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Hermite–Hadamard-type inequalities via *n*-polynomial exponential-type convexity and their applications

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Abstract

In this paper, we give and study the concept of n-polynomial (s,m)-exponential-type convex functions and some of their algebraic properties. We prove new generalization of Hermite–Hadamard-type inequality for the n-polynomial (s,m)-exponential-type convex function ψ. We also obtain some refinements of the Hermite–Hadamard inequality for functions whose first derivatives in absolute value at certain power are n-polynomial (s,m)-exponential-type convex. Some applications to special means and new error estimates for the trapezoid formula are given.

Keywords: Hermite–Hadamard inequality; Hölder inequality; Power mean inequality; (s,m)-exponential-type convexity; n-polynomial

1 Introduction

The theory of convexity played a significant role in the development of the theory of inequalities. Many famously known results in the theory of inequalities can be obtained by using the convexity property of the functions. Hermite–Hadamard's double inequality is one of the most intensively studied results involving convex functions. This result provides us a necessary and sufficient condition for a function to be convex. It is also known as a classical equation of Hermite–Hadamard inequality.

The Hermite–Hadamard inequality asserts that if a function $\psi : J \subset \Re \to \Re$ is convex in J for $a_1, a_2 \in J$ and $a_1 < a_2$, then

$$\psi\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \psi(\chi)\, d\chi \le \frac{\psi(a_1) + \psi(a_2)}{2}. \tag{1.1}$$

Interested readers can refer to [1–20].

Definition 1 ([21]) A function $\psi : [0, +\infty) \to \Re$ is said to be s-convex in the second sense for a real number $s \in (0, 1]$, or ψ belongs to the class of K_s^2, if

$$\psi\big(\chi a_1 + (1 - \chi)a_2\big) \le \chi^s \psi(a_1) + (1 - \chi)^s \psi(a_2) \tag{1.2}$$

holds $\forall a_1, a_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

An s-convex function was introduced in Breckner's article [21], and a number of properties and connections with s-convexity in the first sense are discussed in [5]. Usually, convexity means s-convexity when $s = 1$. Dragomir et al. proved a variant of Hadamard's inequality in [3], which holds for s-convex functions in the second sense.

G. Toader introduced the class of m-convex functions in [10].

Definition 2 ([10]) A function $\psi : [0, a_2] \to \Re$, $a_2 > 0$, is said to be m-convex, where $m \in (0, 1]$, if

$$\psi\big(\chi\theta_1 + m(1-\chi)\theta_2\big) \le \chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2) \tag{1.3}$$

holds $\forall\theta_1, \theta_2 \in [0, a_2]$ and $\chi \in [0, 1]$. Otherwise, ψ is m-concave if $(-\psi)$ is m-convex.

In a recent paper, Eftekhari [4] defined the class of (s, m)-convex functions in the second sense as follows:

Definition 3 A function $\psi : [0, +\infty) \to \Re$ is said to be (s, m)-convex for some fixed real numbers $s, m \in (0, 1]$, if

$$\psi\big(\chi a_1 + m(1-\chi)a_2\big) \le \chi^s\psi(a_1) + m(1-\chi)^s\psi(a_2) \tag{1.4}$$

holds $\forall a_1, a_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

Motivated by the above results and literature, we will give first in Sect. 2 the concept of an n-polynomial (s, m)-exponential-type convex function and we will study some of its algebraic properties. In Sect. 3, we will prove new generalization of Hermite–Hadamard-type inequality for an n-polynomial (s, m)-exponential-type convex function ψ. In Sect. 4, we will obtain some refinements of the Hermite–Hadamard inequality for functions whose first derivatives in absolute value at certain power are n-polynomial (s, m)-exponential-type convex. In Sect. 5, some applications to special means and new error estimates for the trapezoid formula are given. In Sect. 6, a brief conclusion will be provided as well.

2 Some algebraic properties of n-polynomial (s, m)-exponential-type convex functions

In this section, we will give a new definition of an n-polynomial (s, m)-exponential-type convex function, and we will study some of its basic algebraic properties.

Definition 4 A nonnegative function $\psi : J \to \Re$ is called (s, m)-exponential-type convex for some fixed $s, m \in (0, 1]$, if

$$\psi\big(\chi\theta_1 + m(1-\chi)\theta_2\big) \le \big(e^{s\chi} - 1\big)\psi(\theta_1) + m\big(e^{(1-\chi)s} - 1\big)\psi(\theta_2) \tag{2.1}$$

holds $\forall\theta_1, \theta_2 \in J$ and $\chi \in [0, 1]$.

Remark 1 For $m = s = 1$, we get exponential-type convexity given by Kadakal and İşcan in [6].

Remark 2 The range of the (s,m)-exponential-type convex functions for some fixed $m \in (0,1]$ and $s \in [\ln 2, 1]$ is $[0,+\infty)$.

Proof Let $\theta \in J$ be arbitrary for some fixed $m \in (0,1]$ and $s \in [\ln 2, 1]$. Using Definition 4 for $\chi = 1$, we have

$$\psi(\theta) \leq \left(e^{s}-1\right)\psi(\theta) \quad \Longrightarrow \quad \left(e^{s}-2\right)\psi(\theta) \geq 0 \quad \Longrightarrow \quad \psi(\theta) \geq 0. \qquad \square$$

Definition 5 ([22]) A nonnegative function $\psi : J \to \Re$ is called an n-polynomial convex function if for every $\theta_1, \theta_2 \in J$, $n \in \mathbb{N}$, and $\chi \in (0,1]$, we have

$$\psi\left(\chi\theta_1 + (1-\chi)\theta_2\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\chi)^{s}\right]\psi(\theta_1) + \frac{1}{n}\sum_{s=1}^{n}\left[1-\chi^{s}\right]\psi(\theta_2). \tag{2.2}$$

We can give now a new definition of an n-polynomial (s,m)-exponential-type convex function as follows:

Definition 6 A nonnegative function $\psi : J \to \Re$ is called n-polynomial (s,m)-exponential-type convex function for some fixed $s, m \in (0,1]$ if

$$\psi\left(\chi\theta_1 + m(1-\chi)\theta_2\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^{i}\psi(\theta_1) + \frac{1}{n}\sum_{i=1}^{n}m^{i}\left(e^{(1-\chi)s}-1\right)^{i}\psi(\theta_2) \tag{2.3}$$

holds for all $\theta_1, \theta_2 \in J$, $n \in \mathbb{N}$, and $\chi \in [0,1]$.

We discuss some connections between the class of n-polynomial (s,m)-exponential-type convex functions and other classes of generalized convex functions.

Lemma 1 *For all* $\chi \in [0,1]$ *and for some fixed* $m \in (0,1]$ *and* $s \in [\ln 2, 1]$, *the inequalities* $(e^{s\chi}-1) \geq \chi^{s}$ *and* $(e^{(1-\chi)s}-1) \geq (1-\chi)^{s}$ *hold.*

Proof The proof is evident. $\square$

Proposition 1 *Every nonnegative* (s,m)*-convex function is an n-polynomial* (s,m)*-exponential-type convex function for some fixed* $m \in (0,1]$ *and* $s \in [\ln 2, 1]$.

Proof By using Lemma 1, for some fixed $m \in (0,1]$ and $s \in [\ln 2, 1]$, we have

$$\begin{aligned}\psi\left(\chi\theta_1 + m(1-\chi)\theta_2\right) &\leq \chi^{s}\psi(\theta_1) + m(1-\chi)^{s}\psi(\theta_2)\\ &\leq \frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^{i}\psi(\theta_1) + \frac{1}{n}\sum_{i=1}^{n}m^{i}\left(e^{(1-\chi)s}-1\right)^{i}\psi(\theta_2). \qquad \square\end{aligned}$$

Remark 3 If we put $n = 1$ in Proposition 1, then we have

$$\begin{aligned}\psi\left(\chi\theta_1 + m(1-\chi)\theta_2\right) &\leq \chi^{s}\psi(\theta_1) + m(1-\chi)^{s}\psi(\theta_2)\\ &\leq \left(e^{s\chi}-1\right)\psi(\theta_1) + m\left(e^{(1-\chi)s}-1\right)\psi(\theta_2).\end{aligned}$$

Theorem 1 *Let $\psi, \phi : [a_1, a_2] \to \Re$. If ψ and ϕ are n-polynomial (s, m)-exponential-type convex functions for some fixed $s, m \in (0, 1]$, then*

1. *$\psi + \phi$ is an n-polynomial (s, m)-exponential-type convex function;*
2. *For nonnegative real number c, $c\psi$ is an n-polynomial (s, m)-exponential-type convex function.*

Proof By Definition 6 for some fixed $s, m \in (0, 1]$, the proof is obvious. □

Theorem 2 *Let $\psi : [0, a_2] \to J$ be an m-convex function for $a_2 > 0$ and some fixed $m \in (0, 1]$, and let $\phi : J \to \Re$ be nondecreasing and n-polynomial (s, m)-exponential-type convex function for some fixed $s \in (0, 1]$. Then for the same fixed numbers $s, m \in (0, 1]$, the function $\phi \circ \psi : [0, a_2] \to \Re$ is n-polynomial (s, m)-exponential-type convex.*

Proof For all $\theta_1, \theta_2 \in [0, a_2]$ and $\chi \in [0, 1]$, and for the some fixed numbers $s, m \in (0, 1]$, we have

$$
\begin{aligned}
&(\phi \circ \psi)\big(\chi\theta_1 + m(1-\chi)\theta_2\big) \\
&\quad = \phi\big(\psi\big(\chi\theta_1 + m(1-\chi)\theta_2\big)\big) \le \phi\big(\chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2)\big) \\
&\quad \le \frac{1}{n}\sum_{i=1}^{n}\big(e^{s\chi}-1\big)^i(\phi\circ\psi)(\theta_1) + \frac{1}{n}\sum_{i=1}^{n} m^i\big(e^{(1-\chi)s}-1\big)^i(\phi\circ\psi)(\theta_2).
\end{aligned}
$$
□

Remark 4 If we put $n = 1$ in Theorem 2, then we get

$$
\begin{aligned}
&(\phi \circ \psi)\big(\chi\theta_1 + m(1-\chi)\theta_2\big) \\
&\quad = \phi\big(\psi\big(\chi\theta_1 + m(1-\chi)\theta_2\big)\big) \le \phi\big(\chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2)\big) \\
&\quad \le \big(e^{s\chi}-1\big)(\phi\circ\psi)(\theta_1) + m\big(e^{(1-\chi)s}-1\big)(\phi\circ\psi)(\theta_2).
\end{aligned}
$$

Theorem 3 *Let $\psi_i : [a_1, a_2] \to \Re$ be an arbitrary family of n-polynomial (s, m)-exponential-type convex functions for the same fixed $s, m \in (0, 1]$ and let $\psi(\theta) = \sup_i \psi_i(\theta)$. If $E = \{\theta \in [a_1, a_2] : \psi(\theta) < +\infty\} \neq \emptyset$, then E is an interval and ψ is an n-polynomial (s, m)-exponential-type convex function on E.*

Proof For all $\theta_1, \theta_2 \in E$ and $\chi \in [0, 1]$, and for the same fixed numbers $s, m \in (0, 1]$, we have

$$
\begin{aligned}
&\psi\big(\chi\theta_1 + m(1-\psi)\theta_2\big) \\
&\quad = \sup_i \psi_i\big(\chi\theta_1 + m(1-\chi)\theta_2\big) \\
&\quad \le \sup_i\left[\frac{1}{n}\sum_{i=1}^{n}\big(e^{s\chi}-1\big)^i\psi_i(\theta_1) + \frac{1}{n}\sum_{i=1}^{n} m^i\big(e^{(1-\chi)s}-1\big)^i\psi_i(\theta_2)\right] \\
&\quad \le \frac{1}{n}\sum_{i=1}^{n}\big(e^{s\chi}-1\big)^i\sup_i\psi_i(\theta_1) + \frac{1}{n}\sum_{i=1}^{n} m^i\big(e^{(1-\chi)s}-1\big)^i\sup_i\psi_i(\theta_2) \\
&\quad = \frac{1}{n}\sum_{i=1}^{n}\big(e^{s\chi}-1\big)^i\psi(\theta_1) + \frac{1}{n}\sum_{i=1}^{n} m^i\big(e^{(1-\chi)s}-1\big)^i\psi(\theta_2) < +\infty.
\end{aligned}
$$

This shows simultaneously that E is an interval, since it contains every point between any two of its points, and that ψ is an n-polynomial (s,m)-exponential-type convex function on E. □

Theorem 4 *If the function $\psi : [a_1,a_2] \to \Re$ is an n-polynomial (s,m)-exponential-type convex for some fixed $s,m \in (0,1]$, then ψ is bounded on $[a_1, ma_2]$.*

Proof Let $L = \max\{\psi(a_1), \psi(\frac{a_2}{m})\}$ and $x \in [a_1,a_2]$ be an arbitrary point for some fixed $m \in (0,1]$. Then there exists $\chi \in [0,1]$ such that $x = \chi a_1 + (1-\chi)a_2$. Thus, since $e^{s\chi} \leq e^s$ and $e^{(1-\chi)s} \leq e^s$ for some fixed $s \in (0,1]$, we have

$$\begin{aligned}\psi(x) = \psi\big(\chi a_1 + (1-\chi)a_2\big) &\leq \frac{1}{n}\sum_{i=1}^{n}\big(e^{s\chi}-1\big)^i\psi(a_1) + \frac{1}{n}\sum_{i=1}^{n}m^i\big(e^{(1-\chi)s}-1\big)^i\psi\left(\frac{a_2}{m^i}\right)\\ &\leq \frac{1}{n}\sum_{i=1}^{n}\big(e^s-1\big)^iL + \frac{1}{n}\sum_{i=1}^{n}m^i\big(e^s-1\big)^iL = \frac{L}{n}\sum_{i=1}^{n}\big(m^i+1\big)\big(e^s-1\big)^i = M.\end{aligned}$$

We have shown that ψ is bounded above by a real number M. The interested reader can also prove the fact that ψ is bounded below using the same idea as in Theorem 2.4 in [6]. □

3 New generalization of Hermite–Hadamard-type inequality using n-polynomial (s,m)-exponential-type convex functions

The aim of this section is to find new generalization of Hermite–Hadamard-type inequality for the n-polynomial (s,m)-exponential-type convex function ψ.

Theorem 5 *Let $\psi : [a_1, ma_2] \to \Re$ be an n-polynomial (s,m)-exponential-type convex function for some fixed $s,m \in (0,1]$ and $a_1 < ma_2$. If $\psi \in L_1([a_1,ma_2])$, then*

$$\begin{aligned}&\frac{1}{\frac{1}{n}\sum_{i=1}^{n}(e^{\frac{s}{2}}-1)^i}\psi\left(\frac{a_1+ma_2}{2}\right)\\ &\quad\leq \frac{1}{(ma_2-a_1)}\left\{\int_{a_1}^{ma_2}\psi(x)\,dx + m\int_{\frac{a_1}{m}}^{a_2}\psi(x)\,dx\right\}\\ &\quad\leq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^i\left[\psi(a_1)+\psi(a_2)+m^i\left(\psi\left(\frac{a_1}{m^{i+1}}\right)+\psi(a_2)\right)\right]. \qquad (3.1)\end{aligned}$$

Proof Denote

$$\theta_1 = \chi a_1 + m(1-\chi)a_2, \qquad \theta_2 = (1-\chi)\frac{a_1}{m} + \chi a_2, \quad \forall\chi \in [0,1].$$

By using n-polynomial (s,m)-exponential-type convexity of ψ, we have

$$\begin{aligned}\psi\left(\frac{a_1+ma_2}{2}\right) = \psi\left(\frac{\theta_1+m\theta_2}{2}\right)\\ &= \psi\left(\frac{[\chi a_1 + m(1-\chi)a_2] + [(1-\chi)a_1 + m\chi a_2]}{2}\right)\\ &\leq \frac{1}{n}\sum_{i=1}^{n}\big(e^{\frac{s}{2}}-1\big)^i\big[\psi\big(\chi a_1 + m(1-\chi)a_2\big) + \psi\big((1-\chi)a_1 + m\chi a_2\big)\big].\end{aligned}$$

Now, integrating on both sides in the last inequality with respect to χ over $[0,1]$, we get

$$\begin{aligned}\psi\left(\frac{a_1+ma_2}{2}\right) &\leq \frac{1}{n}\sum_{i=1}^{n}\left(e^{\frac{s}{2}}-1\right)^i \\ &\quad\times\left[\int_0^1 \psi\left(\chi a_1+m(1-\chi)a_2\right)d\chi+\int_0^1 \psi\left((1-\chi)\frac{a_1}{m}+\chi a_2\right)d\chi\right] \\ &= \frac{\frac{1}{n}\sum_{i=1}^{n}(e^{\frac{s}{2}}-1)^i}{(ma_2-a_1)}\left\{\int_{a_1}^{ma_2}\psi(x)\,dx+m\int_{\frac{a_1}{m}}^{a_2}\psi(x)\,dx\right\},\end{aligned}$$

which proves the left-hand side inequality. For the right-hand side inequality, using n-polynomial (s,m)-exponential-type convexity of ψ, we obtain

$$\begin{aligned}&\frac{1}{(ma_2-a_1)}\left\{\int_{a_1}^{ma_2}\psi(x)\,dx+m\int_{\frac{a_1}{m}}^{a_2}\psi(x)\,dx\right\} \\ &= \int_0^1 \psi\left(\chi a_1+m(1-\chi)a_2\right)d\chi+\int_0^1 \psi\left((1-\chi)\frac{a_1}{m}+\chi a_2\right)d\chi \\ &\leq \int_0^1\left[\frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^i\psi(a_1)+\frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{(1-\chi)s}-1\right)^i\psi(a_2)\right]d\chi \\ &\quad+\int_0^1\left[\frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^i\psi(a_2)+\frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{(1-\chi)s}-1\right)^i\psi\left(\frac{a_1}{m^{i+1}}\right)\right]d\chi \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^i\left[\psi(a_1)+\psi(a_2)+m^i\left(\psi\left(\frac{a_1}{m^{i+1}}\right)+\psi(a_2)\right)\right],\end{aligned}$$

which gives the right-hand side inequality and the proof is completed. □

Corollary 1 *By choosing* $m=s=n=1$ *in Theorem* 5, *we get Theorem* 3.1 *of* [6].

Remark 5 If we put $n=1$ in Theorem 5, then we obtain

$$\begin{aligned}&\frac{1}{(e^{\frac{s}{2}}-1)}\psi\left(\frac{a_1+ma_2}{2}\right) \\ &\leq \frac{1}{(ma_2-a_1)}\left\{\int_{a_1}^{ma_2}\psi(x)\,dx+m\int_{\frac{a_1}{m}}^{a_2}\psi(x)\,dx\right\} \\ &\leq \left(\frac{e^s-s-1}{s}\right)\left[\psi(a_1)+\psi(a_2)+m\left(\psi\left(\frac{a_1}{m^2}\right)+\psi(a_2)\right)\right].\end{aligned} \tag{3.2}$$

4 Refinements of Hermite–Hadamard-type inequality via *n*-polynomial (*s*, *m*)-exponential-type convex functions

To obtain some refinements of the Hermite–Hadamard inequality for functions whose first derivative in absolute value at certain power is an n-polynomial (s,m)-exponential-type convex, we need some new useful lemmas.

Lemma 2 *Suppose* $0 < k \leq 1$ *and consider a mapping* $\psi : [a_1, \frac{a_2}{k}] \to \Re$ *which is differentiable on* $(a_1, \frac{a_2}{k})$ *with* $0 < a_1 < a_2$. *If* $\psi' \in L_1[a_1, \frac{a_2}{k}]$, *then*

$$\begin{aligned} &\frac{\psi(a_1) + \psi(\frac{a_2}{k})}{2} - \frac{k}{a_2 - ka_1} \int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\,d\theta \\ &= \left(\frac{a_2 - ka_1}{2k}\right) \int_0^1 (1 - 2\chi)\psi'\left(\chi a_1 + (1 - \chi)\frac{a_2}{k}\right) d\chi. \end{aligned} \tag{4.1}$$

Proof Using integration by parts, we have

$$\begin{aligned} &\left(\frac{a_2 - ka_1}{2k}\right) \int_0^1 (1 - 2\chi)\psi'\left(\chi a_1 + (1 - \chi)\frac{a_2}{k}\right) d\chi \\ &= \left(\frac{a_2 - ka_1}{2k}\right) \left\{ \frac{(1 - 2\chi)\psi(\chi a_1 + (1 - \chi)\frac{a_2}{k})}{a_1 - \frac{a_2}{k}} \bigg|_0^1 - \int_0^1 \frac{\psi(\chi a_1 + (1 - \chi)\frac{a_2}{k})}{a_1 - \frac{a_2}{k}} (-2)\,d\chi \right\} \\ &= \left(\frac{a_2 - ka_1}{2k}\right) \left\{ \frac{-\psi(a_1) - \psi(\frac{a_2}{k})}{\frac{ka_1 - a_2}{k}} + \frac{2k}{ka_1 - a_2} \int_0^1 \psi\left(\chi a_1 + (1 - \chi)\frac{a_2}{k}\right) d\chi \right\} \\ &= \left(\frac{a_2 - ka_1}{2k}\right) \left\{ \frac{k(\psi(a_1) + \psi(\frac{a_2}{k}))}{a_2 - ka_1} - \frac{2k}{a_2 - ka_1} \int_0^1 \psi\left(\chi a_1 + (1 - t)\frac{a_2}{k}\right) d\chi \right\} \\ &= \frac{\psi(a_1) + \psi(\frac{a_2}{k})}{2} - \frac{k}{a_2 - ka_1} \int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\,d\theta, \end{aligned}$$

which completes the proof. □

Lemma 3 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : [ka_1, a_2] \to \Re$ *is differentiable on* (ka_1, a_2) *with* $0 < a_1 < a_2$. *If* $\psi' \in L_1[ka_1, a_2]$, *then*

$$\begin{aligned} &\frac{\psi(ka_1) + \psi(a_2)}{2} - \frac{1}{a_2 - ka_1} \int_{ka_1}^{a_2} \psi(\theta)\,d\theta \\ &= \left(\frac{a_2 - ka_1}{2}\right) \int_0^1 (2\chi - 1)\psi'\bigl(k(1 - \chi)a_1 + \chi a_2\bigr)\,d\chi. \end{aligned} \tag{4.2}$$

Proof Using the integration by parts, we have

$$\begin{aligned} &\left(\frac{a_2 - ka_1}{2}\right) \int_0^1 (2\chi - 1)\psi'\bigl(k(1 - \chi)a_1 + \chi a_2\bigr) \\ &= \left(\frac{a_2 - ka_1}{2}\right) \\ &\quad \times \left\{ \frac{(2\chi - 1)\psi(k(1 - \chi)a_1 + \chi a_2)}{a_2 - ka_1} \bigg|_0^1 - \int_0^1 \frac{\psi(k(1 - \chi)a_1 + \chi a_2)}{a_2 - ka_1} (2)\,d\chi \right\} \\ &= \left(\frac{a_2 - ka_1}{2}\right) \left\{ \frac{\psi(a_2) + \psi(ka_1)}{a_2 - ka_1} - \frac{2}{a_2 - ka_1} \int_0^1 \psi\bigl(k(1 - \chi)a_1 + \chi a_2\bigr)\,d\chi \right\} \\ &= \left(\frac{a_2 - ka_1}{2}\right) \left\{ \frac{\psi(a_2) + \psi(ka_1)}{a_2 - ka_1} - \frac{2}{a_2 - ka_1} \int_0^1 \psi\bigl(k(1 - \chi)a_1 + \chi a_2\bigr)\,d\chi \right\} \\ &= \frac{\psi(ka_1) + \psi(a_2)}{2} - \frac{1}{a_2 - ka_1} \int_{ka_1}^{a_2} \psi(\theta)\,d\theta, \end{aligned}$$

which completes the proof. □

Lemma 4 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : [ka_1, a_2] \to \Re$ *is differentiable on* (ka_1, a_2) *with* $0 < a_1 < a_2$. *If* $\psi' \in L_1[ka_1, a_2]$, *then*

$$\begin{aligned} &\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\,d\theta \\ &\quad = \left(\frac{a_2 - ka_1}{k+1}\right) \int_0^1 (2\chi - 1)\psi'\bigl(k(1-\chi)a_1 + \chi a_2\bigr)\,d\chi. \end{aligned} \tag{4.3}$$

Proof Using the integration by parts, we have

$$\begin{aligned} &\left(\frac{a_2 - ka_1}{k+1}\right) \int_0^1 (2\chi - 1)\psi'\bigl(k(1-\chi)a_1 + \chi a_2\bigr) \\ &\quad = \left(\frac{a_2 - ka_1}{k+1}\right) \\ &\qquad \times \left\{ \frac{(2\chi - 1)\psi(k(1-\chi)a_1 + \chi a_2)}{a_2 - ka_1}\bigg|_0^1 - \int_0^1 \frac{\psi(k(1-\chi)a_1 + \chi a_2)}{a_2 - ka_1}(2)\,d\chi \right\} \\ &\quad = \left(\frac{a_2 - ka_1}{k+1}\right) \left\{ \frac{\psi(ka_1) + \psi(a_2)}{a_2 - ka_1} - \frac{2}{a_2 - ka_1} \int_0^1 \psi\bigl(k(1-\chi)a_1 + \chi a_2\bigr)\,d\chi \right\} \\ &\quad = \frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{k+1} \int_0^1 \psi\bigl(k(1-\chi)a_1 + \chi a_2\bigr)\,d\chi \\ &\quad = \frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\,d\theta, \end{aligned}$$

which completes the proof. □

Lemma 5 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : [a_1, \frac{a_2}{k}] \to \Re$ *is differentiable on* $(a_1, \frac{a_2}{k})$ *with* $0 < a_1 < a_2$. *If* $\psi' \in L_1[a_1, \frac{a_2}{k}]$, *then*

$$\begin{aligned} &\frac{k}{a_2 - ka_1} \int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\,d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right) \\ &\quad = \left(\frac{a_2 - ka_1}{k}\right) \\ &\qquad \times \left\{ \int_0^1 \chi \psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right) d\chi - \int_{\frac{1}{2}}^1 \psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right) d\chi \right\}. \end{aligned} \tag{4.4}$$

Proof Using the integration by parts, we have

$$\begin{aligned} &\left(\frac{a_2 - ka_1}{k}\right) \left\{ \int_0^1 \chi \psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right) d\chi - \int_{\frac{1}{2}}^1 \psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right) d\chi \right\} \\ &\quad = \left(\frac{a_2 - ka_1}{k}\right) \\ &\qquad \times \left\{ \frac{\chi \psi(\chi a_1 + (1-\chi)\frac{a_2}{k})}{a_1 - \frac{a_2}{k}}\bigg|_0^1 - \int_0^1 \frac{\psi(\chi a_1 + (1-\chi)\frac{a_2}{k})}{a_1 - \frac{a_2}{k}}\,d\chi \right. \\ &\qquad \left. - \frac{\psi(\chi a_1 + (1-\chi)\frac{a_2}{k})}{a_1 - \frac{a_2}{k}}\bigg|_{\frac{1}{2}}^1 \right\} \end{aligned}$$

$$= \left(\frac{a_2 - ka_1}{k}\right)\left\{\frac{k\psi(a_1)}{ka_1 - a_2} - \frac{k}{ka_1 - a_2}\int_0^1 \psi\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)d\chi - \frac{k}{ka_1 - a_2}\left(\psi(a_1) - \psi\left(\frac{a_1 + a_2}{2k}\right)\right)\right\}$$

$$= \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\,d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right),$$

which completes the proof. □

Lemma 6 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : [ka_1, a_2] \to \Re$ *is differentiable on* (ka_1, a_2) *with* $0 < a_1 < a_2$. *If* $\psi' \in L_1[ka_1, a_2]$, *then*

$$\frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2} \psi(\theta)\,d\theta - \psi\left(\frac{ka_1 + a_2}{2}\right) = (a_2 - ka_1) \times \left\{\int_0^1 (-\chi)\psi'\big(k(1-\chi)a_1 + \chi a_2\big)\,d\chi + \int_{\frac{1}{2}}^1 \psi'\big(k(1-\chi)a_1 + \chi a_2\big)\,d\chi\right\}. \tag{4.5}$$

Proof Using the integration by parts, we have

$$(a_2 - ka_1)\left\{\int_0^1 -\chi\psi'\big(k(1-\chi)a_1 + \chi a_2\big)\,d\chi + \int_{\frac{1}{2}}^1 \psi'\big(k(1-\chi)a_1 + \chi a_2\big)\,d\chi\right\}$$

$$= (a_2 - ka_1)\left\{\frac{-\chi\psi(k(1-\chi)a_1 + \chi a_2)}{a_2 - ka_1}\bigg|_0^1 - \int_0^1 \frac{\psi(k(1-\chi)a_1 + \chi a_2)}{a_2 - ka_1}(-1)\,d\chi + \frac{\psi(k(1-\chi)a_1 + \chi a_2)}{a_2 - ka_1}\bigg|_{\frac{1}{2}}^1\right\}$$

$$= (a_2 - ka_1) \times \left\{\frac{-\psi(a_2)}{a_2 - ka_1} + \frac{1}{a_2 - ka_1}\int_0^1 \psi\big(k(1-\chi)a_1 + \chi a_2\big)\,d\chi + \frac{\psi(a_2) - \psi(\frac{ka_1 + a_2}{2})}{a_2 - ka_1}\right\}$$

$$= \frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2} \psi(\theta)\,d\theta - \psi\left(\frac{ka_1 + a_2}{2}\right),$$

which completes the proof. □

Theorem 6 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : (0, \frac{a_2}{mk}] \to \Re$ *is differentiable on* $(0, \frac{a_2}{mk})$ *with* $0 < a_1 < a_2$. *If* $|\psi'|^q$ *is an n-polynomial* (s, m)*-exponential-type convex function on* $(0, \frac{a_2}{mk}]$ *for* $q > 1$ *and* $q^{-1} + p^{-1} = 1$, *then for some fixed* $s, m \in (0, 1]$, *the following inequality holds*:

$$\left|\frac{\psi(a_1) + \psi(\frac{a_2}{k})}{2} - \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\,d\theta\right| \leq \left(\frac{a_2 - ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{\sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \left(\frac{1}{n}\left|\psi'(a_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i k}\right)\right|^q\right)\right\}^{\frac{1}{q}}. \tag{4.6}$$

Proof From Lemma 2, Hölder's inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(\frac{a_2}{k})}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta\right| \\
&\leq\left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\chi|^p\,d\chi\right)^{\frac{1}{p}}\left\{\int_0^1\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right\}^{\frac{1}{q}} \\
&\leq\left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\chi|^p\,d\chi\right)^{\frac{1}{p}} \\
&\quad\times\left\{\int_0^1\left[\frac{1}{n}\sum_{i=1}^n\left(e^{s\chi}-1\right)^i\left|\psi'(a_1)\right|^q+\frac{1}{n}\sum_{i=1}^n m^i\left(e^{(1-\chi)s}-1\right)^i\left|\psi'\left(\frac{a_2}{m^ik}\right)\right|^q\right]d\chi\right\}^{\frac{1}{q}} \\
&=\left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\sum_{i=1}^n\left(\frac{e^s-s-1}{s}\right)^i\left(\frac{1}{n}\left|\psi'(a_1)\right|^q+\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^ik}\right)\right|^q\right)\right\}^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Remark 6 If we take $n=1$ in Theorem 6, we have

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(\frac{a_2}{k})}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta\right| \\
&\leq\left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(a_1)\right|^q+m\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}\tag{4.7}$$

Theorem 7 *Suppose* $0<k\leq 1$ *and a mapping* $\psi:(0,\frac{a_2}{mk}]\rightarrow\Re$ *is differentiable on* $(0,\frac{a_2}{mk})$ *with* $0<a_1<a_2$. *If* $|\psi'|^q$ *is an* n*-polynomial* (s,m)*-exponential-type convex function on* $(0,\frac{a_2}{mk}]$ *for* $q\geq 1$, *then for some fixed* $s,m\in(0,1]$, *the following inequality holds*:

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(\frac{a_2}{k})}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta\right| \\
&\leq\left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\sum_{i=1}^n\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)^i\right. \\
&\quad\left.\times\left(\frac{1}{n}\left|\psi'(a_1)\right|^q+\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^ik}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}\tag{4.8}$$

Proof From Lemma 2, power mean inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(\frac{a_2}{k})}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta\right| \\
&\leq\left(\frac{a_2-ka_1}{2k}\right)\left\{\int_0^1|1-2\chi|\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|d\chi\right\} \\
&\leq\left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\chi|\,d\chi\right)^{1-\frac{1}{q}}\left\{\int_0^1|1-2\chi|\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right\}^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{a_2 - ka_1}{2k}\right)\left(\int_0^1 |1-2\chi|\,d\chi\right)^{1-\frac{1}{q}} \\
&\quad \times \left\{\int_0^1 |1-2\chi|\left[\frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^i\left|\psi'(a_1)\right|^q\right.\right. \\
&\quad \left.\left. + \frac{1}{n}\sum_{i=1}^{n} m^i\left(e^{(1-\chi)s}-1\right)^i\left|\psi'\left(\frac{a_2}{m^i k}\right)\right|^q\right]d\chi\right\}^{\frac{1}{q}} \\
&= \left(\frac{a_2 - ka_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\quad \times \left\{\sum_{i=1}^{n}\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^i\left(\frac{1}{n}\left|\psi'(a_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i k}\right)\right|^q\right)\right\}^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof. □

Remark 7 If we take $n = 1$ in Theorem 7, we get

$$
\begin{aligned}
&\left|\frac{\psi(a_1) + \psi(\frac{a_2}{k})}{2} - \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\,d\theta\right| \\
&\leq \left(\frac{a_2 - ka_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\quad \times \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(\left|\psi'(a_1)\right|^q + m\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{4.9}
$$

Theorem 8 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : (0, \frac{a_2}{m}] \to \Re$ *is differentiable on* $(0, \frac{a_2}{m})$ *with* $0 < a_1 < a_2$. *If* $|\psi'|^q$ *is an n-polynomial* (s, m)*-exponential-type convex function on* $(0, \frac{a_2}{m}]$ *for* $q > 1$ *and* $q^{-1} + p^{-1} = 1$, *then for some fixed* $s, m \in (0, 1]$, *the following inequality holds*:

$$
\begin{aligned}
&\left|\frac{\psi(ka_1) + \psi(a_2)}{2} - \frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2} \psi(\theta)\,d\theta\right| \\
&\leq \left(\frac{a_2 - ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{i=1}^{n}\left(\frac{e^s - s - 1}{s}\right)^i\left(\frac{1}{n}\left|\psi'(ka_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{4.10}
$$

Proof From Lemma 3, Hölder's inequality, and n-polynomial (s, m)-exponential-type convexity of $|\psi'|^q$, we have

$$
\begin{aligned}
&\left|\frac{\psi(ka_1) + \psi(a_2)}{2} - \frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2} \psi(\theta)\,d\theta\right| \\
&\leq \left(\frac{a_2 - ka_1}{2}\right)\left(\int_0^1 |2\chi - 1|^p\,d\chi\right)^{\frac{1}{p}}\left\{\int_0^1 \left|\psi'\left(k(1-\chi)a_1 + \chi a_2\right)\right|^q\,d\chi\right\}^{\frac{1}{q}} \\
&\leq \left(\frac{a_2 - ka_1}{2}\right)\left(\int_0^1 |2\chi - 1|^p\,d\chi\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left\{\int_0^1\left[\frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{s\chi}-1\right)^i\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q+\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s}-1\right)^i\left|\psi'(ka_1)\right|^q\right]d\chi\right\}^{\frac{1}{q}}$$
$$=\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^i\left(\frac{1}{n}\left|\psi'(ka_1)\right|^q+\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}^{\frac{1}{q}},$$

which completes the proof. □

Remark 8 If we take $n=1$ in Theorem 8, we obtain

$$\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(ka_1)\right|^q+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}. \tag{4.11}$$

Theorem 9 *Suppose* $0<k\leq 1$ *and a mapping* $\psi:(0,\frac{a_2}{m}]\to\Re$ *is differentiable on* $(0,\frac{a_2}{m})$ *with* $0<a_1<a_2$. *If* $|\psi'|^q$ *is an* n*-polynomial* (s,m)*-exponential-type convex function on* $(0,\frac{a_2}{m}]$ *for* $q\geq 1$, *then for some fixed* $s,m\in(0,1]$, *the following inequality holds*:

$$\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\sum_{i=1}^{n}\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)^i\right.$$
$$\left.\times\left(\frac{1}{n}\left|\psi'(ka_1)\right|^q+\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}^{\frac{1}{q}}. \tag{4.12}$$

Proof From Lemma 3, power mean inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left\{\int_0^1|2\chi-1|\left|\psi'\left(k(1-\chi)a_1+\chi a_2\right)\right|d\chi\right\}$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\chi-1|\,d\chi\right)^{1-\frac{1}{q}}$$
$$\times\left\{\int_0^1|2\chi-1|\left|\psi'\left(k(1-\chi)a_1+\chi a_2\right)\right|^q d\chi\right\}^{\frac{1}{q}}$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\chi-1|\,d\chi\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1|2\chi-1|\left\{\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s}-1\right)^i\left|\psi'(ka_1)\right|^q\right.\right.$$

$$+\frac{1}{n}\sum_{i=1}^{n}m^{i}\left(e^{s\chi}-1\right)^{i}\left|\psi'\left(\frac{a_2}{m^{i}}\right)\right|^{q}\biggr\}\,d\chi\biggr]^{\frac{1}{q}}$$

$$=\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\sum_{i=1}^{n}\left(\frac{2(s-2)e^{s}+8e^{\frac{s}{2}}-s^{2}-2s-4}{2s^{2}}\right)^{i}\left(\frac{1}{n}\left|\psi'(ka_1)\right|^{q}+\frac{m^{i}}{n}\left|\psi'\left(\frac{a_2}{m^{i}}\right)\right|^{q}\right)\right\}^{\frac{1}{q}},$$

which completes the proof. □

Remark 9 If we take $n=1$ in Theorem 9, we have

$$\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{2(s-2)e^{s}+8e^{\frac{s}{2}}-s^{2}-2s-4}{2s^{2}}\right)\left(\left|\psi'(ka_1)\right|^{q}+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^{q}\right)\right\}^{\frac{1}{q}}. \tag{4.13}$$

Theorem 10 *Suppose* $0<k\leq 1$ *and a mapping* $\psi:(0,\frac{a_2}{m}]\to\Re$ *is differentiable on* $(0,\frac{a_2}{m})$ *with* $0<a_1<a_2$. *If* $|\psi'|^{q}$ *is an n-polynomial* (s,m)*-exponential-type convex function on* $(0,\frac{a_2}{m}]$ *for* $q>1$ *and* $q^{-1}+p^{-1}=1$, *then for some fixed* $s,m\in(0,1]$, *the following inequality holds*:

$$\left|\frac{\psi(ka_1)+\psi(a_2)}{k+1}-\frac{2}{(k+1)(a_2-ka_1)}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq\left(\frac{a_2-ka_1}{k+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\sum_{i=1}^{n}\left(\frac{e^{s}-s-1}{s}\right)^{i}\left(\frac{1}{n}\left|\psi'(ka_1)\right|^{q}+\frac{m^{i}}{n}\left|\psi'\left(\frac{a_2}{m^{i}}\right)\right|^{q}\right)\right\}^{\frac{1}{q}}. \tag{4.14}$$

Proof From Lemma 4, Hölder's inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^{q}$, we have

$$\left|\frac{\psi(ka_1)+\psi(a_2)}{k+1}-\frac{2}{(k+1)(a_2-ka_1)}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq\left(\frac{a_2-ka_1}{k+1}\right)\left(\int_0^1|2\chi-1|^{p}\,d\chi\right)^{\frac{1}{p}}\left\{\int_0^1\left|\psi'\bigl(k(1-\chi)a_1+\chi a_2\bigr)\right|^{q}\,d\chi\right\}^{\frac{1}{q}}$$
$$\leq\left(\frac{a_2-ka_1}{k+1}\right)\left(\int_0^1|2\chi-1|^{p}\,d\chi\right)^{\frac{1}{p}}$$

$$\times \left\{ \int_0^1 \left[\frac{1}{n} \sum_{i=1}^{n} \left(e^{(1-\chi)s} - 1\right)^i \left|\psi'(ka_1)\right|^q + \frac{1}{n} \sum_{i=1}^{n} m^i \left(e^{s\chi} - 1\right)^i \left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q \right] d\chi \right\}^{\frac{1}{q}}$$

$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \sum_{i=1}^{n} \left(\frac{e^s - s - 1}{s}\right)^i \left(\frac{1}{n}\left|\psi'(ka_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right) \right\}^{\frac{1}{q}},$$

which completes the proof. □

Remark 10 If we take $n = 1$ in Theorem 10, we get

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)$$
$$\times \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right) \right\}^{\frac{1}{q}}. \tag{4.15}$$

Theorem 11 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : (0, \frac{a_2}{m}] \rightarrow \Re$ *is differentiable on* $(0, \frac{a_2}{m})$ *with* $0 < a_1 < a_2$. *If* $|\psi'|^q$ *is an n-polynomial* (s,m)*-exponential-type convex function on* $(0, \frac{a_2}{m}]$ *for* $q \geq 1$, *then for some fixed* $s, m \in (0,1]$, *the following inequality holds*:

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \sum_{i=1}^{n} \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^i \right.$$
$$\left. \times \left(\frac{1}{n}\left|\psi'(ka_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right) \right\}^{\frac{1}{q}}. \tag{4.16}$$

Proof From Lemma 4, power mean inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left[\int_0^1 |2\chi - 1|\left|\psi'\left(k(1-\chi)a_1 + \chi a_2\right)\right| d\chi\right]$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\chi - 1|\, d\chi\right)^{1-\frac{1}{q}} \left\{\int_0^1 |2\chi - 1|\left|\psi'\left(k(1-\chi)a_1 + \chi a_2\right)\right|^q d\chi\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\chi - 1|\, d\chi\right)^{1-\frac{1}{q}}$$
$$\times \left[\int_0^1 |2\chi - 1| \left\{\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s} - 1\right)^i \left|\psi'(ka_1)\right|^q \right.\right.$$
$$\left.\left. + \frac{1}{n}\sum_{i=1}^{n} m^i\left(e^{s\chi} - 1\right)^i \left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q \right\} d\chi\right]^{\frac{1}{q}}$$

$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times \left\{\sum_{i=1}^{n}\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^i \left(\frac{1}{n}\left|\psi'(ka_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}^{\frac{1}{q}},$$

which completes the proof. □

Remark 11 If we take $n = 1$ in Theorem 11, we obtain

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}. \tag{4.17}$$

Theorem 12 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : (0, \frac{a_2}{km}] \to \Re$ *is differentiable on* $(0, \frac{a_2}{km})$ *with* $0 < a_1 < a_2$. *If* $|\psi'|^q$ *is an n-polynomial* (s,m)*-exponential-type convex function on* $(0, \frac{a_2}{km}]$ *for* $q > 1$ *and* $q^{-1} + p^{-1} = 1$, *then for some fixed* $s, m \in (0,1]$, *the following inequality holds*:

$$\left|\frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right)\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\sum_{i=1}^{n}\left(\frac{e^s - s - 1}{s}\right)^i\left(\frac{1}{n}\left|\psi'(a_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.$$
$$+ \left(\frac{1}{n}\left|\psi'(a_1)\right|^q\sum_{i=1}^{n}\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)^i\right.$$
$$\left.\left. + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)^i\right)\right]. \tag{4.18}$$

Proof From Lemma 5, Hölder's inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\left|\frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right)\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\int_0^1 \chi^p\,d\chi\right)^{\frac{1}{p}}\left\{\int_0^1\left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right\}^{\frac{1}{q}}\right.$$
$$\left. + \int_{\frac{1}{2}}^{1}\left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right| d\chi\right]$$
$$\leq \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\int_0^1 \chi^p\,d\chi\right)^{\frac{1}{p}}\right.$$

$$\times \left\{ \int_0^1 \left\{ \frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^i \left|\psi'(a_1)\right|^q + \sum_{i=1}^{n} m^i \left(e^{(1-\chi)s}-1\right)^i \left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q \right\} d\chi \right\}^{\frac{1}{q}}$$

$$+ \int_{\frac{1}{2}}^1 \left\{ \frac{1}{n}\sum_{i=1}^{n}\left(e^{s\chi}-1\right)^i \left|\psi'(a_1)\right|^q + \frac{1}{n}\sum_{i=1}^{n} m^i \left(e^{(1-\chi)s}-1\right)^i \left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q \right\} d\chi \Bigg]$$

$$= \left(\frac{a_2-ka_1}{k}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^i \left(\frac{1}{n}\left|\psi'(a_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{1}{n}\left|\psi'(a_1)\right|^q \sum_{i=1}^{n}\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)^i + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q \sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)^i\right)\right],$$

which completes the proof. □

Remark 12 If we take $n = 1$ in Theorem 12, we have

$$\left|\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta - \psi\left(\frac{a_1+a_2}{2k}\right)\right|$$
$$\leq \left(\frac{a_2-ka_1}{k}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(a_1)\right|^q + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|\psi'(a_1)\right|^q\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right) + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\right)\right]. \tag{4.19}$$

Theorem 13 *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : (0, \frac{a_2}{km}] \to \Re$ *is differentiable on* $(0, \frac{a_2}{km})$ *with* $0 < a_1 < a_2$. *If* $|\psi'|^q$ *is an n-polynomial* (s,m)*-exponential-type convex function on* $(0, \frac{a_2}{km}]$ *for* $q \geq 1$, *then for some fixed* $s, m \in (0,1]$, *the following inequality holds*:

$$\left|\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta - \psi\left(\frac{a_1+a_2}{2k}\right)\right|$$
$$\leq \left(\frac{a_2-ka_1}{k}\right)\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\left|\psi'(a_1)\right|^q\sum_{i=1}^{n}\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)^i\right.\right.$$
$$\left. + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\sum_{i=1}^{n}\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)^i\right)^{\frac{1}{q}}$$
$$+ \left(\frac{1}{n}\left|\psi'(a_1)\right|^q\sum_{i=1}^{n}\left(\frac{e^s-e^{\frac{s}{2}}}{s}-\frac{1}{2}\right)^i\right.$$
$$\left.\left. + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)^i\right)\right]. \tag{4.20}$$

Proof From Lemma 5, power mean inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\left|\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta - \psi\left(\frac{a_1+a_2}{2k}\right)\right|$$
$$\leq \left(\frac{a_2-ka_1}{k}\right)\left\{\int_0^1 \chi\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|d\chi + \int_{\frac{1}{2}}^1\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|d\chi\right\}$$

$$\le \left(\frac{a_2 - ka_1}{k}\right)\left\{\left(\int_0^1 \chi \, d\chi\right)^{1-\frac{1}{q}}\right.$$

$$\left.\times \left(\int_0^1 \chi \left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right)^{\frac{1}{q}} + \int_{\frac{1}{2}}^1 \left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right| d\chi\right\}$$

$$\le \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\int_0^1 \chi \, d\chi\right)^{1-\frac{1}{q}}\right.$$

$$\times \left(\int_0^1 \chi \left\{\frac{1}{n}\sum_{i=1}^n (e^{\chi} - 1)^i \left|\psi'(a_1)\right|^q + \frac{1}{n}\sum_{i=1}^n m^i \left(e^{(1-\chi)s} - 1\right)^i \left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\right\} d\chi\right)^{\frac{1}{q}}$$

$$\left. + \int_{\frac{1}{2}}^1 \left\{\frac{1}{n}\sum_{i=1}^n (e^{s\chi} - 1)^i \left|\psi'(a_2)\right|^q + \frac{1}{n}\sum_{i=1}^n m^i \left(e^{(1-\chi)s} - 1\right)^i \left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q\right\} d\chi\right]$$

$$= \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{1}{n}\left|\psi'(a_1)\right|^q \sum_{i=1}^n \left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right)^i\right.\right.$$

$$\left. + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\right)^{\frac{1}{q}}$$

$$\left. + \left(\frac{1}{n}\left|\psi'(a_1)\right|^q \sum_{i=1}^n \left(\frac{e^s - e^{\frac{s}{2}}}{s} - \frac{1}{2}\right)^i + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{km^i}\right)\right|^q \sum_{i=1}^n \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)^i\right)\right],$$

which completes the proof. □

Remark 13 If we take $n = 1$ in Theorem 13, we get

$$\left|\frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\, d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right)\right|$$

$$\le \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left|\psi'(a_1)\right|^q \left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right)\right.\right.$$

$$\left. + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q \left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)\right)^{\frac{1}{q}}$$

$$\left. + \left(\left|\psi'(a_1)\right|^q \left(\frac{e^s - e^{\frac{s}{2}}}{s} - \frac{1}{2}\right) + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)\right)\right]. \tag{4.21}$$

Theorem 14 *Suppose* $0 < k \le 1$ *and a mapping* $\psi : (0, \frac{a_2}{m}] \to \Re$ *is differentiable on* $(0, \frac{a_2}{m})$ *with* $0 < a_1 < a_2$. *If* $|\psi'|^q$ *is an n-polynomial* (s,m)*-exponential-type convex function on* $(0, \frac{a_2}{m}]$ *for* $q > 1$ *and* $q^{-1} + p^{-1} = 1$, *then for some fixed* $s, m \in (0,1]$, *the following inequality holds*:

$$\left|\frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2} \psi(\theta)\, d\theta - \psi\left(\frac{ka_1 + a_2}{2}\right)\right|$$

$$\le (a_2 - ka_1)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{\sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \left(\frac{1}{n}\left|\psi'(ka_1)\right|^q + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{n}\left|\psi'(ka_1)\right|^q\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)^i + \frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\sum_{i=1}^{n}\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)^i\right)\Bigg]. \tag{4.22}$$

Proof From Lemma 6, Hölder's inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right)\right| \\
&\le (a_2-ka_1)\left\{\left(\int_0^1\chi^p\,d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\big(k(1-\chi)a_1+\chi a_2\big)\right|^q d\chi\right)^{\frac{1}{q}} \right.\\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left|\psi'\big(k(1-\chi)a_1+\chi a_2\big)\right|d\chi\right\} \\
&\le (a_2-ka_1)\left\{\left(\int_0^1\chi^p\,d\chi\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\int_0^1\left(\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s}-1\right)^i\left|\psi'(ka_1)\right|^q+\frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{s\chi}-1\right)^i\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)d\chi\right)^{\frac{1}{q}} \\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left(\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s}-1\right)^i\left|\psi'(ka_1)\right|^q+\frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)d\chi\right\} \\
&=(a_2-ka_1) \\
&\quad\times\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^i\left(\frac{1}{n}\left|\psi'(ka_1)\right|^q+\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{n}\left|\psi'(ka_1)\right|^q\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)^i+\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\sum_{i=1}^{n}\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)^i\right)\right],
\end{aligned}$$

which completes the proof. □

Remark 14 If we take $n=1$ in Theorem 14, we obtain

$$\begin{aligned}
&\left|\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right)\right| \\
&\le (a_2-ka_1)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(ka_1)\right|^q+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\left|\psi'(ka_1)\right|^q\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)\right)\right].
\end{aligned} \tag{4.23}$$

Theorem 15 *Suppose* $0<k\le 1$ *and a mapping* $\psi:(0,\frac{a_2}{m}]\to\Re$ *is differentiable on* $(0,\frac{a_2}{m})$ *with* $0<a_1<a_2$. *If* $|\psi'|^q$ *is an* n*-polynomial* (s,m)*-exponential-type convex function on*

$(0, \frac{a_2}{m}]$ *for* $q \geq 1$, *then for some fixed* $s, m \in (0,1]$, *the following inequality holds*:

$$\begin{aligned}
&\left|\frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta - \psi\left(\frac{ka_1 + a_2}{2}\right)\right| \\
&\quad\leq (a_2 - ka_1)\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\sum_{i=1}^{n}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\frac{1}{n}\left|\psi'(ka_1)\right|^q\right.\right. \\
&\qquad\left.+ \sum_{i=1}^{n}\left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right)^i\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)^{\frac{1}{q}} \\
&\qquad+ \left(\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)^i\frac{1}{n}\left|\psi'(ka_1)\right|^q\right. \\
&\qquad\left.\left.+ \sum_{i=1}^{n}\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)^i\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\}. \qquad (4.24)
\end{aligned}$$

Proof From Lemma 6, power mean inequality, and n-polynomial (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta - \psi\left(\frac{ka_1 + a_2}{2}\right)\right| \\
&\quad\leq (a_2 - ka_1)\left\{\int_0^1 |-\chi|\left|\psi'\big(k(1-\chi)a_1 + \chi a_2\big)\right|d\chi + \int_{\frac{1}{2}}^1\left|\psi'\big(k(1-\chi)a_1 + \chi a_2\big)\right|d\chi\right\} \\
&\quad\leq (a_2 - ka_1)\left\{\left(\int_0^1 \chi\,d\chi\right)^{1-\frac{1}{q}}\right. \\
&\qquad\left.\times\left(\int_0^1 \chi\left|\psi'\big(k(1-\chi)a_1 + \chi a_2\big)\right|^q d\chi\right)^{\frac{1}{q}} + \int_{\frac{1}{2}}^1\left|\psi'\big(k(1-\chi)a_1 + \chi a_2\big)\right|d\chi\right\} \\
&\quad\leq (a_2 - ka_1)\left\{\left(\int_0^1 \chi\,d\chi\right)^{1-\frac{1}{q}}\right. \\
&\qquad\times\left(\int_0^1 \chi\left(\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s} - 1\right)^i\left|\psi'(ka_1)\right|^q + \frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{s\chi} - 1\right)^i\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)d\chi\right)^{\frac{1}{q}} \\
&\qquad\left.+ \int_{\frac{1}{2}}^1\left(\frac{1}{n}\sum_{i=1}^{n}\left(e^{(1-\chi)s} - 1\right)^i\left|\psi'(ka_1)\right|^q + \frac{1}{n}\sum_{i=1}^{n}m^i\left(e^{s\chi} - 1\right)^i\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)d\chi\right\} \\
&\quad= (a_2 - ka_1)\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\sum_{i=1}^{n}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\frac{1}{n}\left|\psi'(ka_1)\right|^q\right.\right. \\
&\qquad\left.+ \sum_{i=1}^{n}\left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right)^i\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)^{\frac{1}{q}} \\
&\qquad\left.+ \left(\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)^i\frac{1}{n}\left|\psi'(ka_1)\right|^q + \sum_{i=1}^{n}\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)^i\frac{m^i}{n}\left|\psi'\left(\frac{a_2}{m^i}\right)\right|^q\right)\right\},
\end{aligned}$$

which completes the proof. □

Remark 15 If we take $n = 1$ in Theorem 15, we have

$$\begin{aligned}&\left|\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right)\right|\\&\quad\le(a_2-ka_1)\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\left|\psi'(ka_1)\right|^q\right.\right.\\&\quad\quad\left.+m\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\\&\quad\quad\left.+\left(\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\left|\psi'(ka_1)\right|^q+m\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}.\end{aligned}\tag{4.25}$$

5 Applications to some means

Consider two special means for different positive real numbers a_1 and a_2 as follows:

1. The arithmetic mean

$$A(a_1,a_2)=\frac{a_1+a_2}{2}.$$

2. The generalized log-mean

$$L_l(a_1,a_2)=\left[\frac{a_2^{l+1}-a_1^{l+1}}{(l+1)(a_2-a_1)}\right]^{\frac{1}{l}};\quad l\in\mathbb{R}\setminus\{-1,0\}.$$

Dragomir et al. [3] have proved that for $s\in(0,1)$, where $1\le l\le\frac{1}{s}$, the function $f(x)=x^{ls}$, $x>0$ is an s-convex function. Then from Proposition 1, it is also an s-exponential convex function for some fixed $s\in[\ln 2,1)$.

Now, using the theoretical results in Sect. 4, we give some applications to above special means for positive different real numbers.

Proposition 2 *Let* $0<a_1<a_2$, $0<k\le 1$, *and* $q>1$ *be such that* $p^{-1}+q^{-1}=1$. *Then for some fixed* $s\in[\ln 2,1)$, *where* $1\le l\le\frac{1}{s}$, *we have*

$$\begin{aligned}&\left|A\left(a_1^{ls},\left(\frac{a_2}{k}\right)^{ls}\right)-\frac{k}{a_2-ka_1}L_{ls}^{ls}\left(a_1,\frac{a_2}{k}\right)\right|\\&\quad\le\frac{ls(a_2-ka_1)}{k\sqrt[p]{2}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{1}{\sqrt[q]{n}}\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^{i\frac{1}{q}}A^{\frac{1}{q}}\left(a_1^{(ls-1)q},\left(\frac{a_2}{k}\right)^{(ls-1)q}\right).\end{aligned}\tag{5.1}$$

Proof Considering the n-polynomial s-exponential-type convex function $\psi(x)=x^{ls}$, $x>0$, and using Theorem 6 with $m=1$, we obtain the required result. □

Proposition 3 *Let* $0<a_1<a_2$, $0<k\le 1$, *and* $q\ge 1$. *Then for some fixed* $s\in[\ln 2,1)$, *where* $1\le l\le\frac{1}{s}$, *we have*

$$\begin{aligned}&\left|A\left(a_1^{ls},\left(\frac{a_2}{k}\right)^{ls}\right)-\frac{k}{a_2-ka_1}L_{ls}^{ls}\left(a_1,\frac{a_2}{k}\right)\right|\\&\quad\le\frac{ls(a_2-ka_1)}{4^{(1-\frac{1}{q})}k}\end{aligned}$$

$$\times \frac{1}{\sqrt[q]{n}} \sum_{i=1}^{n} \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2} \right)^{i\frac{1}{q}} A^{\frac{1}{q}} \left(a_1^{(ls-1)q}, \left(\frac{a_2}{k} \right)^{(ls-1)q} \right). \tag{5.2}$$

Proof Considering the n-polynomial s-exponential-type convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 7 with $m = 1$, we obtain the required result. □

Proposition 4 *Let* $0 < a_1 < a_2$, $0 < r \leq 1$ *and* $q > 1$ *such that* $p^{-1} + q^{-1} = 1$. *Then for some fixed* $s \in [\ln 2, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned} &\left| A\left((ka_1)^{ls}, a_2^{ls}\right) - L_{ls}^{ls}(ka_1, a_2) \right| \\ &\quad \leq \frac{ls(a_2 - ka_1)}{\sqrt[p]{2}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \frac{1}{\sqrt[q]{n}} \sum_{i=1}^{n} \left(\frac{e^s - s - 1}{s} \right)^{i\frac{1}{q}} A^{\frac{1}{q}} \left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q} \right). \end{aligned} \tag{5.3}$$

Proof Considering the n-polynomial s-exponential-type convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 8 with $m = 1$, we obtain the required result. □

Proposition 5 *Let* $0 < a_1 < a_2$, $0 < k \leq 1$ *and* $q \geq 1$. *Then for some fixed* $s \in [\ln 2, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned} &\left| A\left((ka_1)^{ls}, a_2^{ls}\right) - L_{ls}^{ls}(ka_1, a_2) \right| \\ &\quad \leq \frac{ls(a_2 - ka_1)}{4^{(1-\frac{1}{q})}} \\ &\qquad \times \frac{1}{\sqrt[q]{n}} \sum_{i=1}^{n} \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2} \right)^{i\frac{1}{q}} A^{\frac{1}{q}} \left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q} \right). \end{aligned} \tag{5.4}$$

Proof Considering the n-polynomial s-exponential-type convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 9 with $m = 1$, we obtain the required result. □

Proposition 6 *Let* $0 < a_1 < a_2$, $0 < k \leq 1$, *and* $q > 1$ *be such that* $p^{-1} + q^{-1} = 1$. *Then for some fixed* $s \in [\ln 2, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned} &\left| \frac{2}{k+1} \left(A\left((ka_1)^{ls}, a_2^{ls}\right) - L_{ls}^{ls}(ka_1, a_2) \right) \right| \\ &\quad \leq \sqrt[q]{2} \frac{ls(a_2 - ka_1)}{k+1} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \frac{1}{\sqrt[q]{n}} \sum_{i=1}^{n} \left(\frac{e^s - s - 1}{s} \right)^{i\frac{1}{q}} A^{\frac{1}{q}} \left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q} \right). \end{aligned} \tag{5.5}$$

Proof Considering the n-polynomial s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 10 with $m = 1$, we obtain the required result. □

Proposition 7 *Let* $0 < a_1 < a_2$, $0 < k \leq 1$, *and* $q \geq 1$. *Then for some fixed* $s \in [\ln 2, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned} &\left| \frac{2}{k+1} \left(A\left((ka_1)^{ls}, a_2^{ls}\right) - L_{ls}^{ls}(ka_1, a_2) \right) \right| \\ &\quad \leq \frac{ls(a_2 - ka_1)}{2(k+1)} \end{aligned}$$

$$\times \frac{1}{\sqrt[q]{n}} \sum_{i=1}^{n} \left(\frac{2(s-2)e^{s} + 8e^{\frac{s}{2}} - s^{2} - 2s - 4}{2s^{2}} \right)^{i\frac{1}{q}} A^{\frac{1}{q}} \left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q} \right). \tag{5.6}$$

Proof Considering the n-polynomial s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 11 with $m = 1$, we obtain the required result. □

Proposition 8 *Let* $0 < a_1 < a_2$, $0 < k \leq 1$, *and* $q > 1$ *be such that* $p^{-1} + q^{-1} = 1$. *Then for some fixed* $s \in [\ln 2, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned} &\left| L_{ls}^{ls}(ka_1, a_2) - A^{ls}(ka_1, a_2) \right| \\ &\leq ls(a_2 - ka_1) \left\{ \sqrt[q]{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \frac{1}{\sqrt[q]{n}} \sum_{i=1}^{n} \left(\frac{e^{s} - s - 1}{s} \right)^{i\frac{1}{q}} A^{\frac{1}{q}} \left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q} \right) \right. \\ &\quad \left. + (ka_1)^{(ls-1)q} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s} \right)^{i} + a_2^{(ls-1)q} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{2e^{s} - 2e^{\frac{s}{2}} - s}{2s} \right)^{i} \right\}. \end{aligned} \tag{5.7}$$

Proof Considering the n-polynomial s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 14 with $m = 1$, we obtain the required result. □

Proposition 9 *Let* $0 < a_1 < a_2$, $0 < k \leq 1$, *and* $q \geq 1$. *Then for some fixed* $s \in [\ln 2, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned} &\left| L_{ls}^{ls}(ka_1, a_2) - A^{ls}(ka_1, a_2) \right| \\ &\leq ls(a_2 - ka_1) \left\{ \frac{2^{1-\frac{1}{q}}}{\sqrt[q]{n}} \left((ka_1)^{(ls-1)q} \sum_{i=1}^{n} \left(\frac{2e^{s} - s^{2} - 2s - 2}{2s^{2}} \right)^{i} \right. \right. \\ &\quad \left. + a_2^{(ls-1)q} \sum_{i=1}^{n} \left(\frac{(2s-2)e^{s} - s^{2} + 2}{2s^{2}} \right)^{i} \right)^{\frac{1}{q}} \\ &\quad \left. + \frac{1}{n} \left((ka_1)^{(ls-1)q} \sum_{i=1}^{n} \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s} \right)^{i} + a_2^{(ls-1)q} \sum_{i=1}^{n} \left(\frac{2e^{s} - 2e^{\frac{s}{2}} - s}{2s} \right)^{i} \right) \right\}. \end{aligned} \tag{5.8}$$

Proof Considering the n-polynomial s-exponential-type convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 15 with $m = 1$, we obtain the required result. □

At the end, let us consider some applications of the integral inequalities obtained above, to find new error estimates for the trapezoidal and midpoint formula.

For $a_2 > 0$, let $\mathcal{U} : 0 = \chi_0 < \chi_1 < \cdots < \chi_{n-1} < \chi_n = a_2$ be a partition of $[0, a_2]$.

We denote

$$\mathcal{T}(\mathcal{U}, \psi) = \sum_{j=0}^{n-1} \left(\frac{\psi(\chi_j) + \psi(\chi_{j+1})}{2} \right) h_j, \qquad \mathcal{M}(\mathcal{U}, \psi) = \sum_{j=0}^{n-1} \psi \left(\frac{\chi_j + \chi_{j+1}}{2} \right) h_j,$$

and

$$\int_0^{a_2} \psi(x)\, dx = \mathcal{T}(\mathcal{U}, \psi) + \mathcal{R}(\mathcal{U}, \psi), \qquad \int_0^{a_2} \psi(x)\, dx = \mathcal{M}(\mathcal{U}, \psi) + \mathcal{R}^{*}(\mathcal{U}, \psi),$$

where $\mathcal{R}(\mathcal{U},\psi)$ and $\mathcal{R}^*(\mathcal{U},\psi)$ are the remainder terms, and $h_j = \chi_{j+1} - \chi_j$ for $j = 0,1,2,\ldots, n-1$.

Using above notations, we are in a position to prove the following error estimates.

Proposition 10 *Let $\psi : (0,a_2] \to \Re$ be a differentiable mapping on $(0,a_2)$ with $a_2 > 0$. If $|\psi'|^q$ is n-polynomial s-exponential-type convex on $(0,a_2]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s \in (0,1]$, the remainder term satisfies the following error estimate*:

$$\begin{aligned}\bigl|\mathcal{R}(\mathcal{U},\psi)\bigr| &\le \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{1}{\sqrt[q]{n}}\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^{i\frac{1}{q}}\\&\quad\times\sum_{j=0}^{n-1}h_j^2\bigl[\bigl|\psi'(\chi_j)\bigr|^q + \bigl|\psi'(\chi_{j+1})\bigr|^q\bigr]^{\frac{1}{q}}.\end{aligned} \tag{5.9}$$

Proof Using Theorem 6 on the subinterval $[\chi_j,\chi_{j+1}]$ of the closed interval $[0,a_2]$, for all $j = 0,1,2,\ldots,n-1$, we have

$$\begin{aligned}&\left|\left(\frac{\psi(\chi_j)+\psi(\chi_{j+1})}{2}\right)h_j - \int_{\chi_j}^{\chi_{j+1}}\psi(x)\,dx\right|\\&\quad\le \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{1}{\sqrt[q]{n}}\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^{i\frac{1}{q}}h_j^2\bigl[\bigl|\psi'(\chi_j)\bigr|^q + \bigl|\psi'(\chi_{j+1})\bigr|^q\bigr]^{\frac{1}{q}}.\end{aligned} \tag{5.10}$$

Summing inequality (5.10) over j from 0 to $n-1$ and using the property of modulus, we obtain the desired inequality (5.9). The proof of Proposition 10 is completed. □

Proposition 11 *Let $\psi : (0,a_2] \to \Re$ be a differentiable mapping on $(0,a_2)$ with $a_2 > 0$. If $|\psi'|^q$ is n-polynomial s-exponential-type convex on $(0,a_2]$ for $q \geq 1$, then for some fixed $s \in (0,1]$, the remainder term satisfies the following error estimate*:

$$\begin{aligned}\bigl|\mathcal{R}(\mathcal{U},\psi)\bigr| &\le \left(\frac{1}{2}\right)^{2-\frac{1}{q}}\frac{1}{\sqrt[q]{n}}\sum_{i=1}^{n}\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)^{i\frac{1}{q}}\\&\quad\times\sum_{j=0}^{n-1}h_j^2\bigl[\bigl|\psi'(\chi_j)\bigr|^q + \bigl|\psi'(\chi_{j+1})\bigr|^q\bigr]^{\frac{1}{q}}.\end{aligned} \tag{5.11}$$

Proof We prove the claim by applying the same technique as in the proof of Proposition 10 but instead using Theorem 7. □

Proposition 12 *Let $\psi : (0,a_2] \to \Re$ be a differentiable mapping on $(0,a_2)$ with $a_2 > 0$. If $|\psi'|^q$ is n-polynomial s-exponential-type convex on $(0,a_2]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s \in (0,1]$, the remainder term satisfies the following error estimate*:

$$\begin{aligned}\bigl|\mathcal{R}^*(\mathcal{U},\psi)\bigr| &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{1}{\sqrt[q]{n}}\sum_{i=1}^{n}\left(\frac{e^s-s-1}{s}\right)^{i\frac{1}{q}}\\&\quad\times\sum_{j=0}^{n-1}h_j^2\bigl[\bigl|\psi'(\chi_j)\bigr|^q + \bigl|\psi'(\chi_{j+1})\bigr|^q\bigr]^{\frac{1}{q}}\end{aligned}$$

$$+\frac{1}{n}\sum_{j=0}^{n-1}h_j^2\Bigg[\left|\psi'(\chi_j)\right|^q\sum_{i=1}^{n}\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)^i$$
$$+\left|\psi'(\chi_{j+1})\right|^q\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)^i\Bigg]. \tag{5.12}$$

Proof The claim is proved by applying the same technique as in the proof of Proposition 10 but instead using Theorem 12. □

Proposition 13 *Let* $\psi:(0,a_2]\to\Re$ *be a differentiable mapping on* $(0,a_2)$ *with* $a_2>0$. *If* $|\psi'|^q$ *is n-polynomial s-exponential-type convex on* $(0,a_2]$ *for* $q\geq 1$, *then for some fixed* $s\in(0,1]$, *the remainder term satisfies the following error estimate*:

$$\left|\mathcal{R}^*(\mathcal{U},\psi)\right|\leq\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{1}{\sqrt[q]{n}}\sum_{j=0}^{n-1}h_j^2\Bigg[\left|\psi'(\chi_j)\right|^q\sum_{i=1}^{n}\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)^i$$
$$+\left|\psi'(\chi_{j+1})\right|^q\sum_{i=1}^{n}\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)^i\Bigg]^{\frac{1}{q}}$$
$$+\frac{1}{n}\sum_{j=0}^{n-1}h_j^2\Bigg[\left|\psi'(\chi_j)\right|^q\sum_{i=1}^{n}\left(\frac{e^s-e^{\frac{s}{2}}}{s}-\frac{1}{2}\right)^i$$
$$+\left|\psi'(\chi_{j+1})\right|^q\sum_{i=1}^{n}\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)^i\Bigg]. \tag{5.13}$$

Proof We apply the same technique as in the proof of Proposition 10 but use Theorem 13 instead. □

6 Conclusion

In this article, the authors showed new generalizations of Hermite–Hadamard-type inequality for the new class of functions, the so-called n-polynomial (s,m)-exponential-type convex functions. We have obtained refinements of the Hermite–Hadamard inequality for functions whose first derivatives in absolute value at certain power are n-polynomial (s,m)-exponential-type convex. Some applications to special means and new error estimates for the trapezoid formula were given as well. We hope that our new ideas and techniques may inspire many researchers in this fascinating field.

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References

1. Alomari, M., Darus, M., Kirmaci, U.S.: Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means. Comput. Math. Appl. **59**, 225–232 (2010)
2. Chen, F.X., Wu, S.H.: Several complementary inequalities to inequalities of Hermite–Hadamard type for *s*-convex functions. J. Nonlinear Sci. Appl. **9**(2), 705–716 (2016)
3. Dragomir, S.S., Fitzpatrik, S.: The Hadamard's inequality for *s*-convex functions in the second sense. Demonstr. Math. **32**(4), 687–696 (1999)
4. Eftekhari, N.: Some remarks on (s,m)-convexity in the second sense. J. Math. Inequal. **8**, 489–495 (2014)
5. Hudzik, H., Maligranda, L.: Some remarks on *s*-convex functions. Aequ. Math. **48**, 100–111 (1994)
6. Kadakal, M., İşcan, İ.: Exponential type convexity and some related inequalities. J. Inequal. Appl. **2020**(1), 82 (2020)
7. Kashuri, A., Liko, R.: Some new Hermite–Hadamard type inequalities and their applications. Studia Sci. Math. Hung. **56**(1), 103–142 (2019)
8. Omotoyinbo, O., Mogbodemu, A.: Some new Hermite–Hadamard integral inequalities for convex functions. Int. J. Sci. Innov. Tech. **1**(1), 1–12 (2014)
9. Set, E., Noor, M.A., Awan, M.U., Gözpinar, A.: Generalized Hermite–Hadamard type inequalities involving fractional integral operators. J. Inequal. Appl. **2017**, 169 (2017)
10. Toader, G.: Some generalizations of the convexity. In: Proceedings of the Colloquium on Approximation and Optimization, pp. 329–338. Univ. Cluj-Napoca, Cluj-Napoca (1985)
11. Xi, B.Y., Qi, F.: Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means. J. Funct. Spaces Appl. **2012**, Article ID 980438 (2012)
12. Zhang, X.M., Chu, Y.-M., Zhang, X.H.: The Hermite–Hadamard type inequality of *GA*-convex functions and its applications. J. Inequal. Appl. **2010**, Article ID 507560 (2010)
13. Liao, J.G., Wu, S.H., Du, T.S.: The Sugeno integral with respect to α-preinvex functions. Fuzzy Sets Syst. **379**, 102–114 (2020)
14. Du, T.S., Awan, M.U., Kashuri, A., Zhao, S.S.: Some *k*-fractional extensions of the trapezium inequalities through generalized relative semi-$(m;h)$-preinvexity. Appl. Anal. **2019**, 1–21 (2019)
15. Du, T.S., Wang, H., Khan, M.A., Zhang, Y.: Certain integral inequalities considering generalized *m*-convexity on fractal sets and their applications. Fractals **27**(7), Article ID 1950117 (2019)
16. Fernandez, A., Mohammed, P.: Hermite–Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6188
17. Mohammed, P.O., Brevik, I.: A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals. Symmetry **12**, 610 (2020). https://doi.org/10.3390/sym12040610
18. Mohammed, P.O., Sarikaya, M.Z.: On generalized fractional integral inequalities for twice differentiable convex functions. J. Comput. Appl. Math. **372**, 112740 (2020)
19. Mohammed, P.O., Sarikaya, M.Z., Baleanu, D.: On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals. Symmetry **12**, 595 (2020). https://doi.org/10.3390/sym12040595
20. Mohammed, P.O., Abdeljawad, T.: Modification of certain fractional integral inequalities for convex functions. Adv. Differ. Equ. **2020**, 69 (2020)
21. Breckner, W.W.: Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Raumen. Publ. Inst. Math. **23**, 13–20 (1978)
22. Toplu, T., Kadakal, M., İşcan, İ.: On *n*-polynomial convexity and some related inequalities. AIMS Math. **5**(2), 1304–1318 (2020)