Nwaeze et al. *Advances in Difference Equations* (2020) 2020:507
https://doi.org/10.1186/s13662-020-02977-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Fractional inclusions of the Hermite–Hadamard type for m-polynomial convex interval-valued functions

Eze R. Nwaeze[1], Muhammad Adil Khan[2] and Yu-Ming Chu[3*]

*Correspondence:
chuyuming2005@126.com
[3]Department of Mathematics, Huzhou University, Huzhou, China
Full list of author information is available at the end of the article

Abstract

The notion of m-polynomial convex interval-valued function $\Psi = [\psi^-, \psi^+]$ is hereby proposed. We point out a relationship that exists between Ψ and its component real-valued functions ψ^- and ψ^+. For this class of functions, we establish loads of new set inclusions of the Hermite–Hadamard type involving the ρ-Riemann–Liouville fractional integral operators. In particular, we prove, among other things, that if a set-valued function Ψ defined on a convex set $\mathbf{S}$ is m-polynomial convex, $\rho, \epsilon > 0$ and $\zeta, \eta \in \mathbf{S}$, then

$$\begin{aligned}\frac{m}{m+2^{-m}-1}\Psi\left(\frac{\zeta+\eta}{2}\right) &\supseteq \frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta) +{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right]\\ &\supseteq \frac{\Psi(\zeta)+\Psi(\eta)}{m}\sum_{p=1}^{m} S_p(\epsilon;\rho),\end{aligned}$$

where Ψ is Lebesgue integrable on $[\zeta, \eta]$, $S_p(\epsilon;\rho) = 2 - \frac{\epsilon}{\epsilon+\rho p} - \frac{\epsilon}{\rho}\mathcal{B}(\frac{\epsilon}{\rho}, p+1)$ and $\mathcal{B}$ is the beta function. We extend, generalize, and complement existing results in the literature. By taking $m \geq 2$, we derive loads of new and interesting inclusions. We hope that the idea and results obtained herein will be a catalyst towards further investigation.

MSC: 26D15; 26E25; 28B20

Keywords: Hermite–Hadamard; m-polynomial convex; Interval-valued function; ρ-Riemann–Liouville

1 Introduction

A set $\mathbf{S} \subset \mathbb{R}$ is called a convex set if $\xi w + (1-\xi)z \in \mathbf{S}$ for all $w, z \in \mathbf{S}$ and $\xi \in [0,1]$. We call a function $\psi : \mathbf{S} \to \mathbb{R}$ convex if

$$\psi\bigl(\xi w + (1-\xi)z\bigr) \leq \xi\psi(z) + (1-\xi)\psi(w)$$

for all $w,z \in \mathbf{S}$ and $\xi \in [0,1]$. It is generally known that if $\psi : [\zeta,\eta] \to \mathbb{R}$ is convex, then

$$\psi\left(\frac{\zeta+\eta}{2}\right) \le \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(r)\,dr \le \frac{\psi(\zeta)+\psi(\eta)}{2}. \tag{1}$$

Inequality (1) is today known as the Hermite–Hadamard inequality. It was named after two French mathematicians, Charles Hermite and Jacques Hadamard. The former [17] first established the result in 1883, and a decade later it was rediscovered by the latter [16].

There are loads of articles in the literature on generalizations and extensions of (1) for different kinds of convexities. Examples of such can be found in [1–5, 10, 11, 14, 15, 18–26, 33, 34, 38] and the references cited therein. Recently, Toplu et al. [39] proposed and defined an m-polynomial convex function as follows: a real-valued function $\psi : \mathbf{S} \to \mathbb{R}^+ := (0,\infty)$ is m-polynomial convex (concave) if

$$\psi\bigl(\xi w + (1-\xi)z\bigr) \le (\ge) \frac{1}{m}\sum_{p=1}^{m}\bigl[1-(1-\xi)^p\bigr]\psi(w) + \frac{1}{m}\sum_{p=1}^{m}\bigl[1-\xi^p\bigr]\psi(z)$$

for all $w,z \in \mathbf{S}$ and $\xi \in [0,1]$. In this paper, we shall denote the sets of all m-polynomial convex and m-polynomial concave functions from $\mathbf{S}$ into $\mathbb{R}^+$ by $\mathbf{XP}_m(\mathbf{S},\mathbb{R}^+)$ and $\mathbf{VP}_m(\mathbf{S},\mathbb{R}^+)$, respectively. In the same paper, the authors established the following Hermite–Hadamard type inequality for this class of functions.

Theorem 1 ([39]) *Let $\psi : [\zeta,\eta] \to \mathbb{R}^+$ be an m-polynomial convex function. If $\zeta < \eta$ and ψ is Lebesgue integrable on $[\zeta,\eta]$, then*

$$\frac{2^{-1}m}{m+2^{-m}-1}\psi\left(\frac{\zeta+\eta}{2}\right) \le \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(r)\,dr \le \frac{\psi(\zeta)+\psi(\eta)}{m}\sum_{p=1}^{m}\frac{p}{p+1}. \tag{2}$$

Now, recall that the left- and right-sided ρ-Riemann–Liouville fractional integral operators ${}_{\rho}\mathcal{J}_{\zeta^+}^{\epsilon}$ and ${}_{\rho}\mathcal{J}_{\eta^-}^{\epsilon}$ of order $\epsilon > 0$, for a real-valued continuous function $\psi(w)$, are defined as follows:

$${}_{\rho}\mathcal{J}_{\zeta^+}^{\epsilon}\psi(w) = \frac{1}{\rho\Gamma_{\rho}(\epsilon)}\int_{\zeta}^{w}(w-\xi)^{\frac{\epsilon}{\rho}-1}\psi(\xi)\,d\xi, \quad w > \zeta, \tag{3}$$

and

$${}_{k}\mathcal{J}_{\eta^-}^{\epsilon}\psi(w) = \frac{1}{\rho\Gamma_{\rho}(\epsilon)}\int_{w}^{\eta}(\xi-w)^{\frac{\epsilon}{\rho}-1}\psi(\xi)\,d\xi, \quad w < \eta, \tag{4}$$

where $\rho > 0$, and Γ_{ρ} is the ρ-gamma function given by

$$\Gamma_{\rho}(w) := \int_0^{\infty}\xi^{w-1}e^{-\frac{\xi^{\rho}}{\rho}}\,d\xi, \qquad \mathrm{Re}(w) > 0,$$

with the properties $\Gamma_{\rho}(w+\rho) = w\Gamma_{\rho}(w)$ and $\Gamma_{\rho}(\rho) = 1$. If $\rho = 1$, we simply write

$${}_{1}\mathcal{J}_{\zeta^+}^{\epsilon}\psi(w) = \mathcal{J}_{\zeta^+}^{\epsilon}\psi(w) \quad \text{and} \quad {}_{1}\mathcal{J}_{\eta^-}^{\epsilon}\psi(w) = \mathcal{J}_{\eta^-}^{\epsilon}\psi(w).$$

The beta function $\mathcal{B}$ is defined by

$$\mathcal{B}(u,v) = \int_0^1 \xi^{u-1}(1-\xi)^{v-1}\,d\xi \quad \text{for } \operatorname{Re}(u) > 0, \operatorname{Re}(v) > 0. \tag{5}$$

Using these fractional integral operators, Sarikaya et al. [37] established the following fractional version of (1).

Theorem 2 ([37]) *Let $\psi : [\zeta,\eta] \to \mathbb{R}^+$ be a convex function. If $0 \le \zeta < \eta$ and ψ is Lebesgue integrable on $[\zeta,\eta]$, then the following double inequalities for the Riemann–Liouville fractional integrals hold*:

$$\psi\left(\frac{\zeta+\eta}{2}\right) \le \frac{\Gamma(\epsilon+1)}{2(\eta-\zeta)^{\epsilon}}\left[\mathcal{J}^{\epsilon}_{\zeta^+}\psi(\eta) + \mathcal{J}^{\epsilon}_{\eta^-}\psi(\zeta)\right] \le \frac{\psi(\zeta)+\psi(\eta)}{2}, \tag{6}$$

where $\epsilon > 0$.

The theory of interval analysis [29] was initiated by the late American mathematician Ramon E. Moore in 1966. Since its advent, this field has received ample amount of attention from different researchers in the mathematical community. Experts have found applications of interval analysis in global optimization and constraint solution algorithms. It has since grown steadily in popularity over the past decades. Interval analysis has been found to be valuable to engineers and scientists interested in scientific computation, especially in reliability, effects of round-off error, and automatic verification of results, see [8, 9, 12, 13]. With the birth of interval analysis, mathematicians, those who work in the field of mathematical inequalities, want to know if the inequalities in the above-mentioned results can be replaced with inclusions. In some cases, the answer to the question is in the affirmative. In this light, Sadowska (see also [28]) established the following result for a given interval-valued function.

Theorem 3 ([36]) *Let Ψ be a nonnegative continuous convex set-valued function on $[\zeta,\eta]$. Then*

$$\Psi\left(\frac{\zeta+\eta}{2}\right) \supset \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\Psi(r)\,dr \supset \frac{\Psi(\zeta)+\Psi(\eta)}{2}. \tag{7}$$

Results related to (7), for different families of set-valued convex functions, have been established. For example, see the papers [6, 8, 9, 12, 13, 27, 32, 35, 40, 41]. Recently, Budak et al. [7] established the following interval counterpart of (6).

Theorem 4 ([7]) *Let Ψ be a convex interval-valued function defined on $[\zeta,\eta]$ such that $\Psi = [\psi^-,\psi^+]$. If $0 \le \zeta < \eta$ and $\epsilon > 0$, then*

$$\Psi\left(\frac{\zeta+\eta}{2}\right) \supseteq \frac{\Gamma(\epsilon+1)}{2(\eta-\zeta)^{\epsilon}}\left[\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta) + \mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right] \supseteq \frac{\Psi(\zeta)+\Psi(\eta)}{2}. \tag{8}$$

This work is inspired by the above-mentioned articles. It is our purpose in this article to propose a new class of interval-valued functions called the m-polynomial convex functions and then obtain the interval-valued counterpart of (2). This result involves the

ρ-Riemann–Liouville fractional integral operators and generalizes Theorem 4. In addition, we establish four more results in this direction. Our results complement and extend known results in [7] and others in the literature. The paper is arranged as follows: in Sect. 2, we present a quick overview of the theory of interval analysis. Section 3 contains our main results with detailed justifications. Interesting corollaries are also pointed out. A brief introduction follows thereafter.

2 Preliminaries

Interval analysis is roughly described as an analysis of interval-valued functions. It is an annex of numerical analysis where instead of real numbers intervals are used as its operating element. In this section, we collate some basic terms and essentials of the theory of interval analysis from the books [29–31]. In the sequel, let $\mathbb{K}_c$ represent the class of all bounded closed nonempty intervals in $\mathbb{R}$, i.e.,

$$\mathbb{K}_c := \{[\zeta^-,\zeta^+] | \zeta^-,\zeta^+ \in \mathbb{R} \text{ and } \zeta^- \leq \zeta^+\}.$$

The numbers ζ^- and ζ^+ are called the left and right endpoints of $[\zeta^-,\zeta^+]$, respectively. The interval $[\zeta^-,\zeta^+]$ is called degenerated if $\zeta^- = \zeta^+$; positive if $\zeta^- > 0$; and negative if $\zeta^+ < 0$. We denote the sets of all negative intervals and positive intervals in $\mathbb{R}$ by $\mathbb{K}_c^-$ and $\mathbb{K}_c^+$, respectively. That is,

$$\mathbb{K}_c^- := \{[\zeta^-,\zeta^+] \in \mathbb{K}_c | \zeta^+ < 0\}$$

and

$$\mathbb{K}_c^+ := \{[\zeta^-,\zeta^+] \in \mathbb{K}_c | \zeta^- > 0\}.$$

Let $A = [\zeta^-,\zeta^+]$, $B = [\eta^-,\eta^+] \in \mathbb{K}_c$, and $\gamma \in \mathbb{R}$. We say $A \subseteq B$ (or $B \supseteq A$) if and only if $\eta^- \leq \zeta^-$ and $\zeta^+ \leq \eta^+$. The following arithmetic operations are defined thus:

$$\gamma A = \begin{cases} [\gamma\zeta^-,\gamma\zeta^+] & \text{if } \gamma > 0, \\ \{0\} & \text{if } \gamma = 0, \\ [\gamma\zeta^+,\gamma\zeta^-] & \text{if } \gamma < 0; \end{cases}$$

$$A + B = [\zeta^-,\zeta^+] + [\eta^-,\eta^+] := [\zeta^- + \eta^-, \zeta^+ + \eta^+];$$

$$A - B = [\zeta^-,\zeta^+] - [\eta^-,\eta^+] := [\zeta^- - \eta^+, \zeta^+ - \eta^-];$$

$$A \cdot B := [\min\{\zeta^-\eta^-, \zeta^-\eta^+, \zeta^+\eta^-, \zeta^+\eta^+\}, \max\{\zeta^-\eta^-, \zeta^-\eta^+, \zeta^+\eta^-, \zeta^+\eta^+\}];$$

$$\frac{A}{B} := \left[\min\left\{\frac{\zeta^-}{\eta^-}, \frac{\zeta^-}{\eta^+}, \frac{\zeta^+}{\eta^-}, \frac{\zeta^+}{\eta^+}\right\}, \max\left\{\frac{\zeta^-}{\eta^-}, \frac{\zeta^-}{\eta^+}, \frac{\zeta^+}{\eta^-}, \frac{\zeta^+}{\eta^+}\right\}\right]; \quad 0 \notin B.$$

Interval addition is commutative, associative and $\mathbf{0} = [0,0]$ is the identity element. Additive inverses do not exist, but the cancelation law holds. Also, interval multiplication is commutative, associative and $\mathbf{1} = [1,1]$ is the identity element. Multiplicative inverses do not exist and the cancelation law does not hold either. The distributive rule is not valid in general. It is important to also note that the interval arithmetic is said to be inclusion isotonic (see [31, p. 34]). By this, we mean that if A, B, C, and D are intervals such that

$$A \subseteq B \quad \text{and} \quad C \subseteq D,$$

then

$$A \boxtimes C \subseteq B \boxtimes D,$$

where $\boxtimes$ stands for interval addition, subtraction, multiplication, or division. It follows therefore that if $\zeta \leq \eta$ and $C \subseteq D$, then with $A = [\zeta, \zeta]$ and $B = [\eta, \eta]$, we have that $\zeta C \subseteq \eta D$.

The Pompeiu–Hausdorff distance $d_H : \mathbb{K}_c \times \mathbb{K}_c \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$d_H := \max\Bigl\{\max_{\zeta\in A} d(\zeta, B), \max_{\eta\in B} d(\eta, A)\Bigr\} \quad \text{with } d(\eta, A) = \min_{\zeta\in A} |\eta - \zeta|.$$

It is generally known that $(\mathbb{K}_c, d_H)$ is a complete metric space. The concept of a convergent sequence of intervals $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathbb{K}_c$ is considered in the complete metric space $\mathbb{K}_c$, endowed with the d_H distance: We say that $\lim_{n\to\infty} A_n = A$ if and only if for any real number $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$d_H(A_n, A) < \epsilon \quad \text{for all } n > N_\epsilon.$$

Next, we turn our attention to interval-valued functions.

Definition 5 An interval-valued function is defined to be any $\Psi : [\zeta, \eta] \to \mathbb{K}_c$ with $\Psi(w) = [\psi^-(w), \psi^+(w)] \in \mathbb{K}_c$ and $\psi^-(w) \leq \psi^+(w)$ for all $w \in [\zeta, \eta]$. We say that Ψ is Lebesgue integrable on $[\zeta, \eta]$ if the real-valued functions ψ^- and ψ^+ are Lebesgue integrable on $[\zeta, \eta]$, and then we write

$$\int_\zeta^\eta \Psi(r)\,dr = \biggl[\int_\zeta^\eta \psi^-(r)\,dr, \int_\zeta^\eta \psi^+(r)\,dr\biggr].$$

For an interval function $\Psi(w) = [\psi^-(w), \psi^+(w)]$, we define the ρ-Riemann–Liouville integral operators as follows:

$${}_\rho\mathcal{J}^\epsilon_{\zeta^+}\Psi(w) = \bigl[{}_\rho\mathcal{J}^\epsilon_{\zeta^+}\psi^-(w), {}_\rho\mathcal{J}^\epsilon_{\zeta^+}\psi^+(w)\bigr]$$

and

$${}_\rho\mathcal{J}^\epsilon_{\eta^-}\Psi(w) = \bigl[{}_\rho\mathcal{J}^\epsilon_{\eta^-}\psi^-(w), {}_\rho\mathcal{J}^\epsilon_{\eta^-}\psi^+(w)\bigr].$$

3 Main results

We first introduce the notion of m-polynomial convex interval-valued function.

Definition 6 Let $\mathbf{S}$ be a convex set, $\Psi : \mathbf{S} \to \mathbb{K}_c^+$ be an interval-valued function, and $m \in \mathbb{N}$. We say that Ψ is m-polynomial convex (concave) if and only if

$$\frac{1}{m}\sum_{p=1}^{m}\bigl[1 - (1-\xi)^p\bigr]\Psi(w) + \frac{1}{m}\sum_{p=1}^{m}\bigl[1 - \xi^p\bigr]\Psi(z) \subseteq (\supseteq)\Psi\bigl(\xi w + (1-\xi)z\bigr) \tag{9}$$

for all $w, z \in \mathbf{S}$ and $\xi \in [0,1]$. In what follows, we shall denote the sets of all m-polynomial convex and m-polynomial concave interval-valued functions from $\mathbf{S}$ into $\mathbb{K}_c^+$ by $\mathbf{XP}_m(\mathbf{S}, \mathbb{K}_c^+)$ and $\mathbf{VP}_m(\mathbf{S}, \mathbb{K}_c^+)$, respectively.

Remark 7 If we take a particular value of m, then we get a corresponding set inclusion. Take, for instance:

1. If $m = 1$, then we get the definition of a convex interval-valued function
$$\Psi\big(\xi w + (1-\xi)z\big) \supseteq \xi\Psi(w) + (1-\xi)\Psi(z)$$
for all $w, z \in \mathbf{S}$ and $\xi \in [0,1]$;
2. For $m = 2$, we get the following inclusion for a 2-polynomial convex interval-valued function:
$$\Psi\big(\xi w + (1-\xi)z\big) \supseteq \frac{3\xi - \xi^2}{2}\Psi(w) + \frac{2-\xi-\xi^2}{2}\Psi(z)$$
for all $w, z \in \mathbf{S}$ and $\xi \in [0,1]$;
3. For $m = 3$, we deduce the succeeding relation for a 3-polynomial convex interval-valued function:
$$\Psi\big(\xi w + (1-\xi)z\big) \supseteq \frac{6\xi - 4\xi^2 + \xi^3}{3}\Psi(w) + \frac{3-\xi-\xi^2-\xi^3}{3}\Psi(z)$$
for all $w, z \in \mathbf{S}$ and $\xi \in [0,1]$.

We now present a theorem that gives a link between a given interval-valued function Ψ and its component real-valued functions ψ^- and ψ^+.

Theorem 8 *Let $\Psi : \mathbf{S} \to \mathbb{K}_c^+$ be an interval-valued function such that $\Psi(w) = [\psi^-(w), \psi^+(w)] \in \mathbb{K}_c$, and $\psi^-(w) \le \psi^+(w)$ for all $w \in [\zeta, \eta]$. Then $\Psi \in \mathbf{XP}_m(\mathbf{S}, \mathbb{K}_c^+)$ if and only if $\psi^- \in \mathbf{XP}_m(\mathbf{S}, \mathbb{R}^+)$ and $\psi^+ \in \mathbf{VP}_m(\mathbf{S}, \mathbb{R}^+)$.*

Proof Let $w, z \in \mathbf{S}$ and $\xi \in [0,1]$. Then
$$\Psi \in \mathbf{XP}_m\big(\mathbf{S}, \mathbb{K}_c^+\big)$$
if and only if
$$\frac{1}{m}\sum_{p=1}^{m}\big[1-(1-\xi)^p\big]\Psi(w) + \frac{1}{m}\sum_{p=1}^{m}\big[1-\xi^p\big]\Psi(z) \subseteq \Psi\big(\xi w + (1-\xi)z\big)$$
if and only if
$$\begin{aligned}\Bigg[&\frac{1}{m}\sum_{p=1}^{m}\big[1-(1-\xi)^p\big]\psi^-(w) + \frac{1}{m}\sum_{p=1}^{m}\big[1-\xi^p\big]\psi^-(z),\\ &\frac{1}{m}\sum_{p=1}^{m}\big[1-(1-\xi)^p\big]\psi^+(w) + \frac{1}{m}\sum_{p=1}^{m}\big[1-\xi^p\big]\psi^+(z)\Bigg]\end{aligned}$$

$$\subseteq \left[\psi^{-}\left(\xi w+(1-\xi)z\right), \psi^{+}\left(\xi w+(1-\xi)z\right)\right]$$

if and only if

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\xi)^{p}\right]\psi^{-}(w)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\xi^{p}\right]\psi^{-}(z)\geq \psi^{-}\left(\xi w+(1-\xi)z\right),$$

and

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\xi)^{p}\right]\psi^{+}(w)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\xi^{p}\right]\psi^{+}(z)\leq \psi^{+}\left(\xi w+(1-\xi)z\right)$$

if and only if

$$\psi^{-}\in \mathbf{XP}_{m}\left(\mathbf{S},\mathbb{R}^{+}\right) \quad \text{and} \quad \psi^{+}\in \mathbf{VP}_{m}\left(\mathbf{S},\mathbb{R}^{+}\right).$$

That completes the proof in both directions. □

In a similar manner, one can prove the following result.

Theorem 9 *Let $\Psi:\mathbf{S}\to\mathbb{K}_c^{+}$ be an interval-valued function such that $\Psi(w)=[\psi^{-}(w),\psi^{+}(w)]\in\mathbb{K}_c$ and $\psi^{-}(w)\leq\psi^{+}(w)$ for all $w\in[\zeta,\eta]$. Then $\Psi\in\mathbf{VP}_m(\mathbf{S},\mathbb{K}_c^{+})$ if and only if $\psi^{-}\in\mathbf{VP}_m(\mathbf{S},\mathbb{R}^{+})$ and $\psi^{+}\in\mathbf{XP}_m(\mathbf{S},\mathbb{R}^{+})$.*

For the remaining part of this article, we shall assume that $\Psi:\mathbf{S}\to\mathbb{K}_c^{+}$ is always of the form $\Psi(w)=[\psi^{-}(w),\psi^{+}(w)]\in\mathbb{K}_c$ and $\psi^{-}(w)\leq\psi^{+}(w)$ for all $w\in[\zeta,\eta]$. We are now ready to formulate and prove some Hermite–Hadamard type results for m-polynomial convex (concave) interval-valued functions.

Theorem 10 *Let $\Psi:\mathbf{S}\to\mathbb{K}_c^{+}$ be an interval-valued function with $\zeta<\eta$ and $\zeta,\eta\in\mathbf{S}$, and Lebesgue integrable on $[\zeta,\eta]$. If $\Psi\in\mathbf{XP}_m(\mathbf{S},\mathbb{K}_c^{+})$ and $\rho,\epsilon>0$, then*

$$\begin{aligned}\frac{m}{m+2^{-m}-1}\Psi\left(\frac{\zeta+\eta}{2}\right) &\supseteq \frac{\Gamma_{\rho}(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_{\rho}\mathcal{J}^{\epsilon}_{\zeta^{+}}\Psi(\eta)+{}_{\rho}\mathcal{J}^{\epsilon}_{\eta^{-}}\Psi(\zeta)\right]\\ &\supseteq \frac{\Psi(\zeta)+\Psi(\eta)}{m}\sum_{p=1}^{m}S_{p}(\epsilon;\rho),\end{aligned} \tag{10}$$

where

$$S_{p}(\epsilon;\rho)=2-\frac{\epsilon}{\epsilon+\rho p}-\frac{\epsilon}{\rho}\mathcal{B}\left(\frac{\epsilon}{\rho},p+1\right)$$

and $\mathcal{B}$ is the beta function defined by (5). The inclusions are reversed if $\Psi\in\mathbf{VP}_m(\mathbf{S},\mathbb{K}_c^{+})$.

Proof Assuming $\Psi\in\mathbf{XP}_m(\mathbf{S},\mathbb{K}_c^{+})$, we get from (9) the following relation:

$$\Psi\left(\frac{w+z}{2}\right)\supseteq\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^{p}}\right]\Psi(w)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^{p}}\right]\Psi(z).$$

This implies that, for all $w, z \in \mathbf{S}$,

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]\big(\Psi(w)+\Psi(z)\big) \subseteq \Psi\left(\frac{w+z}{2}\right). \tag{11}$$

Now, let $w = \xi\zeta + (1-\xi)\eta$ and $z = \xi\eta + (1-\xi)\zeta$ with $\xi \in [0,1]$. Then (11) becomes

$$\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\big\{\Psi\big(\xi\zeta+(1-\xi)\eta\big)+\Psi\big(\xi\eta+(1-\xi)\zeta\big)\big\} \subseteq \Psi\left(\frac{\zeta+\eta}{2}\right). \tag{12}$$

Multiplying both sides of (12) by $\xi^{\frac{\epsilon}{\rho}-1}$ and then integrating with respect to ξ over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\Psi\left(\frac{\zeta+\eta}{2}\right)d\xi \\
&\quad\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big\{\Psi\big(\xi\zeta+(1-\xi)\eta\big)+\Psi\big(\xi\eta+(1-\xi)\zeta\big)\big\}\,d\xi \\
&\quad= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big\{\psi^-\big(\xi\zeta+(1-\xi)\eta\big)+\psi^-\big(\xi\eta+(1-\xi)\zeta\big)\big\}\,d\xi,\right. \\
&\qquad\left.\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big\{\psi^+\big(\xi\zeta+(1-\xi)\eta\big)+\psi^+\big(\xi\eta+(1-\xi)\zeta\big)\big\}\,d\xi\right].
\end{aligned} \tag{13}$$

Now,

$$\begin{aligned}
&\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big\{\psi^-\big(\xi\zeta+(1-\xi)\eta\big)+\psi^-\big(\xi\eta+(1-\xi)\zeta\big)\big\}\,d\xi \\
&\quad= \frac{1}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[\int_\zeta^\eta (\eta-r)^{\frac{\epsilon}{\rho}-1}\psi^-(r)\,dr+\int_\zeta^\eta (r-\zeta)^{\frac{\epsilon}{\rho}-1}\psi^-(r)\,dr\right] \\
&\quad= \frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[\frac{1}{\rho\Gamma_\rho(\epsilon)}\int_\zeta^\eta (\eta-r)^{\frac{\epsilon}{\rho}-1}\psi^-(r)\,dr+\frac{1}{\rho\Gamma_\rho(\epsilon)}\int_\zeta^\eta (r-\zeta)^{\frac{\epsilon}{\rho}-1}\psi^-(r)\,dr\right] \\
&\quad= \frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^\epsilon_{\zeta^+}\psi^-(\eta)+{}_\rho\mathcal{J}^\epsilon_{\eta^-}\psi^-(\zeta)\big].
\end{aligned} \tag{14}$$

Similarly, one obtains that

$$\begin{aligned}
&\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big\{\psi^+\big(\xi\zeta+(1-\xi)\eta\big)+\psi^+\big(\xi\eta+(1-\xi)\zeta\big)\big\}\,d\xi \\
&\quad= \frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^\epsilon_{\zeta^+}\psi^+(\eta)+{}_\rho\mathcal{J}^\epsilon_{\eta^-}\psi^+(\zeta)\big].
\end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned}
\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\Psi\left(\frac{\zeta+\eta}{2}\right)d\xi &= \left[\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\psi^-\left(\frac{\zeta+\eta}{2}\right)d\xi, \int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\psi^+\left(\frac{\zeta+\eta}{2}\right)d\xi\right] \\
&= \left[\frac{\rho}{\epsilon}\psi^-\left(\frac{\zeta+\eta}{2}\right), \frac{\rho}{\epsilon}\psi^+\left(\frac{\zeta+\eta}{2}\right)\right]
\end{aligned}$$

$$= \frac{\rho}{\epsilon}\Psi\left(\frac{\zeta+\eta}{2}\right). \tag{16}$$

Using (14), (15), and (16) in (13), one gets

$$\begin{aligned}
&\frac{\rho}{\epsilon}\Psi\left(\frac{\zeta+\eta}{2}\right) \\
&\supseteq \frac{m+2^{-m}-1}{m}\frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi^-(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\psi^-(\zeta), {}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi^+(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\psi^+(\zeta)\right] \\
&= \frac{m+2^{-m}-1}{m}\frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left(\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi^-(\eta), {}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi^+(\eta)\right]+\left[{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\psi^-(\zeta), {}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\psi^+(\zeta)\right]\right) \\
&= \frac{m+2^{-m}-1}{m}\frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right].
\end{aligned}$$

This further implies that

$$\frac{m}{m+2^{-m}-1}\Psi\left(\frac{\zeta+\eta}{2}\right)\supseteq\frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right]. \tag{17}$$

Next, we get from (9) the following inclusions:

$$\begin{aligned}
&\Psi\big(\xi\zeta+(1-\xi)\eta\big) \\
&\supseteq \frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\xi)^p\right]\Psi(\zeta)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\xi^p\right]\Psi(\eta)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\Psi\big(\xi\eta+(1-\xi)\zeta\big) \\
&\supseteq \frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\xi)^p\right]\Psi(\eta)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\xi^p\right]\Psi(\zeta).
\end{aligned} \tag{19}$$

Adding (18) and (19) gives

$$\begin{aligned}
&\Psi\big(\xi\zeta+(1-\xi)\eta\big)+\Psi\big(\xi\eta+(1-\xi)\zeta\big) \\
&\supseteq \frac{1}{m}\left\{\sum_{p=1}^{m}\left[1-(1-\xi)^p\right]+\sum_{p=1}^{m}\left[1-\xi^p\right]\right\}\big(\Psi(\zeta)+\Psi(\eta)\big).
\end{aligned} \tag{20}$$

Multiplying (20) by $\xi^{\frac{\epsilon}{\rho}-1}$ and integrating the resulting inclusion with respect to ξ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right] \\
&= \int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\left[\Psi\big(\xi\zeta+(1-\xi)\eta\big)+\Psi\big(\xi\eta+(1-\xi)\zeta\big)\right]d\xi \\
&\supseteq \frac{\Psi(\zeta)+\Psi(\eta)}{m}\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\left\{\sum_{p=1}^{m}\left[1-(1-\xi)^p\right]+\sum_{p=1}^{m}\left[1-\xi^p\right]\right\}d\xi
\end{aligned}$$

$$= \frac{\Psi(\zeta)+\Psi(\eta)}{m}\sum_{p=1}^{m}\left[\frac{2\rho}{\epsilon}-\frac{\rho}{\epsilon+\rho p}-\mathcal{B}\left(\frac{\epsilon}{\rho},p+1\right)\right]. \tag{21}$$

From (21), we obtain

$$\begin{aligned}&\frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right]\\&\quad\supseteq\frac{\Psi(\zeta)+\Psi(\eta)}{m}\sum_{p=1}^{m}\left[2-\frac{\epsilon}{\epsilon+\rho p}-\frac{\epsilon}{\rho}\mathcal{B}\left(\frac{\epsilon}{\rho},p+1\right)\right].\end{aligned} \tag{22}$$

We get the intended result by combining (17) and (22). □

Remark 11 Using Theorem 10, we obtain the following particular cases:

1. For $m = 1$, we deduce the result for convex interval-valued functions

$$\Psi\left(\frac{\zeta+\eta}{2}\right)\supseteq\frac{\Gamma_\rho(\epsilon+\rho)}{2(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right]\supseteq\frac{\Psi(\zeta)+\Psi(\eta)}{2}. \tag{23}$$

   If, in addition, we set $\rho = 1$ in (23), then we recapture (8).

2. If $m = 2$, then we obtain the result for 2-polynomial convex interval-valued functions

$$\begin{aligned}\frac{1}{5}\Psi\left(\frac{\zeta+\eta}{2}\right)&\supseteq\frac{\Gamma_\rho(\epsilon+\rho)}{8(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)\right]\\&\supseteq\frac{\Psi(\zeta)+\Psi(\eta)}{8}\left[1+\frac{\epsilon}{\epsilon+\rho}-\frac{\epsilon}{\epsilon+2\rho}\right].\end{aligned}$$

Theorem 12 *Let $\Psi, G:\mathbf{S}\to\mathbb{K}_c^+$ be two interval-valued functions with $\zeta<\eta$ and $\zeta,\eta\in\mathbf{S}$, and suppose that ΨG is Lebesgue integrable on $[\zeta,\eta]$. If $\rho,\epsilon>0$, $\Psi\in\mathbf{XP}_{m_1}(\mathbf{S},\mathbb{K}_c^+)$, and $G\in\mathbf{XP}_{m_2}(\mathbf{S},\mathbb{K}_c^+)$, then*

$$\begin{aligned}&\frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)G(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)G(\zeta)\right]\\&\quad\supseteq\frac{\epsilon}{\rho}\left\{\mathcal{P}(\zeta,\eta)\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\left[\Delta_1(\xi)+\Delta_4(\xi)\right]d\xi\right.\\&\quad\left.+\mathcal{Q}(\zeta,\eta)\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\left[\Delta_2(\xi)+\Delta_3(\xi)\right]d\xi\right\},\end{aligned}$$

where $\mathcal{P}(\zeta,\eta)=\Psi(\zeta)G(\zeta)+\Psi(\eta)G(\eta)$, $\mathcal{Q}(\zeta,\eta)=\Psi(\zeta)G(\eta)+\Psi(\eta)G(\zeta)$, and

$$\Delta_1(\xi):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-(1-\xi)^p\right]\sum_{p=1}^{m_2}\left[1-(1-\xi)^p\right];$$

$$\Delta_2(\xi):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-(1-\xi)^p\right]\sum_{p=1}^{m_2}\left[1-\xi^p\right];$$

$$\Delta_3(\xi):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-\xi^p\right]\sum_{p=1}^{m_2}\left[1-(1-\xi)^p\right];$$

$$\Delta_4(\xi) := \frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-\xi^p\right]\sum_{p=1}^{m_2}\left[1-\xi^p\right].$$

The inclusions are reversed if $\Psi \in \mathbf{VP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ *and* $G \in \mathbf{VP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$.

Proof Let $\Psi \in \mathbf{XP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $G \in \mathbf{XP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$. Then, for $\xi \in [0,1]$, we have

$$\frac{1}{m_1}\sum_{p=1}^{m_1}\left[1-(1-\xi)^p\right]\Psi(\zeta) + \frac{1}{m_1}\sum_{p=1}^{m_1}\left[1-\xi^p\right]\Psi(\eta) \subseteq \Psi\big(\xi\zeta + (1-\xi)\eta\big)$$

and

$$\frac{1}{m_2}\sum_{p=1}^{m_2}\left[1-(1-\xi)^p\right]G(\zeta) + \frac{1}{m_2}\sum_{p=1}^{m_2}\left[1-\xi^p\right]G(\eta) \subseteq G\big(\xi\zeta + (1-\xi)\eta\big).$$

So,

$$\begin{aligned}
&\Psi\big(\xi\zeta + (1-\xi)\eta\big)G\big(\xi\zeta + (1-\xi)\eta\big)\\
&\quad\supseteq \frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-(1-\xi)^p\right]\sum_{p=1}^{m_2}\left[1-(1-\xi)^p\right]\Psi(\zeta)G(\zeta)\\
&\qquad+ \frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-(1-\xi)^p\right]\sum_{p=1}^{m_2}\left[1-\xi^p\right]\Psi(\zeta)G(\eta)\\
&\qquad+ \frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-\xi^p\right]\sum_{p=1}^{m_2}\left[1-(1-\xi)^p\right]\Psi(\eta)G(\zeta)\\
&\qquad+ \frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}\left[1-\xi^p\right]\sum_{p=1}^{m_2}\left[1-\xi^p\right]\Psi(\eta)G(\eta)\\
&\quad:= \Delta_1(\xi)\Psi(\zeta)G(\zeta) + \Delta_2(\xi)\Psi(\zeta)G(\eta) + \Delta_3(\xi)\Psi(\eta)G(\zeta) + \Delta_4(\xi)\Psi(\eta)G(\eta).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Psi\big(\xi\zeta + (1-\xi)\eta\big)G\big(\xi\zeta + (1-\xi)\eta\big)\\
&\quad\supseteq \Delta_1(\xi)\Psi(\zeta)G(\zeta) + \Delta_2(\xi)\Psi(\zeta)G(\eta) + \Delta_3(\xi)\Psi(\eta)G(\zeta) + \Delta_4(\xi)\Psi(\eta)G(\eta).
\end{aligned} \tag{24}$$

Similarly,

$$\begin{aligned}
&\Psi\big(\xi\eta + (1-\xi)\zeta\big)G\big(\xi\eta + (1-\xi)\zeta\big)\\
&\quad\supseteq \Delta_4(\xi)\Psi(\zeta)G(\zeta) + \Delta_3(\xi)\Psi(\zeta)G(\eta) + \Delta_2(\xi)\Psi(\eta)G(\zeta) + \Delta_1(\xi)\Psi(\eta)G(\eta).
\end{aligned} \tag{25}$$

Adding (24) and (25) gives

$$\begin{aligned}
&\Psi\big(\xi\zeta + (1-\xi)\eta\big)G\big(\xi\zeta + (1-\xi)\eta\big) + \Psi\big(\xi\eta + (1-\xi)\zeta\big)G\big(\xi\eta + (1-\xi)\zeta\big)\\
&\quad\supseteq \big(\Psi(\zeta)G(\zeta) + \Psi(\eta)G(\eta)\big)\big[\Delta_1(\xi) + \Delta_4(\xi)\big]
\end{aligned}$$

$$
\begin{aligned}
&+ \big(\Psi(\zeta)G(\eta) + \Psi(\eta)G(\zeta)\big)\big[\Delta_2(\xi) + \Delta_3(\xi)\big] \\
&:= \mathcal{P}(\zeta,\eta)\big[\Delta_1(\xi) + \Delta_4(\xi)\big] + \mathcal{Q}(\zeta,\eta)\big[\Delta_2(\xi) + \Delta_3(\xi)\big].
\end{aligned}
\tag{26}
$$

Now, multiplying both sides of (26) by $\xi^{\frac{\epsilon}{\rho}-1}$ and integrating the resultant with respect to ξ over $[0,1]$ gives

$$
\begin{aligned}
&\frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)G(\eta) +{}_\rho \mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)G(\zeta)\big] \\
&= \int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\Psi\big(\xi\zeta + (1-\xi)\eta\big)G\big(\xi\zeta + (1-\xi)\eta\big)\,d\xi \\
&\quad + \int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\Psi\big(\xi\eta + (1-\xi)\zeta\big)G\big(\xi\eta + (1-\xi)\zeta\big)\,d\xi \\
&\supseteq \mathcal{P}(\zeta,\eta)\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big[\Delta_1(\xi) + \Delta_4(\xi)\big]\,d\xi + \mathcal{Q}(\zeta,\eta)\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big[\Delta_2(\xi) + \Delta_3(\xi)\big]\,d\xi.
\end{aligned}
$$

Hence, that completes the proof. □

Corollary 13 *Let $\rho,\epsilon > 0$. If $\Psi, G : \mathbf{S} \to \mathbb{K}^+_c$ are two convex interval-valued functions with $\zeta < \eta$, $\zeta,\eta \in \mathbf{S}$ and ΨG is Lebesgue integrable on $[\zeta,\eta]$, then*

$$
\begin{aligned}
&\frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)G(\eta) +{}_\rho \mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)G(\zeta)\big] \\
&\supseteq \mathcal{P}(\zeta,\eta)\left[1 - \left(\frac{2\epsilon}{\epsilon+\rho} - \frac{2\epsilon}{\epsilon+2\rho}\right)\right] + \mathcal{Q}(\zeta,\eta)\left[\frac{2\epsilon}{\epsilon+\rho} - \frac{2\epsilon}{\epsilon+2\rho}\right].
\end{aligned}
$$

Proof Let $m_1 = m_2 = 1$. Then $\Delta_1(\xi) = \xi^2$, $\Delta_2(\xi) = \Delta_1(\xi) = \xi - \xi^2$, and $\Delta_4(\xi) = 1 - 2\xi + \xi^2$. We get the desired inequality by applying Theorem 12. □

Remark 14 Corollary 13 boils down to [7, Theorem 3.5] if we set $\rho = 1$.

Theorem 15 *Let $\Psi, G : \mathbf{S} \to \mathbb{K}^+_c$ be two interval-valued functions with $\zeta < \eta$ and $\zeta,\eta \in \mathbf{S}$, and suppose that ΨG is Lebesgue integrable on $[\zeta,\eta]$. If $\rho,\epsilon > 0$, $\Psi \in \mathbf{XP}_{m_1}(\mathbf{S},\mathbb{K}^+_c)$, and $G \in \mathbf{XP}_{m_2}(\mathbf{S},\mathbb{K}^+_c)$, then*

$$
\begin{aligned}
&\frac{m_1 m_2}{(m_1 + 2^{-m_1} - 1)(m_2 + 2^{-m_2} - 1)}\Psi\left(\frac{\zeta+\eta}{2}\right)G\left(\frac{\zeta+\eta}{2}\right) \\
&\supseteq \frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)G(\eta) +{}_\rho \mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)G(\zeta)\big] \\
&\quad + \frac{\epsilon}{\rho}\int_0^1 \xi^{\frac{\epsilon}{\rho}-1}\big\{\big[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi) + \tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\big]\mathcal{P}(\zeta,\eta) \\
&\quad + \big[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi) + \tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\big]\mathcal{Q}(\zeta,\eta)\big\}\,d\xi,
\end{aligned}
$$

where $\mathcal{P}(\zeta,\eta)$ and $\mathcal{Q}(\zeta,\eta)$ are as defined in Theorem 12, and for $\xi \in [0,1]$,

$$
\Lambda_m(\xi) = \frac{1}{m}\sum_{p=1}^{m}\big[1 - (1-\xi)^p\big],
$$

$$\tilde{\Lambda}_m(\xi) = \frac{1}{m}\sum_{p=1}^{m}\bigl[1-\xi^p\bigr].$$

Proof First, we observe that from the definitions of $\tilde{\Lambda}_m$ and Λ_m given above, we have

$$\tilde{\Lambda}_m\left(\frac{1}{2}\right) = \Lambda_m\left(\frac{1}{2}\right) := L_m := \frac{m+2^{-m}-1}{m}.$$

Hence, from (12), one gets

$$L_{m_1}\bigl\{\Psi\bigl(\xi\zeta+(1-\xi)\eta\bigr)+\Psi\bigl(\xi\eta+(1-\xi)\zeta\bigr)\bigr\} \subseteq \Psi\left(\frac{\zeta+\eta}{2}\right)$$

and

$$L_{m_2}\bigl\{G\bigl(\xi\zeta+(1-\xi)\eta\bigr)+G\bigl(\xi\eta+(1-\xi)\zeta\bigr)\bigr\} \subseteq G\left(\frac{\zeta+\eta}{2}\right).$$

Now,

$$\begin{aligned}
&\Psi\left(\frac{\zeta+\eta}{2}\right)G\left(\frac{\zeta+\eta}{2}\right)\\
&\quad\supseteq L_{m_1}L_{m_2}\bigl[\Psi\bigl(\xi\zeta+(1-\xi)\eta\bigr)G\bigl(\xi\zeta+(1-\xi)\eta\bigr)\\
&\qquad+\Psi\bigl(\xi\eta+(1-\xi)\zeta\bigr)G\bigl(\xi\eta+(1-\xi)\zeta\bigr)\bigr]\\
&\qquad+L_{m_1}L_{m_2}\bigl[\Psi\bigl(\xi\zeta+(1-\xi)\eta\bigr)G\bigl(\xi\eta+(1-\xi)\zeta\bigr)\\
&\qquad+\Psi\bigl(\xi\eta+(1-\xi)\zeta\bigr)G\bigl(\xi\zeta+(1-\xi)\eta\bigr)\bigr]\\
&\quad\supseteq L_{m_1}L_{m_2}\bigl[\Psi\bigl(\xi\zeta+(1-\xi)\eta\bigr)G\bigl(\xi\zeta+(1-\xi)\eta\bigr)\\
&\qquad+\Psi\bigl(\xi\eta+(1-\xi)\zeta\bigr)G\bigl(\xi\eta+(1-\xi)\zeta\bigr)\bigr]\\
&\qquad+L_{m_1}L_{m_2}\bigl\{\bigl[\Lambda_{m_1}(\xi)\Psi(\zeta)+\tilde{\Lambda}_{m_1}(\xi)\Psi(\eta)\bigr]\bigl[\Lambda_{m_2}(\xi)G(\eta)+\tilde{\Lambda}_{m_2}(\xi)G(\zeta)\bigr]\\
&\qquad+\bigl[\Lambda_{m_1}(\xi)\Psi(\eta)+\tilde{\Lambda}_{m_1}(\xi)\Psi(\zeta)\bigr]\bigl[\Lambda_{m_2}(\xi)G(\zeta)+\tilde{\Lambda}_{m_2}(\xi)G(\eta)\bigr]\bigr\}\\
&\quad= L_{m_1}L_{m_2}\bigl[\Psi\bigl(\xi\zeta+(1-\xi)\eta\bigr)G\bigl(\xi\zeta+(1-\xi)\eta\bigr)\\
&\qquad+\Psi\bigl(\xi\eta+(1-\xi)\zeta\bigr)G\bigl(\xi\eta+(1-\xi)\zeta\bigr)\bigr]\\
&\qquad+L_{m_1}L_{m_2}\bigl\{\bigl[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\bigr]\bigl(\Psi(\zeta)G(\zeta)+\Psi(\eta)G(\eta)\bigr)\\
&\qquad+\bigl[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\bigr]\bigl(\Psi(\zeta)G(\eta)+\Psi(\eta)G(\zeta)\bigr)\bigr\}\\
&\quad:= L_{m_1}L_{m_2}\bigl[\Psi\bigl(\xi\zeta+(1-\xi)\eta\bigr)G\bigl(\xi\zeta+(1-\xi)\eta\bigr)\\
&\qquad+\Psi\bigl(\xi\eta+(1-\xi)\zeta\bigr)G\bigl(\xi\eta+(1-\xi)\zeta\bigr)\bigr]\\
&\qquad+L_{m_1}L_{m_2}\bigl\{\bigl[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\bigr]\mathcal{P}(\zeta,\eta)\\
&\qquad+\bigl[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\bigr]\mathcal{Q}(\zeta,\eta)\bigr\}.
\end{aligned}$$

Thus, we get

$$\Psi\left(\frac{\zeta+\eta}{2}\right)G\left(\frac{\zeta+\eta}{2}\right)$$

$$\begin{aligned}
&\supseteq L_{m_1}L_{m_2}\big[\Psi\big(\xi\zeta+(1-\xi)\eta\big)G\big(\xi\zeta+(1-\xi)\eta\big)\\
&\quad+\Psi\big(\xi\eta+(1-\xi)\zeta\big)G\big(\xi\eta+(1-\xi)\zeta\big)\big]\\
&\quad+L_{m_1}L_{m_2}\big\{\big[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\big]\mathcal{P}(\zeta,\eta)\\
&\quad+\big[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\big]\mathcal{Q}(\zeta,\eta)\big\}.
\end{aligned}\tag{27}$$

Multiplying (27) by $\xi^{\frac{\epsilon}{\rho}-1}$ and integrating with respect to ξ over $[0,1]$, we get the following inclusion:

$$\begin{aligned}
&\frac{\rho}{\epsilon}\Psi\left(\frac{\zeta+\eta}{2}\right)G\left(\frac{\zeta+\eta}{2}\right)\\
&=\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\Psi\left(\frac{\zeta+\eta}{2}\right)G\left(\frac{\zeta+\eta}{2}\right)d\xi\\
&\supseteq L_{m_1}L_{m_2}\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\big[\Psi\big(\xi\zeta+(1-\xi)\eta\big)G\big(\xi\zeta+(1-\xi)\eta\big)\\
&\quad+\Psi\big(\xi\eta+(1-\xi)\zeta\big)G\big(\xi\eta+(1-\xi)\zeta\big)\big]\,d\xi\\
&\quad+L_{m_1}L_{m_2}\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\big\{\big[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\big]\mathcal{P}(\zeta,\eta)\\
&\quad+\big[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\big]\mathcal{Q}(\zeta,\eta)\big\}\,d\xi\\
&=L_{m_1}L_{m_2}\frac{\rho\Gamma_\rho(\epsilon)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)G(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)G(\zeta)\big]\\
&\quad+L_{m_1}L_{m_2}\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\big\{\big[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\big]\mathcal{P}(\zeta,\eta)\\
&\quad+\big[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\big]\mathcal{Q}(\zeta,\eta)\big\}\,d\xi,
\end{aligned}$$

from which the desired inequality is obtained. □

Corollary 16 *Let $\rho,\epsilon>0$. If $\Psi,G:\mathbf{S}\to\mathbb{K}_c^+$ are two convex interval-valued functions with $\zeta<\eta$, $\zeta,\eta\in\mathbf{S}$, and ΨG is Lebesgue integrable on $[\zeta,\eta]$, then*

$$\begin{aligned}
&\Psi\left(\frac{\zeta+\eta}{2}\right)G\left(\frac{\zeta+\eta}{2}\right)\\
&\supseteq\frac{\Gamma_\rho(\epsilon+\rho)}{4(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\big[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\Psi(\eta)G(\eta)+{}_\rho\mathcal{J}^{\epsilon}_{\eta^-}\Psi(\zeta)G(\zeta)\big]\\
&\quad+\left[\frac{\epsilon}{2(\epsilon+\rho)}-\frac{\epsilon}{2(\epsilon+2\rho)}\right]\mathcal{P}(\zeta,\eta)+\left[\frac{1}{4}-\left(\frac{\epsilon}{2(\epsilon+\rho)}-\frac{\epsilon}{2(\epsilon+2\rho)}\right)\right]\mathcal{Q}(\zeta,\eta).
\end{aligned}$$

Proof Let $m_1=m_2=1$. Then $\Lambda_{m_1}(\xi)=\Lambda_{m_2}(\xi)=\xi$ and $\tilde{\Lambda}_{m_1}(\xi)=\tilde{\Lambda}_{m_2}(\xi)=1-\xi$. Using Theorem 12, we get the required result. □

Remark 17 If we take $\rho=1$, then Corollary 16 becomes [7, Theorem 3.6].

4 Conclusion

Some new set inclusions of the Hermite–Hadamard types are established for the class of m-polynomial convex interval-valued functions. A relationship between a given m-

polynomial convex (concave) interval-valued function $\Psi = [\psi^-, \psi^+]$ and its component real-valued functions ψ^- and ψ^+ is established. We pointed out some corollaries from which loads of interesting results can be deduced. In addition to these corollaries, if we take $\psi^- = \psi^+ = \psi$, then $\Psi = \psi$ and the inclusions in Theorems 10, 12, and 15 become the following inequalities:

1.

$$\begin{aligned}\frac{m}{m+2^{-m}-1}\psi\left(\frac{\zeta+\eta}{2}\right) &\le \frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi(\eta)+{}_\rho\,\mathcal{J}^{\epsilon}_{\eta^-}\psi(\zeta)\right] \\ &\le \frac{\psi(\zeta)+\psi(\eta)}{m}\sum_{p=1}^{m}S_p(\epsilon;\rho);\end{aligned}$$

2.

$$\begin{aligned}&\frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi(\eta)g(\eta)+{}_\rho\,\mathcal{J}^{\epsilon}_{\eta^-}\psi(\zeta)g(\zeta)\right] \\ &\quad\le \frac{\epsilon\mathcal{P}(\zeta,\eta)}{\rho}\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\left[\Delta_1(\xi)+\Delta_4(\xi)\right]d\xi \\ &\qquad+\frac{\epsilon\mathcal{Q}(\zeta,\eta)}{\rho}\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\left[\Delta_2(\xi)+\Delta_3(\xi)\right]d\xi;\end{aligned}$$

and

3.

$$\begin{aligned}&\frac{m_1m_2}{(m_1+2^{-m_1}-1)(m_2+2^{-m_2}-1)}\psi\left(\frac{\zeta+\eta}{2}\right)g\left(\frac{\zeta+\eta}{2}\right) \\ &\quad\le \frac{\Gamma_\rho(\epsilon+\rho)}{(\eta-\zeta)^{\frac{\epsilon}{\rho}}}\left[{}_\rho\mathcal{J}^{\epsilon}_{\zeta^+}\psi(\eta)g(\eta)+{}_\rho\,\mathcal{J}^{\epsilon}_{\eta^-}\psi(\zeta)g(\zeta)\right] \\ &\qquad+\frac{\epsilon}{\rho}\int_0^1\xi^{\frac{\epsilon}{\rho}-1}\left\{\left[\Lambda_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\Lambda_{m_2}(\xi)\right]\mathcal{P}(\zeta,\eta)\right. \\ &\qquad\left.+\left[\Lambda_{m_1}(\xi)\Lambda_{m_2}(\xi)+\tilde{\Lambda}_{m_1}(\xi)\tilde{\Lambda}_{m_2}(\xi)\right]\mathcal{Q}(\zeta,\eta)\right\}d\xi,\end{aligned}$$

respectively.

Acknowledgements
Many thanks to the referees for their valuable comments and suggestions.

Funding
This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Computer Science, Alabama State University, Montgomery, USA. [2]Department of Mathematics, University of Peshawar, Peshawar, Pakistan. [3]Department of Mathematics, Huzhou University, Huzhou, China.

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Received: 21 July 2020 Accepted: 14 September 2020 Published online: 18 September 2020

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