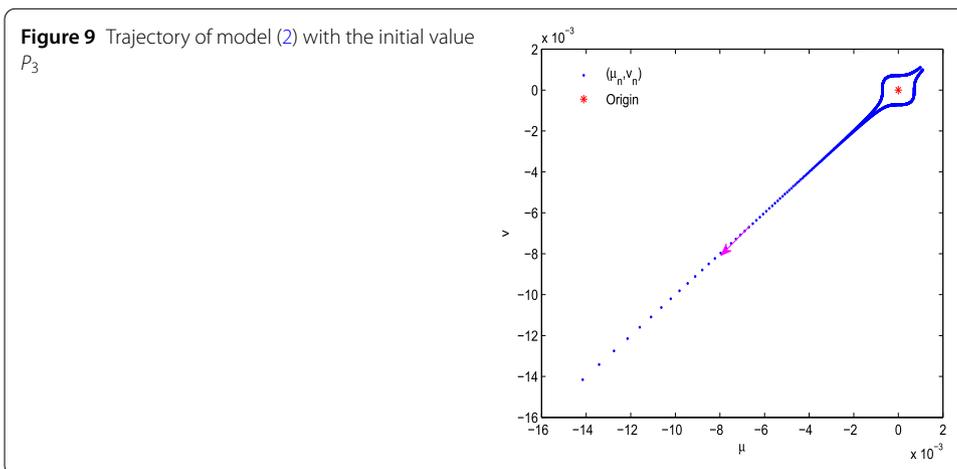
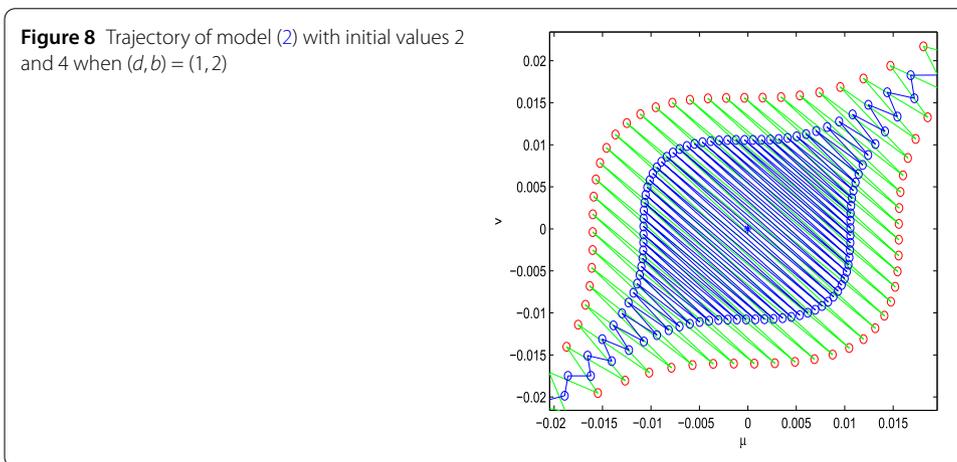
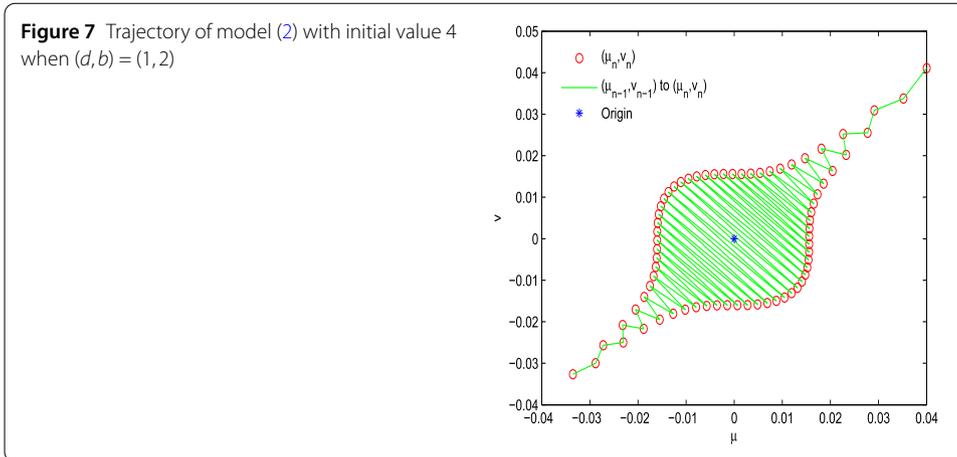
In this section, we select appropriate parameters and initial values for the numerical simulation to verify our results. The details are as follows.



- (1) Comparing Figs. 3 and 4, we find that the phase portraits of the vector field (10) near the origin O and the phase portraits of model (2) near origin O are very “similar”. This indicates that the topological structures of vector field (10) and model (2) near the origin O are “similar” when $(d, b) = (1, 2)$, which confirms Theorem 8, where initial values 1, 2, 3, 4 of Fig. 4 are $(0.03, 0.03111)$, $(0.02, 0.02111)$, $(0.01, 0.0111)$, $(0.04, 0.04111)$, respectively.
- (2) From Figs. 7 and 8 we can find that the sequence $\{T^n(P)\}_0^{+\infty}$ enters the third quadrant from the first quadrant along the straight line $\mu = \nu$. In the process the sequence $\{T^n(P)\}_0^{+\infty}$ is constantly “oscillating” along the straight line $\mu = \nu$, which also confirms Theorem 8.
- (3) Figs. 4, 5, 11, and 12 are very “similar” to subfigures 1, 3, 4–, and $F+$ of Fig. 8 in [8], respectively, which means that model (2) can have a fold-flip bifurcation near the degenerate fixed point \tilde{E}_0 .

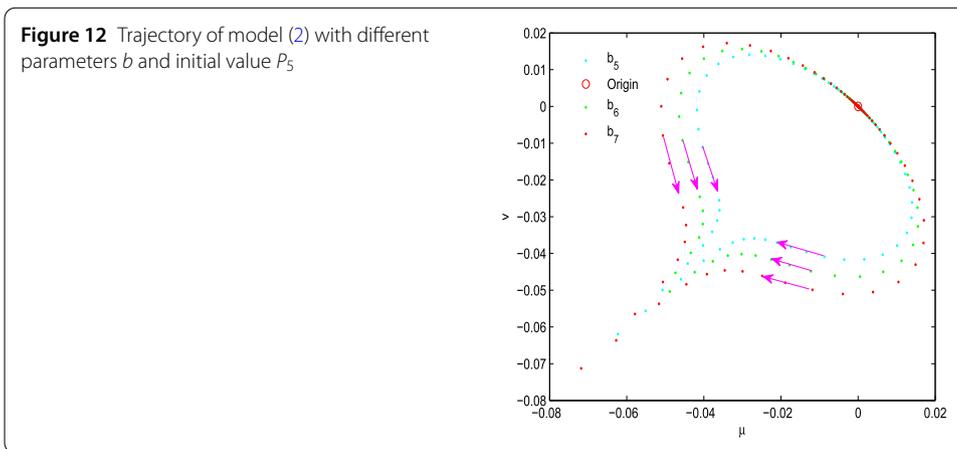
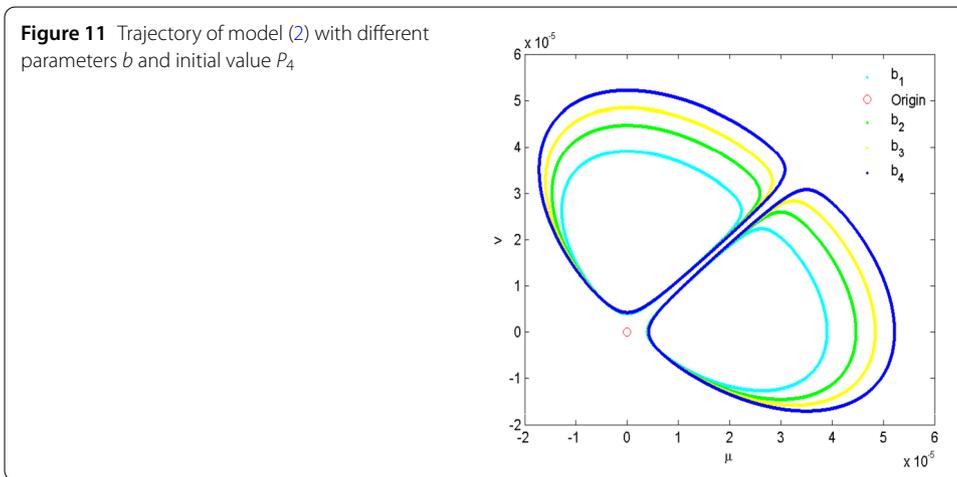
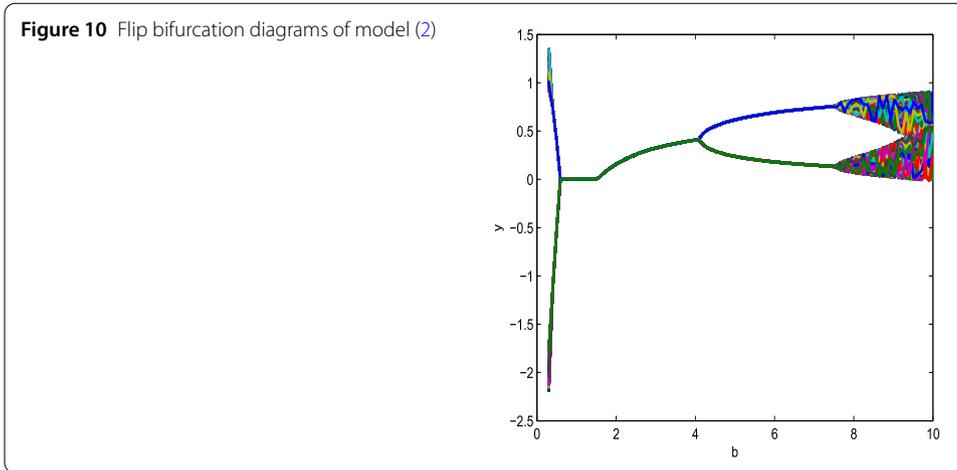
Case 1: Suppose the parameters $b = 2.0001$ and $d = 1$ and the initial value $P_1 := (\mu_0, \nu_0) = (0.00001, 0.0000111)$. Model (2) has fixed points $(0, 0)$ and $(2.499875 \times 10^{-5}, 2.499875 \times 10^{-5})$, and the trajectory of model (2) with initial value P_1 near the fixed point $(0, 0)$ is shown in Fig. 6.

Case 2: Suppose the parameters $b = 2.000434$ and $d = 1.00001$ and the initial value $P_2 := (\mu_0, \nu_0) = (0.00001, 0.00096)$. Model (2) has fixed points $(0, 0)$ and $(1.109748 \times$



10^{-5} , 1.109748×10^{-5}), and the trajectory of model (2) with initial value P_2 near the fixed point $(0, 0)$ is shown in Fig. 5.

Case 3: Suppose the parameter $b = 2$ and $d = 1$ and the initial value $P_3 := (\mu_0, v_0) = (0.001, 0.00111)$. Model (2) has only the fixed point $(0, 0)$, and the trajectory of model (2) with initial value P_3 near the fixed point $(0, 0)$ is shown in Fig. 9.



Case 4: Suppose the parameter $d = 0.432$ and initial value $(\mu_0, v_0) = (0.153, 0.033)$. The flip bifurcation diagram of model (2) is shown in Fig. 10.

Case 5: Suppose the parameters $b_1 = 2.000105$, $b_2 = 2.00012$, $b_3 = 2.00013$, and $b_4 = 2.00014$. The trajectory of model (2) with initial value $P_4 := (\mu_0, v_0) = (0.00001, 0.000111)$ is shown in Fig. 11.

Case 6: Suppose the parameters $b_5 = 1.899$, $b_6 = 1.889$, and $b_7 = 1.879$. The trajectory of model (2) with initial value $P_5 := (\mu_0, \nu_0) = (0.0001, 0.000096)$ is shown in Fig. 12.

7 Conclusion

In this paper, we systematically studied bifurcation properties of model (2) near nonhyperbolic fixed points. We obtained the stability and qualitative structures of the degenerate fixed point \tilde{E}_0 . We used the homogeneous polar blowup to study the local behavior of the vector field near the degenerate equilibrium, which avoids complex calculation. By comparing Figs. 5 with Fig. 8 and Fig. 11 with Fig. 12 we can see that the dynamic properties of model (2) vary greatly near $(d, b) = (1, 2)$ when the initial value is sufficiently small.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, MSL, XLZ, and JMX, contributed to each part of this study equally and read and approved the final version of the manuscript.

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