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A linearized conservative Galerkin–Legendre spectral method for the strongly coupled nonlinear fractional Schrödinger equations

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Abstract

In this paper, based on Galerkin–Legendre spectral method for space discretization and a linearized Crank–Nicolson difference scheme in time, a fully discrete spectral scheme is developed for solving the strongly coupled nonlinear fractional Schrödinger equations. We first prove that the proposed scheme satisfies the conservation laws of mass and energy in the discrete sense. Then a prior bound of the numerical solutions in L^∞-norm is obtained, and the spectral scheme is shown to be unconditionally convergent in L^2-norm, with second-order accuracy in time and spectral accuracy in space. Finally, some numerical results are provided to validate our theoretical analysis.

Keywords: Fractional Schrödinger equation; Legendre spectral method; Conservation law; Unconditional convergence; Spectral accuracy

1 Introduction

The space fractional Schrödinger equation (FSE) is a natural extension of the classic Schrödinger equation, and it has been successfully used to describe the fractional quantum phenomena. Laskin [1, 2] originally derived the Riesz space FSE via replacing the Brownian trajectories with Levy flights in the Feynman path integrals. Some physical applications of the FSE were presented in [3, 4]. For the well-posedness, global attractor, soliton dynamics and ground states related to the FSE, we refer to Refs. [5–7] and the references therein.

The current paper is devoted to deriving a linearized conservative Galerkin–Legendre spectral method for solving the strongly coupled fractional Schrödinger equations (SCFSEs) with extended Dirichlet boundary conditions [8–10]

$$iu_t - \gamma(-\triangle)^{\frac{\alpha}{2}}u + (\kappa|u|^2 + \rho|v|^2)u + \beta u + \varrho v = 0, \quad x \in \Omega, 0 < t \le T, \tag{1}$$

$$iv_t - \gamma(-\triangle)^{\frac{\alpha}{2}}v + (\kappa|v|^2 + \rho|u|^2)v + \beta v + \varrho u = 0, \quad x \in \Omega, 0 < t \le T, \tag{2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega, \tag{3}$$

$$u(x,t) = 0, \qquad v(x,t) = 0, \quad x \in \mathbb{R}\backslash\Omega, 0 \le t \le T, \tag{4}$$

where $i^2 = -1$, $1 < \alpha \le 2$, $\Omega = (a,b)$ with $a \ll 0$ and $b \gg 0$, and the parameters $\gamma > 0$, κ, ρ, β and ϱ are given real constants. $u_0(x)$ and $v_0(x)$ are given initial functions. The Riesz fractional derivative is defined as

$$-(-\Delta)^{\frac{\alpha}{2}} u(x,t) := -\frac{1}{2\cos(\pi\alpha/2)}\left[{}_aD_x^\alpha u(x,t) + {}_xD_b^\alpha u(x,t)\right], \tag{5}$$

where the left and right Riemann–Liouville fractional derivatives [11] are given as

$$ {}_aD_x^\alpha u(x,t) =: \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x \frac{u(\xi,t)\,d\xi}{(x-\xi)^{\alpha-1}}, \tag{6}$$

$$ {}_xD_b^\alpha u(x,t) := \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\xi,t)\,d\xi}{(\xi-x)^{\alpha-1}}. \tag{7}$$

In particular, the Schrödinger system (1)–(4) preserves two invariant quantities, i.e., the mass-conservation law

$$M(t) := \left\|u(\cdot,t)\right\|_{L^2}^2 + \left\|v(\cdot,t)\right\|_{L^2}^2 = M(0), \quad \forall t > 0, \tag{8}$$

and the energy-conservation law

$$\begin{aligned} E(t) := {}& \gamma\left(\left\|(-\triangle)^{\frac{\alpha}{4}} u(\cdot,t)\right\|_{L^2}^2 + \left\|(-\triangle)^{\frac{\alpha}{4}} v(\cdot,t)\right\|_{L^2}^2\right) - \frac{\kappa}{2}\left(\left\|u(\cdot,t)\right\|_{L^4}^4 + \left\|v(\cdot,t)\right\|_{L^4}^4\right) \\ & - \rho\int_\Omega |u|^2|v|^2\,dx - \beta\left(\left\|u(\cdot,t)\right\|_{L^2}^2 + \left\|v(\cdot,t)\right\|_{L^2}^2\right) - 2\varrho\,\mathrm{Re}\int_\Omega u\bar{v}\,dx \\ = {}& E(0), \quad \forall t > 0. \end{aligned} \tag{9}$$

Since it is hard to obtain the analytical solution of the FSE, the idea of developing numerical methods has drawn a growing number of researchers' attention. Up to now, many efforts have been made to develop finite difference methods for the FSE, including the compact difference scheme [12], the mass-preserving schemes [13–15], and the mass- and energy-preserving schemes [16–20]. Li et al. [21–23] investigated a series of Galerkin finite element methods for the FSE, and they discussed the conservation, well-posedness and convergence properties of the discrete systems. In addition, spectral methods have also been applied in solving the nonlocal FSE, including spectral Galerkin schemes [24–30] and collocation schemes [31–35]. On the other hand, numerical studies of the FSE with Caputo fractional derivative in time were considered in [36–39].

The motivations of the current work are as follows. Firstly, since the conservative method performs better than the general goal method in long-time simulation, the discrete scheme which can preserve the invariant quantities of the original system is desirable. Moreover, to avoid time-consuming iterative process at each time step, an interesting topic is to construct a linearly implicit scheme for the SCFSEs. Furthermore, we intend to consider the unconditionally convergent spectral method, which takes advantage of spectral accuracy in space. Based on these considerations, the main objective of this paper is to develop a linearized Galerkin–Legendre spectral scheme for solving the SCFSEs. The derived scheme can preserve both the mass- and the energy-conservation laws in the discrete sense. Based on the discrete energy-conservation law, we show that the numerical solutions are bounded in L^∞-norm. Moreover, the discrete scheme is proved to be unconditionally convergent with second-order accuracy in time and spectral accuracy in space by the energy method.

The outline of this paper is given as follows. In Sect. 2, some useful definitions and lemmas are recalled. In Sect. 3, a linearized Legendre spectral scheme is constructed for the SCFSEs. In Sect. 4, the conservation, boundedness and convergence properties of the proposed scheme are analyzed theoretically. Some numerical results are presented in Sect. 5, and some conclusions are drawn in the last section.

2 Preliminaries

In this section, before deriving the fully discrete Legendre spectral scheme for the SCFSEs, we first introduce some notations, definitions and lemmas which play an important role in subsequent theoretical analysis.

2.1 Notation

Define the inner product in the space $L^2(\Omega)$ as $(v,u) := \int_\Omega v\bar{u}\,dx$ and the associated L^2-norm is denoted by $\|\cdot\|$. Besides, define the L^p-norm ($1 \le p < \infty$) and L^∞-norm as follows:

$$\|v\|_{L^p} := \left(\int_\Omega \left|v(x)\right|^p dx\right)^{\frac{1}{p}}, \qquad \|v\|_{L^\infty} := \operatorname*{ess\,sup}_{x\in\Omega}\left|v(x)\right|. \tag{10}$$

2.2 Fractional derivative spaces

Definition 1 ([40, 41]) For $\alpha > 0$, define the semi-norms and norms of the left, right and symmetric fractional derivative spaces on Ω as

$$|v|_{J_L^\alpha(\Omega)} := \left\|{}_aD_x^\alpha v\right\|, \qquad \|v\|_{J_L^\alpha(\Omega)} := \left(\|v\|^2 + |v|^2_{J_L^\alpha(\Omega)}\right)^{\frac{1}{2}}, \tag{11}$$

$$|v|_{J_R^\alpha(\Omega)} := \left\|{}_xD_b^\alpha v\right\|, \qquad \|v\|_{J_R^\alpha(\Omega)} := \left(\|v\|^2 + |v|^2_{J_R^\alpha(\Omega)}\right)^{\frac{1}{2}}, \tag{12}$$

$$|v|_{J_S^\alpha(\Omega)} := \left|\left({}_aD_x^\alpha v, {}_xD_b^\alpha v\right)\right|^{\frac{1}{2}}, \qquad \|v\|_{J_S^\alpha(\Omega)} := \left(\|v\|^2 + |v|^2_{J_S^\alpha(\Omega)}\right)^{\frac{1}{2}}, \tag{13}$$

and $J_{L,0}^\alpha(\Omega)$, $J_{R,0}^\alpha(\Omega)$, $J_{S,0}^\alpha(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the above norms, respectively.

Definition 2 ([40, 41]) For $\alpha > 0$, define the semi-norm

$$|v|_{H^\alpha(\Omega)} := \left\||\xi|^\alpha \hat{v}(\xi)\right\|_{L^2(\mathbb{R})}, \tag{14}$$

and the norm

$$\|v\|_{H^\alpha(\Omega)} := \left(\|v\|^2 + |v|^2_{H^\alpha(\Omega)}\right)^{\frac{1}{2}}, \tag{15}$$

and $H_0^\alpha(\Omega)$ denotes the closure of $C^\infty(\Omega)$ with respect to $\|\cdot\|_{H^\alpha(\Omega)}$, where ξ and $\hat{v}$ represent the Fourier transform parameter and the Fourier transform of v, respectively.

Next we recall some useful properties of the above semi-norms, norms and spaces.

Lemma 1 ([40, 41]) *For $\alpha > 0$ and $\alpha \neq n - \frac{1}{2}$, $n \in \mathbb{N}$, then $J^{\alpha}_{L,0}(\Omega)$, $J^{\alpha}_{R,0}(\Omega)$, $J^{\alpha}_{S,0}(\Omega)$ and $H^{\alpha}_{0}(\Omega)$ are equal with equivalent norms and semi-norms.*

Lemma 2 (Fractional Poincaré–Friedrichs inequality [40, 41]) *For $v \in J^{\alpha}_{L,0}(\Omega)$, $0 < \mu < \alpha$, we have*

$$\|v\| \leq C|v|_{J^{\alpha}_{L}(\Omega)}, \qquad |v|_{J^{\mu}_{L}(\Omega)} \leq C|v|_{J^{\alpha}_{L}(\Omega)}.$$

Besides, for $v \in J^{\alpha}_{R,0}(\Omega)$, $0 < \mu < \alpha$, we have

$$\|v\| \leq C|v|_{J^{\alpha}_{R}(\Omega)}, \qquad |v|_{J^{\mu}_{R}(\Omega)} \leq C|v|_{J^{\alpha}_{R}(\Omega)}.$$

Similar conclusion can be established for $v \in H^{\alpha}_{0}(\Omega)$ with $\alpha \neq n - \frac{1}{2}$, $n \in \mathbb{N}$.

Lemma 3 ([42]) *Let $1 < \alpha \leq 2$, for $v, w \in J^{\alpha}_{L}(\Omega)$ (or $J^{\alpha}_{R}(\Omega)$), $v|_{\partial\Omega} = 0$, $w|_{\partial\Omega} = 0$, then we have*

$$\left({}_aD^{\alpha}_x v, w\right) = \left({}_aD^{\alpha/2}_x v, {}_xD^{\alpha/2}_b w\right), \qquad \left({}_xD^{\alpha}_b v, w\right) = \left({}_xD^{\alpha/2}_b v, {}_aD^{\alpha/2}_x w\right).$$

3 Fully discrete Legendre spectral scheme

In this section, we will construct a Legendre spectral method for numerically solving the SCFSEs (1)–(4).

3.1 The semi-discrete variational scheme

The Legendre polynomials $L_k(s)$ are determined by the following recurrence relation:

$$\begin{aligned} &L_0(s) = 1, \qquad L_1(s) = s, \\ &L_{k+1}(s) = \frac{2k+1}{k+1} sL_k(s) - \frac{k}{k+1} L_{k-1}(s), \quad s \in [-1, 1], k \geq 1. \end{aligned} \tag{16}$$

Denote

$$\psi_k(x) = L_k(s) - L_{k+2}(s), \quad \text{with } x = \frac{(b-a)s + (a+b)}{2} \in [a, b]. \tag{17}$$

Then the approximate function space V^0_N is given as

$$V^0_N = \operatorname{span}\{\psi_k(x) : k = 0, 1, \ldots, N-2\}. \tag{18}$$

The semi-discrete variational scheme for the SCFSEs (1)–(4) is to find $u_N, v_N : [0, T] \to V^0_N$ such that

$$\begin{aligned} &i(\partial_t u_N, w) - \gamma B(u_N, w) + \left(\left(\kappa|u_N|^2 + \rho|v_N|^2\right)u_N, w\right) + \beta(u_N, w) \\ &\qquad + \varrho(v_N, w) = 0, \quad \forall w \in V^0_N, \end{aligned} \tag{19}$$

$$\begin{aligned} &i(\partial_t v_N, w) - \gamma B(v_N, w) + \left(\left(\kappa|v_N|^2 + \rho|u_N|^2\right)v_N, w\right) + \beta(v_N, w) \\ &\qquad + \varrho(u_N, w) = 0, \quad \forall w \in V^0_N, \end{aligned} \tag{20}$$

$$u^0_N = I_N u_0(x), \qquad v^0_N = I_N v_0(x), \tag{21}$$

where I_N represents the Legendre–Gauss–Lobatto (LGL) interpolation operator [43]. The bilinear form $B(\cdot,\cdot)$ in (19) and (20) is defined as

$$B(v,w) := \big((-\Delta)^{\frac{\alpha}{2}}v, w\big) = \frac{1}{2\cos(\alpha\pi/2)}\big(\big({}_aD_x^{\frac{\alpha}{2}}v, {}_xD_b^{\frac{\alpha}{2}}w\big) + \big({}_xD_b^{\frac{\alpha}{2}}v, {}_aD_x^{\frac{\alpha}{2}}w\big)\big), \tag{22}$$

where Lemma 3 has been used in deriving (22). For convenience of theoretical analysis, one can define the following semi-norm and norm:

$$|v|_{\frac{\alpha}{2}} := \sqrt{B(v,v)}, \qquad \|v\|_{\frac{\alpha}{2}} := \big(\|v\|^2 + |v|_{\frac{\alpha}{2}}^2\big)^{\frac{1}{2}}. \tag{23}$$

By virtue of Lemma 1, $|v|_{\frac{\alpha}{2}}$ and $\|v\|_{\frac{\alpha}{2}}$ are equivalent with the semi-norms and norms of $J_L^{\frac{\alpha}{2}}(\Omega)$, $J_R^{\frac{\alpha}{2}}(\Omega)$, $J_S^{\frac{\alpha}{2}}(\Omega)$ and $H^{\frac{\alpha}{2}}(\Omega)$.

3.2 The fully discrete Galerkin–Legendre spectral scheme

For a given positive constant T and any positive integer M, let $\tau = T/M$ and denote $t_n = n\tau$ $(0 \le n \le M)$. For any function sequence $\{\lambda^n\}$ defined on Ω, when $0 \le n \le M-1$, we denote

$$\delta_t\lambda^{n+\frac{1}{2}} = \frac{\lambda^{n+1}-\lambda^n}{\tau}, \qquad \delta_{\hat{t}}\lambda^n = \frac{\lambda^{n+1}-\lambda^{n-1}}{2\tau},$$
$$\hat{\lambda}^{n+\frac{1}{2}} = \frac{\lambda^{n+1}+\lambda^n}{2}, \qquad \tilde{\lambda}^n = \frac{\lambda^{n+1}+\lambda^{n-1}}{2}.$$

Based on Legendre spectral method for space discretization and a linearized Crank–Nicolson difference scheme in time, we develop a linearized spectral scheme for the Schrödinger system (1)–(4), which is to find $u_N^{n+1}, v_N^{n+1} \in V_N^0$ such that

$$\begin{aligned} &i\big(\delta_{\hat{t}}u_N^n, w\big) - \gamma B\big(\tilde{u}_N^n, w\big) + \big(\big(\kappa\big|u_N^n\big|^2 + \rho\big|v_N^n\big|^2\big)\tilde{u}_N^n, w\big) + \beta\big(\tilde{u}_N^n, w\big) \\ &\quad + \varrho\big(v_N^n, w\big) = 0, \quad \forall w \in V_N^0, 1 \le n \le M-1, \end{aligned} \tag{24}$$

$$\begin{aligned} &i\big(\delta_{\hat{t}}v_N^n, w\big) - \gamma B\big(\tilde{v}_N^n, w\big) + \big(\big(\kappa\big|v_N^n\big|^2 + \rho\big|u_N^n\big|^2\big)\tilde{v}_N^n, w\big) + \beta\big(\tilde{v}_N^n, w\big) \\ &\quad + \varrho\big(u_N^n, w\big) = 0, \quad \forall w \in V_N^0, 1 \le n \le M-1. \end{aligned} \tag{25}$$

To obtain the first step approximate solutions u_N^1 and v_N^1, we employ the following Crank–Nicolson scheme:

$$\begin{aligned} &i\big(\delta_t u_N^{\frac{1}{2}}, w\big) - \gamma B\big(\hat{u}_N^{\frac{1}{2}}, w\big) + \frac{1}{2}\big(\big(\kappa\big(\big|u_N^1\big|^2 + \big|u_N^0\big|^2\big) + \rho\big(\big|v_N^1\big|^2 + \big|v_N^0\big|^2\big)\big)\hat{u}_N^{\frac{1}{2}}, w\big) \\ &\quad + \beta\big(\hat{u}_N^{\frac{1}{2}}, w\big) + \varrho\big(\hat{v}_N^{\frac{1}{2}}, w\big) = 0, \quad \forall w \in V_N^0, \end{aligned} \tag{26}$$

$$\begin{aligned} &i\big(\delta_t v_N^{\frac{1}{2}}, w\big) - \gamma B\big(\hat{v}_N^{\frac{1}{2}}, w\big) + \frac{1}{2}\big(\big(\kappa\big(\big|v_N^1\big|^2 + \big|v_N^0\big|^2\big) + \rho\big(\big|u_N^1\big|^2 + \big|u_N^0\big|^2\big)\big)\hat{v}_N^{\frac{1}{2}}, w\big) \\ &\quad + \beta\big(\hat{v}_N^{\frac{1}{2}}, w\big) + \varrho\big(\hat{u}_N^{\frac{1}{2}}, w\big) = 0, \quad \forall w \in V_N^0, \end{aligned} \tag{27}$$

with the initial conditions

$$u_N^0 = I_N u_0(x), \qquad v_N^0 = I_N v_0(x). \tag{28}$$

4 Theoretical analysis

This section is devoted to discussing the theoretical analysis of the spectral scheme (24)–(28), including the discrete mass- and energy-conservation laws, boundedness and the unconditional convergence.

4.1 Conservative properties of the spectral scheme

Theorem 1 *The fully discrete spectral scheme* (24)–(28) *is conservative in the sense that*

$$M^n = M^0, \quad 0 \le n \le M-1, \tag{29}$$

$$E^n = E^0, 0 \le n \le M-1, \tag{30}$$

where M^n and E^n are defined, respectively, as

$$M^n := \frac{1}{2}\left(\|u_N^{n+1}\|^2 + \|v_N^{n+1}\|^2 + \|u_N^n\|^2 + \|v_N^n\|^2\right) + \tau\varrho \operatorname{Im}\int_\Omega \left(v_N^n \bar{u}_N^{n+1} + u_N^n \bar{v}_N^{n+1}\right) dx, \tag{31}$$

$$\begin{aligned} E^n := {} & \gamma\left(|u_N^{n+1}|_{\frac{\alpha}{2}}^2 + |v_N^{n+1}|_{\frac{\alpha}{2}}^2 + |u_N^n|_{\frac{\alpha}{2}}^2 + |v_N^n|_{\frac{\alpha}{2}}^2\right) - \kappa\int_\Omega \left(|u_N^n|^2 |u_N^{n+1}|^2 \right.\\ & \left. + |v_N^n|^2|v_N^{n+1}|^2\right) dx - \rho\int_\Omega \left(|u_N^n|^2|v_N^{n+1}|^2 + |v_N^n|^2|u_N^{n+1}|^2\right) dx - \beta\left(\|u_N^{n+1}\|^2 \right.\\ & \left. + \|v_N^{n+1}\|^2 + \|u_N^n\|^2 + \|v_N^n\|^2\right) - 2\varrho \operatorname{Re}\int_\Omega \left(u_N^n \bar{v}_N^{n+1} + v_N^n \bar{u}_N^{n+1}\right) dx. \end{aligned} \tag{32}$$

Proof Taking $w = \tilde{u}_N^n$ in (24) gives

$$\begin{aligned} & i\left(\delta_{\hat{t}} u_N^n, \tilde{u}_N^n\right) - \gamma B\left(\tilde{u}_N^n, \tilde{u}_N^n\right) + \left(\left(\kappa|u_N^n|^2 + \rho|v_N^n|^2\right)\tilde{u}_N^n, \tilde{u}_N^n\right) + \beta\left(\tilde{u}_N^n, \tilde{u}_N^n\right) \\ & \quad + \varrho\left(v_N^n, \tilde{u}_N^n\right) = 0. \end{aligned} \tag{33}$$

As a result

$$\operatorname{Re}\left(\delta_{\hat{t}} u_N^n, \tilde{u}_N^n\right) = \frac{1}{4\tau}\left(\|u_N^{n+1}\|^2 - \|u_N^{n-1}\|^2\right), \qquad \operatorname{Im}\left\{B\left(\tilde{u}_N^n, \tilde{u}_N^n\right)\right\} = 0, \tag{34}$$

and

$$\operatorname{Im}\left(\left(\kappa|u_N^n|^2 + \rho|v_N^n|^2\right)\tilde{u}_N^n, \tilde{u}_N^n\right) = 0, \tag{35}$$

then considering the imaginary part of (33) yields

$$\frac{1}{4\tau}\left(\|u_N^{n+1}\|^2 - \|u_N^{n-1}\|^2\right) + \frac{\varrho}{2}\operatorname{Im}\int_\Omega v_N^n\left(\bar{u}_N^{n+1} + \bar{u}_N^{n-1}\right) dx = 0, \quad 1 \le n \le M-1. \tag{36}$$

It further means that

$$\begin{aligned} & \frac{1}{2}\left(\|u_N^{n+1}\|^2 + \|u_N^n\|^2\right) + \tau\varrho \operatorname{Im}\int_\Omega v_N^n \bar{u}_N^{n+1} dx \\ & \quad - \frac{1}{2}\left(\|u_N^n\|^2 + \|u_N^{n-1}\|^2\right) - \tau\varrho \operatorname{Im}\int_\Omega u_N^{n-1}\bar{v}_N^n dx = 0. \end{aligned} \tag{37}$$

Taking $w = \tilde{v}_N^n$ in (25), we arrive at

$$i(\delta_{\hat{t}} v_N^n, \tilde{v}_N^n) - \gamma B(\tilde{v}_N^n, \tilde{v}_N^n) + \left(\left(\kappa\left|v_N^n\right|^2 + \rho\left|u_N^n\right|^2\right)\tilde{v}_N^n, \tilde{v}_N^n\right) + \beta(\tilde{v}_N^n, \tilde{v}_N^n) \\ + \varrho(u_N^n, \tilde{v}_N^n) = 0. \tag{38}$$

Similarly, we take the imaginary part of (38) to get

$$\frac{1}{2}\left(\left\|v_N^{n+1}\right\|^2 + \left\|v_N^n\right\|^2\right) + \tau\varrho \operatorname{Im}\int_\Omega u_N^n \bar{v}_N^{n+1}\,dx \\ - \frac{1}{2}\left(\left\|v_N^n\right\|^2 + \left\|v_N^{n-1}\right\|^2\right) - \tau\varrho \operatorname{Im}\int_\Omega v_N^{n-1}\bar{u}_N^n\,dx = 0. \tag{39}$$

Combining (37) and (39), we can conclude that the discrete mass conservation law (29) holds.

On the other hand, substituting $w = \delta_{\hat{t}} u_N^n$ in (24), we arrive at

$$i(\delta_{\hat{t}} u_N^n, \delta_{\hat{t}} u_N^n) - \gamma B(\tilde{u}_N^n, \delta_{\hat{t}} u_N^n) + \left(\left(\kappa\left|u_N^n\right|^2 + \rho\left|v_N^n\right|^2\right)\tilde{u}_N^n, \delta_{\hat{t}} u_N^n\right) \\ + \beta(\tilde{u}_N^n, \delta_{\hat{t}} u_N^n) + \varrho(v_N^n, \delta_{\hat{t}} u_N^n) = 0. \tag{40}$$

It is easy to get

$$\operatorname{Re}\left\{B(\tilde{u}_N^n, \delta_{\hat{t}} u_N^n)\right\} = \frac{1}{4\tau}\left(\left|u_N^{n+1}\right|_{\frac{\alpha}{2}}^2 - \left|u_N^{n-1}\right|_{\frac{\alpha}{2}}^2\right), \tag{41}$$

$$\operatorname{Re}\left(\left(\kappa\left|u_N^n\right|^2 + \rho\left|v_N^n\right|^2\right)\tilde{u}_N^n, \delta_{\hat{t}} u_N^n\right) \\ = \frac{1}{4\tau}\int_\Omega\left(\kappa\left|u_N^n\right|^2 + \rho\left|v_N^n\right|^2\right)\left(\left|u_N^{n+1}\right|^2 - \left|u_N^{n-1}\right|^2\right)dx, \tag{42}$$

and

$$\operatorname{Re}(v_N^n, \delta_{\hat{t}} u_N^n) = \frac{1}{2\tau}\operatorname{Re}\int_\Omega\left(v_N^n \bar{u}_N^{n+1} - u_N^{n-1}\bar{v}_N^n\right)dx. \tag{43}$$

Taking the real part of (40), and combining with (41)–(43), we have

$$\gamma\left(\left|u_N^{n+1}\right|_{\frac{\alpha}{2}}^2 - \left|u_N^{n-1}\right|_{\frac{\alpha}{2}}^2\right) \\ = \int_\Omega\left(\kappa\left|u_N^n\right|^2 + \rho\left|v_N^n\right|^2\right)\left(\left|u_N^{n+1}\right|^2 - \left|u_N^{n-1}\right|^2\right)dx + \beta\left(\left\|u_N^{n+1}\right\|^2 - \left\|u_N^{n-1}\right\|^2\right) \\ + 2\varrho \operatorname{Re}\int_\Omega\left(v_N^n\bar{u}_N^{n+1} - u_N^{n-1}\bar{v}_N^n\right)dx, \quad 1 \le n \le M-1. \tag{44}$$

Denoting $w = \delta_{\hat{t}} v_N^n$ in (25), we obtain

$$i(\delta_{\hat{t}} v_N^n, \delta_{\hat{t}} v_N^n) - \gamma B(\tilde{v}_N^n, \delta_{\hat{t}} v_N^n) + \left(\left(\kappa\left|v_N^n\right|^2 + \rho\left|u_N^n\right|^2\right)\tilde{v}_N^n, \delta_{\hat{t}} v_N^n\right) \\ + \beta(\tilde{v}_N^n, \delta_{\hat{t}} v_N^n) + \varrho(u_N^n, \delta_{\hat{t}} v_N^n) = 0. \tag{45}$$

Analogously, taking the real part of the above equation yields

$$\begin{aligned}
&\gamma\big(|v_N^{n+1}|_{\frac{\alpha}{2}}^2 - |v_N^{n-1}|_{\frac{\alpha}{2}}^2\big)\\
&\quad= \int_\Omega \big(\kappa|v_N^n|^2 + \rho|u_N^n|^2\big)\big(|v_N^{n+1}|^2 - |v_N^{n-1}|^2\big)\,dx + \beta\big(\|v_N^{n+1}\|^2 - \|v_N^{n-1}\|^2\big)\\
&\qquad+ 2\varrho\,\mathrm{Re}\int_\Omega \big(u_N^n\bar{v}_N^{n+1} - v_N^{n-1}\bar{u}_N^n\big)\,dx,\quad 1\le n\le M-1.
\end{aligned}\tag{46}$$

It is easy to get from (44) and (46)

$$\begin{aligned}
&\gamma\big(|u_N^{n+1}|_{\frac{\alpha}{2}}^2 + |v_N^{n+1}|_{\frac{\alpha}{2}}^2 + |u_N^n|_{\frac{\alpha}{2}}^2 + |v_N^n|_{\frac{\alpha}{2}}^2\big) - \kappa\int_\Omega \big(|u_N^n|^2|u_N^{n+1}|^2 + |v_N^n|^2|v_N^{n+1}|^2\big)\,dx\\
&\qquad- \rho\int_\Omega \big(|u_N^n|^2|v_N^{n+1}|^2 + |v_N^n|^2|u_N^{n+1}|^2\big)\,dx - \beta\big(\|u_N^{n+1}\|^2 + \|v_N^{n+1}\|^2 + \|u_N^n\|^2 + \|v_N^n\|^2\big)\\
&\qquad- 2\varrho\,\mathrm{Re}\int_\Omega \big(u_N^n\bar{v}_N^{n+1} + v_N^n\bar{u}_N^{n+1}\big)\,dx\\
&\quad= \gamma\big(|u_N^n|_{\frac{\alpha}{2}}^2 + |v_N^n|_{\frac{\alpha}{2}}^2 + |u_N^{n-1}|_{\frac{\alpha}{2}}^2 + |v_N^{n-1}|_{\frac{\alpha}{2}}^2\big) - \kappa\int_\Omega \big(|u_N^n|^2|u_N^{n-1}|^2 + |v_N^n|^2|v_N^{n-1}|^2\big)\,dx\\
&\qquad- \rho\int_\Omega \big(|u_N^n|^2|v_N^{n-1}|^2 + |v_N^n|^2|u_N^{n-1}|^2\big)\,dx - \beta\big(\|u_N^n\|^2 + \|v_N^n\|^2 + \|u_N^{n-1}\|^2 + \|v_N^{n-1}\|^2\big)\\
&\qquad- 2\varrho\,\mathrm{Re}\int_\Omega \big(u_N^{n-1}\bar{v}_N^n + v_N^{n-1}\bar{u}_N^n\big)\,dx,\quad 1\le n\le M-1.
\end{aligned}\tag{47}$$

Noticing the definition of E^n, it follows from (47) that $E^n = E^{n-1}$ for $1\le n\le M-1$, which further implies that (30) holds. Therefore, we complete the proof. □

4.2 A prior bound

Lemma 4 ([44]) *If* $\frac{1}{2} - \frac{1}{p} < \alpha \le 1$ *and* $2\le p\le\infty$, *then there exists a positive constant* C_α *such that*

$$\|v\|_{L^p} \le C_\alpha\|v\|_{H^\alpha}.\tag{48}$$

Lemma 5 ([44]) *If* $0\le\alpha_0\le\alpha\le 1$, $\frac{1}{2} - \frac{1}{p} < \alpha_0 \le 1$ *and* $2\le p\le\infty$, *there exists a constant* $C_{\alpha_0} > 0$ *such that*

$$\|v\|_{L^p} \le C_{\alpha_0}\|v\|_{H^\alpha}^{\frac{\alpha_0}{\alpha}}\|v\|^{1-\frac{\alpha_0}{\alpha}}.\tag{49}$$

Based on the discrete mass- and energy-conservation laws, we can establish a prior bound for the numerical solutions of the scheme (24)–(28) in both L^2- and L^∞-norms.

Theorem 2 *The solutions of the fully discrete spectral scheme* (24)–(28) *are bounded in the sense that*

$$\|u_N^n\| \le C,\qquad \|v_N^n\| \le C,\quad 0\le n\le M,\tag{50}$$

$$\|u_N^n\|_{L^\infty} \le C,\qquad \|v_N^n\|_{L^\infty} \le C,\quad 0\le n\le M.\tag{51}$$

Proof It is easy to deduce that

$$\begin{aligned}
&\tau\varrho\,\mathrm{Im}\int_\Omega\left(v_N^n\bar{u}_N^{n+1}+u_N^n\bar{v}_N^{n+1}\right)dx\\
&\quad=-\tau\varrho\,\mathrm{Im}\int_\Omega\left(\bar{v}_N^n u_N^{n+1}+\bar{u}_N^n v_N^{n+1}\right)dx\\
&\quad\ge-\frac{\tau|\varrho|}{2}\left(\|u_N^{n+1}\|^2+\|u_N^n\|^2+\|v_N^{n+1}\|^2+\|v_N^n\|^2\right).
\end{aligned}\tag{52}$$

Combining with the discrete mass conservation law (29), we have

$$M^0\ge\frac{1-\tau|\varrho|}{2}\left(\|u_N^{n+1}\|^2+\|u_N^n\|^2+\|v_N^{n+1}\|^2+\|v_N^n\|^2\right),\quad 0\le n\le M-1.\tag{53}$$

When $\tau\le\frac{1}{2|\varrho|}$, it follows from (53) that (50) holds.

Noticing the energy-conservation law (30), we have

$$\begin{aligned}
&\kappa\int_\Omega\left(|u_N^n|^2|u_N^{n+1}|^2+|v_N^n|^2|v_N^{n+1}|^2\right)dx+\rho\int_\Omega\left(|u_N^n|^2|v_N^{n+1}|^2+|v_N^n|^2|u_N^{n+1}|^2\right)dx\\
&\qquad+\beta\left(\|u_N^{n+1}\|^2+\|v_N^{n+1}\|^2+\|u_N^n\|^2+\|v_N^n\|^2\right)+2\varrho\,\mathrm{Re}\int_\Omega\left(u_N^n\bar{v}_N^{n+1}+v_N^n\bar{u}_N^{n+1}\right)dx\\
&\quad\le\frac{|\kappa|+|\rho|}{2}\left(\|u_N^n\|_{L^4(\Omega)}^4+\|u_N^{n+1}\|_{L^4(\Omega)}^4+\|v_N^n\|_{L^4(\Omega)}^4+\|v_N^{n+1}\|_{L^4(\Omega)}^4\right)\\
&\qquad+\left(|\beta|+|\varrho|\right)\left(\|u_N^{n+1}\|^2+\|v_N^{n+1}\|^2+\|u_N^n\|^2+\|v_N^n\|^2\right)\\
&\quad\le\frac{(|\kappa|+|\rho|)C_{\alpha_0}}{2}\Big(\|u_N^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}\|u_N^n\|^{4-\frac{8\alpha_0}{\alpha}}+\|u_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}\|u_N^{n+1}\|^{4-\frac{8\alpha_0}{\alpha}}\\
&\qquad+\|v_N^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}\|v_N^n\|^{4-\frac{8\alpha_0}{\alpha}}+\|v_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}\|v_N^{n+1}\|^{4-\frac{8\alpha_0}{\alpha}}\Big)+\hat{C}\\
&\quad\le C\Big(\|u_N^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}+\|u_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}+\|v_N^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}+\|v_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}\Big)+\hat{C},
\end{aligned}\tag{54}$$

where the Cauchy–Schwartz inequality, (50) and Lemma 5 have been used in deriving the above inequalities. Since the semi-norm $|\cdot|_{\frac{\alpha}{2}}$ is equivalent to the semi-norm $|\cdot|_{H^{\frac{\alpha}{2}}}$, and noticing Lemma 2, it follows that there exists a positive constant C_1 such that

$$\begin{aligned}
&|u_N^n|_{\frac{\alpha}{2}}^2+|u_N^{n+1}|_{\frac{\alpha}{2}}^2+|v_N^n|_{\frac{\alpha}{2}}^2+|v_N^{n+1}|_{\frac{\alpha}{2}}^2\\
&\quad\ge C_1\left(\|u_N^n\|_{H^{\frac{\alpha}{2}}}^2+\|u_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^2+\|v_N^n\|_{H^{\frac{\alpha}{2}}}^2+\|v_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^2\right).
\end{aligned}\tag{55}$$

In view of (30), (54) and (55), we obtain

$$\begin{aligned}
E^1&\le C_1\gamma\left(\|u_N^n\|_{H^{\frac{\alpha}{2}}}^2+\|u_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^2+\|v_N^n\|_{H^{\frac{\alpha}{2}}}^2+\|v_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^2\right)\\
&\quad-C\Big(\|u_N^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}+\|u_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}+\|v_N^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}+\|v_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\alpha_0}{\alpha}}\Big)-\hat{C}.
\end{aligned}\tag{56}$$

Noticing that $1<\alpha\le 2$, when taking $\frac{1}{4}<\alpha_0<\frac{\alpha}{4}$, it follows from (56) that $E^1\to+\infty$ if $\|u_N^n\|_{H^{\frac{\alpha}{2}}}^2+\|u_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^2+\|v_N^n\|_{H^{\frac{\alpha}{2}}}^2+\|v_N^{n+1}\|_{H^{\frac{\alpha}{2}}}^2\to+\infty$. However, we can conclude that E^1 is bounded by the discrete conservation law (30). It will lead to a contradiction. Therefore, we can deduce that

$$\left\|u_N^n\right\|_{H^{\frac{\alpha}{2}}}^2+\left\|u_N^{n+1}\right\|_{H^{\frac{\alpha}{2}}}^2+\left\|v_N^n\right\|_{H^{\frac{\alpha}{2}}}^2+\left\|v_N^{n+1}\right\|_{H^{\frac{\alpha}{2}}}^2\leq C. \tag{57}$$

According Lemma 4, we can further deduce from (57) that (51) holds, which completes the proof. □

4.3 Convergence analysis

Now we turn to discuss the convergence analysis of the discrete spectral scheme (24)–(28). To this end, we first introduce the projection operator $\Pi_N^{\frac{\alpha}{2},0}:H_0^{\frac{\alpha}{2}}(\Omega)\rightarrow V_N^0$, which satisfies

$$B\bigl(v-\Pi_N^{\frac{\alpha}{2},0}v,w\bigr)=0,\quad \forall w\in V_N^0, v\in H_0^{\frac{\alpha}{2}}(\Omega). \tag{58}$$

The error estimate of the projection operator $\Pi_N^{\frac{\alpha}{2},0}$ is given in the following lemma.

Lemma 6 ([25]) *Let $v\in H_0^{\frac{\alpha}{2}}(\Omega)\bigcap H^s(\Omega)$, we have*

$$\left\|v-\Pi_N^{\frac{\alpha}{2},0}v\right\|\leq CN^{-s}\|v\|_{H^s(\Omega)},\quad \alpha\neq\frac{3}{2}, \tag{59}$$

$$\left\|v-\Pi_N^{\frac{\alpha}{2},0}v\right\|\leq CN^{\sigma-s}\|v\|_{H^s(\Omega)},\quad \alpha=\frac{3}{2},\sigma\in(0,1/2). \tag{60}$$

Lemma 7 ([45]) *For any complex functions V, W, v and w, we have*

$$\bigl||V|^2W-|v|^2w\bigr|\leq\bigl(\max\bigl\{|V|,|W|,|v|,|w|\bigr\}\bigr)^2\bigl(2|V-v|+|W-w|\bigr).$$

Lemma 8 (Grönwall inequality [46]) *Suppose that $\{g_l|l\geq 0\}$ is a nonnegative sequence, $\beta>0$, and the sequence $\{\varepsilon_l|l\geq 0\}$ satisfies*

$$\varepsilon_n\leq\beta+\sum_{l=0}^{n-1}p_l+\sum_{l=0}^{n-1}g_l\varepsilon_l,\quad n\geq 1.$$

If $p_l\geq 0$ for any $l\geq 0$, $\varepsilon_0\leq\beta$, then we have

$$\varepsilon_n\leq\left(\beta+\sum_{l=0}^{n-1}p_l\right)\exp\left(\sum_{l=0}^{n-1}g_l\right).$$

For notation convenience, let $u^{\frac{1}{2}}:=u(x,\frac{1}{2})$ and $v^{\frac{1}{2}}:=v(x,\frac{1}{2})$, and we also use u^n and v^n to represent the analytical solutions $u(x,t_n)$ and $u(x,t_n)$, respectively. In view of (24) and (25), the exact solutions u^n and v^n satisfy the equations

$$\begin{aligned}&i\bigl(\delta_{\hat{t}}u^n,w\bigr)-\gamma B\bigl(\tilde{u}^n,w\bigr)+\bigl(\bigl(\kappa\left|u^n\right|^2+\rho\left|v^n\right|^2\bigr)\tilde{u}^n,w\bigr)+\beta\bigl(\tilde{u}^n,w\bigr)\\&\quad+\varrho\bigl(v^n,w\bigr)=\bigl(R_u^n,w\bigr),\quad \forall w\in V_N^0,1\leq n\leq M-1,\end{aligned} \tag{61}$$

$$
\begin{aligned}
& i\big(\delta_{\hat{t}} v^n, w\big) - \gamma B\big(\tilde{v}_N^n, w\big) + \big(\big(\kappa\big|v^n\big|^2 + \rho\big|u^n\big|^2\big)\tilde{v}^n, w\big) + \beta\big(\tilde{v}^n, w\big) \\
&\quad + \varrho\big(u^n, w\big) = \big(R_v^n, w\big), \quad \forall w \in V_N^0, 1 \le n \le M-1,
\end{aligned} \tag{62}
$$

where the local truncation errors R_u^n and R_v^n are defined as

$$
\begin{aligned}
R_u^n :=& i\big(\delta_{\hat{t}} u^n - u_t^n\big) - \gamma(-\triangle)^{\frac{\alpha}{2}}\big(\tilde{u}^n - u^n\big) + \big(\kappa\big|u^n\big|^2 + \rho\big|v^n\big|^2\big)\big(\tilde{u}^n - u^n\big) \\
& + \beta\big(\tilde{u}^n - u^n\big),
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
R_v^n :=& i\big(\delta_{\hat{t}} v^n - v_t^n\big) - \gamma(-\triangle)^{\frac{\alpha}{2}}\big(\tilde{v}^n - v^n\big) + \big(\kappa\big|v^n\big|^2 + \rho\big|u^n\big|^2\big)\big(\tilde{v}^n - v^n\big) \\
& + \beta\big(\tilde{v}^n - v^n\big).
\end{aligned} \tag{64}
$$

From (26) and (27), we can also deduce that

$$
\begin{aligned}
& i\big(\delta_t u^{\frac{1}{2}}, w\big) - \gamma B\big(\hat{u}^{\frac{1}{2}}, w\big) + \frac{1}{2}\big(\big(\kappa\big(\big|u^1\big|^2 + \big|u_N^0\big|^2\big) + \rho\big(\big|v^1\big|^2 + \big|v^0\big|^2\big)\big)\hat{u}^{\frac{1}{2}}, w\big) \\
&\quad + \beta\big(\hat{u}^{\frac{1}{2}}, w\big) + \varrho\big(\hat{v}^{\frac{1}{2}}, w\big) = \big(R_u^0, w\big), \quad \forall w \in V_N^0,
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
& i\big(\delta_t v^{\frac{1}{2}}, w\big) - \gamma B\big(\hat{v}^{\frac{1}{2}}, w\big) + \frac{1}{2}\big(\big(\kappa\big(\big|v^1\big|^2 + \big|v^0\big|^2\big) + \rho\big(\big|u^1\big|^2 + \big|u^0\big|^2\big)\big)\hat{v}^{\frac{1}{2}}, w\big) \\
&\quad + \beta\big(\hat{v}^{\frac{1}{2}}, w\big) + \varrho\big(\hat{u}^{\frac{1}{2}}, w\big) = \big(R_v^0, w\big), \quad \forall w \in V_N^0,
\end{aligned} \tag{66}
$$

where the local truncation errors R_u^0 and R_v^0 are given as

$$
\begin{aligned}
R_u^0 :=& i\big(\delta_t u^{\frac{1}{2}} - u_t^{\frac{1}{2}}\big) - \gamma(-\triangle)^{\frac{\alpha}{2}}\big(\hat{u}^{\frac{1}{2}} - u^{\frac{1}{2}}\big) + \frac{1}{2}\big(\kappa\big(\big|u^1\big|^2 + \big|u_N^0\big|^2\big) \\
& + \rho\big(\big|v^1\big|^2 + \big|v^0\big|^2\big)\big)\hat{u}^{\frac{1}{2}} - \big(\kappa\big|u^{\frac{1}{2}}\big|^2 + \rho\big|v^{\frac{1}{2}}\big|^2\big)u^{\frac{1}{2}} + \beta\big(\hat{u}^{\frac{1}{2}} - u^{\frac{1}{2}}\big) + \varrho\big(\hat{v}^{\frac{1}{2}} - v^{\frac{1}{2}}\big),
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
R_v^0 :=& i\big(\delta_t v^{\frac{1}{2}} - v_t^{\frac{1}{2}}\big) - \gamma(-\triangle)^{\frac{\alpha}{2}}\big(\hat{v}^{\frac{1}{2}} - v^{\frac{1}{2}}\big) + \frac{1}{2}\big(\kappa\big(\big|v^1\big|^2 + \big|v_N^0\big|^2\big) \\
& + \rho\big(\big|u^1\big|^2 + \big|u^0\big|^2\big)\big)\hat{v}^{\frac{1}{2}} - \big(\kappa\big|v^{\frac{1}{2}}\big|^2 + \rho\big|u^{\frac{1}{2}}\big|^2\big)v^{\frac{1}{2}} + \beta\big(\hat{v}^{\frac{1}{2}} - v^{\frac{1}{2}}\big) + \varrho\big(\hat{u}^{\frac{1}{2}} - u^{\frac{1}{2}}\big).
\end{aligned} \tag{68}
$$

By virtue of a Taylor expansion, we can deduce that

$$
\big\|R_u^n\big\| \le C\tau^2, \qquad \big\|R_v^n\big\| \le C\tau^2, \quad 0 \le n \le M-1. \tag{69}
$$

Next, we focus on a rigorous convergence analysis for the spectral scheme (24)–(28).

Theorem 3 *Assume that the analytical solutions of the Schrödinger system (1)–(4) satisfy $u, v \in C^3(0, T; H_0^{\frac{\alpha}{2}}(\Omega) \bigcap H^s(\Omega))$. Then there exists a positive constant τ_0 such that when $\tau < \tau_0$, the solutions of the fully discrete spectral scheme (24)–(28) satisfy*

$$
\max_{1 \le n \le M}\big(\big\|u^n - u_N^n\big\| + \big\|v^n - v_N^n\big\|\big) \le C\big(\tau^2 + N^{-s}\big), \quad \alpha \ne \frac{3}{2}, \tag{70}
$$

$$
\max_{1 \le n \le M}\big(\big\|u^n - u_N^n\big\| + \big\|v^n - v_N^n\big\|\big) \le C\big(\tau^2 + N^{\sigma - s}\big), \quad \alpha = \frac{3}{2}, 0 < \sigma < 1/2, \tag{71}
$$

where C is a positive constant which is independent of τ and N.

Proof We first consider the case of $\alpha \neq \frac{3}{2}$. To derive the convergence result of the spectral scheme (24)–(28), we split the errors into

$$e_u^n := u^n - u_N^n = \left(u^n - \Pi_N^{\frac{\alpha}{2},0}u^n\right) + \left(\Pi_N^{\frac{\alpha}{2},0}u^n - u_N^n\right) := \phi^n + \theta^n, \quad 0 \leq n \leq M, \tag{72}$$

$$e_v^n := v^n - v_N^n = \left(v^n - \Pi_N^{\frac{\alpha}{2},0}v^n\right) + \left(\Pi_N^{\frac{\alpha}{2},0}v^n - v_N^n\right) := \xi^n + \eta^n, \quad 0 \leq n \leq M. \tag{73}$$

Subtracting (24) from (61) and subtracting (25) from (62), we arrive at

$$i\left(\delta_{\hat{t}}e_u^n, w\right) - \gamma B\left(\tilde{e}_u^n, w\right) + \left(G_u^n, w\right) + \beta\left(\tilde{e}_u^n, w\right) + \varrho\left(e_v^n, w\right) = \left(R_u^n, w\right), \quad \forall w \in V_N^0, \tag{74}$$

$$i\left(\delta_{\hat{t}}e_v^n, w\right) - \gamma B\left(\tilde{e}_v^n, w\right) + \left(G_v^n, w\right) + \beta\left(\tilde{e}_v^n, w\right) + \varrho\left(e_u^n, w\right) = \left(R_v^n, w\right), \quad \forall w \in V_N^0, \tag{75}$$

where

$$G_u^n := \kappa\left(\left|u^n\right|^2\tilde{u}^n - \left|u_N^n\right|^2\tilde{u}_N^n\right) + \rho\left(\left|v^n\right|^2\tilde{u}^n - \left|v_N^n\right|^2\tilde{u}_N^n\right), \tag{76}$$

$$G_v^n := \kappa\left(\left|v^n\right|^2\tilde{v}^n - \left|v_N^n\right|^2\tilde{v}_N^n\right) + \rho\left(\left|u^n\right|^2\tilde{v}^n - \left|u_N^n\right|^2\tilde{v}_N^n\right). \tag{77}$$

By virtue of (58), (72) and (73), the above equations (74) and (75) can be rewritten in the following equivalent form:

$$i\left(\delta_{\hat{t}}\theta^n, w\right) - \gamma B\left(\tilde{\theta}^n, w\right) + \left(G_u^n, w\right) + \beta\left(\tilde{\theta}^n, w\right) + \varrho\left(\eta^n, w\right) = \left(\bar{R}_u^n, w\right), \quad \forall w \in V_N^0, \tag{78}$$

$$i\left(\delta_{\hat{t}}\eta^n, w\right) - \gamma B\left(\tilde{\eta}^n, w\right) + \left(G_v^n, w\right) + \beta\left(\tilde{\eta}^n, w\right) + \varrho\left(\theta^n, w\right) = \left(\bar{R}_v^n, w\right), \quad \forall w \in V_N^0, \tag{79}$$

where

$$\bar{R}_u^n := R_u^n - i\delta_{\hat{t}}\phi^n - \beta\tilde{\phi}^n - \varrho\xi^n, \qquad \bar{R}_v^n := R_v^n - i\delta_{\hat{t}}\xi^n - \beta\tilde{\xi}^n - \varrho\phi^n. \tag{80}$$

Analogously, it follows from (26), (27), (65) and (66) that

$$i\left(\delta_t\theta^{\frac{1}{2}}, w\right) - \gamma B\left(\hat{\theta}^{\frac{1}{2}}, w\right) + \left(G_u^{\frac{1}{2}}, w\right) + \beta\left(\hat{\theta}^{\frac{1}{2}}, w\right) + \varrho\left(\hat{\eta}^{\frac{1}{2}}, w\right) = \left(\bar{R}_u^0, w\right), \quad \forall w \in V_N^0, \tag{81}$$

$$i\left(\delta_t\eta^{\frac{1}{2}}, w\right) - \gamma B\left(\hat{\eta}^{\frac{1}{2}}, w\right) + \left(G_v^{\frac{1}{2}}, w\right) + \beta\left(\hat{\eta}^{\frac{1}{2}}, w\right) + \varrho\left(\hat{\theta}^{\frac{1}{2}}, w\right) = \left(\bar{R}_v^0, w\right), \quad \forall w \in V_N^0, \tag{82}$$

where

$$\begin{aligned} G_u^{\frac{1}{2}} :=\ & \frac{\kappa}{2}\left[\left(\left|u^1\right|^2\hat{u}^{\frac{1}{2}} - \left|u_N^1\right|^2\hat{u}_N^{\frac{1}{2}}\right) + \left(\left|u^0\right|^2\hat{u}^{\frac{1}{2}} - \left|u_N^0\right|^2\hat{u}_N^{\frac{1}{2}}\right)\right] \\ & + \frac{\rho}{2}\left[\left(\left|v^1\right|^2\hat{u}^{\frac{1}{2}} - \left|v_N^1\right|^2\hat{u}_N^{\frac{1}{2}}\right) + \left(\left|v^0\right|^2\hat{u}^{\frac{1}{2}} - \left|v_N^0\right|^2\hat{u}_N^{\frac{1}{2}}\right)\right], \end{aligned} \tag{83}$$

$$\begin{aligned} G_v^{\frac{1}{2}} :=\ & \frac{\kappa}{2}\left[\left(\left|v^1\right|^2\hat{v}^{\frac{1}{2}} - \left|v_N^1\right|^2\hat{v}_N^{\frac{1}{2}}\right) + \left(\left|v^0\right|^2\hat{v}^{\frac{1}{2}} - \left|v_N^0\right|^2\hat{v}_N^{\frac{1}{2}}\right)\right] \\ & + \frac{\rho}{2}\left[\left(\left|u^1\right|^2\hat{v}^{\frac{1}{2}} - \left|u_N^1\right|^2\hat{v}_N^{\frac{1}{2}}\right) + \left(\left|u^0\right|^2\hat{v}^{\frac{1}{2}} - \left|u_N^0\right|^2\hat{v}_N^{\frac{1}{2}}\right)\right], \end{aligned} \tag{84}$$

$$\bar{R}_u^0 := R_u^0 - i\delta_t\phi^{\frac{1}{2}} - \beta\hat{\phi}^{\frac{1}{2}} - \varrho\hat{\xi}^{\frac{1}{2}}, \qquad \bar{R}_v^0 := R_v^0 - i\delta_t\xi^{\frac{1}{2}} - \beta\hat{\xi}^{\frac{1}{2}} - \varrho\hat{\phi}^{\frac{1}{2}}. \tag{85}$$

Thanks to Lemma 6 and (69), we obtain

$$\|\bar{R}_u^n\| + \|\bar{R}_v^n\| \le C(\tau^2 + N^{-s}), \quad 0 \le n \le M-1. \tag{86}$$

Now taking $w = \hat{\theta}^{\frac{1}{2}}$ in (81) and $w = \hat{\eta}^{\frac{1}{2}}$ in (82), and then considering the imaginary part of the resulting equations, we have

$$\frac{1}{2\tau}\left(\|\theta^1\|^2 - \|\theta^0\|^2\right) + \mathrm{Im}\left(G_u^{\frac{1}{2}}, \hat{\theta}^{\frac{1}{2}}\right) + \varrho\,\mathrm{Im}\left(\hat{\eta}^{\frac{1}{2}}, \hat{\theta}^{\frac{1}{2}}\right) = \mathrm{Im}\left(\bar{R}_u^0, \hat{\theta}^{\frac{1}{2}}\right), \tag{87}$$

$$\frac{1}{2\tau}\left(\|\eta^1\|^2 - \|\eta^0\|^2\right) + \mathrm{Im}\left(G_v^{\frac{1}{2}}, \hat{\eta}^{\frac{1}{2}}\right) + \varrho\,\mathrm{Im}\left(\hat{\theta}^{\frac{1}{2}}, \hat{\eta}^{\frac{1}{2}}\right) = \mathrm{Im}\left(\bar{R}_v^0, \hat{\eta}^{\frac{1}{2}}\right). \tag{88}$$

It is obvious that $\mathrm{Im}(\hat{\eta}^{\frac{1}{2}}, \hat{\theta}^{\frac{1}{2}}) + \mathrm{Im}(\hat{\theta}^{\frac{1}{2}}, \hat{\eta}^{\frac{1}{2}}) = 0$, then adding (87) and (88) leads to

$$\begin{aligned} \|\theta^1\|^2 + \|\eta^1\|^2 = {}& \|\theta^0\|^2 + \|\eta^0\|^2 - 2\tau\,\mathrm{Im}\left(G_u^{\frac{1}{2}}, \hat{\theta}^{\frac{1}{2}}\right) - 2\tau\,\mathrm{Im}\left(G_v^{\frac{1}{2}}, \hat{\eta}^{\frac{1}{2}}\right) \\ & + 2\tau\,\mathrm{Im}\left[\left(\bar{R}_u^0, \hat{\theta}^{\frac{1}{2}}\right) + \left(\bar{R}_v^0, \hat{\eta}^{\frac{1}{2}}\right)\right]. \end{aligned} \tag{89}$$

Noticing the definition of $G_u^{\frac{1}{2}}$, and using Lemma 7 as well as Theorem 2, we observe that

$$\begin{aligned} \left|G_u^{\frac{1}{2}}\right| \le {}& \frac{|\kappa|}{2}\left(\left||u^1|^2\hat{u}^{\frac{1}{2}} - |u_N^1|^2\hat{u}_N^{\frac{1}{2}}\right| + \left||u^0|^2\hat{u}^{\frac{1}{2}} - |u_N^0|^2\hat{u}_N^{\frac{1}{2}}\right|\right) \\ & + \frac{|\rho|}{2}\left(\left||v^1|^2\hat{u}^{\frac{1}{2}} - |v_N^1|^2\hat{u}_N^{\frac{1}{2}}\right| + \left||v^0|^2\hat{u}^{\frac{1}{2}} - |v_N^0|^2\hat{u}_N^{\frac{1}{2}}\right|\right) \\ \le {}& C_4\left(|\phi^1| + |\theta^1| + |\phi^0| + |\theta^0| + |\xi^1| + |\eta^1| + |\xi^0| + |\eta^0|\right), \end{aligned} \tag{90}$$

where C_4 denotes a positive constant. Following a similar analysis, we also conclude that

$$\left|G_v^{\frac{1}{2}}\right| \le C_4\left(|\phi^1| + |\theta^1| + |\phi^0| + |\theta^0| + |\xi^1| + |\eta^1| + |\xi^0| + |\eta^0|\right). \tag{91}$$

Therefore, we further deduce that

$$\begin{aligned} & \left|\mathrm{Im}\left(G_u^{\frac{1}{2}}, \hat{\theta}^{\frac{1}{2}}\right)\right| \\ & \quad \le \int_\Omega \left|G_u^{\frac{1}{2}}\right| \cdot \left|\hat{\theta}^{\frac{1}{2}}\right| dx \\ & \quad \le C_4 \int_\Omega \left(|\phi^1| + |\theta^1| + |\phi^0| + |\theta^0| + |\xi^1| + |\eta^1| + |\xi^0| + |\eta^0|\right)\left|\hat{\theta}^{\frac{1}{2}}\right| dx \\ & \quad \le \frac{C_4}{2}\left(\|\phi^1\|^2 + \|\phi^0\|^2 + \|\xi^1\|^2 + \|\xi^0\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + 5\|\theta^1\|^2 + 5\|\theta^0\|^2\right). \end{aligned} \tag{92}$$

Analogously, we find that

$$\begin{aligned} \left|\mathrm{Im}\left(G_v^{\frac{1}{2}}, \hat{\eta}^{\frac{1}{2}}\right)\right| \le {}& \frac{C_4}{2}\left(\|\phi^1\|^2 + \|\phi^0\|^2 + \|\xi^1\|^2 + \|\xi^0\|^2 + \|\theta^1\|^2 \right. \\ & \left. + \|\theta^0\|^2 + 5\|\eta^1\|^2 + 5\|\eta^0\|^2\right). \end{aligned} \tag{93}$$

Obviously, we can also deduce that

$$\begin{aligned}
&\operatorname{Im}\bigl[\bigl(\bar{R}_u^0,\hat{\theta}^{\frac{1}{2}}\bigr)+\bigl(\bar{R}_v^0,\hat{\eta}^{\frac{1}{2}}\bigr)\bigr]\\
&\quad\le\frac{1}{2}\bigl(\|\bar{R}_u^0\|\cdot\|\theta^1+\theta^0\|+\|\bar{R}_v^0\|\cdot\|\eta^1+\eta^0\|\bigr)\\
&\quad\le\frac{1}{2}\biggl[\|\bar{R}_u^0\|^2+\|\bar{R}_v^0\|^2+\frac{1}{2}\bigl(\|\theta^1\|^2+\|\theta^0\|^2+\|\eta^1\|^2+\|\eta^0\|^2\bigr)\biggr].
\end{aligned}\tag{94}$$

Substituting (92)–(94) into (89), we have

$$\begin{aligned}
&\|\theta^1\|^2+\|\eta^1\|^2\\
&\quad=\|\theta^0\|^2+\|\eta^0\|^2+\tau\left(6C_4+\frac{1}{2}\right)\bigl(\|\theta^1\|^2+\|\theta^0\|^2+\|\eta^1\|^2+\|\eta^0\|^2\bigr)\\
&\qquad+2C_4\tau\bigl(\|\phi^1\|^2+\|\phi^0\|^2+\|\xi^1\|^2+\|\xi^0\|^2\bigr)+\tau\bigl(\|\bar{R}_u^0\|^2+\|\bar{R}_v^0\|^2\bigr).
\end{aligned}\tag{95}$$

This, combined with Lemma 6 and (86), gives

$$\begin{aligned}
&\left(1-\tau\left(6C_4+\frac{1}{2}\right)\right)\bigl(\|\theta^1\|^2+\|\eta^1\|^2\bigr)\\
&\quad\le\left(1+\tau\left(6C_4+\frac{1}{2}\right)\right)\bigl(\|\theta^0\|^2+\|\eta^0\|^2\bigr)+C\tau\bigl(\tau^2+N^{-s}\bigr).
\end{aligned}\tag{96}$$

Moreover, one easily gets

$$\|\theta^0\|=\bigl\|\Pi_N^{\frac{\alpha}{2},0}u^0-I_Nu^0\bigr\|\le\bigl\|\Pi_N^{\frac{\alpha}{2},0}u^0-u^0\bigr\|+\bigl\|u^0-I_Nu^0\bigr\|\le CN^{-s},\tag{97}$$

$$\|\eta^0\|=\bigl\|\Pi_N^{\frac{\alpha}{2},0}v^0-I_Nv^0\bigr\|\le\bigl\|\Pi_N^{\frac{\alpha}{2},0}v^0-v^0\bigr\|+\bigl\|v^0-I_Nv^0\bigr\|\le CN^{-s}.\tag{98}$$

Therefore, when the time step τ in (96) is chosen sufficiently small such that $\tau\le\frac{1}{(12C_4+1)}$, it follows from (96)–(98) that

$$\|\theta^1\|^2+\|\eta^1\|^2\le C\bigl(\tau^2+N^{-s}\bigr).\tag{99}$$

This together with Lemma 6 and the triangle inequality implies that (70) holds for $n=1$.

By mathematical induction, we assume that (70) is valid for $1\le n\le m$. Now we turn to a proof that the stated conclusion still holds for $n=m+1$. To this end, taking $w=\tilde{\theta}^n$ in (78) and $w=\tilde{\eta}^n$ in (79), respectively, and considering the imaginary part of the resulting equations, we have

$$\frac{1}{4\tau}\bigl(\|\theta^{n+1}\|^2-\|\theta^{n-1}\|^2\bigr)+\operatorname{Im}\bigl(G_u^n,\tilde{\theta}^n\bigr)+\varrho\operatorname{Im}\bigl(\eta^n,\tilde{\theta}^n\bigr)=\operatorname{Im}\bigl(\bar{R}_u^n,\tilde{\theta}^n\bigr),\tag{100}$$

$$\frac{1}{4\tau}\bigl(\|\eta^{n+1}\|^2-\|\eta^{n-1}\|^2\bigr)+\operatorname{Im}\bigl(G_v^n,\tilde{\eta}^n\bigr)+\varrho\operatorname{Im}\bigl(\theta^n,\tilde{\eta}^n\bigr)=\operatorname{Im}\bigl(\bar{R}_v^n,\tilde{\eta}^n\bigr).\tag{101}$$

Combining (100) and (101) gives

$$\begin{aligned}
&\|\theta^{n+1}\|^2+\|\eta^{n+1}\|^2\\
&\quad=\|\theta^{n-1}\|^2+\|\eta^{n-1}\|^2-4\tau\,\mathrm{Im}\big(G_u^n,\tilde{\theta}^n\big)-4\tau\,\mathrm{Im}\big(G_v^n,\tilde{\eta}^n\big)\\
&\qquad-4\tau\varrho\,\mathrm{Im}\big(\eta^n,\tilde{\theta}^n\big)-4\tau\,\mathrm{Im}\,\varrho\big(\theta^n,\tilde{\eta}^n\big)+4\tau\,\mathrm{Im}\big(\bar{R}_u^n,\tilde{\theta}^n\big)+4\tau\,\mathrm{Im}\big(\bar{R}_v^n,\tilde{\eta}^n\big).
\end{aligned}\tag{102}$$

In view of the definition of G_u^n, Lemma 7 and Theorem 2,

$$\begin{aligned}
\big|G_u^n\big|&\le|\kappa|\big(\big||u^n|^2\tilde{u}^n-|u_N^n|^2\tilde{u}_N^n\big|\big)+|\rho|\big(\big||v^n|^2\tilde{u}^n-|v_N^n|^2\tilde{u}_N^n\big|\big)\\
&\le C_4\big(\big|\phi^{n+1}\big|+\big|\theta^{n+1}\big|+\big|\phi^n\big|+\big|\theta^n\big|+\big|\xi^n\big|+\big|\eta^n\big|+\big|\phi^{n-1}\big|+\big|\theta^{n-1}\big|\big)
\end{aligned}\tag{103}$$

and

$$\big|G_v^n\big|\le C_4\big(\big|\xi^{n+1}\big|+\big|\eta^{n+1}\big|+\big|\phi^n\big|+\big|\theta^n\big|+\big|\xi^n\big|+\big|\eta^n\big|+\big|\xi^{n-1}\big|+\big|\eta^{n-1}\big|\big).\tag{104}$$

Hence, we furthermore obtain

$$\begin{aligned}
&\big|\mathrm{Im}\big(G_u^n,\tilde{\theta}^n\big)\big|\\
&\quad\le\int_\Omega\big|G_u^n\big|\cdot\big|\tilde{\theta}^n\big|\,dx\\
&\quad\le C_4\int_\Omega\big(\big|\phi^{n+1}\big|+\big|\theta^{n+1}\big|+\big|\phi^n\big|+\big|\theta^n\big|+\big|\xi^n\big|+\big|\eta^n\big|+\big|\phi^{n-1}\big|+\big|\theta^{n-1}\big|\big)\big|\tilde{\theta}^n\big|\,dx\\
&\quad\le\frac{C_4}{2}\big(\|\phi^{n+1}\|^2+5\|\theta^{n+1}\|^2+\|\phi^n\|^2+\|\theta^n\|^2+\|\xi^n\|^2+\|\eta^n\|^2\\
&\qquad+\|\phi^{n-1}\|^2+5\|\theta^{n-1}\|^2\big)
\end{aligned}\tag{105}$$

and

$$\begin{aligned}
\big|\mathrm{Im}\big(G_v^n,\tilde{\eta}^n\big)\big|\le{}&\frac{C_4}{2}\big(\|\xi^{n+1}\|^2+5\|\eta^{n+1}\|^2+\|\phi^n\|^2+\|\theta^n\|^2+\|\xi^n\|^2\\
&+\|\eta^n\|^2+\|\xi^{n-1}\|^2+5\|\eta^{n-1}\|^2\big).
\end{aligned}\tag{106}$$

Also, we can conclude that

$$\begin{aligned}
&\mathrm{Im}\big(\eta^n,\tilde{\theta}^n\big)+\mathrm{Im}\,\varrho\big(\theta^n,\tilde{\eta}^n\big)\\
&\quad\le\frac{1}{2}\bigg[\|\eta^n\|^2+\|\theta^n\|^2+\frac{1}{2}\big(\|\theta^{n+1}\|^2+\|\theta^{n-1}\|^2+\|\eta^{n+1}\|^2+\|\eta^{n-1}\|^2\big)\bigg]
\end{aligned}\tag{107}$$

and

$$\begin{aligned}
&\mathrm{Im}\big(\bar{R}_u^n,\tilde{\theta}^n\big)+\mathrm{Im}\big(\bar{R}_v^n,\tilde{\eta}^n\big)\\
&\quad\le\frac{1}{2}\bigg[\|\bar{R}_u^n\|^2+\|\bar{R}_v^n\|^2+\frac{1}{2}\big(\|\theta^{n+1}\|^2+\|\theta^{n-1}\|^2+\|\eta^{n+1}\|^2+\|\eta^{n-1}\|^2\big)\bigg].
\end{aligned}\tag{108}$$

Substituting (105)–(108) into (102), we obtain

$$\begin{aligned}
&\|\theta^{n+1}\|^2 + \|\eta^{n+1}\|^2 \\
&\quad \le \|\theta^{n-1}\|^2 + \|\eta^{n-1}\|^2 + 2\tau\big(\|\bar{R}_u^n\|^2 + \|\bar{R}_v^n\|^2\big) \\
&\qquad + 2C_4\tau\big(\|\phi^{n+1}\|^2 + \|\xi^{n+1}\|^2 + 2\|\phi^n\|^2 + 2\|\xi^n\|^2 + \|\phi^{n-1}\|^2 + \|\xi^{n-1}\|^2\big) \\
&\qquad + \big(10C_4 + 2|\varrho| + 1\big)\tau\big(\|\theta^{n+1}\|^2 + \|\eta^{n+1}\|^2 + \|\theta^n\|^2 \\
&\qquad + \|\eta^n\|^2 + \|\theta^{n-1}\|^2 + \|\eta^{n-1}\|^2\big).
\end{aligned} \tag{109}$$

By virtue of Lemma 6 and (86), it follows from (109) that

$$\begin{aligned}
&\|\theta^{n+1}\|^2 + \|\eta^{n+1}\|^2 \\
&\quad \le \|\theta^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \big(10C_4 + 2|\varrho| + 1\big)\tau\big(\|\theta^{n+1}\|^2 + \|\eta^{n+1}\|^2 \qquad (110)\\
&\qquad + \|\theta^n\|^2 + \|\eta^n\|^2 + \|\theta^{n-1}\|^2 + \|\eta^{n-1}\|^2\big) + C\tau\big(\tau^2 + N^{-s}\big). \qquad (111)
\end{aligned}$$

Summing (111) for n from 1 to m leads to

$$\begin{aligned}
&\|\theta^{m+1}\|^2 + \|\eta^{m+1}\|^2 + \|\theta^m\|^2 + \|\eta^m\|^2 \\
&\quad \le \big(10C_4 + 2|\varrho| + 1\big)\tau\big(\|\theta^{m+1}\|^2 + \|\eta^{m+1}\|^2\big) + \|\theta^1\|^2 + \|\eta^1\|^2 \\
&\qquad + \big(1 + \big(10C_4 + 2|\varrho| + 1\big)\tau\big)\big(\|\theta^0\|^2 + \|\eta^0\|^2\big) \\
&\qquad + 3\big(10C_4 + 2|\varrho| + 1\big)\tau\sum_{n=1}^{m}\big(\|\theta^n\|^2 + \|\eta^n\|^2\big) + mC\tau\big(\tau^2 + N^{-s}\big).
\end{aligned} \tag{112}$$

This combined with (97)–(99) gives

$$\begin{aligned}
&\|\theta^{m+1}\|^2 + \|\eta^{m+1}\|^2 \\
&\quad \le \big(10C_4 + 2|\varrho| + 1\big)\tau\big(\|\theta^{m+1}\|^2 + \|\eta^{m+1}\|^2\big) + C\big(\tau^2 + N^{-s}\big) \\
&\qquad + 3\big(10C_4 + 2|\varrho| + 1\big)\tau\sum_{n=1}^{m}\big(\|\theta^l\|^2 + \|\eta^l\|^2\big) + mC\tau\big(\tau^2 + N^{-s}\big).
\end{aligned} \tag{113}$$

Consequently, when $\tau \le \frac{1}{2(10C_4+2|\varrho|+1)}$, from Lemma 8

$$\|\theta^{m+1}\|^2 + \|\eta^{m+1}\|^2 \le C(1 + mC\tau)\exp(mC\tau)\big(\tau^2 + N^{-s}\big), \tag{114}$$

which further indicates that

$$\|e_u^{m+1}\| + \|e_v^{m+1}\| \le C\big(\tau^2 + N^{-s}\big), \tag{115}$$

where Lemma 6 and the triangle inequality have been used. It means that the conclusion (70) still holds for $n = m + 1$, which completes the proof of Theorem 3 for $\alpha \ne \frac{3}{2}$.

For the case of $\alpha = \frac{3}{2}$, the stated result (71) can be obtained by a similar analysis. Hence, we have completed the proof of Theorem 3. □

5 Numerical experiment

In this section, we present some numerical results to confirm our theoretical analysis of the spectral scheme (24)–(28).

Example 1 Consider the following strongly coupled fractional Schrödinger system:

$$iu_t - (-\triangle)^{\frac{\alpha}{2}} u + 2\big(|u|^2 + |v|^2\big)u + u + v = 0, \quad x \in \Omega, 0 < t \le T, \tag{116}$$

$$iv_t - (-\triangle)^{\frac{\alpha}{2}} v + 2\big(|v|^2 + |u|^2\big)v + v + u = 0, \quad x \in \Omega, 0 < t \le T, \tag{117}$$

subject to the initial conditions

$$\begin{aligned} &u(x,0) = \operatorname{sech}(x+10)\exp(3ix), \\ &v(x,0) = \operatorname{sech}(x-10)\exp(-3ix), \quad x \in \Omega, \end{aligned} \tag{118}$$

and the homogeneous boundary conditions

$$u(x,t) = 0, \qquad v(x,t) = 0, \quad x \in \mathbb{R}\backslash\Omega, 0 \le t \le T, \tag{119}$$

where the computation domain is chosen sufficiently large as $\Omega = (-25, 25)$.

The first objective is to check the convergence behavior of the spectral scheme (24)–(28). Since the analytical solutions of the system (116)–(119) are difficult to find, we take the numerical solutions computed by fixed $\tau = 10^{-5}$ and $N = 512$ as the "exact" solutions. When fixing $N = 512$, we present the L^2-errors with different time steps in Fig. 1. It can be

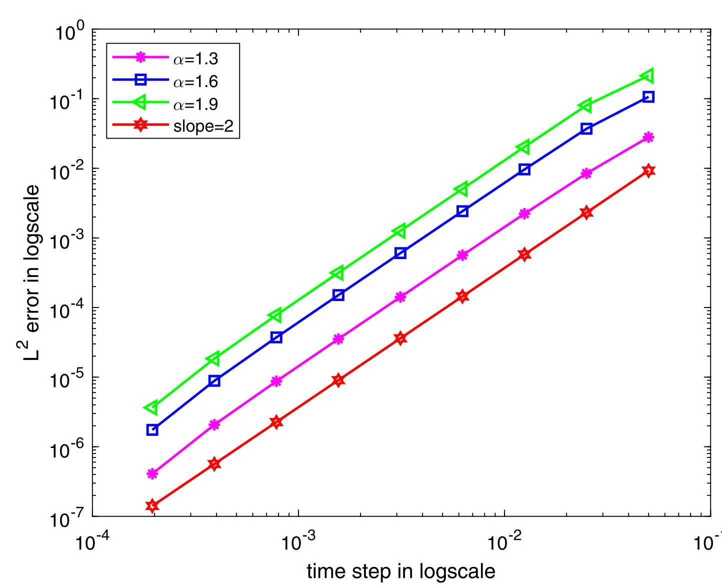

Figure 1 The L^2-error versus time step with $N = 512$. It shows that the derived spectral scheme has second-order temporal accuracy

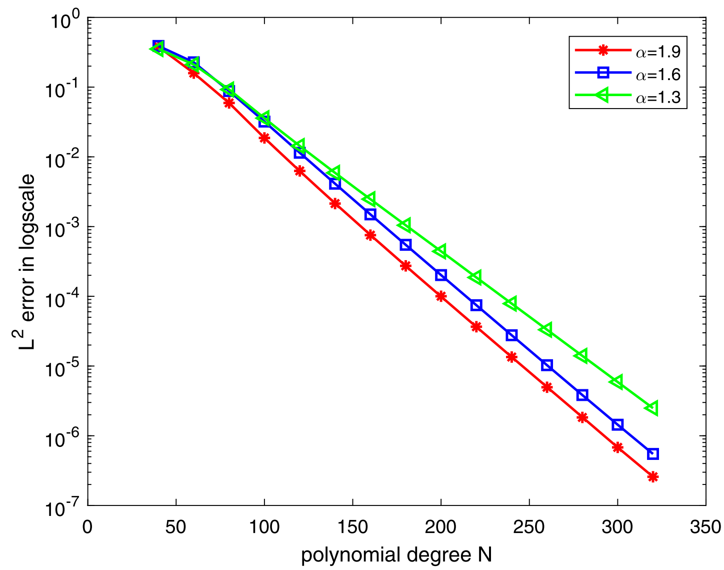

Figure 2 The L^2-error versus N with $\tau = 10^{-5}$. It shows that the derived spectral scheme has spectral accuracy in space

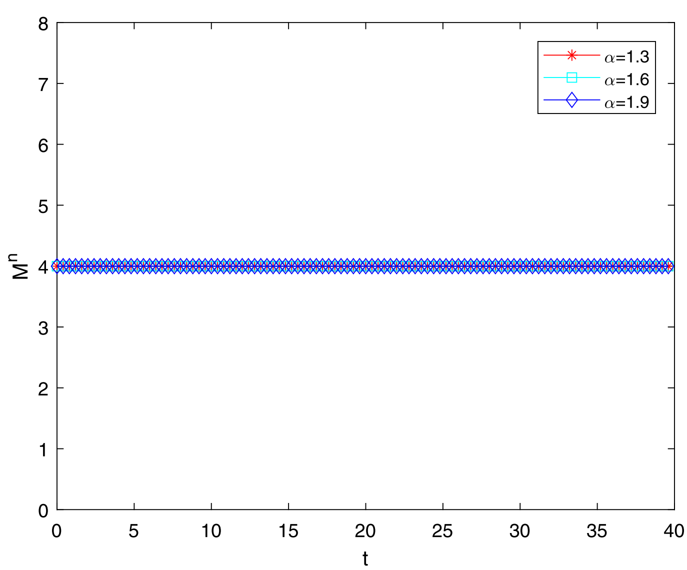

Figure 3 The values of the mass M^n for different α with time evolution. It shows that the spectral scheme preserves the total discrete mass very well and the values of the mass M^n are independent of α

observed that the derived spectral scheme has second-order temporal accuracy. Moreover, we fix $\tau = 10^{-5}$ and plot the L^2-errors with the change of N in Fig. 2. It shows that the errors are exponentially decaying with N increases, and this indicates the spectral accuracy in space.

Now we turn to a validation of the discrete conservation laws of Theorem 1. To the end, we take $\tau = 0.001$ and $N = 256$ and depict the mass M^n and the energy E^n as well as corresponding error functions for different α in Figs. 3–6. It can be found that the spec-

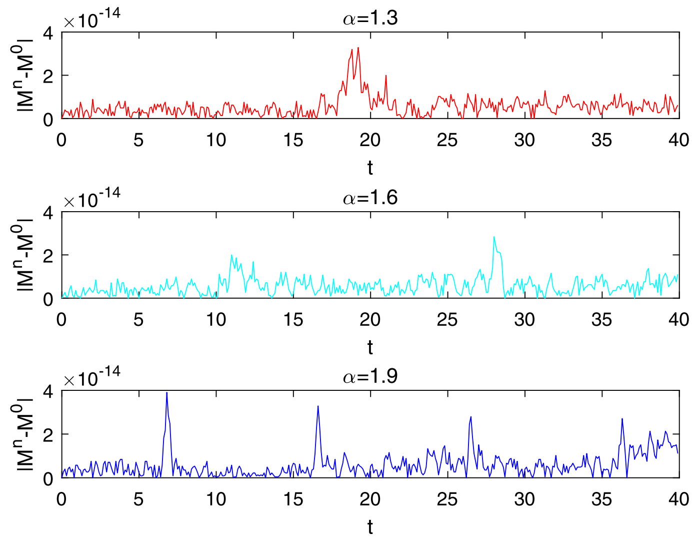

Figure 4 The values of the error function $e_M^n := |M^n - M^0|$ for different α with time evolution. It shows that the spectral scheme preserves the total discrete mass very well

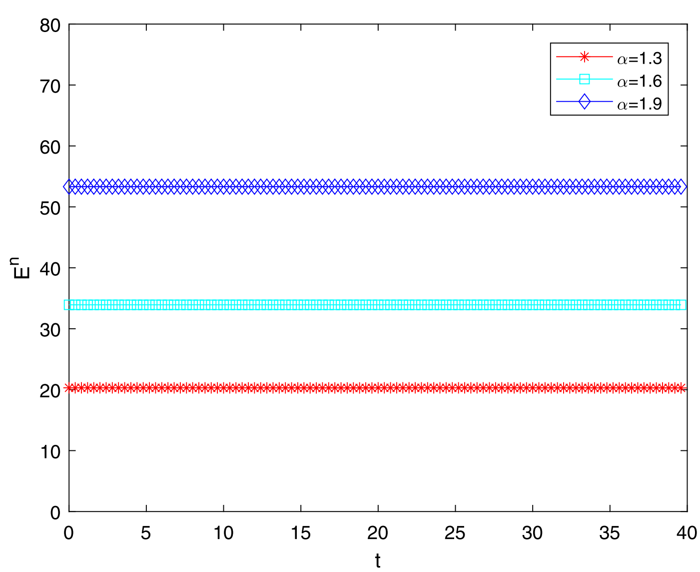

Figure 5 The values of the energy E^n for different α with time evolution. It shows that the spectral scheme preserves the total discrete energy very well and the values of the energy E^n are dependent of α

tral scheme preserves the total discrete mass and energy very well. Moreover, it can be observed that the values of the mass M^n are independent of α, while the values of the energy E^n are dependent of α. These numerical results are all in line with our theoretical analysis. Finally, we plot the graphs of the numerical solutions for $\alpha = 1.6$ and $\alpha = 1.95$ in Figs. 7 and 8. It shows that the value of α affects the shape of wave functions dramatically.

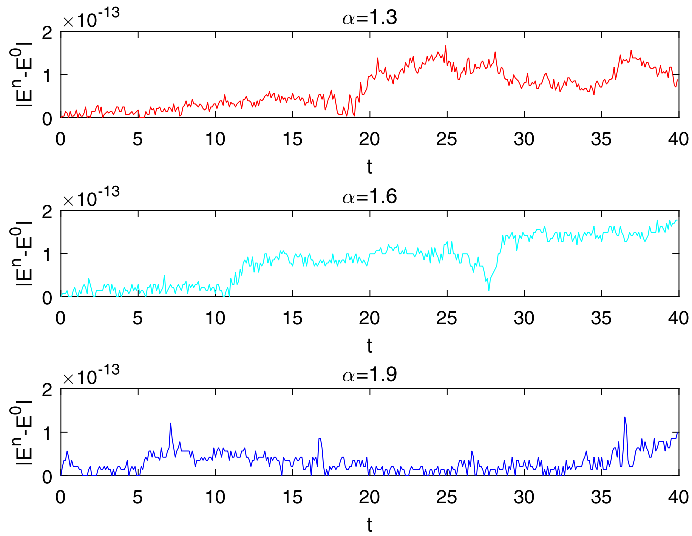

Figure 6 The values of the error function $e_E^n := |E^n - E^1|$ for different $\boldsymbol{\alpha}$ with time evolution. It shows that the spectral scheme preserves the total discrete energy very well

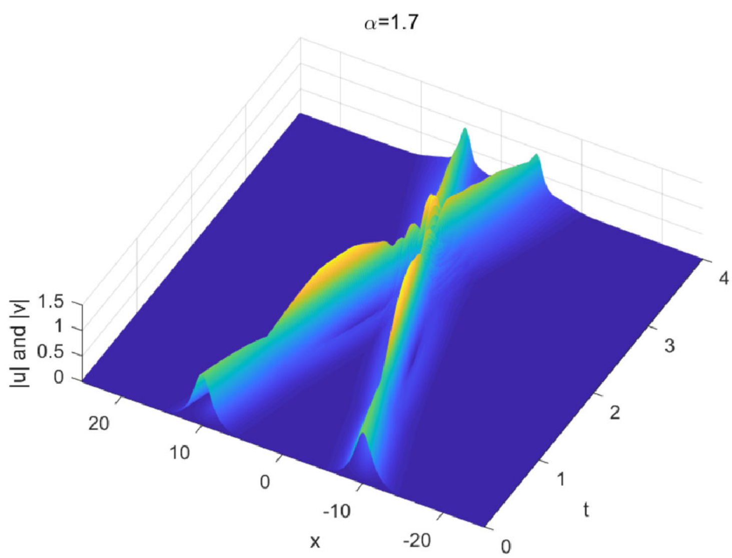

Figure 7 The Graphs of $|u|$ and $|v|$ for $\boldsymbol{\alpha} = 1.7$ with time evolution

6 Conclusion

In the current work, we have constructed a linearized Galerkin–Legendre spectral method for solving the strongly coupled nonlinear fractional Schrödinger equations. The main novelty of this paper is that the proposed scheme can preserve both the mass- and the energy-conservation laws in the discrete sense, and the optimal error estimate is established rigorously without imposing any restriction on the grid ratio. The discrete scheme is efficient in the sense that only a linear system needs to be solved at each time step. Theoretical results show that our scheme is second-order convergent in time and at the same time has the advantage of spectral accuracy in space. Numerical results show that the de-

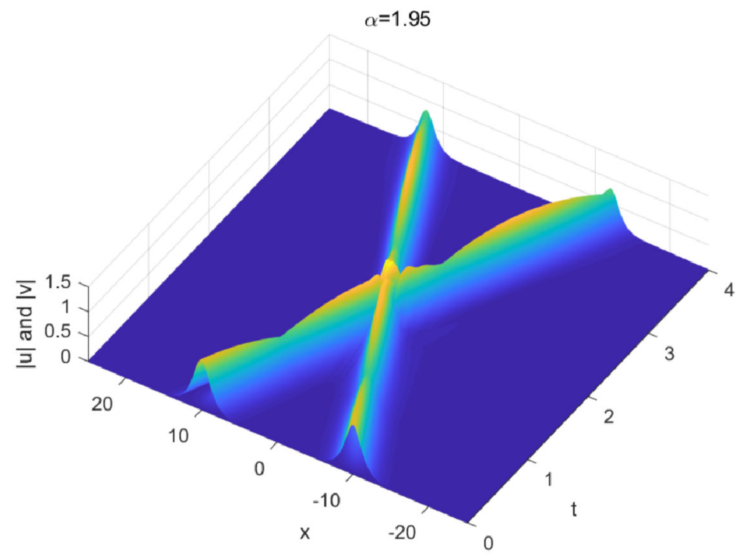

Figure 8 The Graphs of $|u|$ and $|v|$ for $\alpha = 1.95$ with time evolution

rived scheme is quite efficient and exhibits remarkable mass- and energy-preserving properties. The spectral method and corresponding theoretical analysis for high-dimensional SCFSEs is worth of further investigation.

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The authors declare that they have no competing interests.

Authors' contributions

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