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Hermite–Jensen–Mercer type inequalities for conformable integrals and related results

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Abstract

In this paper, certain Hermite–Jensen–Mercer type inequalities are proved via conformable integrals of arbitrary order. We establish some different and new fractional Hermite–Hadamard–Mercer type inequalities for a differentiable function f whose derivatives in the absolute values are convex.

Keywords: Convex functions; Hermite–Hadamard inequalities; Jensen–Mercer inequality; Conformable integrals

1 Introduction

The concept of convex function differs from other function classes with its features such as high application areas in mathematics, statistics, and many other applied sciences. This is due to its special useful definition having geometric interpretation. Moreover, it is one of the indispensable parts of inequality theory and has become the main motivation point of many inequalities.

Although the concept of convex function has a useful place in many fields of mathematical analysis and statistics, it has revealed its main importance and effectiveness in the field of inequality theory with convex analysis. Many classical and analytical inequalities, especially Hadamard's inequality, Jensen's inequality, and Steffensen's inequality, have been achieved with the help of this concept. Detailed information and effectiveness of this function class can be found in [1–6].

Let $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$, and let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be nonnegative weights such that $\sum_{i=1}^n \xi_i = 1$. The famous Jensen inequality (see [7]) in the literature states that if Υ is a convex function on the interval $[a, b]$, then

$$\Upsilon\left(\sum_{i=1}^n \xi_i \mu_i\right) \leq \left(\sum_{i=1}^n \xi_i \Upsilon(\mu_i)\right) \quad (1.1)$$

for all $\mu_i \in [\theta, \vartheta]$ and all $\xi_i \in [0, 1]$ ($i = 1, 2, \dots, n$).

In 2003, Mercer gave a variant of Jensen's inequality (see [8]) as follows.

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Theorem 1.1 *If Υ is a convex function on $[\theta, \vartheta]$, then*

$$\Upsilon\left(\theta + \vartheta - \sum_{i=1}^n \xi_i \mu_i\right) \leq \Upsilon(\theta) + \Upsilon(\vartheta) - \sum_{i=1}^n \xi_i \Upsilon(\mu_i) \quad (1.2)$$

$\forall \mu_i \in [\theta, \vartheta]$ and all $\xi_i \in [0, 1]$ ($i = 1, 2, \dots, n$).

Based on this useful inequality, several papers have been performed. One of them can be stated in Matkovic et al. This study includes some new findings on Jensen's inequality of Mercer type for operators with applications [9]. Then, in 2009, Mercer's result was expanded to higher dimensions by Niezgoda's paper in [10]. Recently, notable contributions have been made on Jensen's Mercer type inequality. In 2014, Kian gave a concept of Jensen's inequality for superquadratic functions [11]. Therefore, Anjidani proved some motivating results on reverse Jensen–Mercer type operator inequalities and Jensen–Mercer operator inequalities for superquadratic functions (see [12, 13]). Ali and Khan generalized integral Mercer's inequality and integral means in [14].

Another important inequality that characterizes convex functions is Hermite–Hadamard inequality, that is, if a mapping $\Upsilon : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on J and $\theta, \vartheta \in J$, $\theta < \vartheta$, then

$$\Upsilon\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} \Upsilon(\lambda) d\lambda \leq \frac{\Upsilon(\theta) + \Upsilon(\vartheta)}{2}$$

(see [15–18] for the fractional setting). Fractional calculus, one of the areas where inequality theory has benefited most in recent years, is an area that continues its development with a high acceleration by defining new fractional derivative and integral operators. Operators' applications in various fields, such as economics, applied mathematics, engineering, and mathematical biology, add strength to fractional analysis (see [19–26]).

Now, we recall the definition of conformable integral of arbitrary order including the higher order case, on which our proven inequalities will be based.

Definition 1.1 ([27]) Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable operator starting at θ if order α is defined by

$$(I_{\alpha}^{\theta} \phi)(\kappa) = \frac{1}{n!} \int_{\theta}^{\kappa} (\kappa - \mu)^n (\mu - \theta)^{\beta-1} \phi(\mu) d\mu, \quad (1.3)$$

and right conformable fractional integral is defined by

$$({}^{\vartheta} I_{\alpha} \phi)(\kappa) = \frac{1}{n!} \int_{\kappa}^{\vartheta} (\mu - \kappa)^n (\vartheta - \mu)^{\beta-1} \phi(\mu) d\mu, \quad (1.4)$$

if $\alpha = n + 1$, then $\beta = \alpha - n = n + 1 - n = 1$, where $n = 0, 1, 2, 3, \dots$, and hence $(I_{\alpha}^{\theta} \phi)(\kappa) = (J_{n+1}^{\theta} \phi)(\kappa)$.

Remark 1.1 Notice that the conformable derivatives of order $\theta > 1$ have memory effect with kernel whose power law is integer.

In this article, by using the Jensen–Mercer inequality, we prove Hermite–Hadamard type inequalities for fractional integrals, and we establish some new conformable integrals connected with the left and right sides of Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute value are convex. Moreover, there will be further equalities for differentiable functions using Hölder inequality and power mean inequality.

2 Hermite–Hadamard–Mercer type inequalities for conformable integrals

By using Jensen–Mercer inequality, Hermite–Hadamard type inequalities can be expressed via conformable integrals as follows.

Theorem 2.1 *Let ϕ be a convex function. Then the following conformable integral inequalities hold:*

$$\begin{aligned} \phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) &\leq \frac{\Gamma(\alpha + 1)}{2(\nu - \mu)^\alpha \Gamma(\alpha - n)} \\ &\quad \times \left\{ I_\alpha^{(\theta + \vartheta - \nu)} \phi(\theta + \vartheta - \mu) + I_\alpha^{(\theta + \vartheta - \mu)} \phi(\theta + \vartheta - \nu) \right\} \\ &\leq \phi(\theta) + \phi(\vartheta) - \left(\frac{\phi(\mu) + \phi(\nu)}{2} \right) \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} \phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) &\leq \phi(\theta) + \phi(\vartheta) - \frac{\Gamma(\alpha + 1)}{2(\nu - \mu)^\alpha \Gamma(\alpha - n)} \left\{ I_\alpha^\mu \phi(\nu) + I_\alpha^\nu \phi(\mu) \right\} \\ &\leq \phi(\theta) + \phi(\vartheta) - \phi\left(\frac{\mu + \nu}{2}\right) \end{aligned} \quad (2.2)$$

$\forall \mu, \nu \in [\theta, \vartheta]$, $\alpha > 0$, and $\Gamma(\cdot)$ is the gamma function.

Proof Using the convexity of ϕ , we can write

$$\phi\left(\theta + \vartheta - \frac{\tau + \omega}{2}\right) = \phi\left(\frac{\theta + \vartheta - \tau + \theta + \vartheta - \omega}{2}\right) \leq \frac{1}{2}(\phi(\theta + \vartheta - \tau) + \phi(\theta + \vartheta - \omega))$$

for all $\tau, \omega \in [\theta, \vartheta]$. By changing the variables $\tau = \kappa\mu + (1 - \kappa)\nu$ and $\omega = (1 - \kappa)\mu + \kappa\nu$, $\kappa \in [0, 1]$, we have

$$2\phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) \leq \phi(\theta + \vartheta - (\kappa\mu + (1 - \kappa)\nu)) + \phi(\theta + \vartheta - ((1 - \kappa)\mu + \kappa\nu)).$$

Multiplying both sides by $\frac{1}{n!} \kappa^n (1 - \kappa)^{\alpha - n - 1}$ and then integrating the resulting inequality over $\kappa \in [0, 1]$. Let $x = (\theta + \vartheta - (\kappa\mu + (1 - \kappa)\nu))$, also let $w = (\theta + \vartheta - ((1 - \kappa)\mu + \kappa\nu))$, we get

$$\begin{aligned} \phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) &\frac{2}{n!} \int_0^1 \kappa^n (1 - \kappa)^{\alpha - n - 1} d\kappa \\ &\leq \frac{1}{n!} \int_0^1 \kappa^n (1 - \kappa)^{\alpha - n - 1} \end{aligned}$$

$$\begin{aligned} & \times (\phi(\theta + \vartheta - (\kappa\mu + (1-\kappa)v)) + \phi(\theta + \vartheta - ((1-\kappa)\mu + \kappa v))) d\kappa \\ & \leq \frac{\Gamma(\alpha+1)}{2(v-\mu)^\alpha \Gamma(\alpha-n)} \{I_\alpha^{(\theta+\vartheta-v)} \phi(\theta + \vartheta - \mu) + {}^{(\theta+\vartheta-\mu)} I_\alpha \phi(\theta + \vartheta - v)\}, \end{aligned}$$

where $\int_0^1 \kappa^n (1-\kappa)^{\alpha-n-1} d\kappa = B(n+1, \alpha-n) = \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}$, and so the first inequality of (2.1) is proved.

Now, to prove the second inequality of (2.1), we first remind that if ϕ is a convex function, then for $\kappa \in [0, 1]$ it gives

$$\phi(\theta + \vartheta - (\kappa\mu + (1-\kappa)v)) \leq \phi(\theta) + \phi(\vartheta) - [\kappa\phi(\mu) + (1-\kappa)\phi(v)] \quad (2.3)$$

and

$$\phi(\theta + \vartheta - ((1-\kappa)\mu + \kappa v)) \leq \phi(\theta) + \phi(\vartheta) - [(1-\kappa)\phi(\mu) + \kappa\phi(v)]. \quad (2.4)$$

By adding the inequalities of (2.3) and (2.4), we have

$$\begin{aligned} & \phi(\theta + \vartheta - (\kappa\mu + (1-\kappa)v)) + \phi(\theta + \vartheta - ((1-\kappa)\mu + \kappa v)) \\ & \leq 2(\phi(\theta) + \phi(\vartheta)) - (\phi(\mu) + \phi(v)). \end{aligned}$$

Multiplying both sides by $\frac{1}{n!} \kappa^n (1-\kappa)^{\alpha-n-1}$ and then integrating the resulting inequality over $\kappa \in [0, 1]$, we have

$$\begin{aligned} & \frac{1}{n!} \int_0^1 \kappa^n (1-\kappa)^{\alpha-n-1} (\phi(\theta + \vartheta - (\kappa\mu + (1-\kappa)v)) + \phi(\theta + \vartheta - ((1-\kappa)\mu + \kappa v))) d\kappa \\ & \leq \{2(\phi(\theta) + \phi(\vartheta)) - (\phi(\mu) + \phi(v))\} \frac{1}{n!} \int_0^1 \kappa^n (1-\kappa)^{\alpha-n-1} d\kappa, \\ & \frac{\Gamma(\alpha+1)}{(v-\mu)^\alpha \Gamma(\alpha-n)} \{I_\alpha^{(\theta+\vartheta-v)} \phi(\theta + \vartheta - \mu) + {}^{(\theta+\vartheta-\mu)} I_\alpha \phi(\theta + \vartheta - v)\} \\ & \leq \{2(\phi(\theta) + \phi(\vartheta)) - (\phi(\mu) + \phi(v))\}. \end{aligned} \quad (2.5)$$

Multiplying $\frac{1}{2}$ to (2.5), we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(v-\mu)^\alpha \Gamma(\alpha-n)} \{I_\alpha^{(\theta+\vartheta-v)} \phi(\theta + \vartheta - \mu) + {}^{(\theta+\vartheta-\mu)} I_\alpha \phi(\theta + \vartheta - v)\} \\ & \leq (\phi(\theta) + \phi(\vartheta)) - \frac{\phi(\mu) + \phi(v)}{2}. \end{aligned}$$

After further simplification we get the required result. Now, in order to prove (2.2), we employ Jensen–Mercer's inequality as follows:

$$\phi\left(\theta + \vartheta - \frac{\tau + \omega}{2}\right) \leq \phi(\theta) + \phi(\vartheta) - \frac{\phi(\tau) + \phi(\omega)}{2} \quad (2.6)$$

$\forall \tau, \omega \in [\theta, \vartheta]$.

Now, by change of variables $\tau = \kappa\mu + (1 - \kappa)v$ and $\omega = \kappa v + (1 - \kappa)\mu$, $\forall \mu, v \in [a, b]$ and $\kappa \in [0, 1]$ in (2.6), we have

$$\phi\left(\theta + \vartheta - \frac{\mu + v}{2}\right) \leq \phi(\theta) + \phi(\vartheta) - \frac{\phi(\kappa\mu + (1 - \kappa)v) + \phi(\kappa v + (1 - \kappa)\mu)}{2}.$$

Multiplying both sides by $\frac{1}{n!}\kappa^n(1 - \kappa)^{\alpha-n-1}$ and then integrating the resulting inequality over $\kappa \in [0, 1]$, we have

$$\begin{aligned} & \phi\left(\theta + \vartheta - \frac{\mu + v}{2}\right) \frac{1}{n!} \int_0^1 \kappa^n (1 - \kappa)^{\alpha-n-1} \\ & \leq \frac{1}{n!} \int_0^1 \kappa^n (1 - \kappa)^{\alpha-n-1} [\phi(\theta) + \phi(\vartheta)] d\kappa \\ & \quad - \frac{\int_0^1 \kappa^n (1 - \kappa)^{\alpha-n-1} (\phi(\kappa\mu + (1 - \kappa)v) + \phi(\kappa v + (1 - \kappa)\mu)) d\kappa}{2}, \\ & \phi\left(\theta + \vartheta - \frac{\mu + v}{2}\right) \leq \phi(\theta) + \phi(\vartheta) - \frac{\Gamma(\alpha + 1)}{2(v - \mu)^\alpha \Gamma(\alpha - n)} \{I_\alpha^\mu \phi(v) + {}^v I_\alpha \phi(\mu)\}, \end{aligned}$$

and so the first inequality of (2.2) is proved.

Now, for the proof of the second inequality of (2.2), we first note that if ϕ is a convex function, then for $\kappa \in [0, 1]$

$$\begin{aligned} \phi\left(\frac{\mu + v}{2}\right) &= \phi\left(\frac{\kappa\mu + (1 - \kappa)v + \kappa v + (1 - \kappa)\mu}{2}\right) \\ &\leq \frac{\phi(\kappa\mu + (1 - \kappa)v) + \phi(\kappa v + (1 - \kappa)\mu)}{2}. \end{aligned}$$

Multiplying both sides by $\frac{1}{n!}\kappa^n(1 - \kappa)^{\alpha-n-1}$ and then integrating the resulting inequality over $\kappa \in [0, 1]$, we have

$$\phi\left(\frac{\mu + v}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(v - \mu)^\alpha \Gamma(\alpha - n)} \{I_\alpha^\mu \phi(v) + {}^v I_\alpha \phi(\mu)\}.$$

Multiplying by (-1) , then adding $\phi(\theta) + \phi(\vartheta)$ on both sides of the inequality, we get the desired result. \square

Remark 2.1 For $\alpha = n + 1$ in Theorem 2.1, we get Theorem 3 proved in [28] in the integer case order.

Theorem 2.2 Let $0 \leq \theta < \vartheta$, $\phi : [\theta, \vartheta] \rightarrow \mathbb{R}$ be a positive function and $\phi \in L_1[\theta, \vartheta]$. Also, suppose that ϕ is a convex function on $[\theta, \vartheta]$, ϕ' on (θ, ϑ) and $\alpha \in (0, 1)$. Then the following conformable integral inequalities hold:

$$\begin{aligned} \phi\left(\theta + \vartheta - \frac{\mu + v}{2}\right) &\leq \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(v - \mu)^\alpha \Gamma(\alpha - n)} \\ &\quad \times \left\{ \left(I_\alpha^{\theta+\vartheta-\frac{\mu+v}{2}}\right)(\phi(\theta + \vartheta - \mu)) + \left({}^{\theta+\vartheta-\frac{\mu+v}{2}}I_\alpha\right)(\phi(\theta + \vartheta - v)) \right\} \\ &\leq \phi(\theta) + \phi(\vartheta) - \left(\frac{\phi(\mu) + \phi(v)}{2}\right) \end{aligned} \quad (2.7)$$

$\forall \mu, v \in [\theta, \vartheta]$, $\alpha > 0$, and $\Gamma(\cdot)$ is the gamma function.

Proof To prove the first part of the inequality, by using the Jensen–Mercer’s inequality and by changing the variables $\tau = \frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu$ and $\omega = \frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu$, $\kappa \in [0, 1]$, we can write the following inequality for $\forall \tau, \omega \in [\theta, \vartheta]$:

$$2\phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) \leq \phi\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu\right)\right) + \phi\left(\theta + \vartheta - \left(\frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu\right)\right).$$

Multiplying both sides by $\frac{1}{n!}\kappa^n(1-\kappa)^{\alpha-n-1}$ and then integrating the resulting inequality over $\kappa \in [0, 1]$, let $w = (\theta + \vartheta - (\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu))$ and $x = (\theta + \vartheta - (\frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu))$, we have

$$\begin{aligned} & \frac{2}{n!}\phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) \int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1} d\kappa \\ & \leq \frac{1}{n!} \int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1} \\ & \quad \times \left(\phi\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu\right)\right) + \phi\left(\theta + \vartheta - \left(\frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu\right)\right) \right) d\kappa, \\ & \phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) \\ & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\nu-\mu)^\alpha\Gamma(\alpha-n)} \left\{ (I_\alpha^{\theta+\vartheta-\frac{\mu+\nu}{2}})(\phi(\theta + \vartheta - \mu)) + (I_\alpha^{\theta+\vartheta-\frac{\mu+\nu}{2}})(\phi(\theta + \vartheta - \nu)) \right\}, \end{aligned}$$

and so the first inequality of (2.7) is proved.

Now, for the proof of the second inequality of the theorem, we first note that if ϕ is a convex function, then for $\kappa \in [0, 1]$ it gives

$$\phi\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu\right)\right) \leq \phi(\theta) + \phi(\vartheta) - \left[\frac{\kappa}{2}\phi(\mu) + \frac{2-\kappa}{2}\phi(\nu)\right] \quad (2.8)$$

and

$$\phi\left(\theta + \vartheta - \left(\frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu\right)\right) \leq \phi(\theta) + \phi(\vartheta) - \left[\frac{2-\kappa}{2}\phi(\mu) + \frac{\kappa}{2}\phi(\nu)\right]. \quad (2.9)$$

By adding the inequalities of (2.8) and (2.9), we have

$$\begin{aligned} & \phi\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu\right)\right) + \phi\left(\theta + \vartheta - \left(\frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu\right)\right) \\ & \leq 2(\phi(\theta) + \phi(\vartheta)) - (\phi(\mu) + \phi(\nu)). \end{aligned}$$

Multiplying both sides by $\frac{1}{n!}\kappa^n(1-\kappa)^{\alpha-n-1}$ and then integrating the resulting inequality over $\kappa \in [0, 1]$, we have

$$\begin{aligned} & \frac{1}{n!} \int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1} \\ & \quad \times \left(\phi\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu\right)\right) + \phi\left(\theta + \vartheta - \left(\frac{2-\kappa}{2}\mu + \frac{\kappa}{2}\nu\right)\right) \right) d\kappa \\ & \leq \{2(\phi(a) + \phi(b)) - (\phi(\mu) + \phi(\nu))\} \frac{1}{n!} \int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1} d\kappa, \end{aligned}$$

$$\begin{aligned} & \frac{2^\alpha \Gamma(\alpha)}{(v-\mu)^\alpha \Gamma(\alpha-n)} \left\{ (I_\alpha^{\theta+\vartheta-\frac{\mu+v}{2}})(\phi(\theta+\vartheta-\mu)) + (I_\alpha^{\theta+\vartheta-\frac{\mu+v}{2}})(\phi(\theta+\vartheta-v)) \right\} \\ & \leq \{2(\phi(\theta) + \phi(\vartheta)) - (\phi(\mu) + \phi(v))\}. \end{aligned}$$

Multiplying the above inequality by $\frac{1}{2}$, we obtain

$$\begin{aligned} & \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(v-\mu)^\alpha \Gamma(\alpha-n)} \left\{ (I_\alpha^{\theta+\vartheta-\frac{\mu+v}{2}})(\phi(\theta+\vartheta-\mu)) + (I_\alpha^{\theta+\vartheta-\frac{\mu+v}{2}})(\phi(\theta+\vartheta-v)) \right\} \\ & \leq (\phi(\theta) + \phi(\vartheta)) - \frac{\phi(\mu) + \phi(v)}{2}. \end{aligned}$$

After further simplifications, we get the required inequality. \square

Lemma 2.1 Let $\phi : [\theta, \vartheta] \rightarrow \Re$ be a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $\phi' \in L[\theta, \vartheta]$, then the following equation holds:

$$\begin{aligned} & B(n+1, \alpha-n) \frac{\phi(\theta+\vartheta-\mu) + \phi(\theta+\vartheta-v)}{2} - \frac{n!}{2(v-\mu)^\alpha} \\ & \quad \times \left\{ (I_\alpha^{\theta+\vartheta-v})(\phi(\theta+\vartheta-\mu)) + (I_\alpha^{\theta+\vartheta-\mu})(\phi(\theta+\vartheta-v)) \right\} \\ & = \frac{v-\mu}{2} \int_0^1 [B_\kappa(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n)] \phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)v)) d\kappa \end{aligned} \quad (2.10)$$

$\forall \mu, v \in [\theta, \vartheta], \alpha > 0, \kappa \in [0, 1]$, and $\beta(\cdot)$ is the beta function.

Proof It suffices to note that

$$I = \frac{v-\mu}{2} \{I_2 - I_1\}, \quad (2.11)$$

where

$$\begin{aligned} I_1 &= \int_0^1 B_{1-\kappa}(n+1, \alpha-n) \phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)v)) d\kappa \\ &= \int_0^1 \left(\int_0^{1-\kappa} \mu^n (1-x)^{\alpha-n-1} \right) \phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)v)) d\kappa \\ &= B(n+1, \alpha-n) \frac{-\phi(\theta+\vartheta-v)}{v-\mu} + \frac{1}{(v-\mu)^{\alpha+1}} \\ & \quad \times \int_{\theta+\vartheta-v}^{\theta+\vartheta-\mu} ((\theta+\vartheta-v)-w)^n (w-(\theta+\vartheta-v))^{\alpha-n-1} \phi(w) dw \\ &= B(n+1, \alpha-n) \frac{-\phi(\theta+\vartheta-v)}{v-\mu} + \frac{n!}{(v-\mu)^{\alpha+1}} \{I_\alpha^{\theta+\vartheta-v} \phi(\theta+\vartheta-\mu)\} \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} I_2 &= \int_0^1 B_\kappa(n+1, \alpha-n) \phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)v)) d\kappa \\ &= \int_0^1 \left(\int_0^\kappa \mu^n (1-\mu)^{\alpha-n-1} \right) \phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)v)) d\kappa \end{aligned}$$

$$\begin{aligned}
&= B(n+1, \alpha-n) \frac{\phi(\theta+\vartheta-\mu)}{v-\mu} - \frac{1}{(v-\mu)^{\alpha+1}} \\
&\quad \times \int_{\theta+\vartheta-\mu}^{\theta+\vartheta-v} (w-(\theta+\vartheta-\mu))^n ((\theta+\vartheta-v)-w)^{\alpha-n-1} \phi(w) dw \\
&= B(n+1, \alpha-n) \frac{\phi(\theta+\vartheta-\mu)}{v-\mu} - \frac{n!}{(v-\mu)^{\alpha+1}} \left\{ {}^{\theta+\vartheta-\mu} I_{\alpha} \phi(\theta+\vartheta-v) \right\}. \quad (2.13)
\end{aligned}$$

By combining (2.12) and (2.13) with (2.11), we get (2.10). \square

Remark 2.2 If we set $\mu = a$ and $v = b$ in Lemma 2.1, we will get Lemma 3.1 in [29].

Theorem 2.3 Suppose that $\phi : [\theta, \vartheta] \rightarrow R$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$ and $\phi \in L[\theta, \vartheta]$. If $|\phi'|$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned}
&\left| B(n+1, \alpha-n) \frac{\phi(\theta+\vartheta-\mu) + \phi(\theta+\vartheta-v)}{2} - \frac{n!}{2(v-\mu)^{\alpha}} \right. \\
&\quad \times \left. \left\{ (I_{\alpha}^{\theta+\vartheta-v})(\phi(\theta+\vartheta-\mu)) + ({}^{\theta+\vartheta-\mu} I_{\alpha})(\phi(\theta+\vartheta-v)) \right\} \right| \\
&\leq \frac{(v-\mu)}{2} B(n+1, \alpha-n) \left\{ |\phi'(\theta)| + |\phi'(\vartheta)| - \left(\frac{|\phi'(\mu)| + |\phi'(v)|}{2} \right) \right\}, \quad (2.14)
\end{aligned}$$

where $\forall \mu, v \in [\theta, \vartheta]$, $\alpha > 0$, $\kappa \in [0, 1]$, and $B(\cdot, \cdot)$ is the Euler beta function.

Proof By using Lemma 2.1 and Jensen–Mercer’s inequality, we have

$$\begin{aligned}
&\left| B(n+1, \alpha-n) \frac{\phi(\theta+\vartheta-\mu) + \phi(\theta+\vartheta-v)}{2} - \frac{n!}{2(v-\mu)^{\alpha}} \right. \\
&\quad \times \left. \left\{ (I_{\alpha}^{\theta+\vartheta-v})(\phi(\theta+\vartheta-\mu)) + ({}^{\theta+\vartheta-\mu} I_{\alpha})(\phi(\theta+\vartheta-v)) \right\} \right| \\
&\leq \frac{(v-\mu)}{2} \left[\int_0^1 B_{\kappa}(n+1, \alpha-n) |\phi'(\theta+\vartheta-(\kappa\mu+(1-\kappa)v))| d\kappa \right. \\
&\quad \left. - \int_0^1 B_{1-\kappa}(n+1, \alpha-n) |\phi'(\theta+\vartheta-(\kappa\mu+(1-\kappa)v))| d\kappa \right] \\
&\leq \frac{v-\mu}{2} [I_1 - I_2],
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} B_{\kappa}(n+1, \alpha-n) \{ |\phi'(\theta)| + |\phi'(\vartheta)| - (\kappa|\phi'(\mu)| + (1-\kappa)|\phi'(v)|) \} \\
&\quad + \int_{\frac{1}{2}}^1 B_{\kappa}(n+1, \alpha-n) \{ |\phi'(\theta)| + |\phi'(\vartheta)| - (\kappa|\phi'(\mu)| + (1-\kappa)|\phi'(v)|) \} d\kappa, \\
I_2 &= - \int_0^{\frac{1}{2}} B_{1-\kappa}(n+1, \alpha-n) \{ |\phi'(\theta)| + |\phi'(\vartheta)| - (\kappa|\phi'(\mu)| + (1-\kappa)|\phi'(v)|) \} \\
&\quad - \int_{\frac{1}{2}}^1 B_{1-\kappa}(n+1, \alpha-n) \{ |\phi'(\theta)| + |\phi'(\vartheta)| - (\kappa|\phi'(\mu)| + (1-\kappa)|\phi'(v)|) \} d\kappa.
\end{aligned}$$

On the other hand, using the property of incomplete beta function, we have

$$\begin{aligned} & B_{1-\kappa}(n+1, \alpha-n) - B_{\kappa}(n+1, \alpha-n) \\ &= \int_0^{1-\kappa} \mu^n (1-\mu)^{\alpha-n-1} d\mu - \int_0^{\kappa} \mu^n (1-\mu)^{\alpha-n-1} d\mu = \int_{\kappa}^{1-\kappa} \mu^n (1-\mu)^{\alpha-n-1} d\mu, \end{aligned}$$

where $0 \leq \kappa \leq \frac{1}{2}$;

$$\begin{aligned} & B_{\kappa}(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n) \\ &= \int_0^{\kappa} \mu^n (1-\mu)^{\alpha-n-1} d\mu - \int_0^{1-\kappa} \mu^n (1-\mu)^{\alpha-n-1} d\mu = \int_{1-\kappa}^{\kappa} \mu^n (1-\mu)^{\alpha-n-1} d\mu, \end{aligned}$$

where $\frac{1}{2} \leq \kappa \leq 1$;

$$\begin{aligned} & \leq \frac{(v-\mu)}{2} \left[\int_0^1 \kappa^{\alpha} \left\{ |\phi'(\theta)| + |\phi'(\vartheta)| - \left(\frac{1+\kappa}{2} |\phi'(\mu)| + \frac{(1-\kappa)}{2} |\phi'(\nu)| \right) \right\} dt \right. \\ & \quad \left. + \int_0^1 \kappa^{\alpha} \left\{ |\phi'(\theta)| + |\phi'(\vartheta)| - \left(\frac{(1-\kappa)}{2} |\phi'(\mu)| + \frac{1+\kappa}{2} |\phi'(\nu)| \right) \right\} d\kappa \right] \\ & \leq \frac{(v-\mu)}{2} B(n+1, \alpha-n) \left\{ |\phi'(\theta)| + |\phi'(\vartheta)| - \left(\frac{|\phi'(\mu)| + |\phi'(\nu)|}{2} \right) \right\}, \end{aligned}$$

which completes the proof. \square

Remark 2.3 If we choose $\mu = \theta$ and $\nu = \vartheta$ in Theorem 2.3, we get Theorem 3.1 for the case of $s = 1$ in [29].

Theorem 2.4 Suppose that $\phi : [\theta, \vartheta] \rightarrow R$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$ and $\phi \in L[\theta, \vartheta]$. If $|\phi'|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned} & \left| B(n+1, \alpha-n) \frac{\phi(\theta + \vartheta - \mu) + \phi(\theta + \vartheta - \nu)}{2} - \frac{n!}{2(v-\mu)^{\alpha}} \right. \\ & \quad \left. \times \left\{ (I_{\alpha}^{\theta+\vartheta-\nu})(\phi(\theta + \vartheta - \mu)) + ({}^{\theta+\vartheta-\mu}I_{\alpha})(\phi(\theta + \vartheta - \nu)) \right\} \right| \\ & \leq \frac{(v-\mu)}{2} \Psi^{\frac{1}{p}} \left\{ |\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \left(\frac{|\phi'(\mu)|^q + |\phi'(\nu)|^q}{2} \right) \right\} \end{aligned} \quad (2.15)$$

for all $\mu, \nu \in [\theta, \vartheta]$, $\alpha > 0$, $\kappa \in [0, 1]$, and $B(\cdot, \cdot)$ is the Euler beta function and $\Psi = 2 \int_0^{\frac{1}{2}} (\int_0^{1-\kappa} \mu^n (1-\mu)^{\alpha-n-1})^p$.

Proof By using Lemma 2.1 and Jensen–Mercer’s inequality, we have

$$\begin{aligned} & \left| B(n+1, \alpha-n) \frac{\phi(\theta + \vartheta - \mu) + \phi(\theta + \vartheta - \nu)}{2} - \frac{n!}{2(v-\mu)^{\alpha}} \right. \\ & \quad \left. \times \left\{ (I_{\alpha}^{\theta+\vartheta-\nu})(\phi(\theta + \vartheta - \mu)) + ({}^{\theta+\vartheta-\mu}I_{\alpha})(\phi(\theta + \vartheta - \nu)) \right\} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(v-\mu)}{2} \left[\int_0^1 |B_\kappa(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n)| \right. \\
&\quad \left. \times |\phi'(\theta + \vartheta - (\kappa\mu + (1-\kappa)v))| d\kappa \right] \\
&\leq \frac{(v-\mu)}{2} \left[\int_0^1 |B_\kappa(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n)|^p \right]^{\frac{1}{q}} \\
&\quad \times \left[|\phi'(\theta + \vartheta - (\kappa\mu + (1-\kappa)v))|^q d\kappa \right]^{\frac{1}{q}}, \\
\Psi &= \int_0^1 |B_\kappa(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n)|^p \\
&= \int_0^{\frac{1}{2}} (B_\kappa(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n))^p \\
&\quad + \int_{\frac{1}{2}}^1 (B_\kappa(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n))^p \\
&= 2 \int_0^{\frac{1}{2}} \left(\int_0^{1-\kappa} \mu^n (1-\mu)^{\alpha-n-1} \right)^p \int_0^1 |\phi'(\theta + \vartheta - (\kappa\mu + (1-\kappa)v))|^q d\kappa.
\end{aligned}$$

After simplifications, we get the required result. \square

Remark 2.4 If we select $\mu = \theta$ and $v = \vartheta$ in Theorem 2.4, we get Theorem 3.2 for the case of $s = 1$ in [29].

Lemma 2.2 Let $\phi : [\theta, \vartheta] \rightarrow \Re$ be a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $\phi' \in L[\theta, \vartheta]$, then the following equality for conformable integrals holds:

$$\begin{aligned}
&\frac{2^{\alpha-1}n!}{(v-\mu)^\alpha} \left[I_{(\theta+\vartheta-\frac{\mu+v}{2})+}^\alpha (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+v}{2})-}^\alpha (\phi(\theta + \vartheta - v)) \right] \\
&\quad - B(n+1, \alpha-n) \phi\left(\theta + \vartheta - \frac{\mu+v}{2}\right) \\
&= \frac{v-\mu}{4} \left[\int_0^1 B_\kappa(n+1, \alpha-n) \phi'\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}v\right)\right) d\kappa \right. \\
&\quad \left. - \int_0^1 B_\kappa(n+1, \alpha-n) \phi'\left(\theta + \vartheta - \left(\frac{\kappa}{2}v + \frac{2-\kappa}{2}\mu\right)\right) d\kappa \right] \quad (2.16)
\end{aligned}$$

with $\alpha \in (n, n+1]$, $n = 0, 1, 2, 3, \dots$, where $B_\kappa(a, b)$ is an incomplete beta function and Γ is the Euler gamma function.

Proof Integrating by parts and changing the variables with $u = \theta + \vartheta - (\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}v)$, we get the following results via conformable integrals:

$$\begin{aligned}
I_1 &= \int_0^1 B_\kappa(n+1, \alpha-n) \phi'\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}v\right)\right) d\kappa \\
&= \frac{2}{v-\mu} B(n+1, \alpha-n) \phi'\left(\theta + \vartheta - \frac{\mu+v}{2}\right) \\
&\quad \times \int_0^1 \kappa^n (1-\kappa)^{\alpha-n-1} \phi\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}v\right)\right) d\kappa
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{v-\mu} B(n+1, \alpha-n) \phi' \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \\
&\quad \times \left(\frac{2}{v-\mu} \right)^{\alpha+1} \int_{\theta+\vartheta-\nu}^{\theta+\vartheta-\frac{\mu+\nu}{2}} [u - (\theta + \vartheta - \nu)]^n \left[\left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) - u \right]^{\alpha-n-1} \phi(u) du \\
&= \frac{2}{v-\mu} B(n+1, \alpha-n) \phi' \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \\
&\quad - \frac{2^{\alpha+1}}{(v-\mu)^{\alpha+1}} n! (I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^{\alpha} (\theta + \vartheta - \nu)). \tag{2.17}
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= -\frac{2}{v-\mu} B(n+1, \alpha-n) \phi' \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \\
&\quad + \frac{2^{\alpha+1}}{(v-\mu)^{\alpha+1}} n! (I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^{\alpha} (\theta + \vartheta - \mu)). \tag{2.18}
\end{aligned}$$

Adding equations (2.17), (2.18) and multiplying with $\frac{v-\mu}{4}$, the proof is completed. \square

Remark 2.5 If we set $\mu = \theta$ and $\nu = \vartheta$ in Lemma 2.2, we get Lemma 2.1 in [30].

Theorem 2.5 Suppose that $\phi : [\theta, \vartheta] \rightarrow R$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $|\phi'|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned}
&\left| \frac{2^{\alpha-1} n!}{(v-\mu)^{\alpha}} \left[I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^{\alpha} (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^{\alpha} (\phi(\theta + \vartheta - \nu)) \right] \right. \\
&\quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \right| \\
&\leq \frac{v-\mu}{4} \Pi^{1-\frac{1}{q}} \left\{ (\Pi |\phi'(\theta)|^q + \Pi |\phi'(\vartheta)|^q - \Pi_1 |\phi'(\mu)|^q - \Pi_2 |\phi'(\nu)|^q)^{\frac{1}{q}} \right. \\
&\quad \left. + (\Pi |\phi'(\theta)|^q + \Pi |\phi'(\vartheta)|^q - \Pi_1 |\phi'(\nu)|^q - \Pi_2 |\phi'(\mu)|^q)^{\frac{1}{q}} \right\}, \tag{2.19}
\end{aligned}$$

where

$$\begin{aligned}
\Pi &= [B(n+1, \alpha-n) - B(n+2, \alpha-n)], \\
\Pi_1 &= \frac{1}{4} [B(n+1, \alpha-n) - B(n+3, \alpha-n)], \\
\Pi_2 &= \frac{1}{4} [3B(n+1, \alpha-n) - 4B(n+2, \alpha-n) + B(n+3, \alpha-n)]
\end{aligned}$$

with $q \geq 1$.

Proof Taking modulus in Lemma 2.2 and using the well-known power mean inequality with convexity of $|\phi'|^q$ and Jensen–Mercer’s inequality, we have

$$\begin{aligned}
&\left| \frac{2^{\alpha-1} n!}{(v-\mu)^{\alpha}} \left[I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^{\alpha} (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^{\alpha} (\phi(\theta + \vartheta - \nu)) \right] \right. \\
&\quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\nu - \mu)}{4} \left[\left| \int_0^1 B_\kappa(n+1, \alpha - n) \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) d\kappa \right| \right. \\
&\quad \left. + \left| \int_0^1 B_\kappa(n+1, \alpha - n) \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) d\kappa \right| \right] \\
&\leq \frac{(\nu - \mu)}{4} \left[\left(\int_0^1 B_\kappa(n+1, \alpha - n) d\kappa \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left(\int_0^1 B_\kappa(n+1, \alpha - n) \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 B_\kappa(n+1, \alpha - n) d\kappa \right)^{1-\frac{1}{q}} \\
&\quad \times \left. \left(\int_0^1 B_\kappa(n+1, \alpha - n) \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \right] \\
&\leq \frac{(\nu - \mu)}{4} \left\{ \left(\int_0^1 B_\kappa(n+1, \alpha - n) d\kappa \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left[\left(|\phi'(\theta)|^q \int_0^1 B_\kappa(n+1, \alpha - n) d\kappa + |\phi'(\vartheta)|^q \int_0^1 B_\kappa(n+1, \alpha - n) d\kappa \right. \right. \\
&\quad \left. \left. - |\phi'(\mu)|^q \int_0^1 B_\kappa(n+1, \alpha - n) \frac{\kappa}{2} d\kappa - |\phi'(\nu)|^q \int_0^1 B_\kappa(n+1, \alpha - n) \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right. \\
&\quad + \left(|\phi'(\theta)|^q \int_0^1 B_\kappa(n+1, \alpha - n) d\kappa + |\phi'(\vartheta)|^q \int_0^1 B_\kappa(n+1, \alpha - n) d\kappa \right. \\
&\quad \left. \left. - |\phi'(\nu)|^q \int_0^1 B_\kappa(n+1, \alpha - n) \frac{\kappa}{2} d\kappa - |\phi'(\mu)|^q \int_0^1 B_\kappa(n+1, \alpha - n) \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right] \Big\}. \tag{2.20}
\end{aligned}$$

Integrating by parts, we get the following equalities:

$$\begin{aligned}
\Pi &= \int_0^1 B_\kappa(n+1, \alpha - n) d\kappa \\
&= [B(n+1, \alpha - n) - B(n+2, \alpha - n)], \tag{2.21}
\end{aligned}$$

$$\begin{aligned}
\Pi_1 &= \int_0^1 B_\kappa(n+1, \alpha - n) \frac{\kappa}{2} d\kappa \\
&= \frac{1}{4} [B(n+1, \alpha - n) - B(n+3, \alpha - n)], \tag{2.22}
\end{aligned}$$

$$\begin{aligned}
\Pi_2 &= \int_0^1 B_\kappa(n+1, \alpha - n) \frac{2-\kappa}{2} d\kappa \\
&= \frac{1}{4} [3B(n+1, \alpha - n) - 4B(n+2, \alpha - n) + B(n+3, \alpha - n)]. \tag{2.23}
\end{aligned}$$

Thus, combining (2.21) to (2.23) in (2.20), the proof is completed. \square

Remark 2.6 If we choose $\mu = \theta$ and $\nu = \vartheta$ in Theorem 2.5, we get Theorem 2.1 in [30].

Theorem 2.6 Suppose that $\phi : [\theta, \vartheta] \rightarrow R$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $|\phi'|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned} & \left| \frac{2^{\alpha-1} n!}{(\nu - \mu)^\alpha} \left[I_{(\theta + \vartheta - \frac{\mu + \nu}{2})+}^\alpha (\phi(\theta + \vartheta - \mu)) + I_{(\theta + \vartheta - \frac{\mu + \nu}{2})-}^\alpha (\phi(\theta + \vartheta - \nu)) \right] \right. \\ & \quad \left. - B(n+1, \alpha - n) \phi \left(\theta + \vartheta - \frac{\mu + \nu}{2} \right) \right| \\ & \leq \frac{\nu - \mu}{4} \Omega^{1-\frac{1}{q}} \left\{ \left(|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \frac{1}{4} |\phi'(\mu)|^q - \frac{3}{4} |\phi'(\nu)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \frac{1}{4} |\phi'(\nu)|^q - \frac{3}{4} |\phi'(\mu)|^q \right)^{\frac{1}{q}} \right\}, \end{aligned} \quad (2.24)$$

where

$$\Omega = \int_0^1 (B_\kappa(n+1, \alpha - n))^p d\kappa$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$.

Proof Taking modulus in Lemma 2.2 and using the well-known Hölder inequality with convexity of $|\phi'|^q$ and Jensen–Mercer’s inequality, we have

$$\begin{aligned} & \left| \frac{2^{\alpha-1} n!}{(\nu - \mu)^\alpha} \left[I_{(\theta + \vartheta - \frac{\mu + \nu}{2})+}^\alpha (\phi(\theta + \vartheta - \mu)) + I_{(\theta + \vartheta - \frac{\mu + \nu}{2})-}^\alpha (\phi(\theta + \vartheta - \nu)) \right] \right. \\ & \quad \left. - B(n+1, \alpha - n) \phi \left(\theta + \vartheta - \frac{\mu + \nu}{2} \right) \right| \\ & \leq \frac{(\nu - \mu)}{4} \left[\left| \int_0^1 B_\kappa(n+1, \alpha - n) \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) d\kappa \right| \right. \\ & \quad \left. + \left| \int_0^1 B_\kappa(n+1, \alpha - n) \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) d\kappa \right| \right] \\ & \leq \frac{(\nu - \mu)}{4} \left[\left(\int_0^1 (B_\kappa(n+1, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \right. \\ & \quad \times \left(\int_0^1 \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_0^1 (B_\kappa(n+1, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \left(\left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \right] \\ & \leq \frac{(\nu - \mu)}{4} \left\{ \left(\int_0^1 (B_\kappa(n+1, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \right. \\ & \quad \times \left[\left(|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - |\phi'(\mu)|^q \int_0^1 \frac{\kappa}{2} d\kappa - |\phi'(\nu)|^q \int_0^1 \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left. + \left(|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - |\phi'(\nu)|^q \int_0^1 \frac{\kappa}{2} d\kappa - |\phi'(\mu)|^q \int_0^1 \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right] \right\}. \end{aligned}$$

After some basic calculations, we get the required result. \square

Remark 2.7 If we set $\mu = \theta$ and $v = \vartheta$ in Theorem 2.6, we get Theorem 2.2 in [30].

Lemma 2.3 Let $\phi : [\theta, \vartheta] \rightarrow \mathbb{R}$ be a twice differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $\phi' \in L[\theta, \vartheta]$, then the following equality for conformable integrals holds:

$$\begin{aligned} & \frac{2^{\alpha-1}n!}{(v-\mu)^\alpha} \left[I_{(\theta+\vartheta-\frac{\mu+v}{2})+}^\alpha (\phi(\theta+\vartheta-\mu)) + I_{(\theta+\vartheta-\frac{\mu+v}{2})-}^\alpha (\phi(\theta+\vartheta-v)) \right] \\ & - B(n+1, \alpha-n) \phi\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \\ & = \frac{(v-\mu)^2}{8} \int_0^1 \left[\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n) \right] \\ & \times \left[\phi''\left(\theta+\vartheta-\left(\frac{\kappa}{2}\mu+\frac{2-\kappa}{2}v\right)\right) + \phi''\left(\theta+\vartheta-\left(\frac{\kappa}{2}v+\frac{2-\kappa}{2}\mu\right)\right) \right] d\kappa \quad (2.25) \end{aligned}$$

with $\alpha \in (n, n+1]$, $n = 0, 1, 2, 3, \dots$, where $B_t(\theta, \vartheta)$ is an incomplete beta function and Γ is the Euler gamma function.

Proof Integrating by parts and changing the variables with $u = \theta + \vartheta - (\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}v)$, we get the following results via conformable integrals:

$$\begin{aligned} I_1 &= \int_0^1 [\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n)] \phi''\left(\theta+\vartheta-\left(\frac{\kappa}{2}\mu+\frac{2-\kappa}{2}v\right)\right) d\kappa \\ &= \frac{2}{v-\mu} [B(n+1, \alpha-n) - B(n+2, \alpha-n)] \phi'\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \\ &\quad - \frac{2}{v-\mu} [B(n+1, \alpha-n) \phi\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \frac{2}{v-\mu} \\ &\quad - \int_0^1 \kappa^n (1-\kappa)^{\alpha-n-1} \phi\left(\theta+\vartheta-\left(\frac{\kappa}{2}\mu+\frac{2-\kappa}{2}v\right)\right) d\kappa \\ &= \frac{2}{v-\mu} [B(n+1, \alpha-n) - B(n+2, \alpha-n)] \phi'\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \\ &\quad - \frac{2}{v-\mu} [B(n+1, \alpha-n) \phi\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \frac{2}{v-\mu} \\ &\quad - \frac{2}{v-\mu} \int_{\theta+\vartheta-v}^{\theta+\vartheta-\frac{\mu+v}{2}} [u - (\theta+\vartheta-v)]^n \left[\left(\theta+\vartheta-\frac{\mu+v}{2}\right) - u \right]^{\alpha-n-1} \phi(u) \frac{2^\alpha}{(v-\mu)^\alpha} du \\ &= \frac{2}{v-\mu} [B(n+1, \alpha-n) - B(n+2, \alpha-n)] \phi'\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \\ &\quad - B(n+1, \alpha-n) \phi\left(\theta+\vartheta-\frac{\mu+v}{2}\right) \frac{4}{(v-\mu)^2} \\ &\quad + \frac{2^{\alpha+2}}{(v-\mu)^{\alpha+2}} n! (I_{(\theta+\vartheta-\frac{\mu+v}{2})-}^\alpha (\theta+\vartheta-v)). \quad (2.26) \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 = & -\frac{2}{v-\mu} [B(n+1, \alpha-n) - B(n+2, \alpha-n)] \phi' \left(\theta + \vartheta - \frac{\mu+v}{2} \right) \\ & - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+v}{2} \right) \frac{4}{(v-\mu)^2} \\ & + \frac{2^{\alpha+2}}{(v-\mu)^{\alpha+2}} n! (I_{(\theta+\vartheta-\frac{\mu+v}{2})-}^{\alpha} (\theta + \vartheta - v)). \end{aligned} \quad (2.27)$$

Adding equations (2.26), (2.27) and multiplying with $\frac{(v-\mu)^2}{8}$, we get the desired result. \square

Remark 2.8 If we select $\mu = \theta$ and $v = \vartheta$ in Lemma 2.3, we get Lemma 2.1 in [31].

Theorem 2.7 Suppose that $\phi : [\theta, \vartheta] \rightarrow R$ is a twice differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $|\phi''|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned} & \left| \frac{2^{\alpha-1} n!}{(v-\mu)^{\alpha}} [I_{(\theta+\vartheta-\frac{\mu+v}{2})+}^{\alpha} (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+v}{2})-}^{\alpha} (\phi(\theta + \vartheta - v))] \right. \\ & \quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+v}{2} \right) \right| \\ & \leq \frac{(v-\mu)^2}{8} \Lambda^{1-\frac{1}{q}} \{ (\Lambda |\phi''(\theta)|^q + \Lambda |\phi''(\vartheta)|^q - \Lambda_1 |\phi''(\mu)|^q - \Lambda_2 |\phi''(v)|^q)^{\frac{1}{q}} \\ & \quad + (\Lambda |\phi''(\theta)|^q + \Lambda |\phi''(\vartheta)|^q - \Lambda_1 |\phi''(v)|^q - \Lambda_2 |\phi''(\mu)|^q)^{\frac{1}{q}} \}, \end{aligned} \quad (2.28)$$

where

$$\begin{aligned} \Lambda &= \frac{1}{2} [B(n+1, \alpha-n) - 2B(n+2, \alpha-n) + B(n+3, \alpha-n)], \\ \Lambda_1 &= \frac{1}{12} [2B(n+1, \alpha-n) - 3B(n+2, \alpha-n) + B(n+4, \alpha-n)], \\ \Lambda_2 &= \frac{1}{12} [4B(n+1, \alpha-n) - 9B(n+2, \alpha-n) + 6B(n+3, \alpha-n) - B(n+4, \alpha-n)]. \end{aligned}$$

Proof Taking modulus in Lemma 2.3 and using the well-known power mean inequality with convexity of $|\phi''|^q$ and Jensen–Mercer’s inequality, we have

$$\begin{aligned} & \left| \frac{2^{\alpha-1} n!}{(v-\mu)^{\alpha}} [I_{(\theta+\vartheta-\frac{\mu+v}{2})+}^{\alpha} (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+v}{2})-}^{\alpha} (\phi(\theta + \vartheta - v))] \right. \\ & \quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+v}{2} \right) \right| \\ & \leq \frac{(v-\mu)^2}{8} \left[\left| \int_0^1 [\kappa B_{\kappa}(n+1, \alpha-n) - B_{\kappa}(n+2, \alpha-n)] \right. \right. \\ & \quad \times \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} v \right) \right) d\kappa \left. \right| \\ & \quad + \left| \int_0^1 [\kappa B_{\kappa}(n+1, \alpha-n) - B_{\kappa}(n+2, \alpha-n)] \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} v + \frac{2-\kappa}{2} \mu \right) \right) d\kappa \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\nu - \mu)^2}{8} \left[\left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \right. \\
&\quad \times \left. \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \right. \\
&\quad \times \left. \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \Bigg] \\
&\leq \frac{(\nu - \mu)^2}{8} \left\{ \left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left[\left(|\phi''(\theta)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \right. \right. \\
&\quad + |\phi''(\vartheta)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \\
&\quad - |\phi''(\mu)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \frac{\kappa}{2} d\kappa \\
&\quad - |\phi''(\nu)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \frac{2-\kappa}{2} d\kappa \Bigg)^{\frac{1}{q}} \\
&\quad + \left(|\phi''(\theta)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \right. \\
&\quad + |\phi''(\vartheta)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \\
&\quad - |\phi''(\nu)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \frac{\kappa}{2} d\kappa \\
&\quad \left. \left. - |\phi''(\mu)|^q \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right] \Bigg\}. \quad (2.29)
\end{aligned}$$

By using Lemma 2.2 in [31], we get the following equalities:

$$\begin{aligned}
\Lambda &= \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] d\kappa \\
&= \frac{1}{2} [B(n+1, \alpha - n) - 2B(n+2, \alpha - n) + B(n+3, \alpha - n)], \quad (2.30)
\end{aligned}$$

$$\begin{aligned}
\Lambda_1 &= \int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)] \frac{\kappa}{2} d\kappa \\
&= \frac{1}{12} [2B(n+1, \alpha - n) - 3B(n+2, \alpha - n) + B(n+4, \alpha - n)], \quad (2.31)
\end{aligned}$$

$$\begin{aligned}
\Lambda_2 &= \int_0^1 [\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n)] \frac{2-t}{2} d\kappa \\
&= \frac{1}{12} [4B(n+1, \alpha-n) - 9B(n+2, \alpha-n) + 6B(n+3, \alpha-n) \\
&\quad - B(n+4, \alpha-n)].
\end{aligned} \tag{2.32}$$

Thus, by combining (2.30) to (2.32) in (2.29), the proof is completed. \square

Remark 2.9 If we choose $\mu = \theta$ and $\nu = \vartheta$ in Theorem 2.7, we get Theorem 2.1 for the case of $m = 1$ in [31].

Theorem 2.8 Suppose that $\phi : [\theta, \vartheta] \rightarrow R$ is a twice differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $|\phi''|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned}
&\left| \frac{2^{\alpha-1} n!}{(\nu - \mu)^\alpha} [I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^\alpha (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^\alpha (\phi(\theta + \vartheta - \nu))] \right. \\
&\quad \left. - B(n+1, \alpha-n) \phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) \right| \\
&\leq \frac{(\nu - \mu)^2}{8} \Psi^{1-\frac{1}{q}} \left\{ \left(|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - \frac{1}{4} |\phi''(\mu)|^q - \frac{3}{4} |\phi''(\nu)|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - \frac{1}{4} |\phi''(\nu)|^q - \frac{3}{4} |\phi''(\mu)|^q \right)^{\frac{1}{q}} \right\},
\end{aligned} \tag{2.33}$$

where

$$\Psi = \int_0^1 [\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n)]^p d\kappa$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$.

Proof Taking modulus in Lemma 2.3 and using the well-known Hölder inequality with convexity of $|\phi''|^q$ and Jensen–Mercer’s inequality, we have

$$\begin{aligned}
&\left| \frac{2^{\alpha-1} n!}{(\nu - \mu)^\alpha} [I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^\alpha (\phi(\theta + \vartheta - \mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^\alpha (\phi(\theta + \vartheta - \nu))] \right. \\
&\quad \left. - B(n+1, \alpha-n) \phi\left(\theta + \vartheta - \frac{\mu + \nu}{2}\right) \right| \\
&\leq \frac{(\nu - \mu)^2}{8} \left[\left| \int_0^1 [\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n)] \right. \right. \\
&\quad \times \phi''\left(\theta + \vartheta - \left(\frac{\kappa}{2}\mu + \frac{2-\kappa}{2}\nu\right)\right) d\kappa \left. \right| \\
&\quad + \left| \int_0^1 [\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n)] \right. \\
&\quad \times \phi''\left(\theta + \vartheta - \left(\frac{\kappa}{2}\nu + \frac{2-\kappa}{2}\mu\right)\right) d\kappa \left. \right| \Big]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\nu - \mu)^2}{8} \left[\left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)]^p d\kappa \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 \left| f'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)]^p d\kappa \right)^{\frac{1}{p}} \\
&\quad \times \left. \left(\int_0^1 \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \right] \\
&\leq \frac{(\nu - \mu)^2}{8} \left\{ \left(\int_0^1 [\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n)]^p d\kappa \right)^{\frac{1}{p}} \right. \\
&\quad \times \left[\left(|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - |\phi''(\mu)|^q \int_0^1 \frac{\kappa}{2} d\kappa - |\phi''(\nu)|^q \int_0^1 \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right. \\
&\quad \left. \left. + \left(|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - |\phi''(\nu)|^q \int_0^1 \frac{\kappa}{2} d\kappa - |\phi''(\mu)|^q \int_0^1 \frac{2-\kappa}{2} d\kappa \right)^{\frac{1}{q}} \right] \right\}.
\end{aligned}$$

After computing the above integrals, we get the required result. \square

Remark 2.10 If we set $\mu = \theta$ and $\nu = \vartheta$ in Theorem 2.8, we get Theorem 2.2 for the case of $m = 1$ in [31].

3 New inequalities via improved Hölder's inequality

Theorem 3.1 Suppose that $\phi : [\theta, \vartheta] \rightarrow \mathbb{R}$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$ and $\phi \in L[\theta, \vartheta]$. If $|\phi'|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned}
&\left| B(n+1, \alpha - n) \frac{\phi(\theta + \vartheta - \mu) + \phi(\theta + \vartheta - \nu)}{2} - \frac{n!}{2(\nu - \mu)^\alpha} \right. \\
&\quad \times \left. \left\{ (I_\alpha^{\theta+\vartheta-\nu})(\phi(\theta + \vartheta - \mu)) + (I_\alpha^{\theta+\vartheta-\mu})(\phi(\theta + \vartheta - \nu)) \right\} \right| \\
&\leq \frac{\nu - \mu}{2} \left\{ \left(\int_0^1 (1 - \kappa) (B_\kappa(n+1, \alpha - n) - B_{1-\kappa}(n+1, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\frac{|\phi'(\theta)|^q + |\phi'(\vartheta)|^q}{2} - \left(\frac{1}{6} |\phi'(\mu)|^q + \frac{1}{3} |\phi'(\nu)|^q \right) \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \kappa (B_\kappa(n+1, \alpha - n) - B_{1-\kappa}(n+1, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad \times \left. \left(\frac{|\phi'(\theta)|^q + |\phi'(\vartheta)|^q}{2} - \left(\frac{1}{3} |\phi'(\mu)|^q + \frac{1}{6} |\phi'(\nu)|^q \right) \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

for all $\mu, \nu \in [\theta, \vartheta]$, $\alpha > 0$, $\kappa \in [0, 1]$, and $\beta(\cdot)$ is the Euler beta function.

Proof By using Lemma 2.1 with Jensen–Mercer's inequality, the convexity of $|\phi'|^q$ and applying the Hölder–İşcan integral inequality that is given in (Theorem 2.1, [32]), we can

write

$$\begin{aligned}
& \left| B(n+1, \alpha-n) \frac{\phi(\theta+\vartheta-\mu) + \phi(\theta+\vartheta-\nu)}{2} - \frac{n!}{2(\nu-\mu)^\alpha} \right. \\
& \quad \times \left. \left\{ \left(I_{\alpha}^{\theta+\vartheta-\nu} \right) (\phi(\theta+\vartheta-\mu)) + \left(I_{\alpha}^{\theta+\vartheta-\mu} \right) (\phi(\theta+\vartheta-\nu)) \right\} \right| \\
& \leq \frac{(\nu-\mu)}{2} \left\{ \left(\int_0^1 (1-\kappa) (B_{\kappa}(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\int_0^1 (1-\kappa) |\phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)\nu))|^q d\kappa \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \kappa (B_{\kappa}(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\int_0^1 \kappa |\phi'(\theta+\vartheta - (\kappa\mu + (1-\kappa)\nu))|^q d\kappa \right)^{\frac{1}{q}} \right\} \\
& \leq \frac{(\nu-\mu)}{2} \left\{ \left(\int_0^1 (1-\kappa) (B_{\kappa}(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\int_0^1 (1-\kappa) [|\phi'(a)|^q + |\phi'(b)|^q - (\kappa|\phi'(\mu)|^q + (1-\kappa)|\phi'(\nu)|^q)] d\kappa \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \kappa (B_{\kappa}(n+1, \alpha-n) - B_{1-\kappa}(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\int_0^1 \kappa [|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - (\kappa|\phi'(\mu)|^q + (1-\kappa)|\phi'(\nu)|^q)] d\kappa \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

By making use of the computations, one can have the required result. \square

Theorem 3.2 Suppose that $\phi : [\theta, \vartheta] \rightarrow \mathbb{R}$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $|\phi'|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} n!}{(\nu-\mu)^\alpha} \left[I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^{\alpha} (\phi(\theta+\vartheta-\mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^{\alpha} (\phi(\theta+\vartheta-\nu)) \right] \right. \\
& \quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \right| \\
& \leq \frac{\nu-\mu}{4} \left[\left\{ \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (B_{\kappa}(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \right. \\
& \quad \times \left(\frac{3|\phi'(\theta)|^q + 3|\phi'(\vartheta)|^q}{4} - \left(\frac{1}{6} |\phi'(\mu)|^q + \frac{7}{12} |\phi'(\nu)|^q \right) \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \frac{\kappa}{2} (B_{\kappa}(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\frac{|\phi'(\theta)|^q + |\phi'(\vartheta)|^q}{4} - \left(\frac{1}{12} |\phi'(\mu)|^q + \frac{1}{6} |\phi'(\nu)|^q \right) \right)^{\frac{1}{q}} \right\} \\
& \quad \times \left. \left(\frac{|\phi'(\theta)|^q + |\phi'(\vartheta)|^q}{4} - \left(\frac{1}{12} |\phi'(\mu)|^q + \frac{1}{6} |\phi'(\nu)|^q \right) \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
& \times \left(\frac{3|\phi'(\theta)|^q + 3|\phi'(\vartheta)|^q}{4} - \left(\frac{1}{6} |\phi'(\nu)|^q + \frac{7}{12} |\phi'(\mu)|^q \right) \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \frac{\kappa}{2} (B_t(n+1, \alpha-n))^p dt \right)^{\frac{1}{p}} \\
& \times \left. \left(\frac{|\phi'(\theta)|^q + |\phi'(\vartheta)|^q}{4} - \left(\frac{1}{12} |\phi'(\nu)|^q + \frac{1}{6} |\phi'(\mu)|^q \right) \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof By using Lemma 2.2 with Jensen–Mercer’s inequality, the convexity of $|\phi'|^q$ and applying the Hölder–İşcan integral inequality that is given in (Theorem 2.1, [32]), we can write

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} n!}{(\nu-\mu)^\alpha} \left[I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^\alpha (\phi(\theta+\vartheta-\mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^\alpha (\phi(\theta+\vartheta-\nu)) \right] \right. \\
& \quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \right| \\
& \leq \frac{(\nu-\mu)}{4} \left[\left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
& \quad \times \left(\int_0^1 \frac{\kappa}{2} (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad + \left(\int_0^1 \frac{\kappa}{2} \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2-\kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
& \quad \times \left(\int_0^1 \frac{\kappa}{2} (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad + \left. \left(\int_0^1 \frac{\kappa}{2} \left| \phi' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2-\kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \right] \\
& \leq \frac{(\nu-\mu)}{4} \left[\left\{ \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \right. \\
& \quad \times \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) \left[|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \left(\frac{\kappa}{2} |\phi'(\mu)|^q + \frac{2-\kappa}{2} |\phi'(\nu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \frac{\kappa}{2} (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\int_0^1 \frac{\kappa}{2} \left[|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \left(\frac{\kappa}{2} |\phi'(\mu)|^q + \frac{2-\kappa}{2} |\phi'(\nu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) \left[|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \left(\frac{\kappa}{2} |\phi'(\nu)|^q + \frac{2-\kappa}{2} |\phi'(\mu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \frac{\kappa}{2} (B_\kappa(n+1, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \frac{\kappa}{2} \left[|\phi'(\theta)|^q + |\phi'(\vartheta)|^q - \left(\frac{2-\kappa}{2} |\phi'(\mu)|^q + \frac{\kappa}{2} |\phi'(\nu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}}.
\end{aligned}$$

By making use of the computations, one can have the required result. \square

Theorem 3.3 Suppose that $\phi : [\theta, \vartheta] \rightarrow \mathbb{R}$ is a differentiable mapping on (θ, ϑ) with $\theta < \vartheta$. If $|\phi''|^q$ is a convex function on $[\theta, \vartheta]$, then the following inequality for conformable integrals holds:

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} n!}{(\nu-\mu)^\alpha} \left[I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^\alpha (\phi(\theta+\vartheta-\mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^\alpha (\phi(\theta+\vartheta-\nu)) \right] \right. \\
& \quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \right| \\
& \leq \frac{\nu-\mu}{4} \left[\left\{ \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \right. \\
& \quad \times \left(\frac{3|\phi'(\theta)|^q + 3|\phi'(\vartheta)|^q}{4} - \left(\frac{1}{6} |\phi''(\mu)|^q + \frac{7}{12} |\phi''(\nu)|^q \right) \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \frac{\kappa}{2} (\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left(\frac{|\phi''(\theta)|^q + |\phi''(\vartheta)|^q}{4} - \left(\frac{1}{12} |\phi''(\mu)|^q + \frac{1}{6} |\phi''(\nu)|^q \right) \right)^{\frac{1}{q}} \Big\} \\
& \quad + \left\{ \left(\int_0^1 \left(\frac{2-\kappa}{2} \right) (\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\frac{3|\phi''(\theta)|^q + 3|\phi''(\vartheta)|^q}{4} - \left(\frac{1}{6} |\phi''(\nu)|^q + \frac{7}{12} |\phi''(\mu)|^q \right) \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \frac{\kappa}{2} (\kappa B_\kappa(n+1, \alpha-n) - B_\kappa(n+2, \alpha-n))^p d\kappa \right)^{\frac{1}{p}} \\
& \quad \times \left(\frac{|\phi''(\theta)|^q + |\phi''(\vartheta)|^q}{4} - \left(\frac{1}{12} |\phi''(\nu)|^q + \frac{1}{6} |\phi''(\mu)|^q \right) \right)^{\frac{1}{q}} \Big\} \Big].
\end{aligned}$$

Proof By using Lemma 2.3 with Jensen–Mercer’s inequality, the convexity of $|\phi''|^q$ and applying the Hölder–İşcan integral inequality that is given in (Theorem 2.1, [32]), we can write

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} n!}{(\nu-\mu)^\alpha} \left[I_{(\theta+\vartheta-\frac{\mu+\nu}{2})+}^\alpha (\phi(\theta+\vartheta-\mu)) + I_{(\theta+\vartheta-\frac{\mu+\nu}{2})-}^\alpha (\phi(\theta+\vartheta-\nu)) \right] \right. \\
& \quad \left. - B(n+1, \alpha-n) \phi \left(\theta + \vartheta - \frac{\mu+\nu}{2} \right) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\nu - \mu)}{4} \left[\left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2 - \kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \frac{\kappa}{2} (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad + \left(\int_0^1 \frac{\kappa}{2} \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \mu + \frac{2 - \kappa}{2} \nu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2 - \kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \frac{\kappa}{2} (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad + \left. \left(\int_0^1 \frac{\kappa}{2} \left| \phi'' \left(\theta + \vartheta - \left(\frac{\kappa}{2} \nu + \frac{2 - \kappa}{2} \mu \right) \right) \right|^q d\kappa \right)^{\frac{1}{q}} \right] \\
&\leq \frac{(\nu - \mu)}{4} \left[\left\{ \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \right. \right. \\
&\quad \times \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) \left[|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - \left(\frac{\kappa}{2} |\phi''(\mu)|^q + \frac{2 - \kappa}{2} |\phi''(\nu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \frac{\kappa}{2} (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad \times \left. \left(\int_0^1 \frac{\kappa}{2} \left[|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - \left(\frac{\kappa}{2} |\phi''(\mu)|^q + \frac{2 - \kappa}{2} |\phi''(\nu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \right\} \\
&\quad + \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left(\frac{2 - \kappa}{2} \right) \left[|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - \left(\frac{\kappa}{2} |\phi''(\nu)|^q + \frac{2 - \kappa}{2} |\phi''(\mu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \frac{\kappa}{2} (\kappa B_\kappa(n+1, \alpha - n) - B_\kappa(n+2, \alpha - n))^p d\kappa \right)^{\frac{1}{p}} \\
&\quad \times \left. \left(\int_0^1 \frac{\kappa}{2} \left[|\phi''(\theta)|^q + |\phi''(\vartheta)|^q - \left(\frac{2 - \kappa}{2} |\phi''(\mu)|^q + \frac{\kappa}{2} |\phi''(\nu)|^q \right) \right] d\kappa \right)^{\frac{1}{q}} \right].
\end{aligned}$$

By making use of the simple computations for the above integrals, one can have the required result. \square

4 Conclusion

Conformable integrals act as inverse operators for conformable derivatives, which are related to a class of local derivatives. Conformable integrals of order between 0 and 1 have been used to generate nonlocal fractional integrals with kernel depending on a function

$\psi(t) = \frac{(t-a)^\rho}{\rho}$ [33], so that certain sequential conformable integrals become special cases of them. However, higher order conformable integrals, for which we have proved Hermite–Jensen–Mercer type inequalities in this work, have a different structure and cannot be considered as special cases of the nonlocal fractional ones. This observation, besides the fact that the conformable integrals with order larger than 1 have kernels of integer power law, adds more interest to the proven results in this article. In fact, this inequality work, to the best of our knowledge, is one among few for such higher order extension.

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Competing interests

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Authors' contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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