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Some fixed point results on $\mathcal{N}_b$-cone metric spaces over Banach algebra

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Abstract

The main aim of this paper is to introduce the concept of $\mathcal{N}_b$-cone metric spaces over a Banach algebra as a generalization of $\mathcal{N}$-cone metric spaces over a Banach algebra and b-metric spaces. Also, we study some coupled common fixed point theorems for generalized Lipschitz mappings in this framework. Finally, we give an example and an application to the existence of solutions of integral equations to illustrate the effectiveness of our generalizations. Some results in the literature are special cases of our results.

Keywords: $\mathcal{N}_b$-cone metric spaces over Banach algebra; c-sequence; Iterative algorithm; Coupled fixed points

1 Introduction

The concept of b-metric space has been introduced by Bakhtin [2] and Czerwik [4]. Via this notion the Banach contraction principle has been extended in b-metric spaces by Czerwik. Many authors have focused on generalizations of metric spaces and have investigated many new results in these new structures (see details in [10, 16–20]).

The concept of cone metric spaces has been introduced by Huang and Zhang [9]; this is a generalization of a metric space in the sense that the codomain of the metric function has been substituted from the set of real numbers to a real Banach space.

The study of $\mathcal{N}$-cone metric spaces was started by Malviya and Fischer [15]. They generalized the concept of $\mathcal{D}^*$-metric spaces [1] to form a new space and defined asymptotically regular maps in $\mathcal{N}$-cone metric spaces.

On the other hand, some authors have investigated fixed point results which are proved in cone metric spaces coinciding with the results of ordinary metric spaces, where the function d^* is defined by a nonlinear scalarization function ξ_e (see [5]) or by a Minkowski functional q_e (see [11]).

Afterwards, the concept of cone metric spaces over Banach algebras proposed by Liu and Xu [13] where cone metric spaces replaced by cone metric spaces over Banach algebras which was interesting for many researchers. Their work contains some fixed point results for generalized Lipschitz mapping in cone metric spaces over Banach algebras. In 2014, Xu and Radenovic [25] showed that the results of Liu and Xu remain valid if we use only solid cones instead of normal cones. The notion of $\mathcal{N}$-cone metric space over a Banach algebra presented by Fernandez *et al.* [6], who generalized the concept of $\mathcal{N}$-cone metric space and proved some fixed point results for generalized Lipschitz mapping in this new framework.

The notion of a coupled fixed point has been presented by Guo and Laksmikantham [7]. Later, Bhaskar and Laksmikantham [3] recommended the mixed monotone property for contractive operators of the type $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$, where $\mathcal{M}$ is a partially ordered metric space. Some coupled coincidence fixed point theorems in non-normal cone metric spaces have been obtained by Shatanawi [23]. For some new coincidence point and common fixed point theorems in the product spaces of complete quasi-ordered metric spaces we refer to [24]. Also, some coincidence point and common fixed point results for hybrid pairs of mappings on a cone b-metric space over a Banach algebra have been stated and proved by Malhotra *et al.* [14].

Inspired by the above works, we organize this article in the following way.

In Sect. 2, we recall some definitions and results from [9] and [13]. In Sect. 3, we present the notion of $\mathcal{N}_b$-cone metric space over a Banach algebra as a generalization of $\mathcal{N}$-cone metric spaces over a Banach algebra and b-metric spaces. Some topological properties of this new structure have been collected in Sect. 4 and the notion of generalized Lipschitz maps have been defined in Sect. 5. We investigate some coupled fixed point problems for generalized Lipschitz maps in Sect. 6. To support our main result we present an example in Sect. 6. Lastly, an application to the existence of solutions for integral equations is given in Sect. 7. The results of [26] are special cases of our results.

2 Preliminaries

Let $\mathcal{A}$ denote a real Banach algebra such that

1. $(\mu\nu)\upsilon = \mu(\nu\upsilon)$,
2. $\mu(\nu+\upsilon) = \mu\nu+\mu\upsilon$ and $(\mu+\nu)\upsilon = \mu\upsilon+\nu\upsilon$,
3. $\alpha(\mu\nu) = (\alpha\mu)\nu = \mu(\alpha\nu)$,
4. $\|\mu\nu\| \le \|\mu\|\|\nu\|$,

for all $\mu,\nu,\upsilon\in\mathcal{A}$ and $\alpha\in\mathbb{R}$.

If $e\mu = \mu e = \mu$, for all $\mu\in\mathcal{A}$, then $e\in\mathcal{A}$ is called unit (i.e., a multiplicative identity). If there is an element $\nu\in\mathcal{A}$ such that $\mu\nu = \nu\mu = e$, then $\mu\in\mathcal{A}$ is said to be invertible. μ^{-1} is the inverse of μ. For more details, we refer to [22].

Proposition 1 ([22]) *Suppose that the spectral radius $\widehat{\rho}(\mu)$ of an element $\mu\in\mathcal{A}$ is less than* 1, *i.e.*

$$\widehat{\rho}(\mu) = \lim_{n\to+\infty}\left\|\mu^n\right\|^{\frac{1}{n}} = \inf_{n\ge1}\left\|\mu^n\right\|^{\frac{1}{n}} < 1,$$

then $e-\mu$ is invertible, where $e\in\mathcal{A}$ is unit. Moreover,

$$(e-\mu)^{-1} = \sum_{i=0}^{\infty}\mu^i.$$

Remark 1 ([22]) The spectral radius $\widehat{\rho}(\mu)$ of μ satisfies $\widehat{\rho}(\mu)\le\|\mu\|$, for all $\mu\in\mathcal{A}$.

Remark 2 ([25]) If $\widehat{\rho}(\mu) < 1$ then $\|\mu^n\|\to 0$ as $n\to+\infty$.

A set $\mathcal{P} \subset \mathcal{A}$ is called a cone if

1. $\mathcal{P} \neq \emptyset$, is closed and $\mathcal{P} \neq \{\theta\}$;
2. $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$ for all non-negative real numbers α, β;
3. $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;
4. $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$,

where θ indicates the null of $\mathcal{A}$. The partial ordering on $\mathcal{A}$ with the help of a cone $\mathcal{P}$ is defined by $\mu \preceq \nu$ if and only if $\nu - \mu \in \mathcal{P}$. We write $\mu \prec \nu$ to indicate that $\mu \preceq \nu$ but $\mu \neq \nu$, while $\mu \ll \nu$ stands for $\nu - \mu \in \operatorname{int}\mathcal{P}$, where $\operatorname{int}\mathcal{P}$ is the interior of $\mathcal{P}$. A cone is called solid if $\operatorname{int}\mathcal{P} \neq \emptyset$.

The cone $\mathcal{P}$ is said to be normal provided that there is a number $\Lambda > 0$ such that

$$\theta \preceq \mu \preceq \nu \quad \text{implies} \quad \|\mu\| \leq \Lambda\|\nu\|, \tag{2.1}$$

for all $\mu, \nu \in \mathcal{A}$.

The least positive number Λ verifying (2.1) is known as the normal constant of $\mathcal{P}$ (see [9]).

Definition 1 ([9, 13]) Let $\mathcal{M}$ be a nonempty set and let $d : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ satisfy

1. $\theta \preceq d(\mu, \nu)$ for all $\mu, \nu \in \mathcal{M}$ and $d(\mu, \nu) = \theta$ if and only if $\mu = \nu$,
2. $d(\mu, \nu) = d(\nu, \mu)$ for all $\mu, \nu \in \mathcal{M}$;
3. $d(\mu, \nu) \preceq d(\mu, \upsilon) + d(\upsilon, \nu)$ for all $\mu, \nu, \upsilon \in \mathcal{M}$.

In this case, d is said to be a cone metric on $\mathcal{M}$ and $(\mathcal{M}, d)$ is said to be a cone metric space over a Banach algebra $\mathcal{A}$.

For more details regarding cone metric spaces over a Banach algebra, we refer to [13].

Definition 2 ([2]) Let $\mathcal{M}$ be a nonempty set and $s \geq 1$ be a constant. A mapping $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is said to be a b-metric if and only if,

1. $d(\mu, \nu) = 0$ if and only if $\mu = \nu$,
2. $d(\mu, \nu) = d(\nu, \mu)$ for all $\mu, \nu \in \mathcal{M}$,
3. $d(\mu, \nu) \leq s[d(\mu, \upsilon) + d(\upsilon, \nu)]$ for all $\mu, \nu, \upsilon \in \mathcal{M}$,

for all $\mu, \nu, \upsilon \in \mathcal{M}$.

Then the pair $(\mathcal{M}, d)$ is called a b-metric space.

Some details on b-metric spaces and their extensions can be found in [10, 18] and [16].

Definition 3 ([15]) Let $\mathcal{M}$ be a nonempty set. A function $\mathcal{N} : \mathcal{M}^3 \to \mathcal{E}$ is called an $\mathcal{N}$-cone metric on $\mathcal{M}$ if

(1) $\theta \leq \mathcal{N}(\mu, \nu, \upsilon)$,
(2) $\mathcal{N}(\mu, \nu, \upsilon) = \theta$ if and only if $\mu = \nu = \upsilon$,
(3) $\mathcal{N}(\mu, \nu, \upsilon) \leq \mathcal{N}(\mu, \mu, a) + \mathcal{N}(\nu, \nu, a) + \mathcal{N}(\upsilon, \upsilon, a)$,

for any $\mu, \nu, \upsilon, a \in \mathcal{M}$.

Then the pair $(\mathcal{M}, \mathcal{N})$ is called a $\mathcal{N}$-cone metric space.

Other definitions and subsequent results on $\mathcal{N}$-cone metric spaces are given in [15].

Definition 4 ([6]) An $\mathcal{N}$-cone metric over a Banach algebra $\mathcal{A}$ on a nonempty set $\mathcal{M}$ is a function $\mathcal{N} : \mathcal{M}^3 \to \mathcal{A}$ for which

($\mathcal{N}_1$) $\theta \preceq \mathcal{N}(\mu,\nu,\upsilon)$,
($\mathcal{N}_2$) $\mathcal{N}(\mu,\nu,\upsilon) = \theta$ if and only if $\mu = \nu = \upsilon$,
($\mathcal{N}_3$) $\mathcal{N}(\mu,\nu,\upsilon) \preceq \mathcal{N}(\mu,\mu,a) + \mathcal{N}(\nu,\nu,a) + \mathcal{N}(\upsilon,\upsilon,a)$,
such that, for all $\mu,\nu,\upsilon,a \in \mathcal{M}$.

Then the pair $(\mathcal{M},\mathcal{N})$ is called a $\mathcal{N}$-cone metric space over a Banach algebra $\mathcal{A}$.

3 $\mathcal{N}_b$-Cone metric space over Banach algebra

We present the definition of $\mathcal{N}_b$-cone metric space over a Banach algebra as follows.

Definition 5 A $\mathcal{N}_b$-cone metric over a Banach algebra $\mathcal{A}$ on a nonempty set $\mathcal{M}$ is a function $\mathcal{N}_b : \mathcal{M}^3 \to \mathcal{A}$ such that, for all $\rho,\varrho,\sigma,a \in \mathcal{M}$:
($\mathcal{N}_{b_1}$) $\theta \preceq \mathcal{N}_b(\rho,\varrho,\sigma)$,
($\mathcal{N}_{b_2}$) $\mathcal{N}_b(\rho,\varrho,\sigma) = \theta$ if and only if $\rho = \varrho = \sigma$,
($\mathcal{N}_{b_3}$) $\mathcal{N}_b(\rho,\rho,\varrho) = \mathcal{N}_b(\varrho,\varrho,\rho)$,
($\mathcal{N}_{b_4}$) $\mathcal{N}_b(\rho,\varrho,\sigma) \preceq s[\mathcal{N}_b(\rho,\rho,a) + \mathcal{N}_b(\varrho,\varrho,a) + \mathcal{N}_b(\sigma,\sigma,a)]$.

The pair $(\mathcal{M},\mathcal{N}_b)$ is called an $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$. The number $s \geq 1$ is called the coefficient of $(\mathcal{M},\mathcal{N}_b)$.

We give an example of $\mathcal{N}_b$-cone metric space over a Banach algebra which is not $\mathcal{N}$-cone metric space over a Banach algebra.

Example 1 Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ with norm defined by $\|\rho\| = \|\rho\|_\infty + \|\rho'\|_\infty$. Under point-wise multiplication, $\mathcal{A}$ is a real unit Banach algebra with unit $e = 1$. Consider a cone $\mathcal{P} = \{\rho \in \mathcal{A} : \rho \geq 0\}$ in $\mathcal{A}$. Moreover, $\mathcal{P}$ is a non-normal cone (see [21]). Let $\mathcal{M} = [0,+\infty)$. Define $\mathcal{N}_b : \mathcal{M}^3 \to \mathcal{A}$ by

$$\mathcal{N}_b(\rho,\varrho,\sigma)(t) = \bigl(|\varrho + \sigma - 2\rho| + |\varrho - \sigma|\bigr)^2 e^t,$$

for all $\rho,\varrho,\sigma \in \mathcal{M}$. Then
(i) $\mathcal{N}_{b_1}$, $\mathcal{N}_{b_2}$ and $\mathcal{N}_{b_3}$ are obvious.
(ii) $\mathcal{N}_{b_4}$: Since for each $\rho,\varrho,\sigma \in \mathcal{M}$

$$\bigl(|\varrho + \sigma - 2\rho| + |\varrho - \sigma|\bigr)^2 e^t \leq 2^2\bigl(4|\rho - a|^2 + 4|\varrho - a|^2 + 4|\sigma - a|^2\bigr)e^t,$$

we obtain

$$\mathcal{N}_b(\rho,\varrho,\sigma)(t) \preceq 4\bigl[\mathcal{N}_b(\rho,\rho,a) + \mathcal{N}_b(\varrho,\varrho,a) + \mathcal{N}_b(\sigma,\sigma,a)\bigr].$$

Therefore, $(\mathcal{M},\mathcal{N}_b)$ is a $\mathcal{N}_b$-cone metric space over a Banach algebra with the coefficient $s = 4$, but as the triangle inequality is not satisfied, it is not a $\mathcal{N}$-cone metric space over a Banach algebra.

4 Topology on $\mathcal{N}_b$-cone metric space over Banach algebra

We now present the concept of topology on $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$.

Definition 6 Let $(\mathcal{M},\mathcal{N}_b)$ be an $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$. Then for a $\rho \in \mathcal{M}$ and $c > \theta$, the $\mathcal{N}_b$-ball with center ρ and radius $c > \theta$ is

$$\mathcal{B}_{\mathcal{N}_b}(\rho, c) = \{\varrho \in \mathcal{M} : \mathcal{N}_b(\rho,\rho,\varrho) \ll c\}.$$

Proposition 2 *Let $(\mathcal{M},\mathcal{N}_b)$ be a $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$ with coefficient $s \geq 4$. The family*

$$\mathcal{B} = \{\mathcal{B}_{\mathcal{N}_b}(\rho, c) : \rho \in \mathcal{M} \text{ and } \theta \ll c\}$$

is a basis for a topology τ on $\mathcal{M}$.

Theorem 1 *Let $(\mathcal{M},\mathcal{N}_b)$ be a $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$ with coefficient s. Then $(\mathcal{M},\mathcal{N}_b)$ is a Hausdorff space.*

Proof Let $(\mathcal{M},\mathcal{N}_b)$ be a $\mathcal{N}_b$-cone metric space over a Banach algebra and let $\rho, \varrho \in \mathcal{M}$ with $\rho \neq \varrho$. Let $\mathcal{N}_b(\rho,\rho,\varrho) = c$. Suppose $\mathcal{U} = \mathcal{B}_{\mathcal{N}_b}(\rho, \frac{c}{4s})$ and $\mathcal{V} = \mathcal{B}_{\mathcal{N}_b}(\varrho, \frac{c}{2s})$. We will show that $\mathcal{U} \cap \mathcal{V} = \emptyset$. Otherwise, there exists a σ such that

$$\sigma \in \mathcal{U} \cap \mathcal{V}.$$

Then we have

$$\mathcal{N}_b(\rho,\rho,\sigma) \prec \frac{c}{4s} \quad \text{and} \quad \mathcal{N}_b(\varrho,\varrho,\sigma) \prec \frac{c}{2s}.$$

So, we get

$$\begin{aligned} c = \mathcal{N}_b(\rho,\rho,\varrho) &\preceq s\big[\mathcal{N}_b(\rho,\rho,\sigma) + \mathcal{N}_b(\rho,\rho,\sigma) + \mathcal{N}_b(\varrho,\varrho,\sigma)\big] \\ &= s\big[2\mathcal{N}_b(\rho,\rho,\sigma) + \mathcal{N}_b(\varrho,\varrho,\sigma)\big] \\ &\prec 2s\frac{c}{4s} + s\frac{c}{2s} \\ &= c, \end{aligned}$$

a contradiction. □

Definition 7 Let $(\mathcal{M},\mathcal{N}_b)$ be a $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$.

a. A sequence (ρ_n) in $(\mathcal{M},\mathcal{N}_b)$ converges to a point $\rho \in \mathcal{M}$ whenever for every $c \gg \theta$ there is a natural number N such that $\mathcal{N}_b(\rho_n,\rho,\rho) \ll c$ for all $n \geq N$. It is denoted by

$$\lim_{n\to+\infty} \rho_n = \rho \quad \text{or} \quad \rho_n \to \rho \quad (n \to +\infty).$$

b. We say that a sequence (ρ_n) in $(\mathcal{M},\mathcal{N}_b)$ is Cauchy, if for every $c \gg \theta$ there exist a natural number N such that, for all $n, m \geq N$, we have

$$\mathcal{N}_b(\rho_n,\rho_n,\rho_m) \ll c.$$

c. $(\mathcal{M},\mathcal{N}_b)$ is said to be complete if every Cauchy sequence (ρ_n) in $\mathcal{M}$ convergent in $\mathcal{M}$.

Definition 8 Let $(\mathcal{X},\mathcal{N}_b)$ be a $\mathcal{N}_b$-cone metric space over a Banach algebra. A $\mathcal{N}_b$-cone metric function $\mathcal{N}_b$ is called continuous if for any $y \in X$ and any $\varepsilon > 0$ there exists $c > \theta$ such that the inequality $\|\mathcal{N}_b(y,y,x) - \mathcal{N}_b(y,y,z)\| < \varepsilon$, provided that $\mathcal{N}_b(x,x,z) < c$.

Definition 9 Let $(\mathcal{M},\mathcal{N}_b)$ and $(\mathcal{M}',\mathcal{N}_b')$ be $\mathcal{N}_b$-cone metric spaces over a Banach algebra $\mathcal{A}$. A function $f : \mathcal{M} \to \mathcal{M}'$ is continuous at $\rho \in \mathcal{M}$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{N}_b(\sigma,\sigma,\rho) < \delta \quad \text{implies} \quad \mathcal{N}_b'\bigl(f(\sigma),f(\sigma),f(\rho)\bigr) < \epsilon.$$

If f is continuous at ρ for all $\rho \in \mathcal{M}$ then f is continuous on $\mathcal{M}$.

5 Generalized Lipschitz maps

First, we introduce the notion of generalized Lipschitz mapping on $\mathcal{N}_b$-cone metric space over a Banach algebra.

Definition 10 Let $(\mathcal{M},\mathcal{N}_b)$ be an $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$ and let $\mathcal{P}$ be a cone in $\mathcal{A}$. A mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is said to be a generalized Lipschitz mapping if there exist a vector $k \in \mathcal{P}$ with $\widehat{\rho}(k) < 1$ such that

$$\mathcal{N}_b(\mathcal{T}\rho,\mathcal{T}\rho,\mathcal{T}\varrho) \preceq k\mathcal{N}_b(\rho,\rho,\varrho),$$

for all $\rho,\varrho \in \mathcal{M}$.

Example 2 Consider a Banach algebra $\mathcal{A}$ and cone $\mathcal{P}$ be as in Example 1 and let $\mathcal{M} = \mathbb{R}^+$. Define a mapping $\mathcal{N}_b : \mathcal{M}^3 \to \mathcal{A}$ by

$$\mathcal{N}_b(\rho,\varrho,\sigma)(t) = \bigl(|\rho - \sigma| + |\varrho - \sigma|\bigr)^2 e^t,$$

for all $\rho,\varrho,\sigma \in \mathcal{M}$. Then $(\mathcal{M},\mathcal{N}_b)$ is a $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$. Define a self-map $\mathcal{T}$ by $\mathcal{T}(\rho) = \ln(1 + \frac{\varrho}{2})$. Since $\ln(1 + \rho) \leq \rho$ for each $\rho \in [0,+\infty)$, we have

$$\begin{aligned}
\mathcal{N}_b(\mathcal{T}\rho,\mathcal{T}\rho,\mathcal{T}\varrho)(t) &= \Biggl[\biggl|\ln\biggl(1+\frac{\rho}{2}\biggr) - \ln\biggl(1+\frac{\varrho}{2}\biggr)\biggr|^2 \\
&\quad + \biggl|\ln\biggl(1+\frac{\rho}{2}\biggr) - \ln\biggl(1+\frac{\varrho}{2}\biggr)\biggr|^2\Biggr]e^t \\
&= \Biggl[2\biggl|\ln\biggl(1+\frac{\rho}{2}\biggr) - \ln\biggl(1+\frac{\varrho}{2}\biggr)\biggr|^2\Biggr]e^t \\
&\preceq \biggl[2\biggl|\frac{\rho}{2} - \frac{\varrho}{2}\biggr|^2\biggr]e^t \\
&\preceq \frac{1}{4}\bigl(2|\rho - \varrho|^2\bigr)e^t \\
&= \frac{1}{2}\mathcal{N}_b(\rho,\rho,\varrho)(t),
\end{aligned}$$

for all $\rho,\varrho \in \mathcal{M}$.

Hence, $\mathcal{T}$ is a generalized Lipschitz mapping on $\mathcal{M}$, where $k = \frac{1}{4}$.

We recall some definitions and results from the literature

Definition 11 ([3]) An element $(\rho,\varrho) \in \mathcal{M} \times \mathcal{M}$ is called a coupled fixed point of the mapping $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ if

$$\mathcal{F}(\rho,\varrho) = \rho \quad \text{and} \quad \mathcal{F}(\varrho,\rho) = \varrho.$$

Definition 12 ([12]) An element $(\rho,\varrho) \in \mathcal{M} \times \mathcal{M}$ is called a coupled coincidence point of the mappings $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $g : \mathcal{M} \to \mathcal{M}$ if

$$\mathcal{F}(\rho,\varrho) = g(\rho) \quad \text{and} \quad \mathcal{F}(\varrho,\rho) = g(\varrho).$$

Definition 13 ([12]) An element $(\rho,\varrho) \in \mathcal{M} \times \mathcal{M}$ is called a coupled common fixed point of the mappings $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $g : \mathcal{M} \to \mathcal{M}$ if

$$\mathcal{F}(\rho,\varrho) = g(\rho) = \rho \quad \text{and} \quad \mathcal{F}(\varrho,\rho) = g(\varrho) = \varrho.$$

Definition 14 ([11]) Let $\mathcal{E}$ be a Banach space and $\mathcal{P}$ be a solid cone. We say that a sequence $\{u_n\} \subset \mathcal{P}$ is a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $\rho_n \ll c$, for all $n > N$.

Lemma 1 ([25]) *A sequence $\{k\rho_n\}$ is a c-sequence in $\mathcal{P}$, $k \in \mathcal{P}$, where $\mathcal{P}$ is a solid cone in $\mathcal{A}$ provided that $\{\rho_n\}$ is a c-sequence in $\mathcal{P}$.*

Lemma 2 ([22]) *Let $\mathcal{A}$ be a unit Banach algebra and $k \in \mathcal{A}$. Then the spectral radius $\rho(k)$ satisfies*

$$\widehat{\rho}(k) = \lim_{n\to+\infty} \|k^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|k^n\|^{\frac{1}{n}}.$$

If $\widehat{\rho}(k) < |\lambda|$, where λ is a complex constant, then $(\lambda e - k)$ is invertible in $\mathcal{A}$. Moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}.$$

Lemma 3 ([22]) *In a unit Banach algebra $\mathcal{A}$, if d commutes with d', for $d, d' \in \mathcal{A}$, then*

$$\widehat{\rho}(d + d') \leq \widehat{\rho}(d) + \widehat{\rho}(d') \quad \text{and} \quad \widehat{\rho}(dd') \leq \widehat{\rho}(d)\widehat{\rho}(d').$$

Lemma 4 ([8]) *Let $\mathcal{A}$ be a Banach algebra and let k be a vector in $\mathcal{A}$. If $\widehat{\rho}(k) < 1$, then*

$$\widehat{\rho}\big((e-k)^{-1}\big) \leq \big(1 - \widehat{\rho}(k)\big)^{-1}.$$

Lemma 5 ([8]) *Consider a unit Banach algebra $\mathcal{A}$ and $\mathcal{P}$ a solid cone. Let $d, k, l \in \mathcal{P}$ satisfying $l \preceq k$ and $d \preceq ld$. If $\widehat{\rho}(k) < 1$, then $d = \theta$.*

Lemma 6 ([8]) *A sequence $\{\rho_n\}$ in a solid cone $\mathcal{P}$ of a real Banach space $\mathcal{E}$ satisfying $\|\rho_n\| \to 0$ $(n \to +\infty)$ is a c-sequence.*

Lemma 7 ([19]) *Let P be a solid cone in a Banach algebra A and let $\{x_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{x_n\}$ is a Cauchy sequence in P. Then $\{kx_n\}$ is a Cauchy sequence too.*

Lemma 8 ([8]) *Let $\mathcal{P}$ be a solid cone in $\mathcal{E}$.*

1. *If $d, d', c \in \mathcal{E}$ and $d \preceq d' \ll c$, then $d \ll c$.*
2. *If $d \in \mathcal{P}$ and $d \ll c$ for each $c \gg \theta$, then $d = \theta$.*

6 Application in fixed point theory

In this section, as an application, we prove a coupled fixed point result for generalized Lipschitz mappings with an example which demonstrates the strength and applicability of our main result.

Theorem 2 *Let $(\mathcal{M}, \mathcal{N}_b)$ be a complete $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 4$ and $\mathcal{P}$ be a solid cone in $\mathcal{A}$. Let $\alpha, \beta \in \mathcal{P}$ be generalized Lipschitz constants with $\widehat{\rho}[\alpha + \beta] < \frac{1}{s}$. Suppose that $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a $\mathcal{T}$-contraction where $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a surjective and one to one mapping that satisfies*

$$\mathcal{N}_b\big(\mathcal{T}\mathcal{F}(\rho, \varrho), \mathcal{T}\mathcal{F}(\rho, \varrho), \mathcal{T}\mathcal{F}(u, v)\big) \preceq \alpha \mathcal{N}_b(\mathcal{T}\rho, \mathcal{T}\rho, \mathcal{T}u) + \beta \mathcal{N}_b(\mathcal{T}\varrho, \mathcal{T}\varrho, \mathcal{T}v), \tag{6.1}$$

for all $\rho, \varrho, u, v \in \mathcal{M}$. Then $\mathcal{F}$ possesses a unique coupled fixed point $(\rho^*, \varrho^*) \in \mathcal{M} \times \mathcal{M}$.

Proof Let ρ_0, ϱ_0 be arbitrary points in $\mathcal{M}$ and let

$$\rho_{n+1} = \mathcal{F}(\rho_n, \varrho_n) = \mathcal{F}^{n+1}(\rho_0, \varrho_0) \quad \text{and} \quad \varrho_{n+1} = \mathcal{F}(\varrho_n, \rho_n) = \mathcal{F}^{n+1}(\varrho_0, \rho_0),$$

for all $n \in \mathcal{N}$. Now according to (6.1), we have

$$\begin{aligned} \mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) &= \mathcal{N}_b\big(\mathcal{T}\mathcal{F}(\rho_{n-1}, \varrho_{n-1}), \mathcal{T}\mathcal{F}(\rho_{n-1}, \varrho_{n-1}), \mathcal{T}\mathcal{F}(\rho_n, \varrho_n)\big) \\ &\preceq \alpha \mathcal{N}_b(\mathcal{T}\rho_{n-1}, \mathcal{T}\rho_{n-1}, \mathcal{T}\rho_n) + \beta \mathcal{N}_b(\mathcal{T}\varrho_{n-1}, \mathcal{T}\varrho_{n-1}, \mathcal{T}\varrho_n) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \mathcal{N}_b(\mathcal{T}\varrho_n, \mathcal{T}\varrho_n, \mathcal{T}\varrho_{n+1}) &= \mathcal{N}_b\big(\mathcal{T}\mathcal{F}(\varrho_{n-1}, \rho_{n-1}), \mathcal{T}\mathcal{F}(\varrho_{n-1}, \rho_{n-1}), \mathcal{T}\mathcal{F}(\varrho_n, \rho_n)\big) \\ &\preceq \alpha \mathcal{N}_b(\mathcal{T}\varrho_{n-1}, \mathcal{T}\varrho_{n-1}, \mathcal{T}\varrho_n) + \beta \mathcal{N}_b(\mathcal{T}\rho_{n-1}, T\rho_{n-1}, \mathcal{T}\rho_n). \end{aligned} \tag{6.3}$$

Let

$$d_n = \mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) + \mathcal{N}_b(\mathcal{T}\varrho_n, \mathcal{T}\varrho_n \mathcal{T}\varrho_{n+1}) \tag{6.4}$$

and $h = \alpha + \beta$. From (6.2) and (6.3), we obtain $d_n \preceq hd_{n-1}$, so

$$\begin{aligned} d_n &\preceq (\alpha + \beta)\big[\mathcal{N}_b(\mathcal{T}\rho_{n-1}, \mathcal{T}\rho_{n-1}, \mathcal{T}\rho_n) + \mathcal{N}_b(\mathcal{T}\varrho_{n-1}, \mathcal{T}\varrho_{n-1}, \mathcal{T}\varrho_n)\big] \\ &\preceq (\alpha + \beta) d_{n-1} \preceq hd_{n-1}, \end{aligned}$$

where $h = \alpha + \beta$ such that $\widehat{\rho}(h) < 1$. So

$$d_n \preceq h d_{n-1} \preceq \cdots \preceq h^n d_0. \tag{6.5}$$

We assume that $d_0 > \theta$. Otherwise (ρ_0, ϱ_0) is a coupled fixed point of $\mathcal{F}$. If $m > n$, then we have

$$\begin{aligned}
&\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_m) \\
&\quad\preceq s\bigl[\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) + \mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) + \mathcal{N}_b(\mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+1}, \mathcal{T}\rho_m)\bigr] \\
&\quad\preceq 2s\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) + s\mathcal{N}_b(\mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+1}, \mathcal{T}\rho_m) \\
&\quad\preceq 2s\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) \\
&\qquad + s^2\bigl[2\mathcal{N}_b(\mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+2}) + \mathcal{N}_b(\mathcal{T}\rho_{n+2}, \mathcal{T}\rho_{n+2}, \mathcal{T}\rho_m)\bigr] \\
&\quad\vdots \\
&\quad\preceq 2s\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) + 2s^2\mathcal{N}_b(\mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+2}) \\
&\qquad + \cdots + 2s^{m-n-1}\mathcal{N}_b(\mathcal{T}\rho_{m-2}, \mathcal{T}\rho_{m-2}, \mathcal{T}\rho_{m-1}) \\
&\qquad + s^{m-n-1}\mathcal{N}_b(\mathcal{T}\rho_{m-1}, \mathcal{T}\rho_{m-1}, \mathcal{T}\rho_m) \\
&\quad\preceq 2s\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_{n+1}) + 2s^2\mathcal{N}_b(\mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+1}, \mathcal{T}\rho_{n+2}) \\
&\qquad + \cdots + 2s^{m-n-1}\mathcal{N}_b(\mathcal{T}\rho_{m-2}, \mathcal{T}\rho_{m-2}, \mathcal{T}\rho_{m-1}) \\
&\qquad + 2s^{m-n}\mathcal{N}_b(\mathcal{T}\rho_{m-1}, \mathcal{T}\rho_{m-1}, \mathcal{T}\rho_m)
\end{aligned} \tag{6.6}$$

and, similarly,

$$\begin{aligned}
&\mathcal{N}_b(\mathcal{T}\varrho_n, \mathcal{T}\varrho_n, \mathcal{T}\varrho_m) \\
&\quad\preceq 2s\mathcal{N}_b(\mathcal{T}\varrho_n, \mathcal{T}\varrho_n, \mathcal{T}\varrho_{n+1}) + 2s^2\mathcal{N}_b(\mathcal{T}\varrho_{n+1}, \mathcal{T}\varrho_{n+1}, \mathcal{T}\varrho_{n+2}) \\
&\qquad + \cdots + 2s^{m-n-1}\mathcal{N}_b(\mathcal{T}\varrho_{m-2}, \mathcal{T}\varrho_{m-2}, \mathcal{T}\varrho_{m-1}) \\
&\qquad + 2s^{m-n}\mathcal{N}_b(\mathcal{T}\varrho_{m-1}, \mathcal{T}\varrho_{m-1}, \mathcal{T}\varrho_m).
\end{aligned} \tag{6.7}$$

From (6.5), (6.6) and (6.7), we obtain

$$\begin{aligned}
&\mathcal{N}_b(\mathcal{T}\rho_n, \mathcal{T}\rho_n, \mathcal{T}\rho_m) + \mathcal{N}_b(\mathcal{T}\varrho_n, \mathcal{T}\varrho_n, \mathcal{T}\varrho_m) \\
&\quad\preceq 2sd_n + 2s^2 d_{n+1} + \cdots + 2s^{m-n} d_{m-1} \\
&\quad\preceq 2sh^n d_0 + 2s^2 h^{n+1} d_0 + \cdots + 2s^{m-n} h^{m-1} d_0 \\
&\quad= 2(sh)^n\bigl[e + sh + \cdots + (sh)^{m-n-1}\bigr] d_0 \\
&\quad\preceq 2(sh)^n (e - sh)^{-1} d_0.
\end{aligned}$$

Since $\widehat{\rho}(h) < \frac{1}{s}$, by Remark 2, we get $\|(sh)^n d_0\| \leq \|(sh)^n\| \cdot \|d_0\| \to 0$. From Lemma 6, it follows that, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$, we have

$$\mathcal{N}_b(\mathcal{T}\rho_n,\mathcal{T}\rho_n,\mathcal{T}\rho_m)+\mathcal{N}_b(\mathcal{T}\varrho_n,\mathcal{T}\varrho_n,\mathcal{T}\varrho_m)\ll c,$$

so $\{\mathcal{T}\rho_n\}$ and $\{\mathcal{T}\varrho_n\}$ are Cauchy sequences in $\mathcal{M}$. The completeness of $\mathcal{M}$ shows that there exist $\rho^*,\varrho^*\in\mathcal{M}$ such that

$$\lim_{n\to+\infty}\mathcal{T}\mathcal{F}^n(\rho_0,\varrho_0)=\rho^*\quad\text{and}\quad\lim_{n\to+\infty}\mathcal{T}\mathcal{F}^n(\varrho_0,\rho_0)=\varrho^*.$$

Using (6.1), we obtain

$$\begin{aligned}
&\mathcal{N}_b\big(\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\rho^*\big)\\
&\quad\preceq s\mathcal{N}_b\big(\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}(\rho_n,\varrho_n)\big)\\
&\qquad+s\mathcal{N}_b\big(\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}(\rho_n,\varrho_n)\big)\\
&\qquad+s\mathcal{N}_b\big(\mathcal{T}\mathcal{F}(\rho_n,\varrho_n),\mathcal{T}\mathcal{F}(\rho_n,\varrho_n),\mathcal{T}\rho^*\big)\\
&\quad\preceq 2s\big[\alpha\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho_n\big)+\beta\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho_n\big)\\
&\qquad+s\mathcal{N}_b(\mathcal{T}\rho_{n+1},\mathcal{T}\rho_{n+1},\mathcal{T}\rho^*\big].
\end{aligned}$$

From the surjectivity property of $\mathcal{T}$, we conclude

$$\mathcal{N}_b\big(\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\rho^*\big)=\theta,$$

that is, $\mathcal{T}\mathcal{F}(\rho^*,\varrho^*)=\mathcal{T}\rho^*$. Since $\mathcal{T}$ is one-to-one, $\mathcal{F}(\rho^*,\varrho^*)=\rho^*$. Similarly, we can get $\mathcal{F}(\varrho^*,\rho^*)=\varrho^*$. Therefore, (ρ^*,ϱ^*) is a coupled fixed point of $\mathcal{F}$. Now, if (ρ',ϱ') is another coupled fixed point of $\mathcal{F}$, then

$$\begin{aligned}
\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big)&=\mathcal{N}_b\big(\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\mathcal{F}\big(\rho^*,\varrho^*\big),\mathcal{T}\big(\rho',\varrho'\big)\big)\\
&\preceq\alpha\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big)+\beta\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)
\end{aligned}\tag{6.8}$$

and

$$\begin{aligned}
\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)&=\mathcal{N}_b\big(\mathcal{T}\mathcal{F}\big(\varrho^*,\rho^*\big),\mathcal{T}\mathcal{F}\big(\varrho^*,\rho^*\big),\mathcal{T}\big(\varrho',\rho'\big)\big)\\
&\preceq\alpha\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)+\beta\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big).
\end{aligned}\tag{6.9}$$

Using (6.8) and (6.9), we have

$$\begin{aligned}
&\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big)+\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)\\
&\quad\preceq(\alpha+\beta)\big[\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big)+\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)\big]\\
&\quad\preceq h\big[\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big)+\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)\big].
\end{aligned}$$

Since $\widehat{\rho}(h)<\frac{1}{s}$, we conclude that

$$\mathcal{N}_b\big(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho'\big)+\mathcal{N}_b\big(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho'\big)=\theta.$$

So, $\mathcal{N}_b(\mathcal{T}\rho^*,\mathcal{T}\rho^*,\mathcal{T}\rho') = \mathcal{N}_b(\mathcal{T}\varrho^*,\mathcal{T}\varrho^*,\mathcal{T}\varrho') = \theta$, that is, $\mathcal{T}\rho^* = \mathcal{T}\rho'$ and $\mathcal{T}\varrho^* = \mathcal{T}\varrho'$. As $\mathcal{T}$ is one-to-one, we have $(\rho^*,\varrho^*) = (\rho',\varrho')$. Thus $\mathcal{F}$ has a unique coupled fixed point. □

Corollary 1 *Let $(\mathcal{M},\mathcal{N}_b)$ be a complete $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 4$ and $\mathcal{P}$ be a solid cone in $\mathcal{A}$ and $\mathcal{T}:\mathcal{M}\to\mathcal{M}$ is a surjective and one-to-one mapping. Then any $\mathcal{T}$-contraction on $\mathcal{M}$ admits a unique fixed point.*

The following example is an illustration of Theorem 2.

Example 3 Let $\mathcal{A} = \mathcal{C}_1^R[0,1]$ and define a norm on $\mathcal{A}$ by $\|\rho\| = \|\rho\|_\infty + \|\rho'\|_\infty$ for all $\rho\in\mathcal{A}$. Let the multiplication in $\mathcal{A}$ be the point wise multiplication. Then $\mathcal{A}$ is a real unit Banach algebra with unit $e = 1$. Set $\mathcal{P} = \{\rho\in\mathcal{A}:\rho\geq 0\}$ which is a cone in $\mathcal{A}$. Moreover, $\mathcal{P}$ is not normal (see [21]). Let $\mathcal{M} = [0,+\infty]$. Define a mapping $\mathcal{N}_b:\mathcal{M}^3\to\mathcal{A}$ by $\mathcal{N}_b(\rho,\varrho,\sigma)(t) = (|\rho-\sigma|+|\varrho-\sigma|)^2e^t$, for all $\rho,\varrho,\sigma\in\mathcal{M}$. Then $(\mathcal{M},\mathcal{N}_b)$ is a complete $\mathcal{N}_b$-cone metric space over a Banach algebra $\mathcal{A}$. Now define a mapping $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ by $\mathcal{F}(\rho,\varrho) = \frac{\rho+\varrho}{8}$, for all $\rho,\varrho\in\mathcal{M}$ and $\mathcal{T}:\mathcal{M}\to\mathcal{M}$ by $\mathcal{T}(\rho) = \frac{\varrho}{3}$, for all $\rho\in\mathcal{M}$. Also,

$$\begin{aligned}
&\mathcal{N}_b\bigl(\mathcal{T}\mathcal{F}(\rho,\varrho),\mathcal{T}\mathcal{F}(\rho,\varrho),\mathcal{T}\mathcal{F}(u,v)\bigr)(t)\\
&\quad= \bigl(\bigl|\mathcal{T}\mathcal{F}(\rho,\varrho)-\mathcal{T}\mathcal{F}(u,v)\bigr|+\bigl|\mathcal{T}\mathcal{F}(\rho,\varrho)-\mathcal{T}\mathcal{F}(u,v)\bigr|\bigr)^2e^t\\
&\quad= 4\left(\left|\frac{\rho+\varrho}{24}-\frac{u+v}{24}\right|\right)^2e^t\\
&\quad= \frac{1}{16}\left(\left|\frac{\rho-u}{3}+\frac{\varrho-v}{3}\right|\right)^2e^t\\
&\quad\preceq \frac{1}{16}\left[2\left|\frac{\rho-u}{3}\right|^2+2\left|\frac{\varrho-v}{3}\right|^2\right]e^t\\
&\quad= \frac{1}{16}\left[2\left|\frac{\rho}{3}-\frac{u}{3}\right|^2+2\left|\frac{\varrho}{3}-\frac{v}{3}\right|^2\right]e^t\\
&\quad= \frac{1}{8}\bigl[\mathcal{N}_b(\mathcal{T}\rho,\mathcal{T}\rho,\mathcal{T}u)(t)+\mathcal{N}_b(\mathcal{T}\varrho,\mathcal{T}\varrho,\mathcal{T}v)(t)\bigr],
\end{aligned}$$

where $\alpha = \beta = \frac{1}{8}$. Thus, the conditions of Theorem 2 hold and $(0,0)$ is a coupled fixed point of $\mathcal{F}$.

7 Application to the existence of solutions of integral equations

Consider $\mathcal{C}([0,T],\mathbb{R})$, the class of continuous functions on $[0,T]$, $T>0$. Let the cone metric $\mathcal{N}_b$ be given by

$$\mathcal{N}_b(\rho,\varrho,\sigma)(t) = \sup_{t\in[0,T]}\bigl(\bigl(|\varrho+\sigma-2\rho|+|\varrho-\sigma|\bigr)^p,\alpha\bigl(|\varrho+\sigma-2\rho|+|\varrho-\sigma|\bigr)^p\bigr)e^t,$$

for all $\rho,\varrho,\sigma\in\mathbb{R}$, where $p>1$ and $\alpha>0$ be arbitrary constants. Note that $\mathcal{C}([0,T],\mathbb{R},\mathcal{N}_b)$ is a complete $\mathcal{N}_b$-cone metric spaces over a Banach algebra with coefficient $s\geq 4$.

Theorem 3 *Let K,f,g and a be the mappings such that*:

(i) $\sup_{t\in[0,T]}|K(t,s)| = M < \frac{1}{T}$, *where* $K\in\mathcal{C}([0,T]\times[0,T],\mathbb{R})$;

(ii) $a\in\mathcal{C}([0,T],\mathbb{R})$;

(iii) $f, g \in \mathcal{C}([0,T] \times \mathbb{R}, \mathbb{R})$;
(iv) $|f(t,r,\rho_1(t)) - f(t,r,\rho_2(t))| + |f(t,\varrho_1(t)) - f(t,\varrho_2(t))| \leq \lambda(|\rho_1(t) - \rho_2(t)| + |\varrho_1(t) - \varrho_2(t)|)$,

for all $\rho_i, \varrho_i \in \mathcal{C}([0,T], \mathbb{R})$, $i = 1,2$, $t \in [0,T]$, *where* $\lambda \in [0, \frac{1}{s})$.

Then the integral equation

$$\rho(t) = \int_0^T K(t,s)\bigl(f\bigl(s,\rho(s)\bigr) + g\bigl(s,\rho(s)\bigr)\bigr)\,ds + a(t), \quad t \in [0,T], \tag{7.1}$$

has a unique solution in $\mathcal{C}([0,T], \mathbb{R})$.

Proof Let $\mathcal{A} = \mathbb{R}^2$ be equipped with the norm $\|(u_1,u_2)\| = |u_1| + |u_2|$ and the multiplication on $\mathbb{R}^2$ be defined by

$$uv = (u_1,u_2)(v_1,v_2) = (u_1v_1, u_1v_2 + u_2v_1).$$

Let $\mathcal{P} = \{u = (u_1,u_2) \in \mathcal{A} : u_1, u_2 \geq 0\}$. Clearly, $\mathcal{P}$ is a normal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = (1,0)$. Let $\mathcal{F} : \mathcal{C}([0,T], \mathbb{R}) \times \mathcal{C}([0,T], \mathbb{R}) \to \mathcal{C}([0,T] \times \mathbb{R})$ be defined by

$$\mathcal{F}(\rho,\varrho)(t) = \int_0^T K(t,s)\bigl(f\bigl(s,\rho(s)\bigr) + g\bigl(s,\varrho(s)\bigr)\bigr)\,ds + a(t),$$

for all $t \in [0,T]$. Using Corollary 1 we obtain

$$\begin{aligned}
&\mathcal{N}_b\bigl(\mathcal{F}(\rho_1,\varrho_1), \mathcal{F}(\rho_1,\varrho_1), \mathcal{F}(\rho_2,\varrho_2)\bigr)(t) \\
&\quad = 2\bigl(\bigl|\mathcal{F}(\rho_2,\varrho_2) - \mathcal{F}(\rho_1,\varrho_1)\bigr|, \alpha\bigl|\mathcal{F}(\rho_2,\varrho_2) - \mathcal{F}(\rho_1,\varrho_1)\bigr|\bigr)^p e^t \\
&\quad \preceq 2\biggl(\int_0^T \bigl|K(t,s)\bigr|\bigl(\bigl|f\bigl(s,\rho_2(s)\bigr) - f\bigl(s,\rho_1(s)\bigr)\bigr|^p + \bigl|g\bigl(s,\varrho_2(s)\bigr) - g\bigl(s,\varrho_1(s)\bigr)\bigr|^p\bigr)\,ds, \\
&\qquad \alpha\int_0^T \bigl|K(t,s)\bigr|\bigl(\bigl|f\bigl(s,\rho_2(s)\bigr) - f\bigl(s,\rho_1(s)\bigr)\bigr|^p + \bigl|g\bigl(s,\varrho_2(s)\bigr) - g\bigl(s,\varrho_1(s)\bigr)\bigr|^p\bigr)\,ds\biggr)e^t \\
&\quad \preceq \biggl(\lambda\int_0^T \bigl|K(t,s)\bigr|\bigl(2\bigl|\rho_2(s) - \rho_1(s)\bigr|^p + 2\bigl|\varrho_2(s) - \varrho_1(s)\bigr|^p\bigr)\,ds, \\
&\qquad \alpha\lambda\int_0^T \bigl|K(t,s)\bigr|\bigl(2\bigl|\rho_2(s) - \rho_1(s)\bigr|^p + 2\bigl|\varrho_2(s) - \varrho_1(s)\bigr|^p\bigr)\,ds\biggr)e^t \\
&\quad \preceq (\lambda,0)\bigl[\mathcal{N}_b(\rho_1,\rho_1,\rho_2)(t) + \mathcal{N}_b(\varrho_1,\varrho_1,\varrho_2)(t)\bigr],
\end{aligned}$$

which implies that

$$\mathcal{N}_b\bigl(\mathcal{F}(\rho_1,\varrho_1), \mathcal{F}(\rho_1,\varrho_1), \mathcal{F}(\rho_2,\varrho_2)\bigr)(t) \preceq (\lambda,0)\bigl[\mathcal{N}_b(\rho_1,\rho_1,\rho_2)(t) + \mathcal{N}_b(\varrho_1,\varrho_1,\varrho_2)(t)\bigr].$$

Since $\|(\lambda,0)^n\|^{\frac{1}{n}} = \|(\lambda^n,0)\|^{\frac{1}{n}} \to \lambda < \frac{1}{s}$ $(n \to +\infty)$, letting $\alpha = (\lambda,0)$ and $\beta = (\lambda,0)$, then all the conditions of Corollary 1 hold and hence there exists a unique coupled fixed point of $\mathcal{F}$. So, the integral equation (7.1) has a unique solution. □

Example 4 Let $C[0,1]$ be the set of all continuous functions on $[0,1]$. Consider the nonlinear integral equation

$$x(t) = \frac{t}{1+\sqrt{t}} + \int_0^1 \frac{\sin s\pi}{8+t}\left[\frac{e^{(-tx(s))}}{9} + \frac{\sin t}{9}.\frac{|x(s)|}{1+|x(s)|}\right]ds, \quad t \in [0,1]. \tag{7.2}$$

Put $a(t) = t/(1+\sqrt{t})$, for all $t \in [0,1]$, $K(s,t) = \frac{\sin s\pi}{8+t}$, for all $t,s \in [0,1]$.
Also, let $f(t,x(s)) = \frac{e^{(-tx(s))}}{9}$ and $g(t,y(s)) = \frac{\sin t}{9}.\frac{|x(s)|}{1+|x(s)|}$ for all $x,y \in C[0,1]$ and $t,s \in [0,1]$.
Then

(1) $a(t) \in C([0,1],R)$,

(2) $f,g \in C([0,1] \times R,R)$,

(3) $\sup_{(t\in[0,1])} |K(t,s)| = M < 1$, where $K \in C([0,1]\times[0,1],R)$,

(4) for all $x_i, y_i \in C([0,1],R)$, $i = 1,2$ and $t \in [0,1]$, we have

$$\left|f\big(t,x_1(t)\big) - f\big(t,x_2(t)\big)\right| + \left|g\big(t,y_1(t)\big) - g\big(t,y_2(t)\big)\right| \le \lambda\left[\left|x_1(t)-x_2(t)\right| + \left|y_1(t)-y_2(t)\right|\right].$$

Proof (1) Since $a(t) = \frac{t}{1+\sqrt{t}}$ for all $t \in [0,1]$, we have $a(t) \in C([0,1],R)$.

(2) Define the mappings $f,g : C([0,1]\times R) \to R$ by $f(t,x(s)) = \frac{e^{(-tx(s))}}{9}$ and $g(t,y(s)) = \frac{\sin t}{9}.\frac{|x(s)|}{1+|x(s)|}$ for all $x,y \in C[0,1]$ and for all $t \in [0,1]$. Then $f,g \in C([0,1]\times R,R)$.

(3) As $K(s,t) = \frac{\sin s\pi}{8+t}$, it is easy to see that $K(t,s)$ is continuous on $[0,1]\times[0,1]$ and $\sup_{(t\in[0,1])} |k(t,s)| = \frac{1}{8} < 1$.

(4) Now, let $x,y \in C([0,1],R)$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}
&\left|f\big(t,x_1(s)\big) - f\big(t,x_2(s)\big)\right| + \left|g\big(t,y_1(s)\big) - g\big(t,y_2(s)\big)\right| \\
&\quad = \left|\frac{e^{(-tx_1(s))}}{9} - \frac{e^{(-tx_2(s))}}{9}\right| + \left|\frac{\sin t}{9}.\frac{|y_1(s)|}{1+|y_1(s)|} - \frac{\sin t}{9}.\frac{|y_2(s)|}{1+|y_2(s)|}\right| \\
&\quad \le \left[\left|\frac{-te^{(-t\zeta)}}{9}\big(x_1(s)-x_2(s)\big)\right| + 1/9\left|\frac{|y_1(s)|}{1+|y_1(s)|} - \frac{|y_2(s)|}{1+|y_2(s)|}\right|\right] \\
&\quad = 1/9\left|x_1(s)-x_2(s)\right| + 1/9\left|\left(1-\frac{1}{1+|y_1(s)|}\right) - \left(1-\frac{1}{1+|y_2(s)|}\right)\right| \\
&\quad = 1/9\left|x_1(s)-x_2(s)\right| + 1/9\left|\frac{1}{1+|y_1(s)|} - \frac{1}{1+|y_2(s)|}\right| \\
&\quad \le 1/9\left|x_1(s)-x_2(s)\right| + 1/9\left|\frac{-1}{(1+\varepsilon)^2}\right|\left|\left|y_1(s)\right| - \left|y_2(s)\right|\right| \\
&\quad \le 1/9\left[\left|x_1(s)-x_2(s)\right| + \left|y_1(s)-y_2(s)\right|\right], \quad \text{as } \left|\big(|x|-|y|\big)\right| \le |x-y|.
\end{aligned}$$

From the above K, a, f and g satisfy all assumptions of Theorem 3. Hence, the integral equation (7.2) has a unique solution in $C([0,1],R)$. □

8 Conclusion

In this paper, we defined a new space named $\mathcal{N}_b$-cone metric spaces over a Banach algebra and proved some coupled fixed point results for two mappings $\mathcal{F}$ and $\mathcal{T}$ satisfying certain contractive conditions. Moreover, we gave an example and an application to the existence of solution of integral equations to validate our results.

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