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An iterative scheme for split monotone variational inclusion, variational inequality and fixed point problems

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Abstract

We propose and analyze a new type iterative algorithm to find a common solution of split monotone variational inclusion, variational inequality, and fixed point problems for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Further, we show that a sequence generated by the algorithm converges strongly to common solution. Furthermore, we list some consequences of our established theorem. Finally, we provide a numerical example to demonstrate the applicability of the algorithm. We emphasize that the result accounted in manuscript unifies and extends various results in this field of study.

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1 Introduction

Throughout the paper, let C_1 be a nonempty subset of a real Hilbert space H_1.

A mapping $S_1 : C_1 \to C_1$ is said to be nonexpansive if

$$\|S_1x_1 - S_1x_2\| \leq \|x_1 - x_2\|, \quad \forall x_1, x_2 \in C_1.$$

Let $\mathrm{Fix}(S_1)$ denote the fixed point of S_1, that is, $\mathrm{Fix}(S_1) = \{x_1 \in C_1 : S_1x_1 = x_1\}$.

The classical scalar nonlinear variational inequality problem (in brief, VIP) is: Find $x_1 \in C_1$ such that

$$\langle Dx_1, x_2 - x_1\rangle \geq 0, \quad \forall x_2 \in C_1, \tag{1.1}$$

where $D : C_1 \to H_1$ is a nonlinear mapping. It was introduced by Hartman and Stampacchia [1].

A mapping $S : H_1 \to H_1$ is said to be

(i) monotone, if

$$\langle Sx_1 - Sx_2, x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in H_1;$$

(ii) γ-inverse strongly monotone (in brief, ism), if

$$\langle Sx_1 - Sx_2, x_1 - x_2 \rangle \geq \gamma \|Sx_1 - Sx_2\|^2, \quad \forall x_1, x_2 \in H_1 \text{ and } \gamma > 0;$$

(iii) firmly nonexpansive, if

$$\langle Sx_1 - Sx_2, x_1 - x_2 \rangle \geq \|Sx_1 - Sx_2\|^2, \quad \forall x_1, x_2 \in H_1;$$

(iv) L-Lipschitz continuous, if

$$\|Sx_1 - Sx_2\| \leq L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in H_1 \text{ and } L > 0.$$

A multi-valued mapping $M_1 : D(M_1) \subseteq H_1 \to 2^{H_1}$ is called monotone if, for all $x_1, x_2 \in D(M_1)$, $u_1 \in M_1x_1$ and $u_2 \in M_1x_2$ such that

$$\langle x_1 - x_2, u_1 - u_2 \rangle \geq 0.$$

And it is maximal if $G(M_1)$, the graph of M_1 defined as

$$G(M_1) = \{(x_1, u_1) : u_1 \in M_1x_1\},$$

is not contained properly in the graph of other. It is well known that a monotone mapping M_1 is maximal iff for $x_1 \in D(M_1)$, $u_1 \in H_1$, $\langle x_1 - x_2, u_1 - u_2 \rangle \geq 0$ for each $(x_2, u_2) \in G(M_1)$ implies that $u_1 \in M_1x_1$.

Let $M_1 : D(M_1) \subseteq H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then the resolvent operator $J_{\rho_1}^{M_1} : H_1 \to D(M_1)$ is defined by

$$J_{\rho_1}^{M_1} x_1 := (1 + \rho_1 M_1)^{-1}(x_1), \quad \forall x_1 \in H_1$$

for $\rho_1 > 0$, where I stands for the identity operator on H_1. We observe that $J_{\rho_1}^{M_1}$ is single-valued nonexpansive and firmly nonexpansive.

Moudafi [2] was first to introduce the split monotone variational inclusion problem: Find $\tilde{x} \in H_1$ such that

$$0 \in f_1(\tilde{x}) + M_1(\tilde{x}), \tag{1.2}$$

and

$$\tilde{y} = B\tilde{x} \in H_2 \quad \text{solves } 0 \in f_2(\tilde{y}) + M_2(\tilde{y}), \tag{1.3}$$

where $f_1 : H_1 \to H_1, f_2 : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to H_2$ is a bounded linear mapping, and $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

The split feasibility, split zero, and split fixed point problems are included as special cases. They have been studied broadly by various authors and solve real life problems essentially in modeling of inverse problems, sensor networks in computerized tomography and radiation therapy; for details, see [3–5].

If $f_1 \equiv 0$ and $f_2 \equiv 0$, then we find a split null point problem (in brief, $\mathrm{S_PNPP}$): Find $\tilde{x} \in H_1$ such that

$$0 \in M_1(\tilde{x}), \tag{1.4}$$

and

$$\tilde{y} = B\tilde{x} \in H_2 \quad \text{solves } 0 \in M_2(\tilde{y}). \tag{1.5}$$

In this paper, we consider the split monotone variational inclusion problem (in brief, $\mathrm{S_PMVIP}$): Find $\tilde{x} \in H_1$ such that

$$0 \in M_1(\tilde{x}), \tag{1.6}$$

and

$$\tilde{y} = B\tilde{x} \in H_2 \quad \text{solves } 0 \in f(\tilde{y}) + M_2(\tilde{y}). \tag{1.7}$$

Let $\Lambda = \{\tilde{x} \in H_1 : \tilde{x} \in \mathrm{Sol(MVIP}(1.6)) \text{ and } B\tilde{x} \in \mathrm{Sol(MVIP}(1.7))\}$ denote the solution of $\mathrm{S_PMVIP}$(1.6)–(1.7).

The iterative algorithm for $\mathrm{S_PMVIP}$(1.2)–(1.3) was introduced and studied by Moudafi [2]:

$$x_0 \in H_1, x_{n+1} = P\bigl(x_n + \eta A^*(Q - I)Ax_n\bigr) \quad \text{for } \rho > 0, \tag{1.8}$$

where $P := J_\rho^{M_1}(I - \lambda f_1)$, $Q := J_\rho^{M_2}(I - \rho f_2)$, A^* is the adjoint operator of A and $0 < \eta < \frac{1}{\varsigma}$, ς is the spectral radius of A^*A.

The convergence analysis was studied by Byrne *et al.* [6] of some iterative algorithm for $\mathrm{S_PNPP}$(1.4)–(1.5). Moreover, Kazmi *et al.* [7] established an iterative method to find a common solution of $\mathrm{S_PNPP}$(1.4)–(1.5) and fixed point problem. For instance, see [8, 9].

Recently, Qin *et al.* [10] proposed an algorithm for an infinite family of nonexpansive mappings as follows:

$$x_0 \in C_1, \quad x_{n+1} = \mu_n \theta g(x_n) + \eta_n x_n + \bigl((1 - \eta_n)I - \mu_n A\bigr)\mathbb{W}_n u_n, \tag{1.9}$$

where g is a contraction mapping on H_1, A is a strongly positive bounded linear operator, W_n is generated by $S_1, S_2, \ldots$ as follows:

$$\begin{aligned}
&\mathbb{V}_{n,n+1} := I,\\
&\mathbb{V}_{n,n} := \lambda_n S_n \mathbb{V}_{n,n+1} + (1 - \lambda_n)I,\\
&\mathbb{V}_{n,n-1} := \lambda_{n-1} S_{n-1} \mathbb{V}_{n,n} + (1 - \lambda_{n-1})I,
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\mathbb{V}_{n,m} := \lambda_m S_m \mathbb{V}_{n,m+1} + (1-\lambda_m)I,\\
&\mathbb{V}_{n,m-1} := \lambda_{m-1} S_{m-1} \mathbb{V}_{n,m} + (1-\lambda_{m-1})I,\\
&\vdots\\
&\mathbb{V}_{n,2} := \lambda_2 S_2 \mathbb{V}_{n,3} + (1-\lambda_3)I,\\
&\mathbb{W}_n \equiv \mathbb{V}_{n,1} := \lambda_1 S_1 \mathbb{V}_{n,2} + (1-\lambda_1)I,
\end{aligned}
\tag{1.10}
$$

where $S_1, S_2, \dots, W_n$ are nonexpansive mappings, $\{\lambda_n\} \subset (0,1]$ for $n \geq 1$; for further work, see [11, 12].

Inspirited by Moudafi [2], Byrne *et al.* [6], Kazmi *et al.* [7, 8], Qin *et al.* [10] and by continuing work, we propose and analyze a new type iterative algorithm to find a common solution of split monotone variational inclusion, variational inequality, and fixed point problems for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Further, we show that the sequence generated by the algorithm converges strongly to common solution. Furthermore, we list some consequences of our established theorem. Finally, we provide a numerical example to demonstrate the applicability of the algorithm. We emphasize that the result accounted in the manuscript unifies and extends various results in this field of study.

2 Preliminaries

This section is devoted to recalling few definitions, entailing mathematical tools, and helpful results that are required in the sequel.

To each $x_1 \in H_1$, there exists a unique nearest point $P_{C_1}x_1$ to x_1 in C_1 such that

$$\|x_1 - P_{C_1}x_1\| \leq \|x_1 - x_2\|, \quad \forall x_2 \in C_1, \tag{2.1}$$

where P_{C_1} is a metric projection of H_1 onto C_1. Also, P_{C_1} is nonexpansive and satisfies

$$\langle x_1 - x_2, P_{C_1}x_1 - P_{C_1}x_2\rangle \geq \|P_{C_1}x_1 - P_{C_1}x_2\|^2, \quad \forall x_1, x_2 \in H_1. \tag{2.2}$$

Moreover, $P_{C_1}x_1$ is characterized by the fact that $P_{C_1}x_1 \in C_1$ and

$$\langle x_1 - P_{C_1}x_1, x_2 - P_{C_1}x_1\rangle \leq 0, \quad \forall x_2 \in C_1. \tag{2.3}$$

This implies that

$$\|x_1 - x_2\|^2 \geq \|x_1 - P_{C_1}x_1\|^2 + \|x_2 - P_{C_1}x_1\|^2, \quad \forall x_1 \in H_1, x_2 \in C_1, \tag{2.4}$$

and

$$\big\|\mu x_1 + (1-\mu)x_2\big\|^2 = \mu\|x_1\|^2 + (1-\mu)\|x_2\|^2 - \mu(1-\mu)\|x_1 - x_2\|^2 \tag{2.5}$$

for all $x_1, x_2 \in H_1$ and $\mu \in [0,1]$.

Also, on H_1 the following inequalities hold:

1. Opial's condition [13], that is, for any $\{x_n\}$ with $x_n \rightharpoonup x_1$ and

$$\liminf_{n\to\infty} \|x_n - x_1\| < \liminf_{n\to\infty} \|x_n - x_2\| \tag{2.6}$$

holds, $\forall x_2 \in H_1$ with $x_2 \neq x_1$;

2.

$$\|x_1 + x_2\|^2 \leq \|x_1\|^2 + 2\langle x_2, x_1 + x_2\rangle, \quad \forall x_1, x_2 \in H_1. \tag{2.7}$$

Definition 2.1 ([14]) A mapping $T_1 : H_1 \to H_1$ is called averaged iff

$$T_1 = (1 - \lambda)I + \lambda S_1,$$

where $\lambda \in (0,1)$, I is the identity mapping on H_1, and $S_1 : H_1 \to H_1$ is a nonexpansive mapping.

Lemma 2.1 ([2])
(i) *If $T_2 = (1 - \lambda)T_1 + \lambda S_1$, where $T_1 : H_1 \to H_1$ is averaged, $S_1 : H_1 \to H_1$ is nonexpansive, and $0 < \lambda < 1$, then T_2 is averaged;*
(ii) *If T_1 is γ-ism, then βT_1 is $\frac{\gamma}{\beta}$-ism for $\beta > 0$;*
(iii) *T_1 is averaged iff $I - T_1$ is γ-ism for some $\gamma > \frac{1}{2}$.*

Lemma 2.2 ([2]) *Let $\rho > 0$, f be a γ-ism, and M be a maximal monotone mapping. If $\rho \in (0, 2\gamma)$, then $J^M_\rho (I - \rho f)$ is averaged.*

Lemma 2.3 ([2]) *Let $\rho_1, \rho_2 > 0$ and M_1, M_2 be maximal monotone mappings. Then*

$$\tilde{x} \quad \text{solves } ((1.2)\text{–}(1.3)) \quad \Leftrightarrow \quad \tilde{x} = J^{M_1}_{\rho_1}(I - \rho_1 f_1)\tilde{x} \quad \text{and} \quad B\tilde{x} = J^{M_2}_{\rho_2}(I - \rho_2 f_2)B\tilde{x}.$$

Lemma 2.4 ([15]) *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in E, a Banach space, and let $0 < \mu_n < 1$ with $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$. Consider $v_{n+1} = (1 - \mu_n)v_n + \mu_n u_n$, $n \geq 0$ and $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leq 0$. Then*

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Lemma 2.5 ([16]) *Assume that B is a strongly positive self-adjoint bounded linear operator on H_1 with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.6 ([17]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers with*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \alpha_n, \quad n \geq 0,$$

where $\lambda_n \in (0,1)$ and $\{\alpha_n\}$ in $\mathbb{R}$ with
(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\alpha_n}{\lambda_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n| < +\infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([18]) *Let $S_1 : C_1 \to H_1$ be a nonexpansive mapping. If S_1 has a fixed point, then $(I - S_1)$, where I is the identity mapping, it is demiclosed, that is, if $x_n \rightharpoonup x_1 \in H_1$ and $x_n - S_1 x_n \to x_2$, then $(I - S_1)x_1 = x_2$.*

Lemma 2.8 ([19]) *Let $C_1 \neq \emptyset$ be a closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$, $\forall i \geq 1$. Then $\lim_{i\to\infty} \mathbb{V}_{i,j}\tilde{x}$ exists, $\forall \tilde{x} \in C_1$ and $j \in \mathbb{N}$.*

Remark 2.9 By Lemma 2.8, define a mapping $\mathbb{W} : C_1 \to C_1$ such that $\mathbb{W}\tilde{x} = \lim_{i\to\infty} \mathbb{W}_i\tilde{x} = \lim_{i\to\infty} \mathbb{V}_{i,1}\tilde{x}$, $\forall \tilde{x} \in C_1$, which is called the $\mathbb{W}$-mapping generated by $S_1, S_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. In the whole paper, we consider $0 < \lambda_i < 1$, $\forall i \geq 1$.

Lemma 2.10 ([19]) *Let $C_1 \neq \emptyset$ be a closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$, $\forall i \geq 1$. Then $\mathrm{Fix}(\mathbb{W}) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.*

Lemma 2.11 ([20]) *Let $C_1 \neq \emptyset$ be a closed convex subset of H_1. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$, $\forall i \geq 1$. For any bounded subset K of C_1, $\lim_{i\to\infty} \sup_{\tilde{x}\in K} \|\mathbb{W}_i\tilde{x} - \mathbb{W}\tilde{x}\| = 0$.*

3 Main result

We study the following convergence result for a new type iterative method to find a common solution of SpMVIP(1.6)–(1.7), VIP(1.1), and fixed point problem.

Theorem 3.1 *Let H_1 and H_2 denote the Hilbert spaces and $C_1 \subset H_1$ be a nonempty closed convex subset of Hilbert space H_1. Let $D : C_1 \to H_1$ be a $\gamma-$ inverse strongly monotone mapping, $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*, $M_1 : C_1 \to 2^{H_1}$, and $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone operators and $f : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Let $g : C_1 \to C_1$ be a contraction mapping with constant $\tau \in (0,1)$, A be a strongly positive bounded linear self-adjoint operator on C_1 with constant $\bar{\theta} > 0$ such that $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$, and $\{S_i\}_{i=1}^{\infty} : C_1 \to C_1$ be an infinite family of nonexpansive mappings such that $\Gamma := \Lambda \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows*:

$$\left.\begin{aligned}
&x_1 \in C_1,\\
&v_n = J_{\rho_1}^{M_1}[x_n + \eta B^*(Q - I)Bx_n],\\
&u_n = P_{C_1}(v_n - \sigma_n D v_n),\\
&x_{n+1} = \mu_n \theta g(\mathbb{W}_n x_n) + \delta_n x_n + ((1 - \delta_n)I - \mu_n A)\mathbb{W}_n u_n, \quad n \geq 1,
\end{aligned}\right\} \tag{3.1}$$

where $\mathbb{W}_n$ is defined in (1.10), $Q = J_{\rho_2}^{f,M_2}(I - \rho_2 f)$, $\{\mu_n\}, \{\delta_n\} \subset (0,1)$ and $\eta \in (0, \frac{1}{\epsilon})$, ϵ is the spectral radius of B^*B. Let the control sequences satisfy the following conditions:

(i) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} \mu_n = \infty$;

(ii) $\rho_1 > 0$, $0 < \rho_2 < 2\alpha$;

(iii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 2\gamma$; $\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$, *where* $\tilde{x} = P_\Gamma(\theta g + (I - A))\tilde{x}$ *which solves*

$$\langle (A - \theta g)\tilde{x}, \nu - \tilde{x} \rangle \geq 0 \quad \text{for all } \nu \in \Gamma. \tag{3.2}$$

Proof For the sake of simplicity, we divide the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

Let $\tilde{x} \in \Gamma$, then $\tilde{x} \in \Lambda$ and thus $J_{\rho_1}^{M_1}\tilde{x} = \tilde{x}$, $J_{\rho_2}^{f,M_2}(I - \rho_2 f)B\tilde{x} = B\tilde{x}$ and $(I + \eta B^*(Q - I)B)\tilde{x} = \tilde{x}$. By Lemma 2.2 and firm nonexpansiveness, $J_{\rho_1}^{M_1}$ and $J_{\rho_2}^{f,M_2}(I - \rho_2 f)$ are averaged. Also, $(I + \eta B^*(Q - I)B)$ is averaged since it is $\frac{\nu}{\epsilon}$-ism for some $\nu > \frac{1}{2}$. From Lemma 2.1(iii), $I - Q$ is ν-ism. Thus, we obtain

$$\begin{aligned} \langle B^*(I - Q)Bx_1 - B^*(I - Q)Bx_2, x_1 - x_2 \rangle &= \langle (I - Q)Bx_1 - (I - Q)Bx_2, \\ &\quad Bx_1 - Bx_2 \rangle \\ &\geq \nu \| (I - Q)Bx_1 - (I - Q)Bx_2 \|^2 \\ &\geq \frac{\nu}{\epsilon} \| B^*(I - Q)Bx_1 \\ &\quad - B^*(I - Q)Bx_2 \|^2. \end{aligned} \tag{3.3}$$

This implies that $\eta B^*(I - Q)B$ is $\frac{\nu}{\eta\epsilon}$-ism. Since $0 < \eta < \frac{1}{\epsilon}$, its complement $(I - \eta B^*(I - Q)B)$ is averaged and hence $J_{\rho_1}^{M_1}[I + \eta B^*(Q - I)B] = \mathbb{R}(\text{say})$. Thus, $I + \eta B^*(Q - I)B$, $J_{\rho_1}^{M_1}$, Q and $\mathbb{R}$ are nonexpansive mappings.

Next, we calculate

$$\begin{aligned} \|\nu_n - \tilde{x}\|^2 &= \| J_{\rho_1}^{M_1}(x_n + \eta B^*(Q - I)Bx_n) - J_{\rho_1}^{M_1}\tilde{x} \|^2 \\ &\leq \| x_n + \eta B^*(Q - I)Bx_n - \tilde{x} \|^2 \\ &= \|x_n - \tilde{x}\|^2 + \eta^2 \| B^*(Q - I)Bx_n \|^2 \\ &\quad + 2\eta \langle x_n - \tilde{x}, B^*(Q - I)Bx_n \rangle, \end{aligned} \tag{3.4}$$

and hence

$$\begin{aligned} \|\nu_n - \tilde{x}\|^2 &\leq \|x_n - \tilde{x}\|^2 + \eta^2 \langle (Q - I)Bx_n, BB^*(Q - I)Bx_n \rangle \\ &\quad + 2\eta \langle x_n - \tilde{x}, B^*(Q - I)Bx_n \rangle. \end{aligned} \tag{3.5}$$

Consider $\Upsilon_1 := \eta^2 \langle (Q - I)Bx_n, BB^*(Q - I)Bx_n \rangle$, and we have

$$\begin{aligned} \Upsilon_1 &= \eta^2 \langle (Q - I)Bx_n, BB^*(Q - I)Bx_n \rangle \\ &\leq \epsilon\eta^2 \langle (Q - I)Bx_n, (Q - I)Bx_n \rangle \\ &= \epsilon\eta^2 \| (Q - I)Bx_n \|^2. \end{aligned} \tag{3.6}$$

Also, let $\Upsilon_2 := 2\eta \langle x_n - \tilde{x}, B^*(Q - I)Bx_n \rangle$, and we calculate

$$\begin{aligned} \Upsilon_2 &= 2\eta \langle x_n - \tilde{x}, B^*(Q - I)Bx_n \rangle \\ &= 2\eta \langle B(x_n - \tilde{x}), (Q - I)Bx_n \rangle \end{aligned}$$

$$
\begin{aligned}
&= 2\eta\big\langle B(x_n-\tilde{x}) + (Q-I)Bx_n - (Q-I)Bx_n, (Q-I)Bx_n\big\rangle \\
&= 2\eta\big(\big\langle QB(x_n - B\tilde{x}), (Q-I)Bx_n\big\rangle - \big\|(Q-I)Bx_n\big\|^2\big) \\
&\le 2\eta(\frac{1}{2}\big\|(Q-I)Bx_n\big\|^2 - \big\|(Q-I)Bx_n\big\|^2 \\
&\le -\eta\big\|(Q-I)Bx_n\big\|^2. && (3.7)
\end{aligned}
$$

By (3.6) and (3.7) in (3.5), we get

$$
\|v_n - \tilde{x}\|^2 \le \|x_n - \tilde{x}\|^2 + \eta(\epsilon\eta - 1)\big\|(Q-I)Bx_n\big\|^2. \qquad (3.8)
$$

Since $0 < \eta < \frac{1}{\epsilon}$, therefore

$$
\|v_n - \tilde{x}\| \le \|x_n - \tilde{x}\|. \qquad (3.9)
$$

Using γ-ism and $0 < \sigma_n < 2\gamma$, we have

$$
\begin{aligned}
\|u_n - \tilde{x}\|^2 &= \|P_{C_1}(v_n - \sigma_n D v_n) - P_{C_1}(v_n - \sigma_n D\tilde{x}\|^2 \\
&\le \|v_n - \sigma_n D v_n - (v_n - \sigma_n D\tilde{x}\|^2 \\
&= \big\|(v_n - \tilde{x}) - \sigma_n(Dv_n - D\tilde{x})\big\|^2 \\
&= \|v_n - \tilde{x}\|^2 - 2\sigma_n\langle Dv_n - D\tilde{x}, v_n - \tilde{x}\rangle + \sigma_n^2\|Dv_n - D\tilde{x}\|^2 \\
&\le \|v_n - \tilde{x}\|^2 - 2\sigma_n\gamma\|Dv_n - D\tilde{x}\|^2 + \sigma_n^2\|Dv_n - D\tilde{x}\|^2 \\
&= \|v_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Dv_n - D\tilde{x}\|^2 \\
&\le \|v_n - \tilde{x}\|^2, && (3.10)
\end{aligned}
$$

this implies

$$
\|u_n - \tilde{x}\| \le \|v_n - \tilde{x}\|. \qquad (3.11)
$$

By using (3.9) and (3.11), we calculate

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \big\|\mu_n\theta g(\mathbb{W}_n x_n) + \delta_n x_n + \big((1-\delta_n)I - \mu_n A\big)\mathbb{W}_n u_n - \tilde{x}\big\| \\
&= \big\|\mu_n\big(\theta g(\mathbb{W}_n x_n) - A\tilde{x}\big) + \delta_n(x_n - \tilde{x}) + \big((1-\delta_n)I - \mu_n A\big)(\mathbb{W}_n u_n - \tilde{x})\big\| \\
&\le \mu_n\big\|\theta g(\mathbb{W}_n x_n) - A\tilde{x}\big\| + \delta_n\|x_n - \tilde{x}\| + \big((1-\delta_n)I - \mu_n\overline{\theta}\big)\|\mathbb{W}_n u_n - \tilde{x}\| \\
&\le \mu_n\big\|\theta g(\mathbb{W}_n x_n) - \theta g(\tilde{x}) + \theta g(\tilde{x}) - A\tilde{x}\big\| \\
&\quad + \delta_n\|x_n - \tilde{x}\| + \big((1-\delta_n)I - \mu_n\overline{\theta}\big)\|u_n - \tilde{x}\| \\
&\le \mu_n\theta\big\|g(\mathbb{W}_n x_n) - g(\tilde{x})\big\| + \mu_n\big\|\theta g(\tilde{x}) - A\tilde{x}\big\| \\
&\quad + \delta_n\|x_n - \tilde{x}\| + \big((1-\delta_n)I - \mu_n\overline{\theta}\big)\|x_n - \tilde{x}\| \\
&\le \mu_n\theta\tau\|x_n - \tilde{x}\| + \mu_n\big\|\theta g(\tilde{x}) - A\tilde{x}\big\| + (1 - \mu_n\overline{\theta})\|x_n - \tilde{x}\| \\
&\le \big(1 - \mu_n(\overline{\theta} - \theta\tau)\big)\|x_n - \tilde{x}\| + \mu_n\big\|\theta g(\tilde{x}) - A\tilde{x}\big\| \\
&\le \max\left\{\|x_n - \tilde{x}\|, \frac{\|\theta g(\tilde{x}) - A\tilde{x}\|}{\overline{\theta} - \theta\tau}\right\}, \quad n \ge 1. && (3.12)
\end{aligned}
$$

Using induction, we get

$$\|x_{n+1}-\tilde{x}\| \leq \max\left\{\|x_1-\tilde{x}\|, \frac{\|\theta g(\tilde{x})-A\tilde{x}\|}{\overline{\theta}-\theta\tau}\right\}.$$

Thus, $\{x_n\}$ is bounded and hence $\{u_n\}$, $\{\mathbb{W}u_n\}$, and $\{g(\mathbb{W}_n x_n)\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\lim_{n\to\infty}\|x_n-\mathbb{W}_n u_n\| = 0$, $\lim_{n\to\infty}\|v_n-x_n\| = 0$, and $\lim_{n\to\infty}\|v_n-u_n\| = 0$.

Since $J_{\rho_1}^{M_1}[I+\eta B^*(Q-I)B]$ is nonexpansive, therefore

$$\begin{aligned}\|v_{n+1}-v_n\| &= \left\|J_{\rho_1}^{M_1}\big[I+\eta B^*(Q-I)B\big]x_{n+1}-J_{\rho_1}^{M_1}\big[I+\eta B^*(Q-I)B\big]x_n\right\| \\ &\leq \|x_{n+1}-x_n\|. \end{aligned} \tag{3.13}$$

Using (3.13), we estimate

$$\begin{aligned}\|u_{n+1}-u_n\| &= \big\|P_C(I-\sigma_{n+1}A)v_{n+1}-P_C(I-\sigma_n S)v_n\big\| \\ &\leq \big\|(I-\sigma_{n+1}A)v_{n+1}-(I-\sigma_n A)v_n\big\| \\ &= \big\|(I-\sigma_{n+1}A)v_{n+1}-(I-\sigma_{n+1}A)v_n+(\sigma_n-\sigma_{n+1})Av_n\big\| \\ &\leq \|v_{n+1}-v_n\|+|\sigma_n-\sigma_{n+1}|\|Av_n\| \\ &\leq \|x_{n+1}-x_n\|+|\sigma_n-\sigma_{n+1}|\|Av_n\| \\ &\leq \|x_{n+1}-x_n\|+\mathbb{N}_1|\sigma_n-\sigma_{n+1}|, \end{aligned} \tag{3.14}$$

where $\mathbb{N}_1 = \sup_{n\geq 1}\|Av_n\|$.

For $i \in 1,2,\ldots,n$, S_i and $\mathbb{V}_{n,i}$ are nonexpansive, therefore from (1.10) we obtain

$$\begin{aligned}\|\mathbb{W}_{n+1}u_n-\mathbb{W}_n u_n\| &= \|\lambda_1 S_1\mathbb{V}_{n+1,2}u_n-\lambda_1 S_1\mathbb{V}_{n,2}u_n\| \\ &\leq \lambda_1\|\mathbb{V}_{n+1,2}u_n-\mathbb{V}_{n,2}u_n\| \\ &\leq \lambda_1\|\lambda_2 S_2\mathbb{V}_{n+1,3}u_n-\lambda_2 S_2\mathbb{V}_{n,3}u_n\| \\ &\leq \lambda_1\lambda_2\|\mathbb{V}_{n+1,3}u_n-\mathbb{V}_{n,3}u_n\| \\ &\vdots \\ &\leq \lambda_1\lambda_2\ldots\lambda_n\|\mathbb{V}_{n+1,n+1}u_n-\mathbb{V}_{n,n+1}u_n\| \\ &\leq \mathbb{N}_2\prod_{i=1}^{n}\lambda_i, \end{aligned} \tag{3.15}$$

where $\mathbb{N}_2 \geq 0$ with $\|\mathbb{V}_{n+1,n+1}u_n-\mathbb{V}_{n,n+1}u_n\| \leq \mathbb{N}_2$, $\forall n\geq 1$.

Setting $x_{n+1} = (1-\delta_n)s_n+\delta_n x_n$, then we have $s_n = \frac{x_{n+1}-\delta_n x_n}{1-\delta_n}$ and

$$\begin{aligned}s_{n+1}-s_n &= \frac{\mu_{n+1}\theta g(\mathbb{W}_{n+1}x_{n+1})+((1-\delta_{n+1})I-\mu_{n+1}A)\mathbb{W}_{n+1}u_{n+1}}{1-\delta_{n+1}} \\ &\quad -\frac{\mu_n\theta g(\mathbb{W}_n x_n)+((1-\delta_n)I-\mu_n A)\mathbb{W}_n u_n}{1-\delta_n}\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\mu_{n+1}}{1-\delta_{n+1}}\right)\bigl(\theta g(\mathbb{W}_{n+1}x_{n+1}) - A\mathbb{W}_{n+1}u_{n+1}\bigr) \\
&\quad + \left(\frac{\mu_n}{1-\delta_n}\right)\bigl(A\mathbb{W}_n u_n - \theta g(\mathbb{W}_n x_n)\bigr) + \mathbb{W}_{n+1}u_{n+1} - \mathbb{W}_n y_n \\
&= \left(\frac{\mu_{n+1}}{1-\delta_{n+1}}\right)\bigl(\theta g(\mathbb{W}_{n+1}x_{n+1}) - A\mathbb{W}_{n+1}u_{n+1}\bigr) \\
&\quad + \left(\frac{\mu_n}{1-\delta_n}\right)\bigl(A\mathbb{W}_n u_n - \theta g(\mathbb{W}_n x_n)\bigr) \\
&\quad + \mathbb{W}_{n+1}u_{n+1} - \mathbb{W}_{n+1}u_n + \mathbb{W}_{n+1}u_n - \mathbb{W}_n u_n.
\end{aligned} \tag{3.16}$$

Hence,

$$\begin{aligned}
\|s_{n+1} - s_n\| &\le \frac{\mu_{n+1}}{1-\delta_{n+1}}\bigl(\bigl\|\theta g(\mathbb{W}_{n+1}x_{n+1})\bigr\| + \|A\mathbb{W}_{n+1}u_{n+1}\|\bigr) \\
&\quad + \frac{\mu_n}{1-\delta_n}\bigl(\|A\mathbb{W}_n u_n\| + \bigl\|\theta g(\mathbb{W}_n x_n)\bigr\|\bigr) \\
&\quad + \|\mathbb{W}_{n+1}u_{n+1} - \mathbb{W}_{n+1}u_n\| + \|\mathbb{W}_{n+1}u_n - \mathbb{W}_n u_n\| \\
&\le \frac{\mu_{n+1}}{1-\delta_{n+1}}\mathbb{N}_3 + \frac{\mu_n}{1-\delta_n}\mathbb{N}_4 \\
&\quad + \|u_{n+1} - u_n\| + \|\mathbb{W}_{n+1}u_n - \mathbb{W}_n u_n\|,
\end{aligned} \tag{3.17}$$

where $\mathbb{N}_3 = \sup_{n\ge 1}(\|\theta g(\mathbb{W}_{n+1}x_{n+1})\| + \|A\mathbb{W}_{n+1}u_{n+1}\|)$ and $\mathbb{N}_4 = \sup_{n\ge 1}(\|A\mathbb{W}_n u_n\| + \|\theta g(\mathbb{W}_n x_n)\|)$.

Using (3.14) and (3.15) in the above inequality

$$\begin{aligned}
\|s_{n+1} - s_n\| &\le \frac{\mu_{n+1}}{1-\delta_{n+1}}\mathbb{N}_3 + \frac{\mu_n}{1-\delta_n}\mathbb{N}_4 + \|x_{n+1} - x_n\| \\
&\quad + \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + \mathbb{N}_2\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{3.18}$$

and thus

$$\begin{aligned}
\|s_{n+1} - s_n\| - \|x_{n+1} - x_n\| &\le \frac{\mu_{n+1}}{1-\delta_{n+1}}\mathbb{N}_3 + \frac{\mu_n}{1-\delta_n}\mathbb{N}_4 \\
&\quad + \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + \mathbb{N}_2\prod_{i=1}^{n}\lambda_i.
\end{aligned} \tag{3.19}$$

Using the given conditions in the above inequality, we have

$$\limsup_{n\to\infty}\bigl(\|s_{n+1} - s_n\| - \|x_{n+1} - x_n\|\bigr) \le 0.$$

By Lemma 2.4, we get

$$\lim_{n\to\infty}\|s_n - x_n\| = 0. \tag{3.20}$$

As $x_{n+1} = (1-\delta_n)s_n + \delta_n x_n$ therefore

$$\|x_{n+1} - x_n\| = \bigl\|(1-\delta_n)(s_n - x_n)\bigr\|,$$

which yields

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.21}$$

Now,

$$\begin{aligned} \|x_n - \mathbb{W}_n u_n\| &= \|x_n - x_{n+1} + x_{n+1} - \mathbb{W}_n u_n\| \\ &\le \|x_{n+1} - x_n\| + \big\|\mu_n \theta g(\mathbb{W}_n x_n) + \delta_n x_n \\ &\quad + \big((1-\delta_n)I - \mu_n A\big)\mathbb{W}_n u_n - \mathbb{W}_n u_n\big\| \\ &= \|x_{n+1} - x_n\| + \big\|\mu_n\big(\theta g(\mathbb{W}_n x_n) - A\mathbb{W}_n u_n\big)\big\| \\ &\quad + \big((1-\delta_n)I - \mu_n A\big)(\mathbb{W}_n u_n - \mathbb{W}_n u_n) + \delta_n(x_n - \mathbb{W}_n u_n) \\ &\le \|x_{n+1} - x_n\| + \mu_n\big\|\theta g(\mathbb{W}_n x_n) - A\mathbb{W}_n u_n\big\| \\ &\quad + \beta_n\|x_n - \mathbb{W}_n u_n\|. \end{aligned} \tag{3.22}$$

Hence,

$$(1-\delta_n)\|x_n - \mathbb{W}_n u_n\| \le \|x_{n+1} - x_n\| + \mu_n\big\|\theta g(\mathbb{W}_n x_n) - A\mathbb{W}_n u_n\big\|.$$

Using the given conditions and (3.21) in (3.22), we get

$$\lim_{n\to\infty} \|x_n - \mathbb{W}_n u_n\| = 0. \tag{3.23}$$

By (3.8) and (3.11), we compute

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &= \big\|\mu_n\big(\theta g(\mathbb{W}_n x_n) - A\tilde{x}\big) + \delta_n(x_n - \mathbb{W}_n u_n) + (1-\mu_n A)(\mathbb{W}_n u_n - \tilde{x})\big\|^2 \\ &\le \big\|(1-\mu_n A)(\mathbb{W}_n u_n - \tilde{x}) + \delta_n(x_n - \mathbb{W}_n u_n)\big\|^2 \\ &\quad + 2\big\langle \mu_n \theta g(\mathbb{W}_n x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\ &\le \big[\big\|(1-\mu_n A)(\mathbb{W}_n u_n - \tilde{x})\big\| + \delta_n\|x_n - \mathbb{W}_n u_n\|\big]^2 \\ &\quad + 2\mu_n\big\langle \theta g(\mathbb{W}_n x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\ &\le \big[(1-\mu_n\bar{\theta})\|u_n - \tilde{x}\| + \delta_n\|x_n - \mathbb{W}_n u_n\|\big]^2 \\ &\quad + 2\mu_n\big\langle \theta g(\mathbb{W}_n x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\ &= (1-\mu_n\bar{\theta})^2\|u_n - \tilde{x}\|^2 + \delta_n^2\|x_n - \mathbb{W}_n u_n\|^2 \\ &\quad + 2(1-\mu_n\bar{\theta})\delta_n\|u_n - \tilde{x}\|\|x_n - \mathbb{W}_n u_n\| \\ &\quad + 2\mu_n\big\langle \theta g(\mathbb{W}_n x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\ &\le (1-\mu_n\bar{\theta})^2\big[\|x_n - \tilde{x}\|^2 + \eta(\epsilon\eta - 1)\big\|(Q-I)Bx_n\big\|^2\big] \\ &\quad + \delta_n^2\|x_n - \mathbb{W}_n u_n\|^2 + 2(1-\mu_n\bar{\theta})\delta_n\|u_n - \tilde{x}\|\|x_n - \mathbb{W}_n u_n\| \\ &\quad + 2\mu_n\big\langle \theta g(\mathbb{W}_n x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\ &= \big(1 - 2\mu_n\bar{\theta} + (\mu_n\bar{\theta})^2\big)\|x_n - \tilde{x}\|^2 + (1-\mu_n\bar{\theta})^2\eta(\epsilon\eta - 1)\big\|(Q-I)Bx_n\big\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\delta_n^2\|x_n-\mathbb{W}_nu_n\|^2+2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_nu_n\|\\
&+2\mu_n\langle\theta g(\mathbb{W}_nx_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
\le{}&\|x_n-\tilde{x}\|^2+(\mu_n\bar{\theta})^2)\|x_n-\tilde{x}\|^2+(1-\mu_n\bar{\theta})^2\eta(\epsilon\eta-1)\big\|(Q-I)Bx_n\big\|^2\\
&+\delta_n^2\|x_n-\mathbb{W}_nu_n\|^2+2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_nu_n\|\\
&+2\mu_n\langle\theta g(\mathbb{W}_nx_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle.
\end{aligned}
\tag{3.24}
$$

Therefore,

$$
\begin{aligned}
(1-\mu_n\bar{\theta})^2\eta(1-\epsilon\eta)\big\|(Q-I)Bx_n\big\|^2\le{}&\|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2\\
&+\delta_n^2\|x_n-\mathbb{W}_nu_n\|^2+\mu_n\bar{\theta}^2\|x_n-\tilde{x}\|^2\\
&+2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_nu_n\|\\
&+2\mu_n\langle\theta g(\mathbb{W}_nx_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
\le{}&\big(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|\big)\|x_n-x_{n+1}\|\\
&+\delta_n^2\|x_n-\mathbb{W}_nu_n\|^2+\mu_n\bar{\theta}^2\|x_n-\tilde{x}\|^2\\
&+2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_nu_n\|\\
&+2\mu_n\big(\theta\big\|g(\mathbb{W}_nx_n)\big\|\\
&+\|A\tilde{x}\|\big)\|x_{n+1}-\tilde{x}\|.
\end{aligned}
\tag{3.25}
$$

Since $\eta(1-\epsilon\eta)>0$, $\lim_{n\to\infty}\mu_n=0$ and $\{x_n\}$, $\{u_n\}$ are bounded, and using (3.21) and (3.23), we have

$$
\lim_{n\to\infty}\big\|(Q-I)Bx_n\big\|^2=0.
\tag{3.26}
$$

Next, we calculate

$$
\begin{aligned}
\|v_n-\tilde{x}\|^2&=\big\|J_{\rho_1}^{M_1}\big(x_n+\eta B^*(Q-I)Bx_n\big)-\tilde{x}\big\|^2\\
&\le\big\|J_{\rho_1}^{M_1}\big(x_n+\eta B^*(Q-I)Bx_n\big)-J_{\rho_1}^{M_1}\tilde{x}\big\|^2\\
&\le\langle v_n-\tilde{x},x_n+\eta B^*(Q-I)Bx_n-\tilde{x}\rangle\\
&=\frac{1}{2}\big\{\|v_n-\tilde{x}\|^2+\big\|x_n+\eta B^*(Q-I)Bx_n-\tilde{x}\big\|^2-\big\|(v_n-\tilde{x})\\
&\quad-\big[x_n+\eta B^*(Q-I)Bx_n-\tilde{x}\big]\big\|^2\big\}\\
&=\frac{1}{2}\big\{\|v_n-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2-\big\|v_n-x_n-\eta B^*(Q-I)Bx_n\big\|^2\big\}\\
&=\frac{1}{2}\big\{\|v_n-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2-\big[\|v_n-x_n\|^2+\eta^2\big\|B^*(Q-I)Bx_n\big\|^2\\
&\quad-2\eta\langle v_n-x_n,B^*(Q-I)Bx_n\rangle\big]\big\}.
\end{aligned}
$$

Hence, we obtain

$$
\|v_n-\tilde{x}\|^2\le\|x_n-\tilde{x}\|^2-\|v_n-x_n\|^2+2\eta\big\|B(v_n-x_n)\big\|\big\|(Q-I)Bx_n\big\|.
\tag{3.27}
$$

By (3.11) and (3.24), we obtain

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\le (1-\mu_n\bar{\theta})^2\|u_n-\tilde{x}\|^2+\delta_n^2\|x_n-\mathbb{W}_n u_n\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_n u_n\| \\
&\quad +2\mu_n\langle\theta g(\mathbb{W}_n x_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le (1-\mu_n\bar{\theta})^2\big[\|x_n-\tilde{x}\|^2-\|v_n-x_n\|^2 \\
&\quad +2\eta\big\|A(u_n-x_n)\big\|\big\|(Q-I)Bx_n\big\|\big] \\
&\quad +\delta_n^2\|x_n-\mathbb{W}_n u_n\|^2+2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_n u_n\| \\
&\quad +2\mu_n\langle\theta g(\mathbb{W}_n x_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le \|x_n-\tilde{x}\|^2+(\mu_n\bar{\theta})^2\|x_n-\tilde{x}\|^2 \\
&\quad -2\mu_n\bar{\theta}\|x_n-\tilde{x}\|^2-(1-\mu_n\bar{\theta})^2\|v_n-x_n\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})^2\eta\big\|A(u_n-x_n)\big\|\big\|(Q-I)Bx_n\big\|+\delta_n^2\|x_n-\mathbb{W}_n u_n\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_n u_n\| \\
&\quad +2\mu_n\langle\theta g(\mathbb{W}_n x_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle. \qquad (3.28)
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-\mu_n\bar{\theta})^2\|v_n-x_n\|^2 &\le \|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2 \\
&\quad +(\mu_n\bar{\theta})^2\|x_n-\tilde{x}\|^2-2\mu_n\bar{\theta}\|x_n-\tilde{x}\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})^2\eta\big\|A(u_n-x_n)\big\|\big\|(Q-I)Bx_n\big\| \\
&\quad +\delta_n^2\|x_n-\mathbb{W}_n u_n\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_n u_n\| \\
&\quad +2\mu_n\langle\theta g(\mathbb{W}_n x_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le \big(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|\big)\|x_n-x_{n+1}\| \\
&\quad +(\mu_n\bar{\theta})^2\|x_n-\tilde{x}\|^2-2\mu_n\bar{\theta}\|x_n-\tilde{x}\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})^2\eta\big\|A(u_n-x_n)\big\|\big\|(Q-I)Bx_n\big\| \\
&\quad +\delta_n^2\|x_n-\mathbb{W}_n u_n\|^2 \\
&\quad +2(1-\mu_n\bar{\theta})\delta_n\|u_n-\tilde{x}\|\|x_n-\mathbb{W}_n u_n\| \\
&\quad +2\mu_n\big(\theta\big\|g(\mathbb{W}_n x_n)\big\|+\|A\tilde{x}\|\big)\|x_{n+1}-\tilde{x}\|. \qquad (3.29)
\end{aligned}$$

As $\{x_n\}$, $\{u_n\}$ are bounded, and using (3.21), (3.23), (3.26) and the given conditions, we have

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \qquad (3.30)$$

Next, we prove that $\lim_{n\to\infty}\|v_n-u_n\|=0$.

We estimate

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &= \bigl\|\mu_n\theta g(\mathbb{W}_n x_n)+\delta_n x_n+\bigl((1-\delta_n)I-\mu_n A\bigr)\mathbb{W}_n u_n-\tilde{x}\bigr\|^2\\ &= \bigl\|(1-\delta_n)(\mathbb{W}_n u_n-\tilde{x})+\delta_n(x_n-\tilde{x})+\mu_n\bigl(\theta g(\mathbb{W}_n x_n)-A\mathbb{W}_n u_n\bigr)\bigr\|^2\\ &\le (1-\delta_n)\|\mathbb{W}_n u_n-\tilde{x}\|^2+\delta_n\|x_n-\tilde{x}\|^2+2\mu_n\langle\kappa_n,x_{n+1}-\tilde{x}\rangle\\ &\le (1-\delta_n)\|\mathbb{W}_n u_n-\tilde{x}\|^2+\delta_n\|x_n-\tilde{x}\|^2+2\omega^2\mu_n\\ &\le (1-\delta_n)\|u_n-\tilde{x}\|^2+\delta_n\|x_n-\tilde{x}\|^2+2\omega^2\mu_n.\end{aligned} \tag{3.31}$$

In the above inequality we set $\kappa_n=\theta g(\mathbb{W}_n x_n)-A\mathbb{W}_n u_n$, and let $\omega>0$ be a suitable constant with $\omega\ge\sup_n\{\|\kappa_n\|,\|x_n-\tilde{x}\|\}$. Thus,

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &\le (1-\delta_n)\|u_n-\tilde{x}\|^2+\delta_n\|x_n-\tilde{x}\|^2+2\omega^2\mu_n\\ &\le (1-\delta_n)\bigl\{\bigl\|P_{C_1}(v_n-\sigma_n Dv_n)-P_{C_1}(\tilde{x}-\sigma_n D\tilde{x})\bigr\|^2\bigr\}\\ &\quad+\delta_n\|x_n-\tilde{x}\|^2+2\omega^2\mu_n\\ &\le (1-\delta_n)\bigl\{\|v_n-\tilde{x}\|^2+\sigma_n(\sigma_n-2\gamma)\|Dv_n-D\tilde{x}\|^2\bigr\}\\ &\quad+\delta_n\|x_n-\tilde{x}\|^2+2\omega^2\mu_n\\ &\le (1-\delta_n)\bigl\{\|x_n-\tilde{x}\|^2+\sigma_n(\sigma_n-2\gamma)\|Dv_n-D\tilde{x}\|^2\bigr\}\\ &\quad+\delta_n\|x_n-\tilde{x}\|^2+2\omega^2\mu_n\\ &\le (1-\delta_n)\sigma_n(\sigma_n-2\omega)\|Dv_n-D\tilde{x}\|^2\\ &\quad+\|x_n-\tilde{x}\|^2+2\omega^2\mu_n,\end{aligned} \tag{3.32}$$

which implies

$$\begin{aligned}(1-\delta_n)\sigma_n(2\omega-\sigma_n)\|Dv_n-D\tilde{x}\|^2 &\le \|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2+2\omega^2\mu_n\\ &\le \bigl(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|\bigr)\|x_n-x_{n+1}\|+2\omega^2\mu_n.\end{aligned}$$

By (3.21) and the given conditions, we get

$$\lim_{n\to\infty}\|Dv_n-D\tilde{x}\|=0. \tag{3.33}$$

From (2.7), we compute

$$\begin{aligned}\|u_n-\tilde{x}\|^2 &= \bigl\|P_{C_1}(v_n-\sigma_n Dv_n)-P_{C_1}(\tilde{x}-\sigma_n D\tilde{x})\bigr\|^2\\ &\le \bigl\langle u_n-\tilde{x},(v_n-\sigma_n Dv_n)-(\tilde{x}-\sigma_n D\tilde{x})\bigr\rangle\\ &\le \frac{1}{2}\bigl\{\|u_n-\tilde{x}\|^2+\bigl\|(v_n-\sigma_n Dv_n)\\ &\quad-(\tilde{x}-\sigma_n D\tilde{x})\bigr\|^2-\bigl\|(u_n-v_n)+\sigma_n(Dv_n-D\tilde{x})\bigr\|^2\bigr\}\\ &\le \frac{1}{2}\bigl\{\|u_n-\tilde{x}\|^2+\|v_n-\tilde{x}\|^2-\bigl\|(u_n-v_n)+\sigma_n(Dv_n-D\tilde{x})\bigr\|^2\bigr\}\\ &\le \|v_n-\tilde{x}\|^2-\|u_n-v_n\|^2-\sigma_n^2\|Dv_n-D\tilde{x}\|^2\end{aligned}$$

$$
\begin{aligned}
&+2\sigma_n\langle u_n - v_n, Du_n - D\tilde{x}\rangle \\
&\le \|v_n - \tilde{x}\|^2 - \|u_n - v_n\|^2 + 2\sigma_n\|u_n - v_n\|\|Dv_n - D\tilde{x}\| \\
&\le \|x_n - \tilde{x}\|^2 - \|u_n - v_n\|^2 + 2\sigma_n\|u_n - v_n\|\|Dv_n - D\tilde{x}\|.
\end{aligned}
$$

By (3.32), we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le (1-\delta_n)\|u_n - \tilde{x}\|^2 + \delta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
&\le (1-\delta_n)\big\{\|x_n - \tilde{x}\|^2 - \|u_n - v_n\|^2 \\
&\quad + 2\sigma_n\|u_n - v_n\|\|Dv_n - D\tilde{x}\|\big\} + \delta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n,
\end{aligned}
$$

which implies

$$
\begin{aligned}
(1-\delta_n)\|u_n - v_n\|^2 &\le \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\
&\quad + 2(1-\delta_n)\sigma_n\|u_n - v_n\|\|Dv_n - D\tilde{x}\| + 2\omega^2\mu_n \\
&\le \big(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|\big)\|x_n - x_{n+1}\| \\
&\quad + 2(1-\delta_n)\sigma_n\|u_n - v_n\|\|Dv_n - D\tilde{x}\| + 2\omega^2\mu_n.
\end{aligned}
$$

Using (3.21), (3.33) and the given conditions, we get

$$
\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.34}
$$

By (3.23), (3.30), and (3.34), we get

$$
\lim_{n\to\infty}\|\mathbb{W}_n u_n - u_n\| = 0. \tag{3.35}
$$

By Lemma 2.11, we have $\lim_{n\to\infty}\|\mathbb{W}u_n - \mathbb{W}_n u_n\| = 0$. Thus,

$$
\lim_{n\to\infty}\|\mathbb{W}u_n - u_n\| = 0. \tag{3.36}
$$

Step 3. We claim that $\tilde{x} \in \Gamma$.

Since $\{x_n\}$ is bounded, therefore consider $\tilde{x} \in H_1$ to be any weak cluster point of $\{x_n\}$. Hence, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with $x_n \rightharpoonup \tilde{x}$. By Lemma 2.7 and (3.35), we have $\tilde{x} \in \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. And $v_{n_j} = J_{\rho_1}^{M_1}[x_{n_j} + \eta B^*(Q-I)Bx_{n_j}]$ can be rewritten as

$$
\frac{(x_{n_j} - v_{n_j}) + B^*(Q-I)Bx_{n_j}}{\rho_1} \in M_1 v_{n_j}. \tag{3.37}
$$

Taking $j \to \infty$ in (3.37) and by (3.26), (3.30) and the concept of the graph of a maximal monotone mapping, we get $0 \in M_1\tilde{x}$, that is, $\tilde{x} \in \mathrm{Sol}(\mathrm{MVIP}(1.6))$. Furthermore, since $\{x_n\}$ and $\{v_n\}$ have the same asymptotical behavior, $Bx_{n_j} \rightharpoonup B\tilde{x}$. As Q is nonexpansive, by (3.26)and Lemma 2.7, we get $(I-Q)B\tilde{x} = 0$. Hence, by Lemma 2.3, $0 \in f(B\tilde{x}) + M_1B\tilde{x}$, that is, $B\tilde{x} \in \mathrm{Sol}(\mathrm{MVIP}(1.7))$. Thus, $\tilde{x} \in \Lambda$.

Next, we prove $\tilde{x} \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Since $\lim_{n\to\infty} \|v_n - u_n\| = 0$ and $\lim_{n\to\infty} \|v_n - x_n\| = 0$, there exist subsequences $\{v_{n_i}\}$ and $\{u_{n_i}\}$ of $\{v_n\}$ and $\{u_n\}$, respectively, such that $v_{n_i} \rightharpoonup \tilde{x}$ and $u_{n_i} \rightharpoonup \tilde{x}$.

Define the mapping $\mathbb{M}$ as

$$\mathbb{M}(z_1) = \begin{cases} D(z_1) + \mathbb{N}_{C_1}(z_1), & \text{if } z_1 \in C_1, \\ \emptyset, & \text{if } z_1 \notin C_1, \end{cases} \tag{3.38}$$

where $\mathbb{N}_{C_1}(z_1) := \{z_2 \in H_1 : \langle z_1 - y, z_2\rangle \geq 0, \forall y \in C_1\}$ is the normal cone to C_1 at $z_1 \in H_1$. Thus, $\mathbb{M}$ is a maximal monotone mapping, and hence $0 \in \mathbb{M}z_1$ if and only if $z_1 \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Let $(z_1, z_2) \in \mathrm{graph}(\mathbb{M})$. Then we have $z_2 \in \mathbb{M}z_1 = Dz_1 + \mathbb{N}_{C_1}(z_1)$, and hence $z_2 - Dz_1 \in \mathbb{N}_{C_1}(z_1)$. So, we have $\langle z_1 - y, z_2 - Dz_1\rangle \geq 0$ for all $y \in C_1$. On the other hand, from $u_n = P_{C_1}(v_n - \sigma_n Dv_n)$ and $z_1 \in C_1$, we have

$$\big\langle (v_n - \sigma_n Dv_n) - u_n, u_n - z_1\big\rangle \geq 0.$$

This implies that

$$\left\langle z_1 - u_n, \frac{u_n - v_n}{\sigma_n} + Dv_n\right\rangle \geq 0.$$

Since $\langle z_1 - y, z_2 - Dz_1\rangle \geq 0$, for all $y \in C_1$ and $u_{n_i} \in C_1$, using the monotonicity of D, we have

$$\begin{aligned} \langle z_1 - u_{n_i}, z_2\rangle &\geq \langle z_1 - u_{n_i}, Dz_1\rangle \\ &\geq \langle z_1 - u_{n_i}, Dz_1\rangle - \left\langle z_1 - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\sigma_n} + Du_{n_i}\right\rangle \\ &= \langle z_1 - u_{n_i}, Dz_1 - Du_{n_i}\rangle + \langle z_1 - u_{n_i}, Du_{n_i} - Dv_{n_i}\rangle \\ &\quad - \left\langle z_1 - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\sigma_n}\right\rangle \\ &\geq \langle z_1 - u_{n_i}, Du_{n_i} - Dv_{n_i}\rangle - \left\langle z_1 - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\sigma_n}\right\rangle. \end{aligned}$$

Since D is continuous, on taking limit $i \to \infty$, we have $\langle z_1 - \tilde{x}, z_2\rangle \geq 0$. Since $\mathbb{M}$ is maximal monotone, we have $\tilde{x} \in \mathbb{M}^{-1}(0)$ and hence $\tilde{x} \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Thus, $\tilde{x} \in \Gamma$.

Step 4. Finally, we prove that $\limsup_{n\to\infty} \langle (\theta g - A)z, x_n - z\rangle \leq 0$, where $z = P_\Gamma(I - A + \theta g)z$ and $x_n \to \tilde{x}$.

By (2.3) and (3.23), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \big\langle (\theta g - A)z, x_n - z\big\rangle &= \limsup_{n\to\infty} \big\langle (\theta g - A)z, \mathbb{W}_n u_n - z\big\rangle \\ &\leq \limsup_{i\to\infty} \big\langle (\theta g - A)z, \mathbb{W}_n u_{n_i} - z\big\rangle \\ &= \big\langle (\theta g - A)z, \tilde{x} - z\big\rangle \\ &\leq 0. \end{aligned} \tag{3.39}$$

Using (3.9) and (3.11), we calculate

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \big\langle \mu_n\big(\theta g(\mathbb{W}_n x_n) - A\tilde{x}\big) \\
&\quad + \delta_n(x_n-\tilde{x}) + \big((1-\delta_n)I - \mu_n A\big)(\mathbb{W}_n u_n - \tilde{x}), x_{n+1}-\tilde{x}\big\rangle \\
&= \mu_n\big\langle \theta g(\mathbb{W}_n x_n) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle + \delta_n\langle x_n-\tilde{x}, x_{n+1}-\tilde{x}\rangle \\
&\quad + \big\langle\big((1-\delta_n)I - \mu_n A\big)(\mathbb{W}_n u_n - \tilde{x}), x_{n+1}-\tilde{x}\big\rangle \\
&\leq \mu_n\big(\theta\big\langle g(\mathbb{W}_n x_n) - g(\tilde{x}), x_{n+1}-\tilde{x}\big\rangle + \big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle\big) \\
&\quad + \delta_n\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
&\quad + \big\|(1-\delta_n)I - \mu_n A\big\|\|\mathbb{W}_n u_n - \tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
&\leq \mu_n\tau\theta\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| + \mu_n\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle \\
&\quad + \delta_n\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| + (1-\delta_n-\mu_n\bar{\theta})\|u_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
&\leq \mu_n\tau\theta\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| + \mu_n\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle \\
&\quad + \delta_n\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| + (1-\delta_n-\mu_n\bar{\theta})\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
&= \big[1-\mu_n(\bar{\theta}-\theta\tau)\big]\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| + \mu_n\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle \\
&\leq \frac{1-\mu_n(\bar{\theta}-\theta\tau)}{2}\big(\|x_n-\tilde{x}\|^2 + \|x_{n+1}-\tilde{x}\|^2\big) \\
&\quad + \mu_n\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle \\
&\leq \frac{1-\mu_n(\bar{\theta}-\theta\tau)}{2}\|x_n-\tilde{x}\|^2 + \frac{1}{2}\|x_{n+1}-\tilde{x}\|^2 \\
&\quad + \mu_n\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle,
\end{aligned} \tag{3.40}$$

which yields that

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\leq \big[1-\mu_n(\bar{\theta}-\theta\tau)\big]\|x_n-\tilde{x}\|^2 \\
&\quad + 2\mu_n(\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle \\
&= \big[1-\mu_n(\bar{\theta}-\theta\tau)\big]\|x_n-\tilde{x}\|^2 + 2\mu_n\big\langle \theta g(\tilde{x}) - A\tilde{x}, x_{n+1}-\tilde{x}\big\rangle.
\end{aligned} \tag{3.41}$$

Thus, by (3.39), (3.41), Lemma 2.6 and using $\lim_{n\to\infty}\mu_n = 0$, we get $x_n \to \tilde{x}$, where $\tilde{x} = P_\Gamma(I+\theta g - A)$. □

Now, we list the following consequences from Theorem 3.1.

Corollary 3.1 *Let H_1 and H_2 denote the Hilbert spaces and $C_1 \subset H_1$ be a nonempty closed convex subset of Hilbert space H_1. Let $D : C_1 \to H_1$ be a $\gamma-$ inverse strongly monotone mapping, $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*, $M_1 : C_1 \to 2^{H_1}$, and $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone operators and $f : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Let $g : C_1 \to C_1$ be a contraction mapping with constant $\tau \in (0,1)$, A be a strongly positive bounded linear self-adjoint operator on C_1 with constant $\bar{\theta} > 0$ such that $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$, and $S : C_1 \to C_1$ be a nonexpansive mapping* *such that* $\Gamma := \Lambda \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \cap \mathrm{Fix}(S)) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated as follows*:

$$\left.\begin{array}{l} x_1 \in C_1, \\ v_n = J_{\rho_1}^{M_1}[x_n + \eta B^*(Q - I)Bx_n], \\ u_n = P_{C_1}(v_n - \sigma_n D v_n), \\ x_{n+1} = \mu_n \theta g(Sx_n) + \delta_n x_n + ((1-\delta_n)I - \mu_n A)Su_n, \end{array}\right\} \tag{3.42}$$

where $Q = J_{\rho_2}^{f,M_2}(I - \rho_2 f)$, $\{\mu_n\}, \{\delta_n\} \subset (0,1)$, *and* $\eta \in (0, \frac{1}{\epsilon})$, ϵ *is the spectral radius of* B^*B. *Let the control sequences satisfy the following conditions*:

(i) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} \mu_n = \infty$;
(ii) $\rho_1 > 0$, $0 < \rho_2 < 2\alpha$;
(iii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 2\gamma$; $\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$, *where* $\tilde{x} = P_\Gamma(\theta g + (I - A))\tilde{x}$, *which solves*

$$\langle (A - \theta g)\tilde{x}, v - \tilde{x} \rangle \geq 0, \quad \forall v \in \Gamma. \tag{3.43}$$

Corollary 3.2 *Let* H_1 *and* H_2 *denote the Hilbert spaces and* $C_1 \subset H_1$ *be a nonempty closed convex subset of Hilbert space* H_1. *Let* $D : C_1 \to H_1$ *be a* $\gamma-$ *inverse strongly monotone mapping*, $B : H_1 \to H_2$ *be a bounded linear operator with its adjoint operator* B^*, $M_1 : C_1 \to 2^{H_1}$, *and* $M_2 : H_2 \to 2^{H_2}$ *be multi-valued maximal monotone operators. Let* $g : C_1 \to C_1$ *be a contraction mapping with constant* $\tau \in (0,1)$, A *be a strongly positive bounded linear self-adjoint operator on* C_1 *with constant* $\bar{\theta} > 0$ *such that* $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$, *and* $S : C_1 \to C_1$ *be a nonexpansive mapping such that* $\Gamma := \mathrm{Sol}(\mathrm{S_P NPP}(1.4) - (1.5)) \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \cap \mathrm{Fix}(S)) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated as follows*:

$$\left.\begin{array}{l} x_1 \in C_1, \\ v_n = J_{\rho_1}^{M_1}[x_n + \eta B^*(J_{\rho_2}^{M_2} - I)Bx_n], \\ u_n = P_{C_1}(v_n - \sigma_n D v_n), \\ x_{n+1} = \mu_n \theta g(Sx_n) + \delta_n x_n + ((1-\delta_n)I - \mu_n A)Su_n, \end{array}\right\} \tag{3.44}$$

where $\{\mu_n\}, \{\delta_n\} \subset (0,1)$ *and* $\eta \in (0, \frac{1}{\epsilon})$, ϵ *is the spectral radius of* B^*B. *Let the control sequences satisfy the following conditions*:

(i) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} \mu_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 2\gamma$; $\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$, *where* $\tilde{x} = P_\Gamma(\theta g + (I - A))\tilde{x}$, *which solves*

$$\langle (A - \theta g)\tilde{x}, v - \tilde{x} \rangle \geq 0, \quad \forall v \in \Gamma. \tag{3.45}$$

4 Numerical example

Example 4.1 Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy$, $\forall x, y \in \mathbb{R}$, and the induced usual norm $|\cdot|$. Let $C_1 = [0, \infty)$; let the mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(y) = y + 6$, $\forall y \in H_2$; let $M_1 : C_1 \to 2^{\mathbb{R}}$, $M_2 : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$M_1(x) = \{3x-2\}$, $\forall x \in C_1$ and $M_2(y) = \{3y\}$, $\forall y \in \mathbb{R}$; let the mapping $B : \mathbb{R} \to \mathbb{R}$ be defined by $B(x) = -\frac{9}{4}x$, $\forall x \in \mathbb{R}$; let the mappings $\{S_i\}_{i=1}^{\infty} : C_1 \to C_1$ be defined by $S_i x = \frac{x+2i}{1+3i}$ for each $i \in \mathbb{N}$, let the mapping $D : C_1 \to \mathbb{R}$ be defined by $Dx = 3x - 2$, $\forall x \in C_1$; let the mapping $g : C_1 \to C_1$ be defined by $g(x) = \frac{x}{5}$, $\forall x \in C_1$ and $Ax = \frac{x}{2}$ with $\theta = \frac{1}{10}$. Setting $\{\mu_n\} = \{\frac{1}{10n}\}$, $\{\delta_n\} = \{\frac{1}{2n^2}\}$, $\{\sigma_n\} = \frac{1}{4}$, and $\{\lambda_n\} = \{\frac{1}{3n^2}\}$, $\forall n \geq 1$. Let $\mathbb{W}_n$ be the $\mathbb{W}$-mapping generated by $S_1, S_2, \ldots$, and $\lambda_1, \lambda_2, \ldots$ which is defined by (1.10). Then there are sequences $\{x_n\}$, $\{u_n\}$, and $\{v_n\}$ as follows: Given x_1,

$$\left.\begin{aligned} &t_n = QBx_n = J_{\rho_2}^{f,M_2}(I - \rho_2 f)Bx_n \\ &y_n = x_n + \eta B^*(t_n - Bx_n) \\ &v_n = J_{\rho_1}^{M_1} y_n \\ &u_n = P_{C_1}(v_n - \sigma_n D v_n), \\ &x_{n+1} = \mu_n \theta g(\mathbb{W}_n x_n) + \delta_n x_n + ((1-\delta_n)I - \mu_n A)\mathbb{W}_n u_n. \end{aligned}\right\} \tag{4.1}$$

Then $\{x_n\}$ converges to $\tilde{x} = \{\frac{2}{3}\} \in \Gamma$.

Proof Obviously, B is a bounded linear operator on $\mathbb{R}$ with adjoint B^* and $\|B\| = \|B^*\| = \frac{9}{4}$, and hence $\eta \in (0, \frac{16}{81})$. Therefore, we choose $\eta = 0.1$. Further, f is 1-ism, $\rho_1 = \frac{1}{4} > 0$ and thus $\rho_2 \subset (0,2)$, so we take $\rho_2 = \frac{1}{4}$. For each i, S_i is nonexpansive with $\mathrm{Fix}(S_i) = \{\frac{2}{3}\}$. Further, D is 3-ism and $\mathrm{Sol}(\mathrm{VIP}(1.1)) = \{\frac{2}{3}\}$. Furthermore, $\mathrm{Sol}(\mathrm{MVIP}(1.6)) = \{\frac{2}{3}\}$ and $\mathrm{Sol}(\mathrm{MVIP}(1.7)) = \{-\frac{3}{2}\}$, and thus $\Lambda = \{\frac{2}{3} \in C_1 : \frac{2}{3} \in \mathrm{Sol}(\mathrm{MVIP}(1.6)) : B(\frac{2}{3}) \in \mathrm{Sol}(\mathrm{MVIP}(1.7))\} = \{\frac{2}{3}\}$. Therefore, $\Gamma := \Lambda \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$. Simplify(4.1)as follows: Given x_1,

$$t_n = \frac{-27x_n - 24}{28}; \qquad y_n = \frac{79x_n - 36t_n}{160};$$

$$v_n = \frac{4}{7}y_n + \frac{2}{7};$$

$$u_n = P_{C_1}(v_n - \sigma_n D v_n);$$

$$= \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 1, \\ \frac{v_n+2}{4} & \text{otherwise}; \end{cases}$$

$$\mathbb{W}_n = x_n;$$

Step 1 :

$i = 1$;

$$\mathbb{W}_n = \left(\frac{1}{3n^2}\right)\frac{(\mathbb{W}_n + 2i)}{1+3i} + \left(1 - \frac{1}{3n^2}\right)x_n;$$

$i = i + 1$;

if $(i \leq N)$ go to Step 1;

$\mathbb{W}'_n = u_n$;

Step $1'$:

$i = 1$;

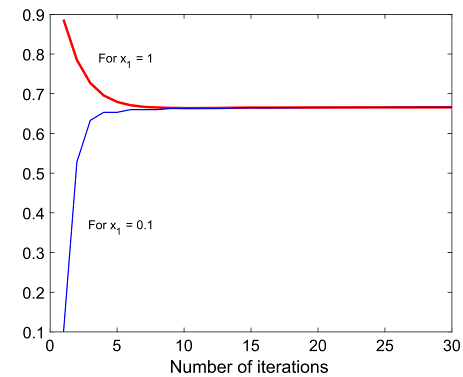

Figure 1 Convergence of $\{x_n\}$

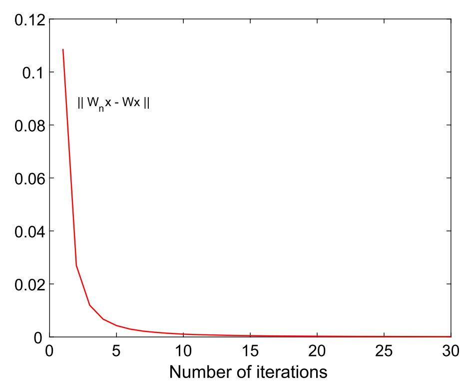

Figure 2 Convergence of $\|W_n x - Wx\|$

$$\mathbb{W}'_n = \left(\frac{1}{3n^2}\right)\frac{(\mathbb{W}'_n + 2i)}{1+3i} + \left(1 - \frac{1}{3n^2}\right)u_n;$$

$$i = i + 1;$$

$$\text{if } (i \leq \mathbb{N}) \quad \text{go to Step } 1';$$

$$x_{n+1} = \mu_n \theta \frac{\mathbb{W}_n x_n}{5} + \delta_n x_n + \bigl((1-\delta_n)I - \mu_n A\bigr)\mathbb{W}'_n u_n,$$

Finally, by the software Matlab 7.8.0, we obtain the Figures 1 and 2 which show that $\{x_n\}$ converges to $\tilde{x} = \frac{2}{3}$ as $n \to +\infty$, and $\lim_{n\to\infty} \|\mathbb{W}_n x - \mathbb{W}x\| = 0$ for each $x \in C_1$. □

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