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Explicit solutions of some equations and systems of mathematical physics

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Abstract

This paper deals at first with a fully integrable evolution system of nonlinear partial differential equations (PDEs) which is a generalization of the classical Heisenberg ferromagnet equation. Then the scalar variant of this system is considered. Looking for solutions of special form, the problem of finding explicit solutions of the above-mentioned equations is reduced to the global solvability of overdetermined real-valued systems of nonlinear PDEs. In many cases particular solutions which are not solitons are expressed by classical functions including some special ones as Jacobi elliptic functions, Legendre elliptic functions, and Weierstrass normal elliptic integrals. A geometrical visualization of several solutions is also proposed.

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1 Introduction

This paper deals at first with a 2×2 fully integrable evolution system of nonlinear partial differential equations. It is a generalization of the classical Heisenberg ferromagnet equation, where the solution S is the unit spin vector (see [2]). A relatively new integrable model was proposed in [6], and two types of soliton solutions, namely quadruplet and doublet solution, were found. In [12] other particular solutions of the same system were constructed. They turn out to be either soliton type ones or quasi-rational ones. In both cases the Zakharov–Shabat dressing method was applied (for this method, see for example [13–15]). The inverse scattering method is a good approach for finding exact solutions of different equations of mathematical physics, too. The above-mentioned methods were intensively used in the last 60 years, and due to them a big progress in the investigation of nonlinear evolution equations and systems was made.

We propose here another classical method on the subject which is applied to the evolution system under consideration (see Sect. 2, Eq. (1)) as well to its scalar variant. The latter is a generalization of the derivative Schrödinger equation studied in [8] from the point of view of the explicit solutions. Some results on the Cauchy problem $iu_t + u_{xx} +$

$F(u,\bar{u},u_x\bar{u}_x) = 0$ can be found in [3]. Here we look for solutions of special form (Ansatz) reducing this way our problem to the global solvability of (over) determined real-valued systems of PDEs. Inverse scattering theory is avoided but rather complicated systems of real PDEs could appear. On the other hand, this simple and direct approach guarantees in several special cases the expression of the corresponding solutions by some well-studied special functions as Jacobi elliptic functions, Legendre elliptic functions, and Weierstrass normal elliptic integrals of first, second, and third kinds (see for details [4, 5, 7]). As one can guess thereby different kinds of non-soliton solutions can be constructed. Theorem 1 of our paper for (1) can be compared with some results of [12] where it is supposed additionally that $u(\pm\infty,t) = 0$, $v(\pm\infty,t) = 1$. Certainly, the solution (u,v) in Theorem 1(b) does not satisfy the latter assumption and is not soliton. Some of the stationary soliton solutions from [12] are $(u = 0, v = e^{i\varphi(x)}, \varphi(\pm\infty) \in 2\pi\mathbf{Z})$, while in Theorem 1(a) the stationary solution is $(\cos k_0 e^{i\varphi(x)}, \sin k_0 e^{i\varphi(x)})$, φ being arbitrary real-valued function, $k_0 = \text{const}$. For a special choice of the rational amplitudes f_1, f_2 in Anzatz (2) the corresponding solution can blow up at some curve in the plane $0xt$. Similar effect is found in [12] on the level of rather complicated example (117), (118) (quasi-rational solution). The solutions of the scalar equation (8) in [6, 12] are of soliton type (see [6], Sect. 3.2, [12] Sects. 2.2, 2.3), while the solutions u from our Theorem 2 are completely different. They are not solitons, of course. The solutions u are expressed by Legendre functions $E(\varphi,k)$, $F(\varphi,k)$, $\Pi(\varphi,\alpha^2,k)$, Jacobi elliptic functions, and Weierstrass functions $\wp$ and ζ.

In what follows, we propose geometrical visualization of u. For example in Theorem 2, 2) the amplitude f is located between two parallel oblique asymptotes and can be expressed by some Jacobi elliptic functions. In Theorem 2, 3) the amplitude $f(k) = \cos k$, $k = k(x)$ and k can be written by Weierstrass $\wp$ and zeta functions. Geometrically, $k(x)$ is possibly periodic cuspon or soliton-cuspon. In the papers devoted to Eq. (8) mainly soliton solutions are proposed due to the used tools—inverse scattering and dressing methods. The appearance of autonomous ordinary differential equations satisfied by Ansatz (9) enables us to enlarge the classes of the particular solutions, to express them by some special functions, and to propose geometrical interpretation. The soliton solutions for (8) in the literature we know are usually written by elementary functions, i.e., exponents, logarithms, and trigonometric ones (see [6, 12], and others).

We do not know papers on the subject in which solutions of other form are found explicitly and studied geometrically. We do not discuss the possible applications of those solutions in physics being concentrated on the mathematical part of the study of the arising differential equations. The results from Theorem 2 are illustrated by four figures.

2 Formulation of the main results

1. Consider the fully integrable nonlinear evolution system

$$\left| \begin{aligned} &iu_t + u_{xx} + [(u\bar{u}_x + v\bar{v}_x)u]_x = 0, \\ &iv_t + v_{xx} + [(u\bar{u}_x + v\bar{v}_x)v]_x = 0. \end{aligned} \right. \tag{1}$$

Usually, the additional condition $|u|^2 + |v|^2 = 1$ is imposed on (1). In vector form the Heisenberg ferromagnet equation is given by $S_t = S \times S_{xx}$, $S(x,t) \in \mathbf{R}^3$, $|S| = 1$.

We look for a solution of (1) having the form

$$\begin{aligned} u &= f_1\bigl(k(x,t)\bigr)e^{i\varphi(x,t)}, \\ v &= f_2\bigl(k(x,t)\bigr)e^{i\varphi(x,t)}, \end{aligned} \tag{2}$$

where $f_1(k), f_2(k), k, \varphi$ are real-valued smooth functions.

Putting (2) into (1), doing the corresponding calculations, and splitting the real and imaginary parts of the expressions, we get the following overdetermined system of four PDEs that should be satisfied by k, φ:

$$-f_1\varphi_t + k_{xx}\left(f_1' + \frac{1}{2}f_1 g'\right) + k_x^2\frac{d}{dk}\left(f_1' + \frac{1}{2}\frac{d}{dk}(f_1 g')\right) + \varphi_x^2 f_1(g-1) = 0, \tag{3}$$

$$f_1'k_t + f_1\varphi_{xx}(1-g) + \varphi_x k_x\left[f_1'(2-g) - \frac{1}{2}f_1 g'\right] = 0, \tag{4}$$

$$-f_2\varphi_t + k_{xx}\left(f_2' + \frac{1}{2}f_2 g'\right) + k_x^2\frac{d}{dk}\left(f_2' + \frac{1}{2}\frac{d}{dk}(f_2 g')\right) + \varphi_x^2 f_2(g-1) = 0, \tag{5}$$

$$f_2'k_t + f_2\varphi_{xx}(1-g) + \varphi_x k_x\left[f_2'(2-g) - \frac{1}{2}f_2 g'\right] = 0, \tag{6}$$

where $g(k) = f_1^2 + f_2^2$.

The simplest case to system (3)–(6) is the following case:

(A) $f_1 = \cos k, f_2(k) = \sin k \Rightarrow g \equiv 1$, i.e., $g'(k) = 0$. Therefore, $|u|^2 + |v|^2 = 1$.

In case (A) system (3)–(6) can be reduced to

$$k_{xx} = 0, \qquad k_t + \varphi_x k_x = 0, \qquad \varphi_t + k_x^2 = 0. \tag{7}$$

Theorem 1 *Consider system* (1), $|u|^2 + |v|^2 = 1$, *under condition* (A). *Then*:

(a) (1) *possesses infinitely many stationary solutions*

$$u = \cos k_0 e^{i\varphi(x)},$$
$$v = \sin k_0 e^{i\varphi(x)},$$

$k_0 = \text{const}$, $\varphi(x)$ *arbitrary smooth real-valued function*.

(b) *Let* $k = A(t)x + B(t)$.

(b1) *If* $A(t) \not\equiv \text{const}$, *the function* $k = C_3 x e^{C_1 t} + \frac{C_2 C_3}{C_1} e^{C_1 t}$, $C_1, C_3 \neq 0$, *while* $\varphi = -\frac{C_3^2}{2C_1} e^{2C_1 t} - C_1 \frac{x^2}{2} - C_2 x$; C_1, C_2, C_3 *are constants*.

(b2) $A(t) \equiv C_1 = \text{const} \neq 0$ *implies that* $k = C_1 x + C_1 C_2 t$, *while* $\varphi = -C_1^2 t - C_2 x$.

(b3) $A \equiv 0$ *coincides with* (a): $k = k_0 = \text{const}$, $\varphi = \varphi(x)$.

The solutions in Theorem 1 are globally defined in $\mathbf{R}^2$. For some rational functions f_1, f_2 smooth in $\mathbf{R}^2$, we can find solutions k that blow up along some curves in the plane. The corresponding example is given at the end of the proof of Theorem 1.

2. Our next step is to study system (1) for $v \equiv 0$, i.e., the nonlinear evolution equation of Schrödinger type, namely

$$iu_t + u_{xx} + 2u|u_x|^2 + u^2\bar{u}_{xx} = 0. \tag{8}$$

u is not obliged to satisfy the condition $|u| = 1$.

As in the previous system, we look for its solution of the form

$$u = f\big(k(x,t)\big)e^{i\varphi(x,t)}, \tag{9}$$

$f(k)$, k, φ being again real-valued smooth functions. Putting (9) into (8) and splitting the real and imaginary parts of the corresponding expression, we conclude that k, φ satisfy the following nonlinear system of PDEs:

$$f\big(-\varphi_t + \varphi_x^2\big(f^2 - 1\big)\big) + f'\big(\big(1 + f^2\big)k_{xx} + 2ff'k_x^2\big) + f''\big(k_x^2 + f^2k_x^2\big) = 0, \tag{10}$$

$$f'k_t + \big(1 - f^2\big)\big(2\varphi_x k_x f' + f\varphi_{xx}\big) = 0. \tag{11}$$

We shall construct solutions of (8) written in the form of (9) in several different cases. They are formulated in what follows.

Theorem 2 *Consider the nonlinear evolution scalar Eq.* (8). *Then we give its solutions of the form* (9) *in the following four cases.*

1) $f \equiv 1$. *Then* $u = e^{i\varphi(x)}$, φ *- arbitrary, is a stationary solution.*

Equation (8) *does not possess nontrivial traveling wave solutions* $\psi(x + ct)$, $|\psi| = 1$ *with velocity* $c \neq 0$. *Each smooth complex-valued function* $\psi(x)$, $|\psi| = 1$ *is a solution of* (8). *Therefore, the Blaschke type functions*

$$\psi(x) = e^{i\gamma}\prod_{n=1}^{m}\frac{z^n - \alpha_j\bar{z}^n}{\bar{z}^n - \bar{\alpha}_j z^n}, \tag{12}$$

where $\gamma \in \mathbf{R}^1$, α_j *with* $|\alpha_j| < 1$ *are arbitrary complex numbers, and the complex-valued function* $z(x) \neq 0$ *everywhere, satisfy* (8). *Let* $A, B \in C^2$, A, B *- real-valued and* $|A(x)| > 0$ *everywhere. Then* (12) *takes the trigonometric form*

$$\psi(x) = e^{i\gamma}\prod_{n=1}^{m}\frac{e^{2\,\mathrm{inarctg}\,\frac{B(x)}{A(x)}} - \alpha_j}{1 - \bar{\alpha}_j e^{2\,\mathrm{inarctg}\,\frac{B(x)}{A(x)}}}. \tag{13}$$

2) *Suppose that* $f \not\equiv 1$ *and* k, φ *are linear functions with respect to* (t, x). *Then there exists a smooth solution* f *of* (10), (11) *possessing two oblique parallel each to other asymptotes and* f *is located between them. Moreover,* f *can be expressed by the Legendre elliptic functions as well as by some Jacobi elliptic functions.*

3) *Suppose that* $f(k) = \cos k$, $k = k(x)$, *and* $\varphi = t + \varphi_1(x)$ *satisfy* (10)–(11). *Then there is a simple link between* $k(x)$, $\varphi_1(x)$, *while* $k(x)$ *satisfies a second order autonomous ODE and can be expressed by Weierstrass normal elliptic integrals of first and second kind, i.e., by* $\wp$ *and zeta functions. Under several conditions* k *turns out to be periodic cuspon or soliton-cuspon.*

4) *Let* $f(k) = e^k$, $k = k(x)$, $\varphi = t + \varphi_1(x)$. *The function* $\varphi_1(x)$ *can be expressed explicitly by* $k(x)$, *while* $k(x)$ *is expressed by the Weierstrass normal elliptic integrals of first and second kinds.*

There are other interesting cases as for example $f(k) = \sin k$, $f(k) = P_m(k)$, P_m being real-valued polynomial of k of order m. We left them to the reader, as they can be studied in the same way as in Theorem 2.

In a similar way one can study the system

$$\left| \begin{array}{l} iu_t + u_{xx} + [(u\bar{u}_x - v\bar{v}_x)u]_x = 0, \\ iv_t + v_{xx} + [(u\bar{u}_x - v\bar{v}_x)v]_x = 0 \end{array} \right.$$

under the condition $|u|^2 - |v|^2 = 1$ looking for a solution of the form (2).

3 Proof of Theorems 1 and 2

1. We shall begin the proof of Theorem 1 by the observation that if $g(k) \equiv 1$ then system (3)–(6) takes the form

$$\begin{aligned} &k_t + k_x\varphi_x = 0, \\ &-f_1\varphi_t + k_{xx}f_1' + k_x^2 f_1'' = 0, \\ &-f_2\varphi_t + k_{xx}f_2' + k_x^2 f_2'' = 0, \end{aligned}$$

i.e.,

$$\begin{aligned} &k_t + k_x\varphi_x = 0, \\ &\varphi_t = k_{xx}\frac{f_1'}{f_1} + k_x^2\frac{f_1''}{f_1} = k_{xx}\frac{f_2'}{f_2} + k_x^2\frac{f_2''}{f_2}, \quad f_1.f_2 \neq 0. \end{aligned}$$

Thus, φ must satisfy the standard condition of total differential $\varphi_{xt} = \varphi_{tx}$, where

$$\begin{aligned} &\varphi_x = -\frac{k_t}{k_x}, \quad k_x \neq 0, \\ &\varphi_t = k_{xx}\frac{f_1'}{f_1} + k_x^2\frac{f_1''}{f_1} \end{aligned}$$

under the assumption that k is a solution of the autonomous ODE with respect to x

$$k_{xx}\left(\frac{f_1'}{f_1} - \frac{f_2'}{f_2}\right) + k_x^2\left(\frac{f_1''}{f_1} - \frac{f_2''}{f_2}\right) = 0,$$

$f = f_1(k), f = f_2(k)$, t being a parameter, $k = k(x,t)$.

The standard change $\frac{dk}{dx} = p(k) \Rightarrow \frac{d^2k}{dx^2} = p\frac{dp}{dk}$ reduces the latter to the following linear ODE:

$$p\frac{dp}{dk}\left(\frac{f_1'}{f_1} - \frac{f_2'}{f_2}\right) + p^2\left(\frac{f_1''}{f_1} - \frac{f_2''}{f_2}\right) = 0.$$

If $p(k) = 0 \Rightarrow k = k(t) \Rightarrow k_x' = 0 \Rightarrow k_t' = 0 \Rightarrow k = \text{const} \Rightarrow \varphi = \varphi(x)$.

If $p(k) \neq 0$ then $p(k) = -A(t)e^{-G(k)} = \frac{-A(t)}{f_1'f_2 - f_2'f_1}$, where

$$G(k) = \int \frac{\frac{f_1''}{f_1} - \frac{f_2''}{f_2}}{\frac{f_1'}{f_1} - \frac{f_2'}{f_2}}\, dk.$$

This way we conclude that

$$F(k) = \int \left(f_1' f_2 - f_2' f_1\right) dk = -A(t)x - B(t).$$

If the inverse function F^{-1} exists, we get that

$$k = F^{-1}\left(-A(t)x - B(t)\right).$$

Our overdetermined system (3)–(6) possesses a solution if the above total differential exists for $k = F^{-1}(-A(t)x - B(t))$, i.e., iff $\varphi_{xt} = \varphi_{tx}$.

In case (A), (b) $F(k) = -k = -A(t)x - B(t) \Rightarrow k = A(t)x + B(t)$, $\varphi_x = -\frac{A'(t)x+B'(t)}{A(t)}$ for $A(t) \not\equiv$ const, $\varphi_t = -A^2(t)$.

Therefore we have total differential iff $\frac{A'(t)}{A(t)} = C_1 = \text{const} \neq 0$, $\frac{B'(t)}{A(t)} = C_2 = \text{const}$. So $A(t) = C_3 e^{C_1 t}$, $C_3 \neq 0$, $B(t) = \frac{C_2 C_3}{C_1} e^{C_1 t}$.

Evidently, $\varphi = -\int A^2(t)dt + p(x) = -\frac{C_3^2}{2C_1} e^{2C_1 t} + p(x)$, while $p'(x) = -C_1 x - C_2 \Rightarrow p(x) = -C_1 \frac{x^2}{2} - C_2 x$ up to an additive constant.

Suppose now that $f_1 = \text{cn}(k,l)$, $f_2 = \text{sn}(k,l)$, $l \in [0,1]$ being the modulus of the corresponding Jacobi elliptic functions. Certainly, $f_1^2 + f_2^2 = g \equiv 1$. Then $F(k) = -\int \text{dn}\, k\, dk = -\arcsin(\text{sn}\, k)$ according to formula 314.01 from [4], i.e.,

$$\text{sn}\, k = \sin\left(A(t)x + B(t)\right) \quad \Rightarrow \quad \text{cn}\, k = \pm\cos\left(A(t)x + B(t)\right).$$

It is well known [4] that if $z = \text{sn}(u,l)$ then

$$u = \text{sn}^{-1}(z,l) = \text{dn}^{-1}\left(\sqrt{1 - l^2 z^2}, l\right) = \text{tn}^{-1}\left(\frac{z}{\sqrt{1-z^2}}, l\right).$$

Consequently, $\varphi_x = -\frac{A'(t)x+B'(t)}{A(t)}$ and after some computations

$$\varphi_t = k_{xx}\frac{f_1'}{f_1} + k_x^2 \frac{f_1''}{f_1} = -C^2(t),$$

as $k = \text{sn}^{-1}(\sin(A(t)x + B(t)))$ and $\frac{d}{dx}\text{sn}^{-1}(x,l) = \frac{1}{\sqrt{(1-x^2)(1-l^2x^2)}}$, $0 < x < 1$. We conclude that the Ansatz $u = \text{cn}(k(x,t),l)e^{i\varphi}$, $v = \text{sn}(k(x,t),l)e^{i\varphi}$ gives us the same solutions as Theorem 1, cases (a) and (b).

It is interesting to point out that (1) can have solutions with singularities if $f_1(k), f_2(k)$ are rational functions. In fact, let $f_1(k) = \frac{k^2-1}{k^2+1}$, $f_2(k) = \frac{2k}{k^2+1}$, i.e., $f_1^2 + f_2^2 = g(k) \equiv 1$ everywhere. Then

$$F(k) = \int \left(f_1' f_2 - f_2' f_1\right) dk = 2\,\text{arctg}\, k = -A(t)x - B(t).$$

We take $2A_1 = -A$, $2B_1 = -B$ that implies

$$k = \text{tg}\left(A_1(t)x + B_1(t)\right), \quad A_1(t) \not\equiv \text{const}.$$

Therefore, $\varphi_x = -\frac{A_1'(t)}{A_1(t)}x - \frac{B_1'(t)}{A_1(t)}$.

Easy calculations show that

$$\varphi_t = k_{xx}\frac{f_1'}{f_1} + k_x^2\frac{f_1''}{f_1} = -4A_1^2(t)$$

as $\frac{f_1'}{f_1} = \frac{4k}{k^4-1}$, $\frac{f_1''}{f_1} = \frac{4(1-3k^2)}{(k^4-1)(k^2+1)}$.

Consequently, $A_1(t) = C_3e^{C_1t}$, $C_1, C_3 \neq 0$, $B_1(t) = \frac{C_2C_3}{C_1}e^{C_1t}$, $\varphi = -2\frac{C_3^2}{C_1}e^{2C_1t} - C_1\frac{x^2}{2} - C_2x$. The solution k is classical (i.e., smooth) if there exists some $n_0 \in \mathbf{Z}$ and such that $C_3e^{C_1t}(x + \frac{C_2}{C_1}) \in (\frac{\pi}{2}(2n_0-1), \frac{\pi}{2}(2n_0+1))$ for (x,t) describing some domain in $\mathbf{R}^2$. As an example we assume that $C_1 > 0$ and $|x| \leq \text{const}$. Then, for $t \ll -1$: $C_3e^{C_1t}(x+\frac{C_2}{C_1}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the function $k(x,t)$ blows up for $C_3e^{C_1t}(x+\frac{C_2}{C_1}) = \frac{\pi}{2}(2n_0-1)$, i.e., on the exponential (smooth) curve $x = \frac{\pi}{2}(2n_0-1)\frac{e^{-C_1t}}{C_3} - \frac{C_2}{C_1}$. It will be interesting to find other cases of amplitude functions $f_1(k)$, $f_2(k)$ for which the solution k of the overdetermined system (3)–(6) develops singularities.

2. We shall prove now Theorem 2, the proof being divided into several steps depending on the cases.

Case 1. Let $f \equiv 1$. Then (11) is identity, while (10) gives $\varphi_t = 0$, i.e., $\varphi = \varphi(x)$.

We are looking now for traveling wave solutions $\psi = \psi(x+ct)$, $\xi = x+ct$, $|\psi| = 1$ of (8). Certainly, $\psi = A(\xi) + iB(\xi)$, where $A(\xi)$, $B(\xi)$ are real-valued functions. Evidently, $ic\psi' + \psi'' + (\psi^2\bar{\psi}')' = 0 \Rightarrow ic\psi(\xi) + \psi'(\xi) + \psi^2\bar{\psi}'(\xi) = d = \text{const}$. The real part of the last expression is

$$-cB + A' + (A^2 - B^2)A' + 2ABB' = \operatorname{Re} d,$$

$A^2 + B^2 = 1$, i.e., $2BB' = \frac{d}{d\xi}B^2 = -2AA'$, $B' = \mp\frac{AA'}{\sqrt{1-A^2}}$.

Therefore, for $c \neq 0$,

$$\mp c\sqrt{1-A^2} = \operatorname{Re} d \quad\Rightarrow\quad 1 - A^2 = \left(\frac{\operatorname{Re} d}{c}\right)^2 \quad\Rightarrow\quad A = \text{const}$$

if it exists $\Rightarrow B = \text{const} \Rightarrow \psi(\xi) = e^{i\varphi_0}$, $\varphi_0 = \text{const}$.

Nontrivial traveling waves exist only for velocity $c = 0$.

It is obvious that if $|\psi(x)|^2 = 1$, then $\psi' = -\frac{\bar{\psi}'}{\bar{\psi}^2} = -\bar{\psi}'\psi^2 \Rightarrow \psi'' + \frac{\partial}{\partial x}(\psi^2\bar{\psi}') = 0$. So $\psi(x)$ satisfies (8) (stationary solution). We can take $\psi(x) = \frac{z(x)}{\bar{z}(x)}$, where $z(x) \neq 0$ everywhere. Then ψ is a solution of (8). Another form (trigonometric) of the stationary solution is the following one: $\psi(x) = e^{2i\operatorname{arctg}\frac{B(x)}{A(x)}}$, $|A(x)| > 0$ everywhere. This way we obtain via the Blaschke function solution (12) of (8). The identity $\frac{z}{\bar{z}} = e^{2i\varphi}$, $\varphi = \operatorname{arctg}\frac{\operatorname{Im} z}{\operatorname{Re} z}$, where $z = |z|e^{i\varphi}$ gives us solution (13).

Case 2. Let $f \not\equiv 1$, $\varphi = A_1x + B_1t$, $A_1^2 + B_1^2 > 0$, $k = A_2x + B_2t$, $A_2^2 + B_2^2 > 0$. We are looking for f from (9). According to (11),

$$f'\big(B_2 + 2A_1A_2(1-f^2)\big) = 0.$$

As the case $f = \text{const}$ is trivial, we concentrate on $f \not\equiv \text{const} \Rightarrow B_2 + 2A_1A_2(1-f^2) = 0$. Consequently, $A_1A_2 = 0 \Rightarrow B_2 = 0 \Rightarrow k = A_2x$ for $A_1 = 0$, $A_2 \neq 0$, $\varphi = B_1t$ for $B_1 \neq 0$, and we suppose that $B_1 > 0$.

According to (10), f satisfies the ODE

$$-B_1 f + 2(f')^2 f A_2^2 + f'' A_2 (1+f^2) = 0. \tag{14}$$

We equip (14) with the Cauchy data $f(k_0) = f_0$, $f'(k_0) = f_0'$. The standard change in (14) $\frac{df}{dk} = f'(k) = p(f) \Rightarrow f''(k) = \frac{1}{2}\frac{d}{df}(p^2)$ and the substitution $p^2 = q(f)$ transform (14) into the linear first order ODE with respect to q:

$$\frac{dq}{df} + \frac{4fq}{1+f^2} - \frac{2B_1}{A_2^2}\frac{f}{1+f^2} = 0, \tag{15}$$

$q(f_0) = p^2(f_0) = (f')^2(k_0) = (f_0')^2$.

Then q can be written as

$$q(f) = \frac{1}{(1+f^2)^2}\left[C + \frac{B_1}{2A_2^2}(1+f^2)^2\right], \tag{16}$$

where $C = [(f_0')^2 - \frac{B_1}{2A_2^2}](1+f_0^2)^2$. We suppose that $C > 0$ (to fix the ideas) if $C = 0 \Rightarrow q = \frac{B_1}{2A_2^2} > 0 \Rightarrow f(k) = \pm\sqrt{\frac{B_1}{2A_2^2}}(k-k_0)$.

After some calculations we obtain for $f(k)$ the relation $\frac{df}{dk} = p \pm \sqrt{q(f)}$ with $\sqrt{q(f)} = \sqrt{\frac{B_1}{2}\frac{1}{A_2}(C_1^2 + (1+f^2)^2)}$, where $C_1^2 = \frac{2A_2^2 C}{B_1} > 0$, $B_1 > 0$. Here we apply some technique from [1].

We shall study the case with the sign + in front of $\sqrt{q}$ as the other case is similar.

Thus,

$$\int_{f_0}^{f(k)} \frac{d\lambda}{\sqrt{q(\lambda)}} = k - k_0. \tag{17}$$

Put

$$F(f) = \int_{f_0}^{f} \frac{d\lambda}{\sqrt{q(\lambda)}} = A_2\sqrt{\frac{2}{B_1}}\int_{f_0}^{f}\frac{(1+\lambda^2)\,d\lambda}{\sqrt{C_1^2 + (1+\lambda^2)^2}}, \tag{18}$$

$A_2 > 0$ - without loss of generality.

Evidently, $F' > 0$ for each $f \in \mathbf{R}^1$, $F(f_0) = 0$, $F(f) > 0$ for $f > f_0$, $F(f) < 0$ for $f < f_0$. Moreover, $F(f)_{f\to\infty} \to \infty$, $F(f)_{f\to-\infty} \to -\infty$, i.e., $F : \mathbf{R}^1 \to \mathbf{R}^1$ is a diffeomorphism. More precisely, $F(f) \sim A_2\sqrt{\frac{2}{B_1}}f$ for $|f| \to \infty$. One can guess that the function $z = F(f)$ possesses two oblique parallel each to other asymptotes

$$I: \quad z_+ = fA_2\sqrt{\frac{2}{B_1}} + A_+, \quad f \to \infty, \qquad II: \quad z_- = fA_2\sqrt{\frac{2}{B_1}} + A_-, \quad f \to -\infty.$$

Therefore, $f = F^{-1}(k-k_0)$, $k \in \mathbf{R}^1$, $f(k_0) = f_0$ and f has the following oblique asymptotes:

$$I': \quad f = \sqrt{\frac{B_1}{2}}\frac{k-k_0-A_+}{A_2}, \quad k \to \infty, \qquad II': \quad f = \sqrt{\frac{B_1}{2}}\frac{k-k_0-A_-}{A_2}, \quad k \to -\infty$$

(see Fig. 1).

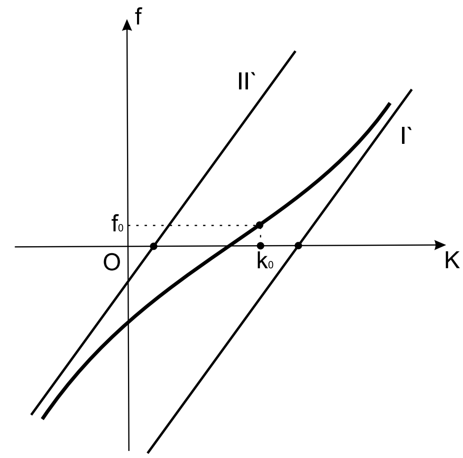

Figure 1 Graph of the solution $f = f(k)$ of Eq. (14) having two parallel asymptotes I' and II'

We will show now that integral (18) can be expressed by the Legendre elliptic functions of first, second, and third kinds. To do this we shall use formulas 267.00, 267.01, 342.05, 342.00, 342.01, 342.03, 342.04 from the Handbook [4]. Thus, consider

$$\int \frac{1+x^2}{\sqrt{(1+x^2)^2 + A_1^2}}, \quad A_1 > 0.$$

The biquadratic polynomial $P_4(x) = x^4 + 2x^2 + (1 + A_1^2)$ possesses the complex roots $\alpha, \bar{\alpha}$, $\beta, \bar{\beta}$ ($\operatorname{Im}\alpha \neq 0$, $\operatorname{Im}\beta \neq 0$)

$\alpha = (1+A_1^2)^{\frac{1}{4}} e^{i\frac{\varphi_1}{2}}$, $0 < \varphi_1 < \pi$, $\bar{\alpha} = (1+A_1^2)^{\frac{1}{4}} e^{-i\frac{\varphi_1}{2}}$, $\beta = -(1+A_1^2)^{\frac{1}{4}} e^{i\frac{\varphi_1}{2}}$, $\bar{\beta} = -(1+A_1^2)^{\frac{1}{4}} e^{-i\frac{\varphi_1}{2}}$.

In fact, $P_4 = (x^2 + Dx + E)(x^2 - Dx + E)$, where $E = \sqrt{1+A_1^2} > 1$, $D = \sqrt{2}\sqrt{\sqrt{1+A_1^2} - 1}$.

Put $\sqrt{(x-\alpha)(x-\bar{\alpha})(x-\beta)(x-\bar{\beta})} = \sqrt{[(x-b_1)^2 + a_1^2][(x-b_2)^2 + a_2^2]}$, where $b_1 = \operatorname{Re}\alpha$, $b_2 = \operatorname{Re}\beta$, $a_1^2 = -\frac{(\alpha-\bar{\alpha})^2}{4}$, $a_2^2 = -\frac{(\beta-\bar{\beta})^2}{4}$, and denote $A^2 = (b_1-b_2)^2 + (a_1+a_2)^2$, $B^2 = (b_1 - b_2)^2 + (a_1 - a_2)^2$, $k^2 = \frac{4AB}{(A+B)^2}$ (this constant k is different from the variable $k, f = f(k)$). It is not confusing in our situation; $k \in (0,1)$ stands for the modulus of the corresponding elliptic function, while $k' = \sqrt{1-k^2}$ is called complimentary modulus, $g = \frac{2}{A+B}$, $g_1^2 = \frac{4a_1^2 - (A-B)^2}{(A+B)^2 - 4a_1^2}$, $y_1 = b_1 - a_1 g_1$, $\varphi = amu_1 = \operatorname{arctg}[\frac{y - b_1 + a_1 g_1}{a_1 + g_1 b_1 - g_1 y}]$, the elliptic tangent $\operatorname{tn} u = \frac{x - b_1 + a_1 g_1}{a_1 + g_1 b_1 - g_1 x}$, $\operatorname{tn} u_1 = \operatorname{tg}\varphi$.

According to formula (267.00) from [4],

$$\int_{y_1}^{y} \frac{dx}{\sqrt{P_4(x)}} = g u_1 = g \operatorname{tn}^{-1}(\operatorname{tg}\varphi, k) = g F(\varphi, k),$$

where $F(\varphi, k)$ is Legendre's elliptic integral of the first kind (see (110.02) from [4] for the definition of $F(\varphi, k)$). By using (267.01) from [4] we have

$$\int_{y_1}^{y} \frac{x^2 dx}{\sqrt{P_4(x)}} = \frac{g(b_1 - g_1 a_1)^2}{g_1^2} \sum_{j=0}^{2} \frac{2\alpha_1^{2-j}(g_1 - \alpha_1)^j}{(2-j)!j!} \int_0^u \frac{du}{(1 + g_1 \operatorname{tn} u)^j},$$

where $\alpha_1 = \frac{a_1 + b_1 g_1}{b_1 - a_1 g_1}$.

The last integral is given explicitly by formula (342.05) and via formulas (342.00), (342.01), (342.02), (342.03), (342.04) from [4]. As the result is rather complicated, we shall mention only that the corresponding formula contains Legendre's elliptic functions of the three kinds $E(\varphi,k)$, $F(\varphi,k)$, $\Pi(\varphi,\alpha^2,k)$ as well as different Jacobi elliptic functions such as $\operatorname{dn} u$, $\operatorname{tn} u$, dcu, ncu.

Case 3. Let $f(k) = \cos k$, $k = k(x)$, $\varphi = t + \varphi_1(x)$. By using (11) we get $2\varphi_1' k' = \operatorname{ctg} k \varphi_1''$. We shall consider here only the case $\varphi_1' \not\equiv 0$. $\varphi_1' \equiv 0$ is simpler to deal with, and we omit it.

In what follows we suppose that $k(x_0) = k_0$, $k'(x_0) = k_0'$, $\sin k_0 \neq 0$, $\cos k_0 \neq 0$, $0 < k_0 < \frac{\pi}{2}$. Evidently,

$$\varphi_1(x) = D\int_{x_0}^{x} \frac{d\lambda}{\cos^2 k(\lambda)} + E, \quad D, E = \text{const}, D \neq 0. \tag{19}$$

Thus, $\varphi_1(x_0) = E$, $\varphi_1'(x_0) = \frac{D}{\cos^2 k_0}$.

Putting $\varphi_1(x)$ in (10) we get

$$\begin{aligned} &-\operatorname{ctg} k\left(1 + \sin^2 k \frac{D^2}{\cos^4 k} + \left(k'\right)^2(x) + \left(k'\right)^2 \cos^2 k(x)\right) \\ &\quad = \left(1 + \cos^2 k\right)k''(x) - \sin 2k\left(k'\right)^2(x), \end{aligned} \tag{20}$$

$k(x_0) = k_0$, $k'(x_0) = k_0'$.

The classical change $k'(x) = p(k) \Rightarrow k'' = \frac{1}{2}\frac{d}{dk}q$, $q = p^2$, $q(k_0) = p^2(k_0) = (k_0{}')^2$ and easy computations lead to the following linear first order ODE satisfied by q:

$$\frac{dq}{dk} + 2q\left(\operatorname{ctg} k - \frac{\sin 2k}{1 + \cos^2 k}\right) + 2\operatorname{ctg} k \frac{\cos^4 k + D^2(1 - \cos^2 k)}{\cos^4 k(1 + \cos^2 k)} = 0, \tag{21}$$

$q(k_0) = (k_0')^2$.

Thus,

$$q(k) = \frac{1}{\sin^2 k(1 + \cos^2 k)^2}\left[C - 2\int \frac{\cos^4 k + D^2(1 - \cos^2 k)}{\cos^3 k}\left(1 + \cos^2 k\right)\sin k\, dk\right], \tag{22}$$

$C = \text{const}$.

The last indefinite integral can be calculated by the change $w = \cos k$. Thus,

$$p^2(k) = q(k) = \frac{1}{\sin^2 k(1 + \cos^2 k)^2}\left[C + \frac{\cos^4 k}{2} + \left(1 - D^2\right)\cos^2 k - \frac{D^2}{\cos^2 k}\right], \tag{23}$$

where $C = (k_0')^2 \sin^2 k_0(1 + \cos^2 k_0)^2 - \frac{\cos^4 k_0}{2} - (1 - D^2)\cos^2 k_0 + \frac{D^2}{\cos^2 k_0}$.

To fix the ideas and by using some technique from [1], we shall investigate only the ODE with separate variables

$$k'(x) = p(k) = \sqrt{q(k)}, \quad k(x_0) = k_0. \tag{24}$$

Then

$$F(k) = \int_{k_0}^{k} \frac{d\lambda}{\sqrt{q(\lambda)}} = x - x_0, \tag{25}$$

where

$$F(k)=\sqrt{2}\int_{k_0}^{k}\frac{\cos\lambda\sin\lambda(1+\cos^2\lambda)\,d\lambda}{\sqrt{2C\cos^2\lambda+\cos^6\lambda+2(1-D^2)\cos^4\lambda-2D^2}}. \tag{26}$$

Our study is in the interval $0<k<\frac{\pi}{2}$, i.e., $0<k_0<\frac{\pi}{2}$.

We observe here that the indefinite integral

$$\sqrt{2}\int\frac{\sin k\cos k(1+\cos^2 k)\,dk}{\sqrt{\cos^6\lambda+2(1-D^2)\cos^4\lambda+2C\cos^2\lambda-2D^2}}$$

after the change $v=\cos^2 k$ takes the form

$$-\frac{\sqrt{2}}{2}\left[\int\frac{dv}{\sqrt{P_3(v)}}+\int\frac{v\,dv}{\sqrt{P_3(v)}}\right], \tag{27}$$

where $P_3(v)=v^3+2(1-D^2)v^2+2Cv-2D^2$. According to the Appendix in [4], the integrals in (27) can be expressed by the Weierstrass elliptic function $\wp$ and the Weierstrass zeta function ζ. Several details on the subject will be given below.

Consider now the function

$$r(\lambda)=\cos^6\lambda+2\left(1-D^2\right)\cos^4\lambda+2C\cos^2\lambda-2D^2$$

in the interval $0\le\lambda\le\frac{\pi}{2}$. Then $r(0)=3-4D^2+2C$. We assume that

$$r(0)>0. \tag{28}$$

On the other hand, $r(\frac{\pi}{2})=-2D^2<0$. Such choice of $r(0)$ is possible in some cases. Assume that there is a link among the initial conditions k_0, k_0', D given by

$$3+2\left(k_0'\right)^2\sin^2 k_0\left(1+\cos^2 k_0\right)^2>\cos^4 k_0+2\cos^2 k_0+2D^2\left(2-\frac{1}{\cos^2 k_0}-\cos^2 k_0\right),$$

where $k_0\in(0,\frac{\pi}{2})$, $D\neq 0$. Then (28) holds. Let $\tilde{\lambda}$ be the first zero of $r(\lambda)$ in $(0,\frac{\pi}{2})$, i.e., $r(\tilde{\lambda})=0$, $0<\lambda<\tilde{\lambda}<\frac{\pi}{2}\Rightarrow r(\lambda)>0$. Then there are two possibilities:

$$r'(\tilde{\lambda})\neq 0, \tag{29}$$

$$r'(\tilde{\lambda})=0. \tag{30}$$

Certainly, we shall take $0<k_0<\tilde{\lambda}$; $0<k<\tilde{\lambda}$; $\tilde{\lambda}$ is a simple (multiple) root of $r(\lambda)=0$ depending on the roots of the biquadratic equation $3\cos^4\lambda+4(1-D^2)\cos^2\lambda+2C=0$ in $(0,\frac{\pi}{2})$. Evidently, $F'(k)>0$ in $(0,\tilde{\lambda})$, $F'(0)=0$, while (29) implies that there exists $F(\tilde{\lambda}-0)\neq\infty$, $F'(\tilde{\lambda}-0)=\infty$ and (30) implies that $F(\tilde{\lambda}-0)=+\infty$. Thus, in case (29) the mapping $F:[0,\tilde{\lambda})\to[F(0),F(\tilde{\lambda}-0))$ is a diffeomorphism on $(0,\tilde{\lambda})$, $k=F^{-1}(x-x_0)$, $k(x_0)=k_0$, $k_0=F^{-1}(0)$. We construct $k(x)$ on the interval $[F(0)+x_0,F(\tilde{\lambda}-0)+x_0]$ and then continue it in an even way with respect to the end point $F(\tilde{\lambda}-0)+x_0$. Put $T=2(F(\tilde{\lambda}-0)-F(0))$. Our last step is to continue $k(x)$ periodically with period T on $\mathbf{R}^1$, obtaining this way periodic cuspon (see Fig. 2). The terminology used here can be found, for example, in [9–11].

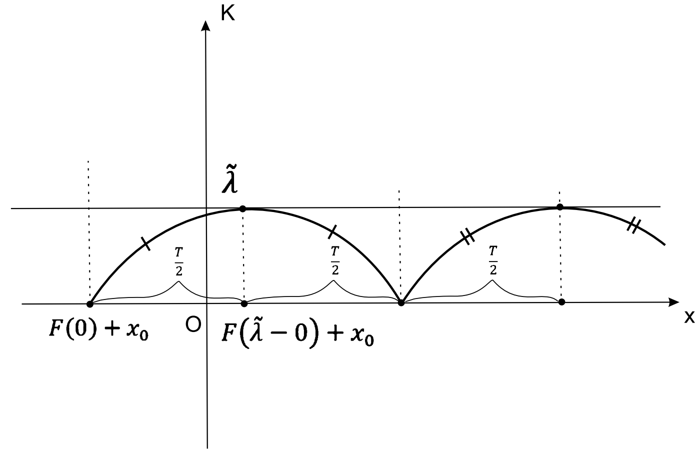

Figure 2 The graph of the solution $k = k(x)$ of Eq. (24) is periodic cuspon under condition (29)

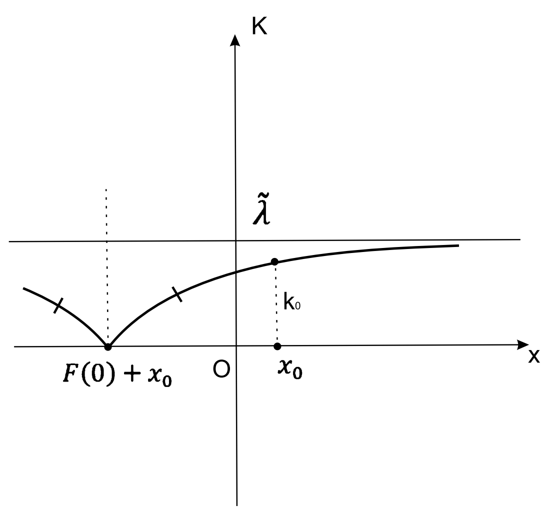

Figure 3 The graph of the solution $k = k(x)$ of Eq. (24) is soliton-cuspon under condition (30), i.e., it has cusp type singularity at $F(0) + x_0$

In case (30) the mapping $F : [0, \tilde{\lambda}) \to [F(0), \infty)$ is a diffeomorphism on $(0, \tilde{\lambda})$ and $k(x) = F^{-1}(x - x_0)$ forms the configuration soliton-cuspon (see Fig. 3).

The solution $k(x)$ is even with respect to $x = F(0) + x_0$.

Below is the last remark concerning Weierstrass elliptic functions. The integrals from (27) have the form $\int \frac{R_1(x_1)\,dx_1}{\sqrt{a_0x_1^3+bx_1^2+cx_1+d}}$, $a_0 \neq 0$, R_1 being a rational function. The standard change in this integral $x_1 = \sqrt[3]{\frac{4}{a_0}}t - \frac{b}{3a_0}$ reduces it to the general elliptic integral

$$\int \frac{R(t)\,dt}{\sqrt{4t^3 - g_2t - g_3}}, \tag{31}$$

where the relative invariants g_2, g_3 are given by $g_2 = (\frac{b^3}{3a_0} - c)\sqrt[3]{\frac{4}{a_0}}$, $g_3 = \frac{cb}{3a_0} - \frac{2b^3}{27a_0^2} - d$. The inverse of the Weierstrass elliptic function $y = \wp(u, g_2, g_3)$ is defined by the formula $u =$

$\wp^{-1}(y) = \int_y^{\infty} \frac{dt_1}{\sqrt{4t_1^3-g_2t_1-g_3}}$ and $\wp$ satisfies the ODE $(\wp')^2 = 4\wp^3(u) - g_2\wp(u) - g_3$. Evidently, $\int_{\infty}^{\wp(u)} \frac{dt}{\sqrt{4t^3-g_2t-g_3}} \equiv u$.

According to (1036.02) from [4] the values of (31) and therefore of (27) can always be expressed by the following normal elliptic integrals of first, second, and third kinds respectively: 1) $\int_{\infty}^{\wp(u)} \frac{dt}{\sqrt{4t^3-g_2t-g_3}}$, 2) $\int_{\wp(u_0)}^{\wp(u)} \frac{tdt}{\sqrt{4t^3-g_2t-g_3}} = \int_{u_0}^{u} \wp(u)\,du = -\zeta(u) + \zeta(u_0)$, 3) $\int_{\infty}^{\wp(u)} \frac{dt}{(t-\alpha^2)\sqrt{4t^3-g_2t-g_3}}$.

In case (27) only the first two integrals are important. We remind that $\zeta(u)$ is the Weierstrass zeta function, $\zeta'(u) = -\wp(u)$.

Case 4. Suppose that $f(k) = e^k$. Then (11) implies $2\varphi_1'k' + \varphi_1'' = 0$.

Again we shall consider the case $\varphi_1' \neq 0$ only.

Therefore,

$$\varphi_1(x) = E\int_{x_0}^{x} e^{-2k(x)}\,dx + F, \quad E = \text{const} \neq 0, F = \text{const}. \tag{32}$$

Applying (10) we obtain the second order autonomous ODE

$$k''(x) + \left(k'\right)^2\frac{(1+3e^{2k(x)})}{1+e^{2k(x)}} + \frac{-1+E^2e^{-4k(x)}(e^{2k}-1)}{1+e^{2k(x)}} = 0. \tag{33}$$

The changes $k' = p(k)$, $p^2 = q$ lead to linear first order ODE for $q = q(k)$. Standard calculations enable us to find the formula

$$q(k) = \frac{1}{e^{2k}(1+e^{2k})^2}\left(C + \frac{(1+e^{2k})^2}{2} - 2E^2\operatorname{ch}2k\right), \quad C = \text{const}. \tag{34}$$

Thus, we can concentrate on the case

$$\frac{dk}{dx} = p(k) = \sqrt{q(k)},$$

i.e., $F(k) = \int_{k_0}^{k} \frac{d\lambda}{\sqrt{q(\lambda)}} = x - x_0$, where more precisely

$$F(k) = \sqrt{2}\int_{k_0}^{k} \frac{e^{\lambda}(1+e^{2\lambda})\,d\lambda}{\sqrt{2C + (1+e^{2\lambda})^2 - 4E^2\operatorname{ch}2\lambda}}. \tag{35}$$

We will show at first that $F(k)$ can be expressed by the Weierstrass elliptic integrals. In fact, the change $\lambda = \frac{1}{2}\ln\gamma$, i.e., $\gamma = e^{2\lambda}$ in the indefinite integral $G(\lambda)$ corresponding to the definite integral (35) leads to

$$G(\lambda) = \frac{\sqrt{2}}{2}\int \frac{(1+\gamma)\,d\gamma}{\sqrt{\gamma^3 + 2\gamma^2(1-E^2) + \gamma(2C+1) - 2E^2}}.$$

The latter integral is of the type (31) and as in case 3 can be written by the Weierstrass elliptic integrals 1), 2).

Remark 1 To explain better the things, put $r(\lambda) = e^{6\lambda} + 2(1-E^2)e^{4\lambda} + e^{2\lambda}(2C+1) - 2E^2$ and assume that $r(0) = 2(C - 2E^2 + 2) > 0$. Having in mind that $r(\lambda) \to_{\lambda\to-\infty} -2E^2 < 0$, we

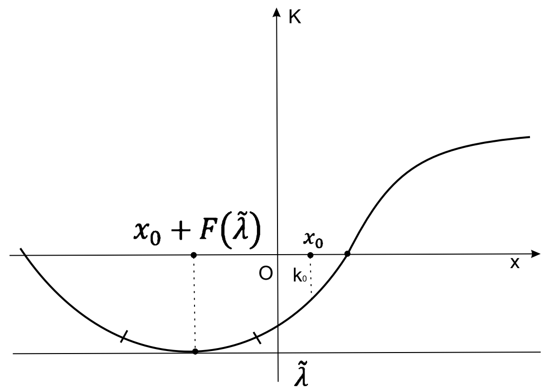

Figure 4 The graph of the function $k = k(x)$ satisfying Eq. (33) has a logarithmic growth at infinity and horizontal tangent at $x_0 + F(\tilde{\lambda})$

denote by $\tilde{\lambda} < 0$ the first negative zero of $r(\lambda) = 0$, i.e., $r(\lambda) \neq 0$ for $\tilde{\lambda} < \lambda \leq 0$ and suppose that $r(\lambda) > 0$ for $\lambda > 0$. Let $\tilde{\lambda} < k$, $\tilde{\lambda} < k_0$. Evidently, $F(k) \sim \sqrt{2}e^k$ for $k \to \infty$. For the sake of simplicity, let $\tilde{\lambda}$ be simple zero. Then $x - x_0 = F(k)$, $k = F^{-1}(x - x_0)$, $k(x) \sim \ln \frac{x-x_0}{\sqrt{2}}$ for $x \to \infty$ and horizontal tangent at $x_0 + F(\tilde{\lambda})$ exists (see Fig. 4).

4 Discussion

1. By using Ansatz (2) we found special solutions of system (21), respectively Ansatz (9) enables us to construct different solutions of (8) via special functions. Possible generalizations can be given for (1) where $[(u\bar{u}_x - v\bar{v}_x)u]_x$ and $[(u\bar{u}_x - v\bar{v}_x)v]_x$ participate instead of $[(u\bar{u}_x + v\bar{v}_x)u]_x$ and $[(u\bar{u}_x + v\bar{v}_x)v]_x$. Then the additional condition is $|u|^2 - |v|^2 = 1$ and (2) holds with $f_1 = \operatorname{ch} k$, $f_2 = \operatorname{sh} k$, $k = C_3 e^{C_1 t}(x + \frac{C_2}{C_1}$ for $C_1, C_3 \neq 0$, $\varphi = \frac{C_3^2}{2C_1} e^{2C_1 t} - C_1 \frac{x^2}{2} - C_2 x$. Certainly, u, v are not solitons.

It is interesting to look for and to find solutions of (1) in another possible form. Under the influence of [12], we look for rather different Ansatz. Put $z = x + iy = Ae^{\Theta} + Be^{-\Theta} + iCe^{-\Theta}$ with $A, B, C \neq 0$ real constants, i.e., $z_{\Theta,\Theta} = z$, denote $\Theta = Dx + Et$, D, E being real constants and suppose that $z = z(\Theta(x,t))$. Instead of (2) we take the Ansatz

$$u = K\frac{z}{\bar{z}^2} e^{-i(Fx+Gt)}, \quad F, G \in \mathbf{R}^1, K \in \mathbf{C}^1, \tag{36}$$

$$v = 1 - L\frac{1}{\bar{z}^2}, \quad L \in \mathbf{C}^1, |u|^2 + |v|^2 = 1. \tag{37}$$

We have above nine unknown constants. Substituting (36), (37) in (1), after tiresome computations we come to a complicated nonlinear algebraic system (overdetermined) satisfied by the parameters $A, B, C, D, E, F, G, K, L$. It should be solved.

So we can formulate the following open problem. Find solutions of (1), (18) in some other form instead of (2), (9) or (36), (37). It is interesting if they are not of soliton type because the latter can be constructed in several cases by the dressing method. The geometrical constraints $|u|^2 + |v|^2 = 1$, $|u|^2 - |v|^2 = 1$ can be omitted or replaced by other ones—algebraic, trigonometrical, etc.

Another open problem is to solve the overdetermined system (3), (4), (5), (6) for larger classes of functions f_1, f_2. In fact, $f_1 = \cos k$, $f_2 = \sin k$, respectively, $f_1 = \operatorname{ch} k$, $f_2 = \operatorname{sh} k$ are very special. Imposing other restrictions on f_1, f_2, we can obtain other classes of solutions of (1). As we mentioned above, the solutions of (1), (8) can develop singularities. It is evident for (8) as then autonomous ODEs appear. It is worth studying the $2\times$ system of PDEs (10), (11) satisfied by k, φ for some fixed smooth amplitude f. In this case there are no constraints imposed on u. What about blow-up of u?

2. Interesting open problem is the interaction of the solutions of (1), (8) which are not of soliton type. The quadruplet soliton interactions in the case of odd dispersion laws were studied in [6]. More precisely, two soliton interaction reduces to shifts of the relative center of mass and phases of each of the solutions. To do this the dressing method was applied. Is it possible to obtain new results on the subject for non-soliton solutions? Qualitative properties of the solutions are also of interest. Some of them are related to (1) containing large parameter R and under the assumption $|u|^2 + |v|^2 = R^2$. Then the number $N(R^2)$ of the integer points located in the disc $|u|^2 + |v|^2 \le R^2$ is given asymptotically by $N(R^2) \sim \pi R^2 + O(R^\nu)$, $R \to \infty$, $1/2 \le \nu \le 2/3$. The reader can propose some other properties of the solution (u_R, v_R) of system (1) with $R \to \infty$.

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Authors' contributions
PP formulated the main theorems. AS was involved in the proofs as well as in geometrical interpretations shown on the figures. All authors read and approved the final manuscript.

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