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Study of hybrid orthonormal functions method for solving second kind fuzzy Fredholm integral equations

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Abstract

The approximate numerical solution of the linear second kind of fuzzy integral Fredholm equations is discussed in this article. A new approach uses hybrid functions, and some useful properties of these functions are proposed to transform linear second type fuzzy integral Fredholm equations into an algebraic equation. The new approach is a mixture of Bernstein polynomials (BPs) and enhanced block-pulse functions (IBPFs) at interval $[0, 1)$. The approach is appealing and very easy to implement computationally. Some numerical tests show the reliability and exactness of the suggested scheme.

MSC: 65T60; 65R20; 45A05; 41A30

Keywords: Fuzzy Fredholm integral equation; Bernstein polynomial; Improved block-pulse function; Absolute error; Accuracy

1 Introduction

There are many fuzzy mathematical models for the analysis of fuzzy systems with applications. The main objective of these models is to focus on how to use the model of a fuzzy system to solve fuzzy mathematics problems. Some mathematical models are based on fuzzy set theory; see [1]. In various cases of data collection one may identify situations, where measurements in a data sample are only partially associated with their underlying population. The presence of such data imposes challenges to any statistical procedure of the comparison of distributions or numerical characteristics of variables. Some work presents procedures to test the identity of distributions on the basis of one key, the so-called fuzzy samples; see for example [2]. Moreover, fuzzy systems and neural network techniques seem very well suited for typical technical problems; for example, see [3, 4] where the generalized net model is addressed to the appraisal of lecturers with intuitionistic fuzzy estimations that represent a model of a digital university.

Integral fuzzy forms equations in applied mathematics are necessary to research and solve a large proportion of the problems in different topics. Fuzzy integral equations (FIEs) topics have been of increasing interest for some time, particularly in physics, geography, medicine, biology, and fuzzy control, and have grown rapidly in recent years. Some of

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the parameters for our problems are usually set for several applications that are typically defined by fuzzy numbers rather than smooth, and therefore it is important to establish mathematical models and computational methods for appropriate care and resolution of general fuzzy integral equations. Those were studied in particular by Nanda [5], Kaleva [6], Ralescu and Adams [7], Goetschel and Voxman [8], Wang [9], and Bede and Gal [10]. Molabahrami et al. [11] recently used the parametric shape of a fuzzy number and in the crisp case they transformed a linear fuzzy integral Fredholm equation to two linear systems of second type integral equations. These equations are usually analytically difficult to solve. Therefore approximate solutions need to be found. A variety of computational methods for solving linear fuzzy Fredholm integral equations have recently been investigated. Block-pulse functions [12–17], hair functions [18], triangular functions [19–21], Chebyshev hybrids with functions of block-pulse form [22, 23], Legendre hybrid with functions of block-pulse form [24–26], hybrid Taylor, block-pulse functions [27], Fourier hybrid with functions of block-pulse form [28] and Bernstein polynomials hybrid with functions of block-pulse form [29–31]. Harmuth first introduced block-pulse functions to electrical engineering, and other scholars addressed the block pulse [32–35]. Bernstein polynomials play a prominent role in various mathematical fields. These polynomials were also used to solve integral equations, differentials and in approximation theory [36–40]. The improved block-pulse function is introduced by Farshid Mirzaee [41]. These modified block-pulse functions are applied to numerical solution of stochastic Volterra integral equations [42]. The Bernstein polynomials (BPs) and improved block-pulse functions (IBPFs) are introduced in [43].

The purpose of this article is the application of the HBIBP method, for the first time, to get an approximate solution for the second kind of linear Fredholm fuzzy equations. Finally, we test the system on a few examples to demonstrate its accuracy and effectiveness.

The paper is organized as follows: Several notes and theorems on fuzzy set structures and analysis of integral equations of fuzzy Fredholm type, improved block-pulse function and polynomials of Bernstein and its properties are briefly discussed in Sect. 2. In Sect. 3, we are launching a new blend of the Bernstein and enhanced block-pulse functions. The linear integral Fredholm equation is solved using a new basis in Sect. 4. The error estimate is given in Sect. 5 for the proposed method. Test problems to demonstrate the reliability and accuracy of the proposed scheme are considered in Sect. 6. Finally, we make our closing remarks.

2 Preliminaries

2.1 Bernstein-type polynomials

Definition ([44]) The Mth degree Bernstein polynomials are defined on the interval $[0, 1]$ by

$$B_{i,M}(x) = \binom{M}{i} x^i (1-x)^{M-i}, \quad i = 0, 1, \dots, M,$$

where $\binom{M}{i} = \frac{M!}{i!(M-i)!}$.

By binomial expansion of $(1-x)^{M-i}$, we get

$$\binom{M}{i} x^i (1-x)^{M-i} = \sum_{k=0}^{M-i} (-1)^k \binom{M}{i} \binom{M-i}{k} x^{i+k},$$

and we normally set for mathematical convenience $B_{i,M} = 0$ if $i < 0$ or $i > M$. One can also use a recursive description to generate the Bernstein polynomials over $[0,1]$ so that the ith Bernstein polynomial of Mth degree takes the form

$$B_{i,M}(x) = (1-x)B_{i,M-1}(x) + xB_{i-1,M-1}(x).$$

The Bernstein polynomials can be proved to be positive and the sum of all the Bernstein polynomials is unity, for all real $x \in [0,1]$, i.e., $\sum_{i=0}^{M} B_{i,M}(x) = 1$ (unity partition property). We can easily prove that every Mth degree polynomial can be extended in terms of these basis functions.

2.2 Improved block-pulse function

Definition An $(n+1)$-set of IBPFs consists of $(n+1)$ functions that are specified over a region D, with a slight change in the definition of function, as follows [41]:

$$\varphi_1(x) = \begin{cases} 1, & x \in [0, \frac{h}{2}), \\ 0, & \text{otherwise}, \end{cases}$$

$$\varphi_i(x) = \begin{cases} 1, & x \in [(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}), \\ 0, & \text{otherwise}, \end{cases} \quad i = 2, 3, \dots, n,$$

$$\varphi_{n+1}(x) = \begin{cases} 1, & x \in [1 - \frac{h}{2}, 1), \\ 0, & \text{otherwise}. \end{cases}$$

Here n is an arbitrary positive integer, and $h = \frac{1}{n}$.

The most important properties for these functions are disjointness, orthogonality, and completeness.

Lemma *Let a set of improved block-pulse functions (IBPFs) $\varphi_i(x)$, $i = 1, 2, \dots, N+1$ be defined on the interval $[0,1)$ such that*

$$\varphi_1(x) = \begin{cases} 1, & x \in [0, \frac{h}{2}), \\ 0, & \textit{otherwise}, \end{cases}$$

$$\varphi_i(x) = \begin{cases} 1, & x \in [(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}), \\ 0, & \textit{otherwise}, \end{cases} \quad i = 2, 3, \dots, n,$$

$$\varphi_{n+1}(x) = \begin{cases} 1, & x \in [1 - \frac{h}{2}, 1), \\ 0, & \textit{otherwise}. \end{cases}$$

The properties of such functions correspond to the following:

(i) *disconnectedness*,
(ii) *orthogonality*,
(iii) *completeness*.

Proof Using the definition of an improved block pulse, the disjointness property can be obtained as follows:

$$\varphi_i(x)\varphi_j(x) = \begin{cases} \varphi_i(x), & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The other property of IBPFs is mutual orthogonality, where $x \in D$,

$$\int_0^1 \varphi_i(x)\varphi_j(x)\,dx = \begin{cases} \frac{h}{2}, & i = j \in \{1, n+1\}, \\ h, & i = j \in \{2, 3, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

For the completeness property: every $f \in L^2[0,1]$. If n goes to infinity, Parseval's identity holds:

$$\int_0^1 f^2(x)\,dx = \sum_{i=0}^{\infty} f_i^2 \|\phi_i(x)\|^2, \quad \text{where } f_i = \frac{1}{h}\int_0^1 f(x)\phi_i(x)\,dx. \qquad \square$$

2.2.1 Vector forms of IBPFS

Consider the first $(n+1)$ terms of IBPFs and write them concisely *as* $(n+1)$-vector

$$\Phi_n(x) = \bigl[\varphi_1(x), \varphi_2(x), \dots, \varphi_{n+1}(x)\bigr]^T, \quad x \in D, \tag{2.1}$$

from disjointness

$$\Phi_n(x)\Phi_n^T(x) = \begin{pmatrix} \varphi_1(x) & 0 & \dots & 0 \\ 0 & \varphi_2(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varphi_{n+1}(x) \end{pmatrix} = \operatorname{diag}\bigl(\Phi_n(x)\bigr).$$

2.2.2 IBPFs expansions

A continuous function $f(x) \in L^2(D)$ may be expanded by the IBPFs as

$$f(x) \simeq f_n(x) = \sum_{i=1}^{n+1} f_i\varphi_i(x) = F_n^T\Phi_n(x) = \Phi_n^T(x)F_n,$$

where F_n is a $(n+1) \times 1$ vector given by

$$F_n = [f_1, f_2, \dots, f_{n+1}]^T,$$

and $\Phi_n(x)$ is defined in (2.1) and f_i is obtained as

$$f_i = \begin{cases} 2n\int_0^{\frac{h}{2}} f(x)\,dx, & i = 1, \\ n\int_{(i-2)h+\frac{h}{2}}^{(i-1)h+\frac{h}{2}} f(x)\,dx, & i = 2, \dots, n, \\ 2n\int_{1-\frac{h}{2}}^{1} f(x)\,dx, & i = n+1. \end{cases}$$

Similarly a function of two variables $K(x,y) \in L^2(D \times D)$ can be approximated by IBPFs as follows:

$$K(x,y) \simeq \Phi_{n+1}^T(x) K_n \Phi_{n+1}(y).$$

Here $\Phi_n(x)$ and $\Phi_n(y)$ are IBPFs vector of dimension $(n+1)$ and $K_n = [k_{i,j}]$ is the $(n+1) \times (n+1)$ IBPFs coefficient matrix of $k(x,y)$.

2.3 Fuzzy functions

Now recall the following definitions needed throughout the paper.

Definition ([45]) A fuzzy number is a fuzzy set $u: R^1 \to [0,1]$ that satisfies following condition:

- u is upper semi-continuous.
- $u(x) = 0$ outside the interval $[c,d]$.
- Here the real numbers a and b: $c \le a \le b \le d$ for which
  - $u(x)$ shows rising monotony on $[c,a]$,
  - $u(x)$ is monotonously declining on $[b,d]$,
  - $u(x) = 1$, $a \le x \le b$.

Definition ([45]) A fuzzy number u is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, that satisfy the following requirements:

- $\underline{u}(r)$ is a bounded monotonic, continuous increasing left function,
- $\overline{u}(r)$ is a bounded monotonous decreasing continuous left function,
- $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$ we describe adding $(u+v)$ and multiplication by k as follows:

- $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) + \overline{v}(r)$,
- $u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,
- $$(\lambda \otimes u) = \begin{cases} (\lambda \underline{u}(r), \lambda \overline{u}(r)) & \lambda \ge 0, \\ (\lambda \overline{u}(r), \lambda \underline{u}(r)) & \lambda < 0. \end{cases}$$

2.4 Fredholm integral equations of fuzzy form

The Fredholm fuzzy second form integral equation is

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^b k(x,t) \tilde{u}(t)\,dt, \tag{2.2}$$

where > 0, $k(x,t)$ is an arbitrary kernel function and $\tilde{f}(x)$ and $\tilde{u}(x)$ are fuzzy functions over $[a,b]$.

The solution of Eq. (2.2) is crisp where $\tilde{f}(x)$ is a crisp function. However, if $\tilde{f}(x)$ is a fuzzy function, then Eq. (2.2) can possess only fuzzy solutions. For the implementation of the numerical method for solving Eq. (2.2), we rewrite it as

$$\underline{u}(x,r) = \underline{f}(x,r) + \lambda \int_a^b \underline{U}(t,r)\,dt,$$

$$\overline{u}(x,r) = \overline{f}(x,r) + \lambda \int_a^b \overline{U}(t,r)dt,$$

where

$$\underline{U}(t,r) = \begin{cases} k(x,t)\underline{u}(t,r), & k(x,t) \geq 0, \\ k(x,t)\overline{u}(t,r), & k(x,t) < 0, \end{cases}$$

$$\overline{U}(t,r) = \begin{cases} k(x,t)\overline{u}(t,r), & k(x,t) \geq 0, \\ k(x,t)\underline{u}(t,r), & k(x,t) < 0. \end{cases}$$

Theorem 1 *Let $k(x,t)$ be continuous for $a \leq x, t \leq b$ and $\tilde{f}(x)$ be a fuzzy continuous of x, $a \leq x \leq b$. If $\lambda < \frac{1}{M(b-a)}$, where $M = \max_{a\leq x,t\leq b} |k(x,t)|$, then the iterative procedure*

$$\tilde{u}_0(x) = \tilde{f}(x),$$

$$\vdots$$

$$\tilde{u}_k(x) = \tilde{f}(x) + \lambda \int_a^b k(x,t)\tilde{u}_{k-1}(t)dt, \quad k \geq 1,$$

converges to the unique solution of (2.2). *Specially,*

$$\sup_{a\leq x\leq b} D\big(\tilde{u}(x), \tilde{u}_k(x)\big) \leq \frac{L^k}{1-L} \sup_{a\leq x\leq b} D\big(\tilde{u}_0(x), \tilde{u}_1(x)\big),$$

where $L = \lambda M(b-a)$.

3 New hybrid Bernstein improved block-pulse functions (HBIBPFs) method

Definition ([43]) $HBIBP_{i,j}(x)$ is the combination of Bernstein polynomials and improved block-pulse functions where both are complete and orthogonal, then the set is a complete orthogonal complete system. Hybrid orthonormal Bernstein and improved block-pulse functions where $j = 0, 1, \dots, M$, $i = 1, 2, \dots, N+1$, $HBIBP_{i,j}(x)$ have two arguments; i and j are the order of IBPFs and degree of BPs, respectively. $HBIBP(x)$ is defined on the interval $[0, 1)$ as follows:

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}(\frac{2x}{h}), & x \in [0, \frac{h}{2}), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i = 1, j = 0, 1, \dots, M, \tag{3.1}$$

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}(\frac{x}{h} + \frac{3}{2} - i), & x \in [(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}), \\ 0, & \text{otherwise,} \end{cases}$$
$$\text{for } i = 2, 3, \dots, N, j = 0, 1, \dots, M, \tag{3.2}$$

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}(\frac{2x}{h} - \frac{2}{h} + 1), & x \in [1 - \frac{h}{2}, 1), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i = N+1, j = 0, 1, \dots, M. \tag{3.3}$$

Thus, our new basis is $\{HBIBP_{1,0}, HBIBP_{1,1}, \dots, HBIBP_{N+1,M}\}$. The following section demonstrates the problem of approximating these functions.

3.1 Function approximation by HBIBPFs

In terms of the HBIBP basis a function $u(x)$ can be expressed as follows:

$$u(x) = \sum_{i=1}^{N+1}\sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x) = C^T HBIBP(x), \tag{3.4}$$

where

$$HBIBP(x) = [HBIBP_{1,0}, HBIBP_{1,1}, \dots, HBIBP_{N+1,M}]^T$$

and

$$C = [c_{1,0}, c_{1,1}, \dots, c_{N+1,M}]^T,$$

we have

$$C^T\bigl\langle HBIBP(x), HBIBP(x)\bigr\rangle = \bigl\langle u(x), HBIBP(x)\bigr\rangle,$$

then

$$C = L^{-1}\bigl\langle u(x), HBIBP\bigr\rangle, \tag{3.5}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product and L is an $((N+1)(M+1)\times(N+1)(M+1))$ matrix that is said to be the dual matrix; that is,

$$\begin{aligned} L &= \bigl\langle HBIBP(x), HBIBP(x)\bigr\rangle \\ &= \int_0^1 HBIBP(x).HBIBP^T(x)\,dx \\ &= \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \cdots & 0 \\ 0 & 0 & L_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L_{n+1} \end{pmatrix}, \end{aligned} \tag{3.6}$$

L_i $(i = 1, 2, \dots, n+1)$ is defined as follows:

$$\begin{aligned} (L_1)_{i+1,j+1} &= \int_0^{\frac{h}{2}} B_{i,M}\left(\frac{2x}{h}\right)B_{j,M}\left(\frac{2x}{h}\right)dx = \frac{h}{2}\int_0^1 B_{i,M}(x)B_{j,M}(x)\,dx \\ &= \frac{h\binom{M}{i}\binom{M}{j}}{2(2M+1)\binom{2M}{i+j}}, \quad \text{for } i,j = 0,\dots,M, \\ (L_r)_{i+1,j+1} &= \int_{(i-2)h+\frac{h}{2}}^{(i-1)h+\frac{h}{2}} B_{i,M}\left(\frac{x}{h}+\frac{3}{2}-i\right)B_{j,M}\left(\frac{x}{h}+\frac{3}{2}-i\right)dx, \quad \text{for } r = 2,\dots,N \\ &= h\int_0^1 B_{i,M}(x)B_{j,M}(x)\,dx = \frac{h\binom{M}{i}\binom{M}{j}}{(2M+1)\binom{2M}{i+j}}, \quad \text{for } i,j = 0,\dots,M, \end{aligned}$$

$$
\begin{aligned}
(L_{N+1})_{i+1,j+1} &= \int_{1-\frac{h}{2}}^{1} B_{i,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) B_{j,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) dx \\
&= \frac{h}{2} \int_0^1 B_{i,M}(x) B_{j,M}(x)\, dx = \frac{h\binom{M}{i}\binom{M}{j}}{2(2M+1)\binom{2M}{i+j}}, \quad \text{for } i,j = 0,\dots,M.
\end{aligned}
$$

We can also approximate the function $k(x,t) \in L^2([0,1] \times [0,1])$ as follows:

$$
k(x,t) = HBIBP^T(x).K.HBIBP(t), \tag{3.7}
$$

where K is an $(M+1)(N+1)$ matrix that we can obtain as follows:

$$
K = L^{-1}\langle HBIBP(x), \langle k(x,t), HBIBP(t)\rangle\rangle L^{-1}.
$$

4 Solving integral Fredholm equation of second kind via HBIBP function

In this section, a *HBIBPF* method is presented to transform the fuzzy Fredholm integral equation of the second kind that is linear (FFIE-2) to two crisp coupled systems. First we consider the fuzzy Fredholm case as the integral Eq. (2.2) in this article with $a = 0$, $b = 1$ and $\lambda = 1$, where $\tilde{u}(x), \tilde{f}(x) \in L^2([0,1))$ and $k(x,t) \in L^2([0,1) \times [0,1))$. Our task is to evaluate *HBIBP* coefficients from the information functions and kernel within the interval $[0,1)$. So, we can add the FFIE-2 parametric form as follows:

$$
\underline{u}(x,r) = \underline{f}(x,r) + \lambda \int_a^b \underline{U}(t,r)dt, \tag{4.1}
$$

$$
\overline{u}(x,r) = \overline{f}(x,r) + \lambda \int_a^b \overline{U}(t,r)dt. \tag{4.2}
$$

Let us expand $\underline{u}(x,r)$, $\overline{u}(x,\ r)$, $\underline{f}(x,r)$, $\overline{f}(x,\ r)$ and $k(x,t)$ by *HBIBPFs* (3.1)–(3.7) as follows:

$$
\begin{aligned}
&k(x,t) = HBIBP^T(x).K.HBIBP(t), \\
&\underline{u}(x,r) = HBIBP^T(x).U1.HBIBP(r), \qquad \overline{u}(x,r) = HBIBP^T(x).U2.HBIBP(r), \\
&\underline{f}(x,r) = HBIBP^T(x).F1.HBIBP(r), \qquad \overline{f}(x,r) = HBIBP^T(x).F2.HBIBP(r).
\end{aligned}
$$

Substituting the above equations into Eq. (4.1) yields

$$
\begin{aligned}
&HBIBP^T(x).U1.HBIBP(r) \\
&\quad = HBIBP^T(x).F1.HBIBP(r) \\
&\qquad + \lambda \int_0^1 HBIBP^T(x).K.HBIBP(t).HBIBP^T(t).U1.HBIBP(r)\, dt, \\
&HBIBP^T(x).U1.HBIBP(r) \\
&\quad = HBIBP^T(x).F1.HBIBP(r) \\
&\qquad + \lambda.HBIBP^T(x).K\left(\int_0^1 .HBIBP(t).HBIBP^T(t)\, dt\right) U1.HBIBP(r)
\end{aligned}
$$

manipulating Eq. (3.6) and the above equation yields

$$\begin{aligned}&HBIBP^T(x).U1.HBIBP(r)\\&\quad= HBIBP^T(x).F1.HBIBP(r) + \lambda.HBIBP^T(x).K.D.U1.HBIBP(r).\end{aligned}$$

Then

$$U1 = F1 + \lambda.K.D.U1 \quad \Rightarrow \quad (I - \lambda.K.D)U1 = F1.$$

Here I is the $(N+1)(M+1)$ identity matrix, then the above equation can be written as follows:

$$U1 = (I - \lambda.K.D)^{-1}F1.$$

For the solution of the above matrix, we can find the matrix $U1$. So $\underline{u}(x,r) = HBIBP^T(x).U1.$ $HBIBP(r)$. The same pattern is true of Eq. (4.2).

5 The convergence characteristics of the proposed method

This section estimates the error regarding the proposed number method.

Theorem 2 *The Fredholm fuzzy integral equation solution using HBIBPFs converges if* $M < 1$; *where* $M = \max_{0\leq x,t\leq 1} |k(x,t)|$.

Proof Assume $\tilde{u}_E(x)$ and $\tilde{u}(x)$ are the exact and approximate solutions of Eq. (2.2), respectively. Then

$$\begin{aligned}D\big(\tilde{u}(x),\tilde{u}_E(x)\big) &= D\left(\int_0^1 k(x,t)\tilde{u}(t)dt, \int_0^1 k(x,t)\sum_{i=1}^{N+1}\sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x)dt\right)\\&\leq M\int_0^1 D\left(\tilde{u}(x),\sum_{i=1}^{N+1}\sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x)\right)dt,\end{aligned}$$

$$D\big(\tilde{u}(x),\tilde{u}_E(x)\big) \leq M\int_0^1 D\big(\tilde{u}(x),\tilde{u}_E(x)\big)\,dt,$$

$$\sup_{0\leq x\leq 1} D\big(\tilde{u}(x),\tilde{u}_E(x)\big) \leq M \sup_{0\leq x\leq 1} D\big(\tilde{u}(x),\tilde{u}_E(x)\big).$$

Therefore if $M < 1$, we will have

$$\lim_{(n+1)(m+1)\to\infty}\left(\sup_{0\leq x\leq 1} D\big(\tilde{u}(x),\tilde{u}_E(x)\big)\right) = 0. \qquad \square$$

6 Numerical examples

The accuracy of the proposed solution is demonstrated by solving the following test problems, where all the computations were carried out using the Matlab program (R2018b).

Example 1 Consider the following Fredholm integral linear fuzzy equation [45, 46]:

$$\underline{f}(x,r) = -\frac{1}{3}x^2 + rx^2 + \frac{1}{3}x + \frac{1}{4}r - \frac{1}{12},$$

Table 1 The numerical results for Example 1 with $x = 0.5$, $M = 2$, $N = 3$

r	Exact solution	HBT method [46]	Block-pulse method [45]	Presented method	Absolute error
	$\underline{u}(x,r)$				
0	0.000000000	0.000000000	0.007956	0.000000000	8.58708660e–11
0.1	0.050000000	0.050000000	0.056347	0.050000000	6.79634155e–11
0.2	0.100000000	0.100000000	0.104737	0.100000000	5.21343024e–11
0.3	0.150000000	0.150000000	0.153128	0.150000000	3.83835130e–11
0.4	0.200000000	0.200000000	0.201519	0.200000000	2.67110889e–11
0.5	0.250000000	0.250000000	0.266040	0.250000000	1.53477231e–12
0.6	0.300000000	0.300000000	0.314430	0.300000000	8.80506779e–13
0.7	0.350000000	0.350000000	0.362820	0.350000000	4.41124914e–13
0.8	0.400000000	0.400000000	0.411210	0.400000000	2.16571205e–13
0.9	0.45000000	0.450000000	0.359603	0.450000000	1.03909548e–11
	$\overline{u}(x,r)$				
0	1.000000000	1.000000000	1.024160	1.000000000	1.00612851e–11
0.1	0.950000000	0.950000000	0.975770	0.950000000	9.72248948e–12
0.2	0.900000000	0.900000000	0.927379	0.900000000	7.29647454e–12
0.3	0.850000000	0.850000000	0.878988	0.850000000	2.78301826e–12
0.4	0.800000000	0.800000000	0.830598	0.800000000	3.81765730e–12
0.5	0.750000000	0.750000000	0.766077	0.750000000	0.00000000e+00
0.6	0.700000000	0.700000000	0.717986	0.700000000	3.09223758e–12
0.7	0.650000000	0.650000000	0.669290	0.650000000	6.18458618e–12
0.8	0.600000000	0.600000000	0.630905	0.600000000	9.27682375e–12
0.9	0.550000000	0.550000000	0.572514	0.550000000	4.72755612e–11

$$\overline{f}(x,r) = \frac{1}{3}x - x^2 r - \frac{1}{4}r + \frac{5}{3}x^2 + \frac{5}{12},$$

and

$$k(x,t) = (2t-1)^2(1-2x), \quad 0 \le x, t \le 1 \text{ and } \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx,$$
$$\overline{u}(x,r) = (2-r)x.$$

Table 1 shows the comparison of the approximate solution for presented method, block-pulse functions [46] for $m = 32$, hybrid block-pulse functions and Taylor series (HBT) with $M = 4$, $N = 2$ (N and M are the order of block-pulse functions and Taylor polynomials, respectively) [45] and the exact solution.

Example 2 Consider the following linear Fredholm integral fuzzy equation [47]:

$$\underline{f}(x,r) = \frac{4}{3}rx - \frac{1}{6}r,$$
$$\overline{f}(x,r) = \frac{8}{3}x - \frac{4}{3}rx + \frac{1}{6}r - \frac{1}{3},$$

and

$$k(x,t) = (2t-1)^2(1-2x), \quad 0 \le x, t \le 1 \text{ and } \lambda = 1.$$

Table 2 The numerical results for Example 2 with $x = 0.5$, $M = 2$, $N = 3$

r	Exact solution	Presented method	Absolute error	Method [47]	
				HBPBP	*BP*
	$\underline{u}(x,r)$				
0	0.000000	0.000000	8.587086597e–11	0.001190	0.104322
0.1	0.050000	0.050000	6.796341545e–11	0.002138	0.024766
0.2	0.100000	0.100000	5.213430243e–11	0.004753	0.025467
0.3	0.150000	0.150000	3.838351303e–11	0.005232	0.030005
0.4	0.200000	0.200000	2.671108890e–11	0.004012	0.085691
0.5	0.250000	0.250000	1.534772309e–12	0.000891	0.070301
0.6	0.300000	0.300000	8.805067786e–13	0.002630	0.084357
0.7	0.350000	0.350000	4.411249144e–13	0.003999	0.099502
0.8	0.400000	0.400000	2.165712054e–13	0.000248	0.054350
0.9	0.450000	0.450000	1.039095476e–11	0.003274	0.066229
1.0	0.500000	0.500000	1.773514668e–11	0.002265	0.012674
	$\overline{u}(x,r)$				
0	1.000000	1.000000	1.006128514e–11	0.002730	0.095432
0.1	0.950000	0.950000	9.722489480e–12	0.001276	0.083485
0.2	0.900000	0.900000	7.296474536e–12	0.001000	0.072324
0.3	0.850000	0.850000	2.783018260e–12	0.008388	0.062431
0.4	0.800000	0.800000	3.817657301e–12	0.001192	0.059004
0.5	0.750000	0.750000	0.000000000e+00	0.002449	0.055920
0.6	0.700000	0.700000	3.092237577e–12	0.008295	0.061354
0.7	0.650000	0.650000	6.184586177e–12	0.002286	0.075632
0.8	0.600000	0.600000	9.276823754e–12	0.000710	0.059959
0.9	0.550000	0.550000	4.727556124e–11	0.005462	0.040971
1.0	0.500000	0.500000	5.854872143e–11	0.001111	0.028821

The exact solution in this case is given by

$$\underline{u}(x,r) = rx,$$
$$\overline{u}(x,r) = (2-r)x.$$

Table 2 shows the comparison of the approximate solution for presented method, two methods in [47] (hybrid block-pulse functions and Bernoulli polynomials (HBPBPs) and Bernoulli polynomials (BPs)) with $M = 10$, $N = 10$ where N and M are the order of block-pulse functions and Bernoulli polynomials, respectively and the exact solution.

Example 3 Consider the linear fuzzy Fredholm integral equation

$$\underline{f}(x,r) = \left(\frac{x}{2} - \frac{1}{3}\right)r,$$
$$\overline{f}(x,r) = \left(\frac{x}{2} - \frac{1}{3}\right)(2-r),$$

and

$$k(x,t) = x + t, \quad 0 \le x, t \le 1 \text{ and } \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx,$$
$$\overline{u}(x,r) = (2-r)x.$$

Table 3 The numerical results for Example 3 with $x = 0.5$, $M = 2$, $N = 3$

r	Exact solution $(\underline{u}(x,r),\overline{u}(x,r))$	Presented method	Absolute error
0	(0.00000, 1.00000)	(0.00000, 1.00000)	(1.11378e–11, 9.60654e–12)
0.1	(0.05000, 0.95000)	(0.05000, 0.95000)	(8.82267e–12, 7.71254e–12)
0.2	(0.10000, 0.90000)	(0.10000, 0.90000)	(6.79151e–12, 5.94123e–12)
0.3	(0.15000, 0.85000)	(0.15000, 0.85000)	(5.04428e–12, 4.29281e–12)
0.4	(0.20000, 0.80000)	(0.20000, 0.80000)	(3.58099e–12, 2.76719e–12)
0.5	(0.25000, 0.75000)	(0.25000, 0.75000)	(1.53477e–12, 0.00000e+00)
0.6	(0.30000, 0.70000)	(0.30000, 0.70000)	(8.80507e–13, 3.09224e–12)
0.7	(0.35000, 0.65000)	(0.35000, 0.65000)	(4.41125e–13, 6.18459e–12)
0.8	(0.40000, 0.60000)	(0.40000, 0.60000)	(2.16571e–13, 9.27682e–12)
0.9	(0.45000, 0.55000)	(0.45000, 0.55000)	(1.80808e–11, 1.17943e–10)

Table 4 The numerical results for Example 4 with $x = 0.5$, $M = 2$, $N = 3$

r	Exact solution $(\underline{u}(x,r),\overline{u}(x,r))$	Presented method	Absolute error
0	(0.00000, 0.50000)	(0.00000, 1.00000)	(3.98881e–12, 4.90985e–12)
0.1	(0.02500, 0.47500)	(0.05000, 0.95000)	(3.11232e–12, 3.77101e–12)
0.2	(0.05000, 0.45000)	(0.10000, 0.90000)	(2.34518e–12, 2.77027e–12)
0.3	(0.07500, 0.42500)	(0.15000, 0.85000)	(1.68740e–12, 1.90765e–12)
0.4	(0.10000, 0.40000)	(0.20000, 0.80000)	(1.13896e–12, 1.18320e–12)
0.5	(0.12500, 0.37500)	(0.25000, 0.75000)	(7.67386e–13, 0.00000e+00)
0.6	(0.15000, 0.35000)	(0.30000, 0.70000)	(4.40253e–13, 1.54612e–12)
0.7	(0.17500, 0.32500)	(0.35000, 0.65000)	(2.20562e–13, 3.09229e–12)
0.8	(0.20000, 0.30000)	(0.40000, 0.60000)	(1.08286e–13, 4.63841e–12)
0.9	(0.22500, 0.27500)	(0.45000, 0.55000)	(1.13459e–12, 1.43827e–10)

Table 3 shows the comparison of the approximate solution for presented method and the exact solution.

Example 4 Consider the linear fuzzy Fredholm integral equation

$$\underline{f}(x,r) = \left(\frac{38}{39}x^2 - \frac{1}{65} + \frac{2}{39}\right)r,$$
$$\overline{f}(x,r) = \left(\frac{38}{39}x^2 - \frac{1}{65} + \frac{2}{39}\right)(2-r),$$

and

$$k(x,t) = \frac{x^2 + t^2 - 2}{13}, \quad 0 \le x, t \le 1 \text{ and } \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx^2,$$
$$\overline{u}(x,r) = (2-r)x^2.$$
$$\bigl(\underline{u}(x,r), \overline{u}(x,r)\bigr).$$

Table 4 shows the comparison of the approximate solution for presented method and the exact solution.

7 Conclusion

Studying many problems in the applied mathematical many topics requires are required for the solution of the fuzzy integral equations (FIEs). This proposed work introduced a new hybrid method for numerically solving the linear fuzzy integral Fredholm equation. The new hybrid method is based on the combination of Bernstein and improved block-pulse functions. The results obtained from the application of the new method illustrate the superior performance and strong precision of the technique proposed as opposed to other approaches such as HBP, BPFs, HBPBP and BP. Finally, the technique presented in the paper would be of the interest of scientists to solve and study many applied science mathematical problems.

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