Ma and Kim *Advances in Difference Equations* (2020) 2020:569
https://doi.org/10.1186/s13662-020-03030-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A note on negative λ-binomial distribution

Yuankui Ma[1] and Taekyun Kim[1,2*]

*Correspondence:
kwangwoonmath@hanmail.net
[1]School of Science, Xi'an Technological University, Xi'an, 710021, Shaanxi, People's Republic of China
[2]Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Abstract
In this paper, we introduce one discrete random variable, namely the negative λ-binomial random variable. We deduce the expectation of the negative λ-binomial random variable. We also get the variance and explicit expression for the moments of the negative λ-binomial random variable.

MSC: 11B83; 11S80

Keywords: Negative λ-binomial random variable; Expectation; Variance; Moments

1 Introduction

In a sequence of independent Bernoulli trials, let the random variable X denote the trial at which the rth success occurs, where r is a fixed nonnegative integer. Then

$$P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r},\quad x=r,r+1,r+2,\ldots,$$

and we say that X has a negative binomial distribution with parameters (r,p) (see [1–3, 12, 13]).

The negative binomial distribution is sometimes defined in terms of the random variable Y, the number of failures before the rth success. This formulation is statistically equivalent to one given above in terms of X denoting the trial at which the rth success occurs, since $Y=X-r$. The alternative form of the negative binomial distribution is

$$p(k)=P(Y=k)=\binom{r+k-1}{k}p^r(1-p)^k,\quad k=0,1,2,\ldots,$$

where p is the probability of success in the trial (see [1, 3, 12, 13]).

It is known that the degenerate exponential function is defined by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\quad \lambda\in\mathbb{R},\tag{1}$$

where

$$(x)_{0,\lambda}=1,\qquad (x)_{n,\lambda}=x(x-\lambda)\cdots\bigl(x-(n-1)\lambda\bigr)\quad (n\geq 1)\text{ (see [5–7, 10, 11]).}\tag{2}$$

Recently, λ-analogue of binomial coefficients was considered by Kim to be

$$\binom{x}{0}_\lambda = 1, \qquad \binom{x}{n}_\lambda = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)\cdots(x-(n-1)\lambda)}{n!} \quad (n \geq 1) \text{ (see [6, 8, 9])}. \tag{3}$$

In this paper, we consider the negative λ-binomial distribution and obtain expressions for its moments.

2 Negative λ-binomial distribution

Definition 2.1 Y_λ is the negative λ-binomial random variable if the probability mass function of Y_λ with parameters (r,p) is given by

$$p_\lambda(k) = P_\lambda(Y_\lambda = k) = \binom{r+(k-1)\lambda}{k}_\lambda e_\lambda^r(p-1)(1-p)^k, \tag{4}$$

where $\lambda \in (0,1)$ and p is the probability of success in the trials.

Note that

$$\binom{r+(k-1)\lambda}{k}_\lambda = (-1)^k \binom{-r}{k}_\lambda, \quad k \geq 0 \text{ (see [4])} \tag{5}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} p_\lambda(k) &= \sum_{n=0}^{\infty} \binom{r+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^r(p-1) \\ &= e_\lambda^r(p-1) e_\lambda^{-r}(p-1) = 1.\end{aligned} \tag{6}$$

From (4), we note that

$$\lim_{\lambda \to 1} p_\lambda(k) \tag{7}$$

is the probability mass function of negative binomial random variable with parameters (r,p), and

$$\lim_{\lambda \to 0} p_\lambda(k) \tag{8}$$

is the probability mass function of Poisson random variable with parameters $r(1-p)$.

Let X be a discrete random variable, and let $f(x)$ be a real-valued function. Then we have

$$E\bigl(f(X)\bigr) = \sum_x f(x)p(x), \tag{9}$$

where $p(x)$ is the probability mass function.

From (9), we note that

$$\begin{aligned} E(Y_\lambda) &= \sum_{k=0}^{\infty} k p_\lambda(k) = \sum_{k=0}^{\infty} k \binom{r+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^r(p-1) \\ &= \frac{r}{e_\lambda^\lambda(p-1)} \sum_{k=1}^{\infty} \frac{(r+(k-1)\lambda)\cdots(r+\lambda)}{(k-1)!}(1-p)^k e_\lambda^{r+\lambda}(p-1) \end{aligned} \tag{10}$$

$$
\begin{aligned}
&= \frac{r}{e_\lambda^\lambda(p-1)} \sum_{k=0}^{\infty} \frac{(r+k\lambda)\cdots(r+\lambda)}{k!}(1-p)^{k+1} e_\lambda^{r+\lambda}(p-1) \\
&= \frac{r(1-p)}{e_\lambda^\lambda(p-1)} \sum_{k=0}^{\infty} \binom{r+\lambda+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^{r+\lambda}(p-1) \\
&= \frac{r(1-p)}{e_\lambda^\lambda(p-1)} e_\lambda^{-(r+\lambda)}(p-1) e_\lambda^{r+\lambda}(p-1) \\
&= \frac{r(1-p)}{e_\lambda^\lambda(p-1)}.
\end{aligned}
$$

Therefore, by (10), we obtain the following theorem.

Theorem 2.1 *Let Y_λ be a negative λ-binomial random variable with parameters (r,p). Then we have*

$$
E(Y_\lambda) = \frac{r(1-p)}{e_\lambda^\lambda(p-1)}.
$$

Note 2.1

$$
\lim_{\lambda\to 1} E(Y_\lambda) = \frac{r(1-p)}{p} = E(Y),
$$

where Y is the negative binomial random variable with parameters (r,p).

Note 2.2

$$
\lim_{\lambda\to 0} E(Y_\lambda) = r(1-p) = E(Y),
$$

where Y is the Poisson random variable with parameter $r(1-p)$.

Now, we observe that

$$
\begin{aligned}
E(Y_\lambda^2) &= \sum_{k=0}^{\infty} k^2 p_\lambda(k) = \sum_{k=0}^{\infty} k(k+1-1)p_\lambda(k) \qquad (11) \\
&= \sum_{k=0}^{\infty} k(k-1)p_\lambda(k) + \sum_{k=0}^{\infty} k p_\lambda(k) \\
&= \sum_{k=0}^{\infty} k(k-1)\binom{r+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^r(p-1) + E(Y_\lambda) \\
&= \frac{r(r+\lambda)}{e_\lambda^{2\lambda}(p-1)} \sum_{k=2}^{\infty} \frac{(r+(k-1)\lambda)\cdots(r+2\lambda)}{(k-2)!}(1-p)^k e_\lambda^{r+2\lambda}(p-1) + E(Y_\lambda) \\
&= \frac{r(r+\lambda)}{e_\lambda^{2\lambda}(p-1)} \sum_{k=0}^{\infty} \binom{r+(k+1)\lambda}{k}_\lambda (1-p)^{k+2} e_\lambda^{r+2\lambda}(p-1) + E(Y_\lambda) \\
&= \frac{r(r+\lambda)(1-p)^2}{e_\lambda^{2\lambda}(p-1)} \sum_{k=0}^{\infty} \binom{r+2\lambda+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^{r+2\lambda}(p-1) + E(Y_\lambda)
\end{aligned}
$$

$$= \frac{r(r+\lambda)(1-p)^2}{e_\lambda^{2\lambda}(p-1)} e_\lambda^{-(r+2\lambda)}(p-1)e_\lambda^{r+2\lambda}(p-1) + E(Y_\lambda)$$
$$= \frac{r(r+\lambda)(1-p)^2}{e_\lambda^{2\lambda}(p-1)} + \frac{r(1-p)}{e_\lambda^{\lambda}(p-1)}.$$

The variance of random variable X is defined by

$$\mathrm{Var}(X) = E\bigl(X^2\bigr) - \bigl[E(X)\bigr]^2 \quad \text{(see [1, 3])}. \tag{12}$$

From Theorem 2.1, (11), and (12), we note that

$$\begin{aligned}\mathrm{Var}(Y_\lambda) &= E\bigl(Y_\lambda^2\bigr) - \bigl[E(Y_\lambda)\bigr]^2 \\ &= \frac{r(r+\lambda)(1-p)^2}{e_\lambda^{2\lambda}(p-1)} + \frac{r(1-p)}{e_\lambda^{\lambda}(p-1)} - \frac{r^2(1-p)^2}{e_\lambda^{2\lambda}(p-1)} \\ &= \frac{r(1-p)^2}{e_\lambda^{2\lambda}(p-1)}(r+\lambda-r) + \frac{r(1-p)}{e_\lambda^{\lambda}(p-1)} \\ &= \lambda\frac{r(1-p)^2}{e_\lambda^{2\lambda}(p-1)} + \frac{r(1-p)}{e_\lambda^{\lambda}(p-1)}.\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.2 *Let Y_λ be a negative λ-binomial random variable with parameters (r,p). Then we have*

$$\mathrm{Var}(Y_\lambda) = \lambda\frac{r(1-p)^2}{e_\lambda^{2\lambda}(p-1)} + \frac{r(1-p)}{e_\lambda^{\lambda}(p-1)}.$$

Note 2.3

$$\lim_{\lambda\to 1}\mathrm{Var}(Y_\lambda) = \frac{r(1-p)}{p^2} = \mathrm{Var}(Y),$$

where Y is the negative binomial random variable with parameters (r,p).

Note 2.4

$$\lim_{\lambda\to 0}\mathrm{Var}(Y_\lambda) = r(1-p) = \mathrm{Var}(Y),$$

where Y is the Poisson random variable with parameter $r(1-p)$.

Note that

$$k^n = \sum_{l=0}^{n} S_2(n,l)(k)_l, \tag{13}$$

where $S_2(n,l)$ is the Stirling number of the second kind, and

$$(k)_0 = 1, \qquad (k)_l = k(k-1)\cdots(k-l+1) \quad (l \ge 1) \text{ (see [14, 15])}.$$

From (13), we note that

$$\begin{aligned}
E\bigl(Y_\lambda^n\bigr) &= \sum_{k=0}^{\infty} k^n p_\lambda(k) = \sum_{l=0}^{n} S_2(n,l) \sum_{k=l}^{\infty} (k)_l p_\lambda(k) \\
&= \sum_{l=0}^{n} S_2(n,l) \sum_{k=l}^{\infty} (k)_l \binom{r+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^r(p-1) \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)}{e_\lambda^{l\lambda}(p-1)} \\
&\quad \times \sum_{k=l}^{\infty} \frac{(r+(k-1)\lambda)\cdots(r+l\lambda)}{(k-l)!}(1-p)^k e_\lambda^{r+l\lambda}(p-1) \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)}{e_\lambda^{l\lambda}(p-1)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(r+(k+l-1)\lambda)\cdots(r+l\lambda)}{k!}(1-p)^{k+l} e_\lambda^{r+l\lambda}(p-1) \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)}{e_\lambda^{l\lambda}(p-1)} \\
&\quad \times \sum_{k=0}^{\infty} \binom{r+(k+l-1)\lambda}{k}_\lambda (1-p)^{k+l} e_\lambda^{r+l\lambda}(p-1) \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e_\lambda^{l\lambda}(p-1)} \\
&\quad \times \sum_{k=0}^{\infty} \binom{r+l\lambda+(k-1)\lambda}{k}_\lambda (1-p)^{k} e_\lambda^{r+l\lambda}(p-1) \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e_\lambda^{l\lambda}(p-1)} e_\lambda^{-r-l\lambda}(p-1) e_\lambda^{r+l\lambda}(p-1) \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e_\lambda^{l\lambda}(p-1)} \\
&= \sum_{l=0}^{n} S_2(n,l) \frac{(r+(l-1)\lambda)_{l,\lambda}(1-p)^l}{e_\lambda^{l\lambda}(p-1)}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.3 *Let Y_λ be a negative λ-binomial random variable with parameters (r,p). Then we have*

$$E\bigl(Y_\lambda^n\bigr) = \sum_{l=0}^{n} S_2(n,l) \frac{(r+(l-1)\lambda)_{l,\lambda}(1-p)^l}{e_\lambda^{l\lambda}(p-1)}.$$

Note 2.5

$$\lim_{\lambda\to 1} E(Y_\lambda^n) = \sum_{l=0}^{n} S_2(n,l)\frac{(r+(l-1))_l(1-p)^l}{p^l} = E(Y^n),$$

where Y is the negative binomial random variable with parameters (r,p) (see [4, 12]).

Note 2.6

$$\lim_{\lambda\to 0} E(Y_\lambda^n) = \sum_{l=0}^{n} S_2(n,l)\big(r(1-p)\big)^l = E(Y^n),$$

where Y is the Poisson random variable with parameter $r(1-p)$ (see [16]).

Note that

$$\begin{aligned}
E(Y_\lambda^n) &= \sum_{k=0}^{\infty} k^n p_\lambda(k) \\
&= \sum_{k=0}^{\infty} k^n \binom{r+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^r(p-1) \\
&= \sum_{k=1}^{\infty} k^{n-1} \frac{(r+(k-1)\lambda)\cdots(r+\lambda)r}{(k-1)!}(1-p)^k e_\lambda^r(p-1) \\
&= \sum_{k=0}^{\infty} (k+1)^{n-1} \frac{(r+k\lambda)\cdots(r+\lambda)r}{k!}(1-p)^{k+1} e_\lambda^r(p-1) \\
&= r(1-p)\sum_{k=0}^{\infty}\sum_{i=0}^{n-1}\binom{n-1}{i}k^i\frac{(r+k\lambda)\cdots(r+\lambda)}{k!}(1-p)^k e_\lambda^r(p-1) \\
&= \frac{r(1-p)}{e_\lambda^\lambda(p-1)}\sum_{i=0}^{n-1}\binom{n-1}{i}\sum_{k=0}^{\infty}k^i\binom{r+\lambda+(k-1)\lambda}{k}_\lambda (1-p)^k e_\lambda^{r+\lambda}(p-1) \\
&= \frac{r(1-p)}{e_\lambda^\lambda(p-1)}\sum_{i=0}^{n-1}\binom{n-1}{i}E(Z_\lambda^i),
\end{aligned}$$

where Z_λ is the negative λ-binomial random variable with parameters $(r+\lambda,p)$.

Therefore, we obtain the following theorem.

Theorem 2.4 *Let Y_λ, Z_λ be two negative λ-binomial random variables with parameters (r,p), $(r+\lambda,p)$ respectively. Then we have*

$$E(Y_\lambda^n) = \frac{r(1-p)}{e_\lambda^\lambda(p-1)}\sum_{i=0}^{n-1}\binom{n-1}{i}E(Z_\lambda^i).$$

3 Conclusion

In this paper, we introduced one discrete random variable, namely the negative λ-binomial random variable. The details and results are as follows. We defined the negative λ-binomial random variable with parameter (r,p) in (4) and deduced its expectation in Theorem 2.1. We also obtained its variance in Theorem 2.2 and derived explicit expression for the moment of the negative λ-binomial random variable in Theorem 2.3.

Acknowledgements
The authors thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding
This research was funded by the National Natural Science Foundation of China (No. 11871317, 11926325, 11926321).

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
All authors reveal that there is no ethical problem in the production of this paper.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
All authors want to publish this paper in this journal.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Publisher's Note

Received: 3 September 2020 Accepted: 4 October 2020 Published online: 08 October 2020

References

1. Alexander, H.W.: Recent publications: introduction to probability and mathematical statistics. Am. Math. Mon. **70**(2), 222–223 (1963)
2. Bayad, A., Chikhi, J.: Apostol–Euler polynomials and asymptotics for negative binomial reciprocals. Adv. Stud. Contemp. Math. (Kyungshang) **24**(1), 33–37 (2014)
3. Carlitz, L.: Comment on the paper "Some probability distributions and their associated structures". Math. Mag. **37**(1), 51–53 (1964)
4. Funkenbusch, W.: On writing the general term coefficient of the binomial expansion to negative and fractional powers, in tri-factorial form. Natl. Math. Mag. **17**(7), 308–310 (1943)
5. Kim, D.S., Kim, T.: A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. **27**(2), 227–235 (2020)
6. Kim, T.: λ-Analogue of Stirling numbers of the first kind. Adv. Stud. Contemp. Math. (Kyungshang) **27**(3), 423–429 (2017)
7. Kim, T., Kim, D.S.: Degenerate Laplace transform and degenerate gamma function. Russ. J. Math. Phys. **24**(2), 241–248 (2017)
8. Kim, T., Kim, D.S.: Degenerate Bernstein polynomials. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(3), 2913–2920 (2019)
9. Kim, T., Kim, D.S.: Correction to: Degenerate Bernstein polynomials. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(3), 2921–2922 (2019)
10. Kim, T., Kim, D.S.: Note on the degenerate gamma function. Russ. J. Math. Phys. **27** (3), 352–358 (2020)
11. Kim, T., Kim, D.S., Jang, L.C., Kim, H.Y.: A note on discrete degenerate random variables. Proc. Jangjeon Math. Soc. **23**(1), 125–135 (2020)
12. Rider, P.R.: Classroom notes: the negative binomial distribution and the incomplete beta function. Am. Math. Mon. **69**(4), 302–304 (1962)
13. Ross, S.M.: Introduction to Probability Models. Twelfth edition of [MR0328973]. Academic Press, London (2019). ISBN 978-0-12-814346-9
14. Simsek, Y.: Identities on the Changhee numbers and Apostol-type Daehee polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **27**(2), 199–212 (2017)
15. Simsek, Y.: Combinatorial inequalities and sums involving Bernstein polynomials and basis functions. J. Inequal. Spec. Funct. **8**(3), 15–24 (2017)
16. Theodorescu, R., Borwein, J.M.: Problems and solutions: solutions: moments of the Poisson distribution: 10738. Am. Math. Mon. **107**(7), 659 (2000)