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Global exponential stabilization of nonlinear systems via width time-dependent periodically intermittent smooth control

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Abstract

In this paper, the global exponential stability and stabilization problems for a class of nonlinear systems are investigated. Some sufficient conditions to guarantee global exponential stable and estimate the minimum admissible value of the control width are presented in virtue of time-dependent width Lyapunov functions. Furthermore, a periodically intermittent smooth controller with variant control width is introduced and theoretical analysis is provided. The smooth index function of periodically intermittent smooth control inputs is defined and the supremum (or least upper bound) of smooth index function set can be solved. On the basis of the analysis, the designed periodically intermittent smooth controller not only can globally exponentially stabilize the nonlinear systems, but also can control the exponential convergence rate of the nonlinear systems. Finally, numerical simulations are given to verify the obtained theoretical results.

Keywords: Global exponential stable; Nonlinear systems; Periodically intermittent smooth control; Smooth index function; Time-dependent width Lyapunov functions

1 Introduction

Stabilization problem of nonlinear dynamic systems has been a topic of focus in recent years. Some important control methods have been developed, which include continuous control [14], impulsive control [16], adaptive control [5], and variable structure control [7]. Moreover, the intermittent control scheme that can effectively reduce control cost and shorten control time in comparison with the continuous time control method has attracted more interest due to its wide applications in engineering, economics, transportation, and communication. The examples refer to economics management [2], complex networks intermittent control [13], vibration reduction of the quarter car body model [3], and so on. Unlike the previous results (see [2, 3, 6, 13, 15, 18] and the references therein), another interesting intermittent control strategy which is more adaptable to the practice application and can be found in Fig. 1 is introduced and studied in this paper. In addition, when applying the Lyapunov approach to the intermittently controlled system, the selected Lyapunov function should be capable of capturing the hybrid structure characteristics of the underlying system and the properties of work time. However, most of the

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work (e.g. [6, 15, 18]) may neglect the properties of work time and the difference of the dynamical properties between the controlled subsystem and the uncontrolled subsystem, and thus the resulting stability criteria may be conservative. Therefore, the stabilization problem of nonlinear systems via intermittent control needs further exploration and improvement.

On the other hand, a time-dependent control law can achieve stability results and transient performances that a controller with a constant matrix gain cannot. However, a time-dependent control law may give rise to the control input bumps often resulting in undesired transients and even instability, which is not acceptable in a practical situation. In order to overcome such a drawback, a variety of bumpless transfer methods have been suggested over the years. For example, a bumpless control solution is provided in [19] for linear systems using an L_2 method. Zhen et al. [20] propose a bumpless control scheme completed by a mismatch compensator taking into account uncertainties. Furthermore, there are a few results that go beyond removing input discontinuities and deal with controller re-initialization, addition of extra dynamics (see [4, 9], and so on). Recently, a new strategy that selects among the nearly optimal controllers the one leading to limited control discontinuities at the switching instants was provided in [1]. Inspired by the main idea of ϵ-suboptimal set of an optimization problem and [1, 19], a novel scheme can be provided, together with a smooth index function.

Motivated by the aforementioned observations, width time-dependent periodically intermittent smooth controller design for a class of nonlinear systems is presented in this paper. Unlike the existing solutions, the proposed periodically intermittent control scheme has time dependence and variant control width. Furthermore, a novel smooth control method, together with the smooth index function can be given by adopting the notion of the ϵ-suboptimal set of an optimization problem. The rest of this paper is organized as follows. In Sect. 2, we formulate the problem of periodically intermittent control law with variant control width of a class of nonlinear systems, and introduce some necessary preliminaries. In Sect. 3, some sufficient criteria of global exponential stabilization for a class of nonlinear systems by means of time-dependent width Lyapunov functions are given. Meanwhile, the exponential convergence rate of systems and the minimum admissible value of control width are also explicitly provided. Moreover, a novel periodically intermittent smooth controller design together with the smooth index function is derived in virtue of rigorous mathematical analysis. As an application, periodically intermittent control of Chua's oscillator system is discussed in Sect. 4. Finally, the conclusion is drawn in Sect. 5.

Notations The notation used throughout this paper is fairly standard. The subscripts "T" and "-1" stand for matrix transposition and inverse, $\mathbb{R}^n$ denotes the n dimensional Euclidean space. The notation $P > 0$ ($P \geq 0$) means that P is symmetric and positive (semi-positive) definite. I and $\mathbf{0}$ represent, respectively, identity matrix and zero matrix. In symmetric block matrices or complex matrix expressions, we use the symbol "$*$" as an ellipsis for the terms that are introduced by symmetry and diag$\{\cdots\}$ stands for a block-diagonal matrix. For real matrices, the Hermitian operator He$\{\cdot\}$ is defined as He$\{M\} = M + M^T$. $\|\cdot\|$ is used to refer to the Euclidean vector norm or the spectral norm for matrices. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues, respectively.

2 Preliminaries and problem formulation

Consider a class of nonlinear systems described as

$$
\begin{cases}
\dot{x}(t) = Ax(t) + f(x(t)) + u(t), \\
x(0) = x_0,
\end{cases}
\tag{2.1}
$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous nonlinear function satisfying $\|f(x)\|^2 \leq x^T L x$ with $L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\} \geq 0, f(0) = 0$. $u(t) \in \mathbb{R}^n$ denotes the control input vector. $A \in \mathbb{R}^{n \times n}$ is known constant matrix.

In this paper, we will investigate the following type of periodically intermittent control strategy (see Fig. 1). In any period, the time is divided into two parts: "work time" and "rest time," and the control input only occurs in the work time. We assume that the time intervals for work time are $[mT, mT + \tau_m)$, where $m = 0, 1, 2, \ldots, 0 < \tau_m < T$, T denotes the control period, and τ_m is named the mth control width (or work width); while $[mT + \tau_m, (m + 1)T)$ are the rest time, and $T - \tau_m$ is called the mth rest width. Then we assume that the control law exposed on the system is of the form
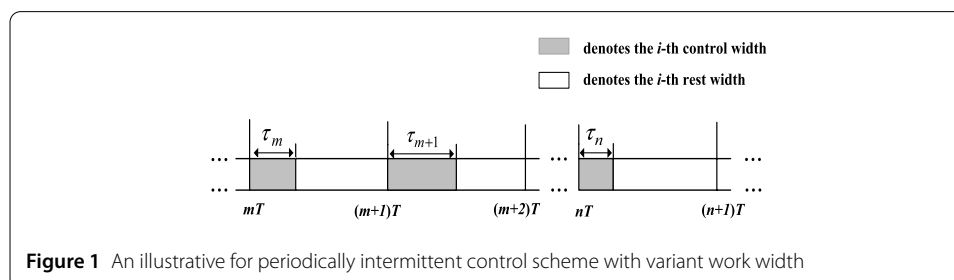
$$
u(t) =
\begin{cases}
K(t)x(t), & t \in [mT, mT + \tau_m), \\
0, & t \in [mT + \tau_m, (m + 1)T),
\end{cases}
\tag{2.2}
$$

where $K(t)$ is the time-dependent control gain matrix. For the control law (2.2), it is not required that the control gain matrix and the control width are fixed compared with that in [6, 8] and so on, which may be adequate in the practice application. So, this strategy can be applied to a wider class of nonlinear systems and is helpful to improve the existing results. Then, under control law (2.2), we can rewrite (2.1) as

$$
\begin{cases}
\dot{x}(t) = Ax(t) + f(x(t)) + K(t)x(t), & t \in [mT, mT + \tau_m), \\
\dot{x}(t) = Ax(t) + f(x(t)), & t \in [mT + \tau_m, (m + 1)T).
\end{cases}
\tag{2.3}
$$

Such a system can be viewed as a switched system consisting of a stable-controlled subsystem and an unstable-uncontrolled subsystem. The stability of (2.3) will rely on the choice of suitable T, τ_m (or the maximum and minimum control width), and $K(t)$. The purpose of this paper is to design a periodically intermittent state feedback control law (2.2) and establish the strategy of smooth control inputs such that the system (2.1) can be locally globally exponentially stable (GES).

In the following, we present the definitions and lemmas, which will be useful throughout this paper.

Figure 1 An illustrative for periodically intermittent control scheme with variant work width

Definition 2.1 ([10]) The equilibrium $x = 0$ of system (2.3) is said to be locally globally exponentially stable (GES), if there exist two positive scalars M and γ such that

$$\left\| x(t,0,x_0) \right\| \leq M\|x_0\|e^{-\gamma t}, \quad t \geq 0,$$

where γ is called the exponential convergence rate.

Definition 2.2 ([12]) For $\epsilon \geq 0$, x is called ϵ-suboptimal if $x \in \mathcal{C}$ and $f(x) \leq P^* + \epsilon$, where f is the objective function, $\mathcal{C}$ is the feasible set and P^* is the optimal value of the general convex optimization problem. let $\mathcal{X}_\epsilon$ denote the set of all ϵ-suboptimal points, i.e.,

$$\mathcal{X}_\epsilon = \left\{ x \in \mathcal{C} | f(x) \leq P^* + \epsilon \right\},$$

which we call the ϵ-suboptimal set.

Lemma 2.3 (Schur complement [17]) *For symmetric matrix R*

$$R_{n \times n} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}, \quad R_{11}, R_{22} \text{ is square-matrix.}$$

The inequalities

$$R < 0,$$
$$R_{11} < 0, \qquad R_{22} - R_{12}^T R_{11}^{-1} R_{12} < 0,$$
$$R_{22} < 0, \qquad R_{11} - R_{12} R_{22}^{-1} R_{12}^T < 0,$$

are equal.

Lemma 2.4 ([11]) *Given any matrices X, Y, Z of appropriate dimensions and a scalar $\varepsilon > 0$ such that $Z^T = Z > 0$. Then the following inequality holds*:

$$X^T Y + Y^T X \preceq \varepsilon X^T Z X + \varepsilon^{-1} Y^T Z^{-1} Y.$$

3 Main results
3.1 Global exponential stability and stabilization
In order to fully capture the hybrid structure characteristics of the considered system and the properties of work time, we choose the following time-dependent width Lyapunov function for system (2.3):

$$\mathcal{V}\big(t, x(t)\big) = x^T(t)P(t)x(t), \tag{3.1}$$

where $P(t)$ can be provided as follows:
  (i) In the work time

$$P(t) \triangleq P\big(d(t)\big),$$

where

$$d(t) \triangleq mT + \theta_{m,n} + \alpha(t)h_m, \quad t \in \mathcal{R}_{m,n}, \tag{3.2}$$

with $[mT, mT + \tau_m)$ is divided into H segments described as $\mathcal{R}_{m,n} \triangleq [mT + \theta_{m,n}, mT + \theta_{m,n+1})$, $n = 0, 1, \ldots, H - 1$, $h_m = \frac{\tau_m}{H}$, $0 \leq \alpha(t) \leq 1$, $\alpha(t) \triangleq (t - mT - \theta_{m,n})/h_m$, $\theta_{m,n} = nh_m$. Let $P_n \triangleq P(mT + \theta_{m,n}) > 0$,

$$
\begin{aligned}
P\big(d(t)\big) &\triangleq P\big(mT + \theta_{m,n} + \alpha(t)h_m\big) \\
&= \big(1 - \alpha(t)\big)P_n + \alpha(t)P_{n+1}.
\end{aligned} \tag{3.3}
$$

(ii) In the rest time

$$P(t) \triangleq S\big(\varphi(t)\big),$$

where

$$\varphi(t) \triangleq \frac{t - mT - \tau_m}{T - \tau_m},$$

$$S\big(\varphi(t)\big) \triangleq \big(1 - \varphi(t)\big)S_1 + \varphi(t)S_2,$$

$$mT + \tau_m \leq t < (m + 1)T, \quad S_i > 0, i = 1, 2.$$

Hence the piecewise continuous matrix function $P(t)$ is described as

$$P(t) \triangleq \begin{cases} P(d(t)), & t \in \mathcal{R}_{m,n}, n = 0, 1, \ldots, H - 1, \\ S(\varphi(t)), & t \in [mT + \tau_m, (m + 1)T). \end{cases} \tag{3.4}$$

The matrix function $P(t)$ is piecewise linear in time t. If it is enforced that $H = 1$, $P(d(t))$ becomes a positive definite constant matrix P in the work time $[mT, mT + \tau_m)$. In addition, as $S_1 \equiv S_2 \equiv P$ in the rest time $[mT + \tau_m, (m + 1)T)$, it leads to $S(\varphi(t)) \equiv P(d(t)) \equiv P$. Thus, the time-dependent width Lyapunov functions are reduced to the traditional Lyapunov function. In this sense, it can give less conservative results in terms of the time-dependent width Lyapunov functions.

Based on the above time-dependent width Lyapunov functions, the main results are stated in the following.

Theorem 3.1 *If there exist a set of matrices $S_i > 0$, $P_n > 0$, Y_n, scalars $\lambda_i > 0$, $\beta_n > 0$ ($i = 1, 2$, $n = 0, 1, \ldots, H$), monotone and bounded real-valued functions $g_i(t) > 0$ ($i = 1, 2$), such that*

$$\begin{bmatrix} \Sigma_{1n} & -P_n \\ * & -\beta_n I \end{bmatrix} < 0, \quad n = 0, 1, \ldots, H - 1, \tag{3.5}$$

$$\begin{bmatrix} \Sigma_{2n} & -P_{n+1} \\ * & -\beta_{n+1}I \end{bmatrix} < 0, \quad n = 0, 1, \ldots, H - 1, \tag{3.6}$$

$$\begin{bmatrix} \Phi_i & -S_i \\ * & -\lambda_i I \end{bmatrix} < 0, \quad i = 1, 2, \tag{3.7}$$

$$\hat{g}_1 \tau_{\min} - (T - \tau_{\min})\bar{g}_2 > 0, \tag{3.8}$$

where

$$\Sigma_{1j} \triangleq \frac{(P_{n+1}-P_n)H}{\tau_{\min}} + \Pi_j, \qquad \Sigma_{2j} \triangleq \frac{(P_{n+1}-P_n)H}{\tau_{\min}} + \Pi_{j+1},$$

$$\Phi_i \triangleq \frac{S_2-S_1}{T-\tau_{\max}} + \Gamma_i,$$

with

$$\Pi_j \triangleq \mathrm{He}\{P_j A + Y_j\} + \beta_j L + \bar{g}_1 P_j,$$

$$\Gamma_i \triangleq \mathrm{He}\{S_i A\} + \lambda_i L - \hat{g}_2 S_i,$$

$$\hat{g}_i \triangleq \min_t\{g_i(t)\}, \qquad \bar{g}_i \triangleq \max_t\{g_i(t)\},$$

$j = 0, 1, \ldots, H - 1$, $i = 1, 2$, *then the origin of the system* (2.3) *is globally exponentially stable in the following sense*:

$$\|x(t)\| \le \sqrt{\frac{\lambda_M}{\lambda_m}} \|x_0\| e^{-\theta(t-\tau_{\max})}, \quad \forall t > 0, \tag{3.9}$$

where

$$\lambda_M \triangleq \max_{n,i}\{\lambda_{\max}(P_n), \lambda_{\max}(S_i)\},$$

$$\lambda_m \triangleq \min_{n,i}\{\lambda_{\min}(P_n), \lambda_{\min}(S_i)\},$$

$$\tau_{\min} \triangleq \min_m\{\tau_m\}, \qquad \tau_{\max} \triangleq \max_m\{\tau_m\},$$

$$\theta \triangleq \frac{\tau_{\min}\hat{g}_1 - (T - \tau_{\min})\bar{g}_2}{2T}.$$

θ *is the exponential convergence rate of system* (2.3), *and the controller gain is given as*

$$K(t) = P^{-1}(d(t))Y(d(t)), \tag{3.10}$$

where $Y(d(t))$ *is stated as follows*:

$$Y(d(t)) \triangleq Y(mT + \theta_{m,n} + \alpha(t)h_m)$$
$$= (1 - \alpha(t))Y_n + \alpha(t)Y_{n+1}.$$

Proof Construct the following Lyapunov function:

$$\mathcal{V}(t, x(t)) = x^T(t)P(t)x(t), \tag{3.11}$$

where $P(t)$ is defined in (3.4).

For $t \in \mathcal{R}_{m,n} = [mT + \theta_{m,n}, mT + \theta_{m,n+1})$, $n = 0, 1, \ldots, H - 1$, according to (3.4), it follows $\dot{P}(t) = (P_{n+1} - P_n)\frac{H}{\tau_m}$.

Based on Lemma 2.4, it yields, for $n = 0, 1, \ldots, H$,

$$f^T\big(x(t)\big)P_n x(t) + x^T(t)P_n f\big(x(t)\big)$$
$$\leq \beta_n^{-1} x^T(t) P_n P_n x(t) + \beta_n x^T(t) L x(t).$$

Thus, the derivative of (3.11) with respect to time along the trajectories of the first subsystem of system (2.3) is calculated and estimated as follows:

$$\dot{\mathcal{V}}\big(t, x(t)\big)$$
$$\leq \big(1 - \alpha(t)\big)x^T(t)\left[\frac{(P_{n+1} - P_n)H}{\tau_m} + \Omega_n\right]x(t)$$
$$\quad + \alpha(t)x^T(t)\left[\frac{(P_{n+1} - P_n)H}{\tau_m} + \Omega_{n+1}\right]x(t) - g_1(t)\mathcal{V}\big(t, x(t)\big),$$

where $\Omega_j \triangleq \Delta_j + \mathrm{He}\{Y_j\}$ with $\Delta_j \triangleq \mathrm{He}\{P_j A\} + \beta_j^{-1} P_j P_j + \beta_j L + g_1(t)P_j$, $Y_j \triangleq P_j K(t)$, $j = 0, 1, \ldots, H$. According to (3.5), (3.6) and Schur's complement lemma, it follows

$$\dot{\mathcal{V}}\big(t, x(t)\big) \leq -g_1(t)\mathcal{V}\big(t, x(t)\big), \quad t \in [mT, mT + \tau_m). \tag{3.12}$$

When $mT + \tau_m \leq t < (m+1)T$, using the same procedure as before, we find

$$\dot{\mathcal{V}}\big(t, x(t)\big)$$
$$\leq \big(1 - \varphi(t)\big)x^T(t)F_1 x(t) + \varphi(t)F_2 x(t) + g_2(t)\mathcal{V}\big(t, x(t)\big),$$

where $F_i \triangleq \frac{S_2 - S_1}{T - \tau_m} + \mathrm{He}\{S_i A\} + \lambda_i^{-1} S_i S_i + \lambda_i L - g_2(t)S_i$, $i = 1, 2$. Based on the Schur complement lemma and (3.7), it implies

$$\dot{\mathcal{V}}\big(t, x(t)\big) \leq g_2(t)\mathcal{V}\big(t, x(t)\big), \quad t \in \big[mT + \tau_m, (m+1)T\big). \tag{3.13}$$

According to (3.12) and (3.13), it follows that
(i) when $0 \leq t < \tau_0$,

$$\mathcal{V}\big(t, x(t)\big) \leq \mathcal{V}(0, x_0)e^{-\int_0^t g_1(s)\,ds} \leq \mathcal{V}(0, x_0)e^{-\hat{g}_1 t},$$
$$\mathcal{V}\big(\tau_0, x(\tau_0)\big) \leq \mathcal{V}(0, x_0)e^{-\int_0^{\tau_0} g_1(s)\,ds} \leq \mathcal{V}(0, x_0)e^{-\hat{g}_1 \tau_{\min}},$$

(ii) when $\tau_0 \leq t < T$,

$$\mathcal{V}\big(t, x(t)\big) \leq \mathcal{V}\big(\tau_0, x(\tau_0)\big)e^{\int_{\tau_0}^t g_2(s)\,ds}$$
$$\leq \mathcal{V}(0, x_0)e^{-\int_0^{\tau_{\min}} g_1(s)\,ds + \int_{\tau_{\min}}^t g_2(s)\,ds}$$
$$\leq \mathcal{V}(0, x_0)e^{-\hat{g}_1 \tau_{\min} + \bar{g}_2(t - \tau_{\min})},$$
$$\mathcal{V}\big(T, x(T)\big) \leq \mathcal{V}(0, x_0)e^{-\hat{g}_1 \tau_{\min} + \bar{g}_2(T - \tau_{\min})},$$

(iii) when $T \leq t < T + \tau_1$,

$$
\begin{aligned}
\mathcal{V}\big(t, x(t)\big) &\leq \mathcal{V}\big(T, x(T)\big) e^{-\int_T^t g_1(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{-\int_0^{\tau_{\min}} g_1(s)\,ds} \\
&\quad \times e^{\int_{\tau_{\min}}^T g_2(s)\,ds - \int_T^t g_1(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{-\hat{g}_1(t - T + \tau_{\min}) + \bar{g}_2(T - \tau_{\min})},
\end{aligned}
$$

$$
\mathcal{V}\big(T + \tau_1, x(T + \tau_1)\big) \leq \mathcal{V}(0, x_0) e^{-2\hat{g}_1 \tau_{\min} + \bar{g}_2(T - \tau_{\min})},
$$

(iv) when $T + \tau_1 \leq t < 2T$,

$$
\begin{aligned}
\mathcal{V}\big(t, x(t)\big) &\leq V\big(T + \tau_1, x(T + \tau_1)\big) e^{\int_{(T+\tau_1)}^t g_2(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{-\int_0^{\tau_{\min}} g_1(s)\,ds + \int_{\tau_{\min}}^T g_2(s)\,ds} \\
&\quad \times e^{-\int_T^{(T+\tau_{\min})} g_1(s)\,ds + \int_{(T+\tau_1)}^t g_2(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{-2\hat{g}_1 \tau_{\min} + \bar{g}_2(T - \tau_{\min}) + \bar{g}_2(t - T - \tau_{\min})},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}\big(2T, x(2T)\big) &\leq \mathcal{V}(0, x_0) e^{-\int_0^{\tau_{\min}} g_1(s)\,ds + \int_{\tau_{\min}}^T g_2(s)\,ds} \\
&\quad \times e^{-\int_T^{(T+\tau_{\min})} g_1(s)\,ds + \int_{(T+\tau_1)}^{2T} g_2(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{-2\hat{g}_1 \tau_{\min} + 2\bar{g}_2(T - \tau_{\min})}.
\end{aligned}
$$

It follows by induction that
(v) when $mT \leq t < mT + \tau_m$,

$$
\begin{aligned}
\mathcal{V}\big(t, x(t)\big) & \\
&\leq \mathcal{V}\big(mT, x(mT)\big) e^{-\int_{mT}^t g_1(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{[-\hat{g}_1 \tau_{\min} + \bar{g}_2(T - \tau_{\min})]m} \\
&\leq \mathcal{V}(0, x_0) e^{-2\theta(t - \tau_{\max})},
\end{aligned}
\tag{3.14}
$$

(vi) when $mT + \tau_m \leq t < (m+1)T$,

$$
\begin{aligned}
\mathcal{V}\big(t, x(t)\big) & \\
&\leq \mathcal{V}\big(mT + \tau_m, x(mT + \tau_m)\big) e^{\int_{(mT+\tau_{\min})}^t g_2(s)\,ds} \\
&\leq \mathcal{V}(0, x_0) e^{-2\theta(t - \tau_{\max})},
\end{aligned}
\tag{3.15}
$$

(3.14) and (3.15) together yield

$$
\mathcal{V}\big(t, x(t)\big) \leq \mathcal{V}(0, x_0) e^{-2\theta(t - \tau_{\max})}, \quad \forall t > 0.
$$

Then we can obtain (3.9). This completes the proof. □

Remark 3.2 It should be mentioning that the larger H is chosen, the denser the division of the interval $\mathcal{R}_{m,n}$ comes to be and, intuitively, a less conservative result can be obtained.

Remark 3.3 Note that Theorem 3.1 provides us a method to estimate the minimum admissible value of control width, which has not been reported in the previous literature mainly because of the analytical complexity. Especially, when $\tau_{\min} \to T$, the intermittent control will become the usual continuous control, while $\tau_{\max} \to 0$ means that the intermittent control will reduce to the general impulsive control. The intermittent control can be viewed as a transition between continuous and impulsive control.

Remark 3.4 The exponential convergence rate θ is explicitly given in Theorem 3.1. Moreover, the exponential convergence rate θ is closely linked to the minimal work width $\tau_{\min}$ and the properties of control inputs. It follows that smooth control inputs have an effect on the exponential convergence rate θ. A numerical example will be presented to characterize the relationship between the exponential convergence rate θ, the minimal work width $\tau_{\min}$ and the control input smooth index function in the next section.

Similarly, we can obtain the following conclusion for the system (2.1) in terms of the time-independent Lyapunov function. Here we omit its proof to avoid repetition.

Corollary 3.5 *If there exist positive definite symmetric matrices P, S and a matrix Y with appropriate dimensions, scalars $\mu_i > 0$ ($i = 1, 2$), monotone and bounded real-valued functions $g_i(t) > 0$ ($i = 1, 2$), such that*

$$\begin{bmatrix} He\{PA + Y\} + \mu_1 L + \bar{g}_1 P & -P \\ * & -\mu_1 I \end{bmatrix} < 0, \tag{3.16}$$

$$\begin{bmatrix} He\{SA\} + \mu_2 L - \hat{g}_2 S & -S \\ * & -\mu_2 I \end{bmatrix} < 0, \tag{3.17}$$

and (3.8) *are satisfied, then the origin of the system* (2.3) *is globally exponentially stable in the following sense*:

$$\|x(t)\| \leq \sqrt{\frac{\lambda_M}{\lambda_m}} \|x_0\| e^{-\theta(t - \tau_{\max})}, \quad \forall t > 0,$$

where $\bar{g}_1, \hat{g}_2, \tau_{\min}, \tau_{\max}$ and θ are defined in Theorem 3.1, and the controller gain is given as $K = P^{-1} Y$.

Remark 3.6 Corollary 3.5 is reduced to Theorem 1 [6] by setting $P = S$, $\tau_{\min} = \tau_{\max}$, $g_i(t) \equiv C_i$ (constant). Moreover, Corollary 3.5 is easily extended to the optimal intermittent controller design like the results in [6].

3.2 Smooth control analysis

It may lead to unwanted chatter behaviors that result in dropping and damaging of the instruments in many practical problems, although time-dependent state feedback control laws are able to improve the transient behaviors that time-invariant state feedback control laws cannot. As is well known, the control laws with a constant matrix gain are not

bumpy [1, 19]. In the case of matrix gain $K(t)$ given in (3.10) this means that one has to fix $P_n = P$ and $Y_n = Y$ for $0 \le n \le H$, which is conservative. Therefore, a control scheme that encompasses all the advantages of the aforementioned methodologies and avoids their drawbacks at the same time is in demand. We deal with the problem by using the main idea of ϵ-suboptimal set of an optimization problem. Instead of adopting a constant gain matrix, we propose to select among the state feedback gain matrices the one satisfying $\|K - K(t)\| < \varepsilon(t)$ where K is a constant matrix and a real-valued function $\varepsilon(t)$ named smooth index function to be determined. This constraint can be easily written by means of the unknown variables.

Theorem 3.7 *Assume that there exist monotone and bounded real-valued functions $\rho_i(t) > 0$ $(i = 1, 2)$, and matrices $P_n^T = P_n > 0$, Y_n $(n = 0, 1, \dots, H)$, and K with appropriate dimensions that satisfy the following inequalities*:

$$
\begin{bmatrix}
\hat{\rho}_1 I & K^T & Y_n^T \\
* & \hat{\rho}_2 I & I \\
* & * & P_n
\end{bmatrix} > 0, \quad n = 0, 1, \dots, H,
\tag{3.18}
$$

where $\hat{\rho}_i \triangleq \min_t\{\rho_i(t)\}$, $i = 1, 2$, then the state feedback control law with the matrix gain (3.10) *satisfies $\|K - K(t)\| < \varepsilon(t)$ with $\varepsilon(t) = \sqrt{\rho_1(t)\rho_2(t)}$.*

Proof Notice that $K(t) = P^{-1}(d(t))Y(d(t))$.

The theorem will be proved by showing that

$$
\left\| K - P^{-1}\big(d(t)\big)Y\big(d(t)\big) \right\| < \sqrt{\rho_1(t)\rho_2(t)}.
$$

According to condition (3.18), it implies for $n = 0, 1, \dots, H - 1$,

$$
\begin{bmatrix}
\rho_1(t)I & K^T & [(1 - \alpha(t))Y_n + \alpha(t)Y_{n+1}]^T \\
* & \rho_2(t)I & I \\
* & * & (1 - \alpha(t))P_n + \alpha(t)P_{n+1}
\end{bmatrix} > 0,
$$

i.e.,

$$
\begin{bmatrix}
\rho_1(t)I & K^T & Y^T(d(t)) \\
* & \rho_2(t)I & I \\
* & * & P(d(t))
\end{bmatrix} > 0,
$$

where $\alpha(t)$ and $d(t)$ are defined in (3.2).

Based on the Schur complement lemma, it follows that

$$
\begin{bmatrix}
\Lambda_1 & \Lambda_2 \\
* & \Lambda_3
\end{bmatrix} > 0,
$$

where

$$
\Lambda_1 \triangleq \rho_1(t)I - Y^T\big(d(t)\big)P^{-1}\big(d(t)\big)Y\big(d(t)\big),
$$

$$\Lambda_2 \triangleq K^T - Y^T\big(d(t)\big)P^{-1}\big(d(t)\big),$$

$$\Lambda_3 \triangleq \rho_2(t)I - P^{-1}\big(d(t)\big).$$

By the fact that $\Lambda_1 \preceq \rho_1(t)I$, $\Lambda_3 \preceq \rho_2(t)I$, it yields

$$\begin{bmatrix} \rho_1(t)I & \Lambda_2 \\ * & \rho_2(t)I \end{bmatrix} > 0.$$

It is evident to see that $\rho_1(t)\rho_2(t)I - \Lambda_2^T\Lambda_2 > 0$.
Consequently, we can infer that

$$\big\| K - P^{-1}\big(d(t)\big)Y\big(d(t)\big)\big\| < \sqrt{\rho_1(t)\rho_2(t)}$$

i.e.,

$$\big\| K - K(t)\big\| < \varepsilon(t).$$

The proof of the theorem is now completed. □

Remark 3.8 Theorem 3.7 implies the minimal value of the smooth index function $\varepsilon(t)$ will become smaller when $\rho_i(t)$ become smaller. Meanwhile, the supremum (or least upper bound) of smooth index function set $\{\varepsilon(t)\}$, which is denoted by $\varepsilon_{\sup}$, can be solved. On the other hand, Theorem 3.7 along with the conditions of Theorem 3.1 can determine the periodically intermittent smooth controller gain. It is convenient for us to obtain the solution by using the proposed method, for which there is not required an additional loop to remove input discontinuities compared with that in [4, 9], and so on.

4 Numerical example

In this section, an example is given to demonstrate the effectiveness of the obtained theoretical results. Consider Chua's oscillator displaying chaotic behavior [6]. The system is described by the following equation:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - \omega(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2, \end{cases} \tag{4.1}$$

where $\omega(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$, and $\alpha = 9.2156$, $\beta = 15.9946$, $a = -1.2495$, $b = -0.75735$. To employ the results given above, we rewrite (4.1) into the following form:
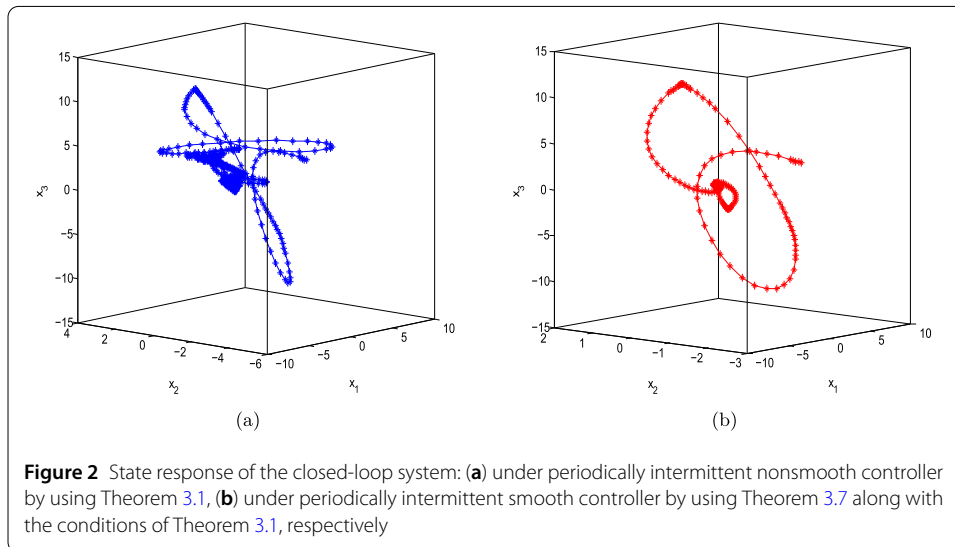
$$\dot{x}(t) = Ax(t) + f(x),$$

where

$$A = \begin{bmatrix} -\alpha - \alpha b & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \qquad f(x) = \begin{bmatrix} \Psi \\ 0 \\ 0 \end{bmatrix},$$

Table 1 The minimum admissible value of control width under period $T = 2$

	Theorem 3.1 ($H = 4$)	Theorem 3.1 ($H = 3$)	Corollary 3.5
$1 \leq g_1, g_2 \leq 3$	0.55	0.6	Infeasible
$0.5 \leq g_1, g_2 \leq 3$	0.35	0.4	Infeasible
$1 \leq g_1, g_2 \leq 5$	0.35	0.4	Infeasible
$0.5 \leq g_1, g_2 \leq 5$	0.25	0.3	Infeasible
$1 \leq g_1, g_2 \leq 10$	0.2	0.25	Infeasible
$0.5 \leq g_1, g_2 \leq 10$	0.1	0.15	Infeasible
$1 \leq g_1, g_2 \leq 15$	0.13	0.15	0.85
$0.5 \leq g_1, g_2 \leq 15$	0.09	0.1	0.84

Figure 2 State response of the closed-loop system: (**a**) under periodically intermittent nonsmooth controller by using Theorem 3.1, (**b**) under periodically intermittent smooth controller by using Theorem 3.7 along with the conditions of Theorem 3.1, respectively
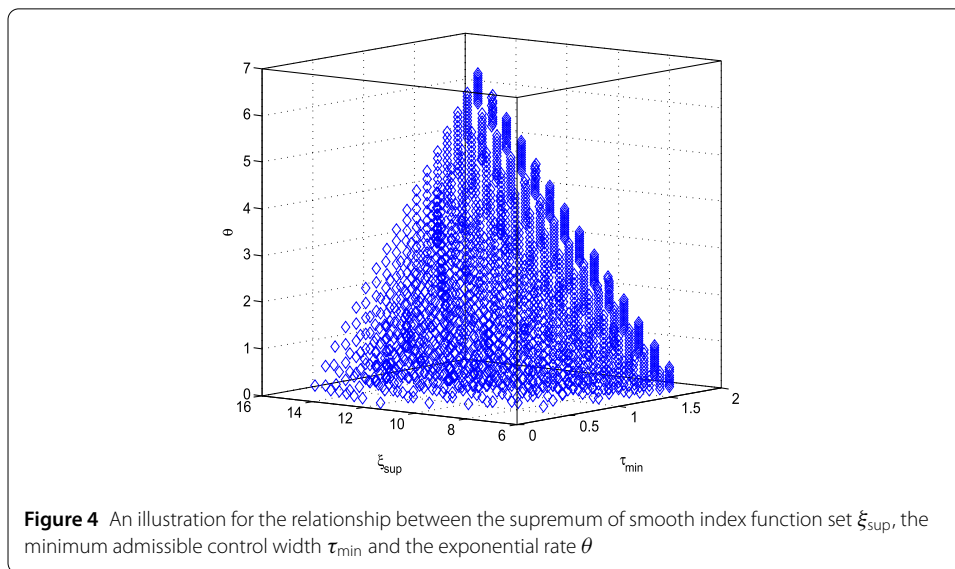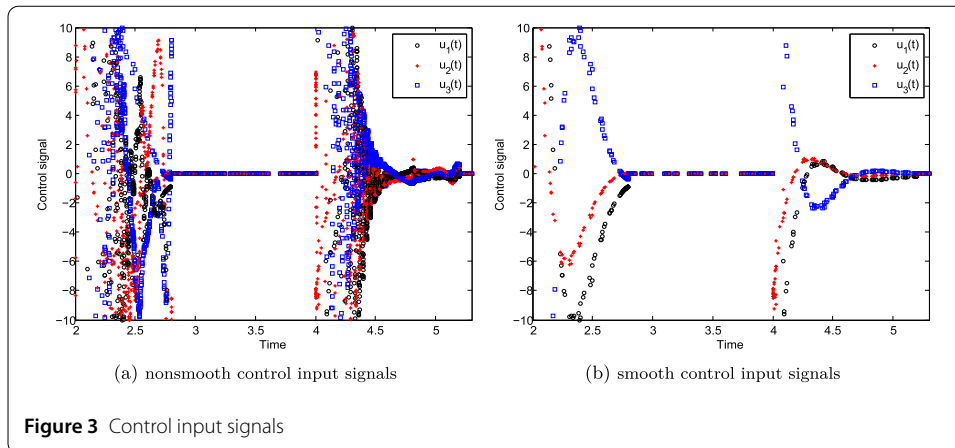
with $\Psi \triangleq -\frac{\alpha(a-b)(|x_1+1|-|x_1-1|)}{2}$. It can easily be verified that $\|f(x)\|^2 \leq x^T L x$, where $L = \text{diag}\{\Upsilon, 0, 0\}$, with $\Upsilon \triangleq \alpha^2(a-b)^2$. In the following, for simplification of the design procedure, we take $g_i(t) \equiv g_i$ (constant) in Theorem 3.1.

Our first purpose is to solve the minimum admissible value of control width for Chua's oscillator (4.1) such that the origin of system (4.1) is globally exponentially stable. By different strategies and setting the relevant parameters appropriately, the computation results for the system are listed in Table 1. It is evident that the time-dependent width Lyapunov function method is less conservative than the time-independent one. Moreover, as mentioned in Remark 3.2, less conservative results can be obtained as the parameter H increases.

Next, we consider the global exponential stabilization of the origin of system (4.1) in terms of the periodically intermittent smooth control law and the periodically intermittent nonsmooth control law, respectively. With the initial condition $x(0) = [6 \ -1 \ 2]^T$, the period $T = 2$, parameters $H = 3$, $0.5 \leq g_1, g_2 \leq 3$, and the control width $0.4 \leq \tau_i \leq 1.9$, $i = 1, 2, \dots$, the simulation results of system (4.1) are plotted in Fig. 2 and the corresponding control input signals are shown in Fig. 3. The results plotted in these figures illustrate the efficiency of the strategy. Not only concerns the periodically intermittent smooth control law less control effort than the periodically intermittent nonsmooth control law, but the periodically intermittent smooth control law also achieves better transients than that.

Finally, for given the period $T = 2$, parameters $H = 3$, $0.1 \leq g_1, g_2 \leq 15$, Fig. 4 shows the relationship between the supremum of smooth index function set ε_{sup}, the minimum

Figure 3 Control input signals

Figure 4 An illustration for the relationship between the supremum of smooth index function set ξ_{sup}, the minimum admissible control width τ_{min} and the exponential rate θ

admissible control width τ_{min} and the exponential rate θ, where the monotonicity can be observed. θ is increasing as ε_{sup} and τ_{min} increase. Such phenomena confirm the fact that a larger control effort for a relatively longer work width leads to a more rapid convergence rate.

5 Conclusion

In this paper, we have formulated the problems of global exponential stability and stabilization for a class of nonlinear systems and have designed a periodically intermittent smooth controller for the nonlinear systems. The global exponential stabilization criteria are established in virtue of time-dependent width Lyapunov functions and linear matrix inequality techniques. Furthermore, the exponential convergence rate of systems and the minimum admissible value of the control width are also given simultaneously. Especially, it has been substantiated that when applying a periodically intermittent width time-dependent control law to a class of nonlinear systems, the deduced minimum admissible values of the work widths are smaller than that in terms of periodically intermittent time-independent control law. On the other hand, Theorem 3.7 in conjunction with the conditions of Theorem 3.1 can obtain the periodically intermittent smooth controller gain.

Meanwhile, the supremum (or least upper bound) of the smooth index function set can be solved. Numerical simulations have showed the validity of our theoretical result.

Availability of data and materials
All of the authors declare that all the data can be accessed in our manuscript in the numerical simulation section.

Competing interests
The authors declare that there is no conflict of interests.

Authors' contributions
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