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# Existence and uniqueness criteria for the higher-order Hilfer fractional boundary value problems at resonance

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# Abstract

This investigation is devoted to the study of a certain class of coupled systems of higher-order Hilfer fractional boundary value problems at resonance. Combining the coincidence degree theory with the Lipschitz-type continuity conditions on nonlinearities, we present some existence and uniqueness criteria. Finally, to practically implement the obtained theoretical criteria, we give an illustrative application.

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**Keywords:** Hilfer fractional boundary value problems; Resonance; Coupled systems; Coincidence degree theory; Existence; Uniqueness

# **1** Introduction

In this paper, we study the coupled systems of the higher-order Hilfer fractional differential equations

$$\begin{cases} D_{a^{+}}^{\alpha,\beta}u(t) = \Phi(t,v, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}v, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v), & a < t < b, \\ D_{a^{+}}^{\alpha,\beta}v(t) = \Psi(t,u, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u), & a < t < b, \end{cases}$$
(1.1)

subject to the boundary conditions

$$\begin{cases} D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta}u(a) = 0, & D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(b), \\ D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta}v(a) = 0, & D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(b), \end{cases}$$
(1.2)

for i = 2, 3, ..., n.

The general assumptions of the coupled system (1.1)-(1.2) are as follows:

- (*A*<sub>1</sub>)  $D_{a^+}^{\alpha,\beta}$  stands for the Hilfer fractional derivative of order  $n 1 < \alpha \le n$ ,  $n \in \mathbb{N}_2$ , and type  $0 \le \beta \le 1$ .
- $(A_2) \ \Phi, \Psi : [a, b] \times \mathbb{R}^n \to \mathbb{R}$  are continuous functions governing the nonlinearities.
- (*A*<sub>3</sub>)  $I_{a^+}^j g(t) = 0$ , j = 1, 2, ..., n, and  $I_{a^+}^{1-\beta(n-\alpha)}g(t) = 0$ , g = u, v, where  $I_{a^+}^{\mu}$  stands for the Riemann–Liouville fractional integral of order  $\mu$ .

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Since the second half of the seventeenth century by now, almost over 370 years, applicability potential of the differential/difference equations and their mutual inverses, integral/finite sum equations in description of the natural phenomena besides theoretical sciences tied up with mathematics have been proved. Meanwhile, the fractional differential calculus that includes arbitrary-order differential/difference and integral/finite sum operators, as a result of more rigorous frame to describe biological, chemical, and physical models of natural phenomena, possesses an elegant position in comparison with the integer-order differential calculus. Reasonability of our claim can be verified in the monographs [4, 6, 7, 18, 26, 27, 33–38].

Restricting ourselves to the investigation of the solvability of fractional-order differential equations opens an independent wide research world in front of us. As some of the generally known techniques to solve a given fractional differential equation, we may mention the fixed point theory (via Green functions, controlling the growth of nonlinearities or measures of noncompactness), the operational calculus, the approximation theory, and the coincidence degree theory. Here we suggest (absolutely not comprehensive) a collection of the pioneering research papers [1–3, 5, 8–14, 17, 21, 23–25, 32, 40, 42–47], and [19, 22, 28–31], respectively, and cited bibliography for more consultation of the interested followers.

Among this variety, we are interested in the investigation of solvability of higher-order generalized fractional differential systems using coincidence degree theory due to Mawhin [33]. More precisely, we are going to study fractional coupled systems at resonance. To specify the concept of resonance, let us consider the differential system

$$\mathfrak{L}u = \mathfrak{F}, \qquad \mathfrak{B}\mathfrak{C}u = 0, \tag{1.3}$$

in which  $\mathfrak{L}$ ,  $\mathfrak{F}$ , and  $\mathfrak{BC}$  denote a differential operator, nonlinearity, and boundary conditions of the system, respectively. We say that the differential system (1.3) is of resonance category if in the corresponding homogenous differential system

$$\mathfrak{L}u = 0, \qquad \mathfrak{B}\mathfrak{C}u = 0, \tag{1.4}$$

the differential operator  $\mathfrak{L}$  is invertible with respect to the boundary conditions  $\mathfrak{BC}$ , that is, there is at least one nontrivial solution for the homogenous differential system (1.4). Otherwise, we are concerned with the nonresonance case. For some of the most motivating research works on the fractional-order resonance phenomenon, we refer to [22, 28–31].

The authors in [16] considered the coupled resonant system of the higher-order Caputotype fractional  $\Delta$ -difference boundary value problems

$$\begin{cases} \Delta_*^{\alpha} y(t) = f(t + N - \alpha - 2, z, \Delta z, \Delta^2 z, \dots, \Delta^{N-1} z), \\ t = a, a + 1, \dots, b, \\ \Delta_*^{\alpha} z(t) = g(t + N - \alpha - 2, y, \Delta y, \Delta^2 y, \dots, \Delta^{N-1} y), \end{cases}$$
(1.5)  
$$\begin{cases} \Delta^{N-i} y(a + N - \alpha - 2) = 0, \qquad \Delta^{N-1} y(a + N - \alpha - 2) = \Delta^{N-1} y(b + N - \alpha - 1), \\ \Delta^{N-i} z(a + N - \alpha - 2) = 0, \qquad \Delta^{N-1} z(a + N - \alpha - 2) = \Delta^{N-1} z(b + N - \alpha - 1), \end{cases}$$
(1.6)

where i = 2, 3, ..., N,  $N - 1 < \alpha \le N$ ,  $N \in \mathbb{N}_2$ , and  $a \in \mathbb{Z}_1$ ,  $b \in \mathbb{Z}_2$  with a < b,  $\Delta^{\alpha}_*$  denotes the Caputo-type fractional  $\Delta$ -difference of order  $\alpha > 0$ , and  $f, g : \mathbb{N}_{a+N-\alpha-1}^{b+N-\alpha-1} \times \mathbb{R}^N \to \mathbb{R}$  are continuous functions. Using the coincidence degree theory, the authors obtained some existence and uniqueness criteria for the discrete fractional coupled system (1.5)–(1.6).

The authors in [19] studied the following higher-order Caputo fractional resonant system:

$$\begin{cases} D_{0^{+}}^{\alpha} u(t) = f(t, v, v', v'', \dots, v^{(n-1)}), & 0 < t < 1, \\ D_{0^{+}}^{\alpha} v(t) = g(t, u, u', u'', \dots, u^{(n-1)}), & 0 < t < 1, \end{cases}$$
(1.7)

subject to the boundary conditions

.

$$\begin{aligned} u(a) &= u'(0) = u''(0) = \dots = u^{(n-2)}(0) = 0, \qquad u^{(n-1)}(0) = u^{(n-1)}(1), \\ v(a) &= v'(0) = v''(0) = \dots = v^{(n-2)}(0) = 0, \qquad v^{(n-1)}(0) = v^{(n-1)}(1). \end{aligned}$$
 (1.8)

Thanks to the coincidence degree theory, they obtained existence criteria for at least one solution of this fractional-order coupled system.

Also, the authors in [48] concentrated on the higher-order Riemann–Liouville fractional resonant boundary value problem

$$\begin{cases} D_{0^{+}}^{\alpha}u(t) = f(t, u(t), D_{0^{+}}^{\alpha-n+1}u(t), D_{0^{+}}^{\alpha-n+2}u(t), \dots, D_{0^{+}}^{\alpha-1}u(t)) + e(t), & 0 < t < 1, \\ I_{0^{+}}^{n-\alpha}u(0) = D_{0^{+}}^{\alpha-n+1}u(0) = \dots = D_{0^{+}}^{\alpha-2}u(0) = 0, & u(1) = \sigma u(\eta), \end{cases}$$
(1.9)

where  $n - 1 < \alpha \le n$ ,  $n \in \mathbb{N}_2$ ,  $\sigma \in (0, +\infty)$ , and  $\eta \in (0, 1)$  with  $\sigma \eta^{\alpha - 1} = 1$ ,  $e \in L[0, 1]$ , and  $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$  is a continuous function. Similarly to the discrete fractional coupled system (1.5)–(1.6) and the Caputo fractional resonant system (1.7)–(1.8), some existence and uniqueness criteria were obtained via coincidence degree theory. The aforementioned research papers are the main motivation that led us to study the Hilfer fractional coupled system (1.1)–(1.2).

The paper is organized as follows. Some standard definitions and technical lemmas related to the fractional calculus operators, a quick overview on the coincidence degree theory, and demonstration why the Hilfer fractional differential system (1.1)–(1.2) is called a resonant differential system are given in the Sect. 2. Section 3, the main body of the investigation, includes some existence and uniqueness criteria for the Hilfer fractional resonant system (1.1)–(1.2). In this section, combining the coincidence degree theory with some controls on the nonlinearities  $\Phi$  and  $\Psi$ , we prove the existence of at least one solution of the fractional coupled system (1.1)–(1.2), and imposing another conditions on these nonlinearities, we present a uniqueness criterion. In Sect. 4, we present an illustrative example to justify the obtained theoretical results. The last section of this paper deals with the novelty and some advantages of studying the Hilfer fractional derivatives in differential equations. Furthermore, we suggest in this section some research lines for the future work on the topic.

## 2 Technical requirements

This section begins with a quick overview on those parts of the fractional calculus that will be needed in this paper. So, we start with the definitions of the Riemann–Liouville fractional integrals and derivatives.

**Definition 2.1** ([26]) The left- and right-sided Riemann–Liouville fractional integrals of order  $\alpha \ge 0$  for a function  $f \in L^1[a, b]$  are given by

$$\mathcal{I}_{a^{+}(b_{-})}^{\alpha}f(t) = \begin{cases} \mathcal{I}_{a^{+}}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1}f(s) \, ds, & \alpha > 0, \\ \mathcal{I}_{b_{-}}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} (s-t)^{\alpha-1}f(s) \, ds, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases}$$
(2.1)

**Definition 2.2** ([26]) The left- and right-sided Riemann–Liouville fractional derivatives of order  $\alpha \ge 0$  for a function  $f \in AC^n(a, b)$  are defined by

$$\mathcal{D}_{a^{+}(b_{-}}^{\alpha}f(t) = \left\{ \begin{array}{l} \mathcal{D}_{a^{+}}^{\alpha}f(t) := \left(\frac{d^{n}}{dt^{n}}\right)\mathcal{I}_{a^{+}}^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d^{n}}{dt^{n}}\right)\int_{a}^{t}(t-s)^{n-\alpha-1}f(s)\,ds, \quad \alpha > 0, \\ \mathcal{D}_{b_{-}}^{\alpha}f(t) := \left(\frac{d^{n}}{dt^{n}}\right)\mathcal{I}_{b_{-}}^{n-\alpha}f(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\left(\frac{d^{n}}{dt^{n}}\right)\int_{t}^{b}(s-t)^{n-\alpha-1}f(s)\,ds, \quad \alpha > 0, \\ f(t), \qquad \alpha = 0, \end{array} \right.$$
(2.2)

where  $n = [\alpha] + 1$ .

Interchanging the affection position of the *n*th-order derivative  $\frac{d^n}{dt^n}$  as follows gives us the left- and right-sided Caputo fractional derivatives

$$= \begin{cases} {}^{c}\mathcal{D}_{a^{+}(b_{-})}^{\alpha}f(t) := \mathcal{I}_{a^{+}}^{n-\alpha}(\frac{d^{n}}{dt^{n}}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}(\frac{d^{n}}{dt^{n}}f)(s)\,ds, \quad \alpha > 0, \\ {}^{c}\mathcal{D}_{b_{-}}^{\alpha}f(t) := \mathcal{I}_{b_{-}}^{n-\alpha}(\frac{d^{n}}{dt^{n}}f)(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{t}^{b}(s-t)^{n-\alpha-1}(\frac{d^{n}}{dt^{n}}f)(s)\,ds, \quad \alpha > 0, \\ {}^{f}(t), \qquad \alpha = 0. \end{cases}$$
(2.3)

The power rules of the Riemann–Liouville fractional operators are given in the following lemma.

**Lemma 2.3** ([26]) Let  $n - 1 < \alpha \le n$ ,  $n \in \mathbb{N}$ . Then for each  $\beta > -1$ , we have

$$\left(\mathcal{I}_{a^+}^{\alpha}(t-a)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(x-a)^{\beta+\alpha},\tag{2.4}$$

$$\left(\mathcal{I}_{b_{-}}^{\alpha}(b-t)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(b-x)^{\beta+\alpha},\tag{2.5}$$

$$\left(\mathcal{D}_{a^+}^{\alpha}(t-a)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha},\tag{2.6}$$

$$\left(\mathcal{D}^{\alpha}_{b_{-}}(b-t)^{\beta}\right)(x) = \frac{(-1)^{n}\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha}.$$
(2.7)

Now we are ready to define the Hilfer fractional derivatives and their basic properties.

**Definition 2.4** ([18, 41]) Let  $n - 1 < \alpha \le n$ ,  $n \in \mathbb{N}$ , and  $0 \le \beta \le 1$ . Then the left- and right-sided Hilfer fractional derivatives of order  $\alpha$  and type  $\beta$  are defined as follows:

$$\mathcal{D}_{a^{+}(b_{-})}^{\alpha,\beta}f(t) = \begin{cases} \mathcal{D}_{a^{+}}^{\alpha,\beta}f(t) := (\mathcal{I}_{a^{+}}^{\beta(n-\alpha)}\mathcal{D}_{a^{+}}^{\alpha+\beta(n-\alpha)}f)(t), & t > a, \\ \mathcal{D}_{b_{-}}^{\alpha,\beta}f(t) := (\mathcal{I}_{b_{-}}^{\beta(n-\alpha)}\mathcal{D}_{b_{-}}^{\alpha+\beta(n-\alpha)}f)(t), & t < b. \end{cases}$$
(2.8)

Note for  $\beta = 0$ , the left- and right-sided Hilfer fractional derivatives (2.8) reduce to the left- and right-sided Riemann–Liouville fractional derivatives (2.2), whereas for  $\beta = 1$ , they reduce to the left- and right-sided Caputo fractional derivatives (2.3).

Next, we present the inversion rules of the Hilfer fractional derivatives.

**Lemma 2.5** ([41]) Let  $n - 1 < \alpha \le n, n \in \mathbb{N}$ ,  $0 \le \beta \le 1$ , and let the assumption (A<sub>3</sub>) hold. If  $f \in L(a, b)$  and  $\mathcal{I}_{a^+}^{(n-\alpha)(1-\beta)} f \in AC^{n-1}[a, b]$ , then

$$\left(\mathcal{D}_{a^{+}}^{\alpha,\beta}\mathcal{I}_{a^{+}}^{\alpha}f\right)(t) = f(t),\tag{2.9}$$

$$\left(\mathcal{I}_{a^{+}}^{\alpha}\mathcal{D}_{a^{+}}^{\alpha,\beta}f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \cdot c_k,$$
(2.10)

where

$$c_k := \lim_{t \to a^+} \frac{d^k}{dx^k} \big( \mathcal{I}_{a^+}^{(n-\alpha)(1-\beta)} f \big)(t).$$

Based on the power rules (2.4)-(2.7) in Lemma 2.3, we give the following power rules for the Hilfer fractional derivatives without proof.

**Lemma 2.6** Let  $n - 1 < \alpha \le n$ ,  $n \in \mathbb{N}$ , and  $0 \le \beta \le 1$ . Then, for each  $\gamma > -1$ ,

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta}(t-a)^{\gamma}\right)(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(x-a)^{\gamma-\alpha},\tag{2.11}$$

$$\left(\mathcal{D}_{b_{-}}^{\alpha,\beta}(b-t)^{\gamma}\right)(x) = \frac{(-1)^{n}\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(b-x)^{\gamma-\alpha}.$$
(2.12)

*Remark* 2.7 Let  $n - 1 < \alpha \le n$ ,  $n \in \mathbb{N}$ ,  $0 \le \beta \le 1$ , k = 0, 1, ..., n - 1, and i = 2, 3, ..., n. Then, we get

$$\left(\mathcal{D}_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta}(t-a)^{k-(n-\alpha)(1-\beta)}\right)(x) = \frac{\Gamma(k-(n-\alpha)(1-\beta)+1)}{\Gamma(k+i-n+1)}(x-a)^{k+i-n}, \quad (2.13)$$

$$\left(\mathcal{I}_{a^{+}}^{n-\alpha}(t-a)^{k-(n-\alpha)(1-\beta)}\right)(x) = \frac{\Gamma(k-(n-\alpha)(1-\beta)+1)}{\Gamma(k+\beta(n-\alpha)+1)}(x-a)^{k+\beta(n-\alpha)}.$$
(2.14)

Note that these particular power rules will play crucial roles in what follows.

Here we begin the second part of this section, which includes a quick overview on the coincidence degree theory. For detailed discussions, we refer to Chapters IV and V of [15, 22], and [29].

**Definition 2.8** Let *X* and *Y* be real normed spaces. A linear mapping  $\mathcal{L} : \operatorname{dom} \mathcal{L} \subset X \to Y$  is called a Fredholm mapping if the following conditions hold:

- (i) ker  $\mathcal{L}$  has a finite dimension,
- (ii)  $\,\, \mbox{Im}\, {\cal L}$  is closed and has a finite codimension.

Let  $\mathcal{L}$  be a Fredholm mapping. Then its *index* is given by

Ind  $\mathcal{L} = \dim \ker \mathcal{L} - \operatorname{codim} \operatorname{Im} \mathcal{L}$ .

Let  $\mathcal{L}$  is a Fredholm mapping with index zero and suppose that there exist continuous projectors  $P: X \to X$  and  $Q: Y \to Y$  such that

 $\operatorname{Im} P = \ker \mathcal{L}, \qquad \ker Q = \operatorname{Im} \mathcal{L}, \qquad X = \ker \mathcal{L} \oplus \ker P, \qquad Y = \operatorname{Im} \mathcal{L} \oplus \operatorname{Im} Q.$ 

It follows that the mapping

 $\mathcal{L}|_{\mathrm{dom}\,\mathcal{L}\cap\ker P}:\mathrm{dom}\,\mathcal{L}\cap\ker P\to\mathrm{Im}\,\mathcal{L}$ 

is invertible. We denote this inverse by  $K_P : \operatorname{Im} \mathcal{L} \to \operatorname{dom} \mathcal{L} \cap \ker P$ . The generalized inverse of  $\mathcal{L}$  denoted by  $K_{P,Q} : Z \to \operatorname{dom} \mathcal{L} \cap \ker P$  is defined by  $K_{P,Q} = K_P(I - Q)$ .

If  $\mathcal{L}$  is a Fredholm mapping of index zero, then for every isomorphism  $J : \text{Im } Q \to \ker \mathcal{L}$ , the mapping  $JQ + K_{P,Q} : Z \to \operatorname{dom} \mathcal{L}$  is an isomorphism, and for every  $u \in \operatorname{dom} \mathcal{L}$ ,

 $(JQ + K_{P,Q})^{-1}u = (\mathcal{L} + J^{-1}P)u.$ 

**Definition 2.9** Let  $\mathcal{L} : \operatorname{dom} \mathcal{L} \subset X \to Y$  be a Fredholm mapping, and let E be a metric space. A mapping  $\mathcal{N} : E \to Y$  is called *L*-compact on E if  $Q\mathcal{N} : E \to Y$  is continuous and  $K_{P,Q}\mathcal{N} : E \to X$  is compact on E. In addition, we say that  $\mathcal{N}$  is *L*-completely continuous if it is *L*-compact on every bounded subset  $E \subset X$ .

**Theorem 2.10** Let  $\Omega \subset X$  be open and bounded, let  $\mathcal{L}$  be a Fredholm mapping of index zero, and let  $\mathcal{N}$  be L-compact on  $\overline{\Omega}$ . Assume that the following conditions are satisfied:

- (i)  $\mathcal{L}u \neq \lambda \mathcal{N}u$  for every  $(u, \lambda) \in ((\operatorname{dom} \mathcal{L} \setminus \ker \mathcal{L}) \cap \partial \Omega) \times (0, 1);$
- (ii)  $\mathcal{N}u \notin \operatorname{Im} \mathcal{L}$  for every  $u \in \ker \mathcal{L} \cap \partial \Omega$ ;
- (iii)  $\deg(JQ\mathcal{N}|_{\ker \mathcal{L}\cap\partial\Omega}, \Omega \cap \ker \mathcal{L}, 0) \neq 0$  with a continuous projector  $Q: Y \to Y$  such that  $\ker Q = \operatorname{Im} \mathcal{L}$  and  $J: \operatorname{Im} Q \to \ker \mathcal{L}$  is an isomorphism.

*Then the equation*  $\mathcal{L}u = \mathcal{N}u$  *has at least one solution in* dom  $\mathcal{L} \cap \overline{\Omega}$ *.* 

At the end of this part, we reveal why the Hilfer fractional coupled system (1.1)-(1.2) is of resonance category. Let us consider the homogeneous Hilfer fractional system

$$\begin{cases} D_{a^{+}}^{\alpha,\beta} u(t) = 0, & a < t < b, \\ D_{a^{+}}^{\alpha,\beta} v(t) = 0, & a < t < b, \end{cases}$$
(2.15)

subject to the boundary conditions

$$\begin{cases} D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta}u(a) = 0, & D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(b), \\ D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta}v(a) = 0, & D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(b), \end{cases}$$
(2.16)

where i = 2, 3, ..., n. Having identity (2.10) in hand and imposing the boundary conditions

$$D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta}u(a) = 0, \quad i = 2, 3, \dots, n,$$

it follows that  $c_k = 0, k = 0, 1, ..., n - 3$ , whereas the coefficients  $c_{n-2}$  and  $c_{n-1}$  are arbitrary. Now imposing the boundary condition

$$D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta}u(a) = D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta}u(b)$$

yields  $c_{n-2} = 0$ . So, we can conclude that the homogeneous Hilfer fractional system (2.15)–(2.16) has a nontrivial solution of the form

$$(u(t), v(t)) = (c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}, c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}).$$

Therefore our situation lies in the resonance category.

We finalize this section by presenting appropriate function spaces and their relevant norms. We first define the following Banach spaces:

$$\begin{split} &\mathcal{B} := \left\{ u | u, D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} u \in C[a,b], i = 1, 2, \dots, n \right\}, \\ &\| u \|_{\mathcal{B}} := \max \left\{ \| u \|, \left\| D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} u \right\|, i = 1, 2, \dots, n \right\}, \\ &\mathcal{E} := \left\{ y | y \in C[a,b] \right\}, \qquad \| y \| := \max_{t \in [a,b]} \left| y(t) \right|. \end{split}$$

Accordingly, the appropriate Banach spaces in this paper are given as follows:

$$X := \mathcal{B} \times \mathcal{B}, \qquad \left\| (u, v) \right\|_{X} := \max\{ \|u\|_{\mathcal{B}}, \|v\|_{\mathcal{B}} \}, Y := \mathcal{E} \times \mathcal{E}, \qquad \left\| (y, z) \right\|_{Y} := \max\{ \|y\|, \|z\| \}.$$
(2.17)

## 3 Main results

In this section, we obtain some existence and uniqueness criteria for the Hilfer fractional resonant system (1.1)–(1.2). To this aim, as explained in the previous section, we are going to apply the coincidence degree theory. So, based on Theorem 2.10, we first transform the Hilfer system (1.1)–(1.2) into an adequate abstract equation such as  $\mathcal{L}(u, v) = \mathcal{N}(u, v)$ . Here we construct required elements of this transformation:

$$\mathcal{L}(u,v) \coloneqq (\mathcal{L}_1 u, \mathcal{L}_2 v), \qquad \begin{cases} \mathcal{L}_1 u \coloneqq D_{a^+}^{\alpha,\beta} u, \\ \mathcal{L}_2 v \coloneqq D_{a^+}^{\alpha,\beta} v, \end{cases}$$
(3.1)

$$\operatorname{dom} \mathcal{L} := (\operatorname{dom} \mathcal{L}_1, \operatorname{dom} \mathcal{L}_2), \tag{3.2}$$

where

$$dom \mathcal{L}_{1} := \left\{ u \in \mathcal{B} | D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta} u(a) = 0, \\ D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(b), i \in \mathbb{N}_{2}^{n} \right\},$$

$$dom \mathcal{L}_{2} := \left\{ v \in \mathcal{B} | D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta} v(a) = 0, \\ D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(b), i \in \mathbb{N}_{2}^{n} \right\}.$$
(3.3)

Here we present the structure of the abstract nonlinearity  $\mathcal{N}$ :

$$\mathcal{N}(u,v) := (\mathcal{N}_{1}v, \mathcal{N}_{2}u),$$

$$\begin{cases}
\mathcal{N}_{1}v := \Phi(t, v, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}v, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}v, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v), \\
\mathcal{N}_{2}u := \Psi(t, u, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u).
\end{cases}$$
(3.4)

Considering the setting (3.1)–(3.3), we come to the conclusion that the Hilfer resonant system (1.1)–(1.2) is equivalent to the abstract problem  $\mathcal{L}(u, v) = \mathcal{N}(u, v)$ . We further have to implement the basis of the coincidence degree theory, step by step as follows.

In the first step, we prove that the operator  $\mathcal{L}(u, v)$  defined by (3.1) is a Fredholm operator of index 0.

**Lemma 3.1** The operator  $\mathcal{L} : \operatorname{dom} \mathcal{L} \cap X \to Y$  is a Fredholm operator of index 0.

*Proof* Thanks to identity (2.10) in Lemma 2.6, focussing on the operator  $\mathcal{L}$  defined by (3.1) with dom  $\mathcal{L}$  defined by (3.3), we get that

$$\ker \mathcal{L} = \left( c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}, d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1} \right),$$

which yields ker  $\mathcal{L} \cong \mathbb{R}^2$ . Now let  $(x, y) \in \text{Im } \mathcal{L}$ . So there is a pair  $(u, v) \in \text{dom } \mathcal{L}$  such that  $\mathcal{L}(u, v) = (x, y)$ , that is,

$$u(t) = I_{a^+}^{\alpha} x(t) + c_0(t-a)^{-(n-\alpha)(1-\beta)} + c_1(t-a)^{1-(n-\alpha)(1-\beta)} + \dots + c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1},$$
  
$$v(t) = I_{a^+}^{\alpha} y(t) + d_0(t-a)^{-(n-\alpha)(1-\beta)} + d_1(t-a)^{1-(n-\alpha)(1-\beta)} + \dots + d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}.$$

According to the dom  $\mathcal{L}$ , we conclude that  $c_i = d_i = 0, i = 0, 1, \dots, n - 2$ . Thus we arrive at

$$u(t) = I_{a^+}^{\alpha} x(t) + c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1},$$
  
$$v(t) = I_{a^+}^{\alpha} y(t) + d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}.$$

Equivalently, we have

$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(t) = I_{a^{+}}^{1-\beta(n-\alpha)}x(t) + c_{n-1}\Gamma(n-(n-\alpha)(1-\beta)),$$

$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(t) = I_{a^{+}}^{1-\beta(n-\alpha)}y(t) + d_{n-1}\Gamma(n-(n-\alpha)(1-\beta)).$$
(3.5)

Next, implementing the boundary conditions

$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(b),$$
  
$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(a) = D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(b)$$

into equalities (3.5) gives us the structure of the Im  $\mathcal{L}$  as follows:

$$\int_{a}^{b} (b-s)^{-\beta(n-\alpha)} x(s) \, ds = 0, \qquad \int_{a}^{b} (b-s)^{-\beta(n-\alpha)} y(s) \, ds = 0. \tag{3.6}$$

In other words,

$$\operatorname{Im} \mathcal{L} := \left\{ (x, y) \, \Big| \, \int_a^b (b-s)^{-\beta(n-\alpha)} x(s) \, ds = 0, \, \int_a^b (b-s)^{-\beta(n-\alpha)} y(s) \, ds = 0 \right\}.$$

To get the desired outcome, we define the operator  $Q: Y \to Y$  with  $Q(x, y) = (Q_1x, Q_2y)$ , where

$$Q_1: \mathcal{E} \to \mathcal{E}, \qquad Q_1 x(t) \coloneqq \frac{1 - \beta(n - \alpha)}{(b - a)^{1 - \beta(n - \alpha)}} \int_a^b (b - s)^{-\beta(n - \alpha)} x(s) \, ds, \tag{3.7}$$

$$Q_2: \mathcal{E} \to \mathcal{E}, \qquad Q_2 y(t) \coloneqq \frac{1 - \beta(n - \alpha)}{(b - a)^{1 - \beta(n - \alpha)}} \int_a^b (b - s)^{-\beta(n - \alpha)} y(s) \, ds. \tag{3.8}$$

It is easy to check that  $\operatorname{Im} \mathcal{L} = \ker Q$  and  $Q_i^2 = Q_i$ , i = 1, 2. So, we deduce that  $Q^2(u, v) = Q(u, v)$ . This is the expected opportunity to complete the proof. Since (x, y) = (x, y) - Q(x, y) + Q(x, y), we have  $Y = \operatorname{Im} \mathcal{L} + \operatorname{Im} Q$ . Furthermore  $\operatorname{Im} \mathcal{L} \cap \operatorname{Im} Q = \{(0, 0)\}$  leads us to the fact that  $Y = \operatorname{Im} \mathcal{L} \oplus \operatorname{Im} Q$ . Next, we define the operator  $P : X \to X$  with  $P(u, v) = (P_1u, P_2v)$ , where

$$P_1: \mathcal{B} \to \mathcal{B}, \qquad P_1 u(t) \coloneqq \frac{D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a)}{\Gamma(n-(n-\alpha)(1-\beta))} (t-a)^{n-(n-\alpha)(1-\beta)-1}, \tag{3.9}$$

$$P_2: \mathcal{B} \to \mathcal{B}, \qquad P_2 \nu(t) := \frac{D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} \nu(a)}{\Gamma(n-(n-\alpha)(1-\beta))} (t-a)^{n-(n-\alpha)(1-\beta)-1}.$$
(3.10)

The definitions of the operators  $P_i$ , i = 1, 2, immediately give us  $P_i^2 = P_i$ , i = 1, 2, that is,  $P^2(u, v) = P(u, v)$ .

Definitions (3.9) and (3.10) imply that

$$\ker P := \left\{ (u,v) \in X | D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) = 0, D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) = 0 \right\}.$$
(3.11)

Since (u, v) = (u, v) - P(u, v) + P(u, v), we can derive that  $X := \ker P + \ker \mathcal{L}$ . On the other hand, since  $\ker P \cap \ker \mathcal{L} = \{(0, 0)\}$ , we have  $X := \ker P \oplus \ker \mathcal{L}$ . So, according to Definition 2.8, and the fact that  $\ker Q = \operatorname{Im} \mathcal{L}$ , we conclude that

dim ker  $\mathcal{L} := \operatorname{codim} \operatorname{Im} \mathcal{L} = 2.$ 

Now the proof is complete.

In this position, we begin the second step proving the *L*-compactness of the operator  $\mathcal{N}$  defined by (3.4). Let us define the operator  $K_P : \operatorname{Im} \mathcal{L} \to \operatorname{dom} \mathcal{L} \cap \ker P$  by

$$K_P(x, y) := (I_{a^+}^{\alpha} x, I_{a^+}^{\alpha} y).$$
(3.12)

So we directly get that for each  $(x, y) \in \text{Im } \mathcal{L}$ ,

$$\mathcal{L}K_{P}(x,y) := \left( D_{a^{+}}^{\alpha,\beta} I_{a^{+}}^{\alpha} x, D_{a^{+}}^{\alpha,\beta} I_{a^{+}}^{\alpha} y \right) = (x,y).$$
(3.13)

Next, let  $(u, v) \in \text{dom } \mathcal{L} \cap \text{ker } P$ . Therefore, in the identities

$$u(t) = I_{a^{+}}^{\alpha} D_{a^{+}}^{\alpha,\beta} u(t) + c_{0}(t-a)^{-(n-\alpha)(1-\beta)} + c_{1}(t-a)^{1-(n-\alpha)(1-\beta)}$$
$$+ \dots + c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1},$$
$$v(t) = I_{a^{+}}^{\alpha} D_{a^{+}}^{\alpha,\beta} v(t) + d_{0}(t-a)^{-(n-\alpha)(1-\beta)} + d_{1}(t-a)^{1-(n-\alpha)(1-\beta)}$$
$$+ \dots + d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1},$$

all the coefficients  $c_i$  and  $d_i$  vanish for i = 0, 1, ..., n - 1, which yields

$$K_P \mathcal{L}(u, v) := \left( I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} u, I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} v \right) = (u, v).$$
(3.14)

Finally, relying on (3.13) and (3.14), we conclude that  $K_P = (\mathcal{L}_{\text{dom } \mathcal{L} \cap \text{ker } P})^{-1}$ .

Here we prove the *L*-compactness of the operator  $\mathcal{N}$  in the following lemma.

**Lemma 3.2** Let  $\Omega$  be an open bounded subset of X such that dom  $\mathcal{L} \cap \overline{\Omega} \neq \emptyset$ . Then the operator  $\mathcal{N}$  defined by (3.4) is L-compact.

*Proof* General assumption  $(A_2)$  emphasizes that the nonlinearities  $\Phi$  and  $\Psi$  both are continuous, that is, the operator  $\mathcal{N}$  is continuous. So, we immediately come to the conclusion that both operators  $Q\mathcal{N}(\overline{\Omega})$  and  $K_P(I-Q)\mathcal{N}(\overline{\Omega})$  are bounded. To prove the *L*-compactness of the operator  $\mathcal{N}$ , according to the Arzelà–Ascoli theorem, it just remains to prove the equicontinuity of the operator  $K_{P,Q} := K_P(I-Q)\mathcal{N}(\overline{\Omega})$ . Recalling the continuity of  $\Phi$  and  $\Psi$  once again, we get that there exist positive constants  $M_i$ , i = 1, 2, such that for all  $t \in [a, b]$  and  $(u, v) \in \overline{\Omega}$ , we have  $|(I-Q)\mathcal{N}u(t)| \leq M_1$  and  $|(I-Q)\mathcal{N}v(t)| \leq M_2$ . Therefore, for each  $a \leq t_1 < t_2 \leq b$ ,  $(u, v) \in \overline{\Omega}$ , and i = 1, 2, ..., n-1, we have

$$\begin{split} |K_{P,Q}u(t_{2}) - K_{P,Q}u(t_{1})| \\ &:= \left|I_{a^{+}}^{\alpha}u(t_{2}) - I_{a^{+}}^{\alpha}u(t_{2})\right| \\ &= \frac{1}{\Gamma(\alpha)} \left|\int_{a}^{t_{2}}(t_{2} - s)^{\alpha - 1}(I - Q)\mathcal{N}u(s)\,ds - \int_{a}^{t_{1}}(t_{1} - s)^{\alpha - 1}(I - Q)\mathcal{N}u(s)\,ds \right| \\ &\leq \frac{M_{1}}{\Gamma(\alpha)}\int_{a}^{t_{1}}\left\{(t_{2} - s)^{\alpha - 1} - (t_{1} - s)^{\alpha - 1}\right\}\,ds + \int_{t_{1}}^{t_{2}}(t_{2} - s)^{\alpha - 1}\,ds \\ &= \frac{M_{1}}{\Gamma(\alpha + 1)}\left\{(t_{2} - a)^{\alpha} - (t_{1} - a)^{\alpha}\right\} \end{split}$$

and, similarly,

$$|K_{P,Q}\nu(t_2) - K_{P,Q}\nu(t_1)| \le \frac{M_2}{\Gamma(\alpha+1)} \{(t_2 - a)^{\alpha} - (t_1 - a)^{\alpha}\}.$$

In addition,

$$\begin{split} \left| D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} K_{P,Q} u(t_2) - D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} K_{P,Q} u(t_2) \right| \\ &:= \left| D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} I_{a^+}^{\alpha} (I-Q) \mathcal{N} u(t_2) - D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} I_{a^+}^{\alpha} (I-Q) \mathcal{N} u(t_1) \right| \\ &= \left| I_{a^+}^{i-\beta(n-\alpha)} (I-Q) \mathcal{N} u(t_2) - I_{a^+}^{i-\beta(n-\alpha)} (I-Q) \mathcal{N} u(t_1) \right| \end{split}$$

$$\begin{split} &= \frac{1}{\Gamma(i-\beta(n-\alpha))} \\ &\times \left| \int_{a}^{t_{2}} (t_{2}-s)^{i-\beta(n-\alpha)-1} (I-Q) \mathcal{N} u(s) \, ds - \int_{a}^{t_{1}} (t_{1}-s)^{i-\beta(n-\alpha)-1} (I-Q) \mathcal{N} u(s) \, ds \right| \\ &\leq \frac{M_{1}}{\Gamma(i-\beta(n-\alpha))} \\ &\times \int_{a}^{t_{1}} \left\{ (t_{2}-s)^{i-\beta(n-\alpha)-1} - (t_{1}-s)^{i-\beta(n-\alpha)-1} \right\} \, ds + \int_{t_{1}}^{t_{2}} (t_{2}-s)^{i-\beta(n-\alpha)-1} \, ds \\ &= \frac{M_{1}}{\Gamma(i-\beta(n-\alpha)+1)} \left\{ (t_{2}-a)^{i-\beta(n-\alpha)} - (t_{1}-a)^{i-\beta(n-\alpha)} \right\} \end{split}$$

and, similarly,

$$\begin{split} \big| D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} K_{P,Q} \nu(t_2) - D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} K_{P,Q} \nu(t_2) \big| \\ & \leq \frac{M_2}{\Gamma(i-\beta(n-\alpha)+1)} \big\{ (t_2-a)^{i-\beta(n-\alpha)} - (t_1-a)^{i-\beta(n-\alpha)} \big\}. \end{split}$$

At the and, because of the uniform continuity of the functions  $\rho(t) := (t - a)^{\alpha}$  and  $\rho_i(t) := (t - a)^{i-\beta(n-\alpha)}$ , i = 1, 2, ..., n - 1 on [a, b], we can derive that  $K_{P,Q}(\overline{\Omega}) \subset C[a, b]$  and  $D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta}K_{P,Q}(\overline{\Omega}) \subset C[a, b]$  are bounded and equicontinuous. This guarantees that the operator  $K_{P,Q} := K_P(I - Q)\mathcal{N}$  is compact on  $\overline{\Omega}$ , which yields the *L*-compactness of the operator  $\mathcal{N}$ .

*Remark* 3.3 Let  $(u, v) \in X$ . So,

$$||P(u,v)||_{X} = ||(P_{1}u, P_{2}v)||_{X} := \max\{||P_{1}u||_{\mathcal{B}}, ||P_{2}v||_{\mathcal{B}}\}.$$

Equivalently, for each i = 1, 2, ..., n - 1, we have

$$\begin{split} \left\| P(u,v) \right\|_{X} &= \max \{ \max \{ \|P_{1}u\|_{\mathcal{B}}, \left\| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta} P_{1}u \right\|_{\mathcal{B}} \}, \\ &\max \{ \|P_{2}v\|_{\mathcal{B}}, \left\| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta} P_{2}v \right\|_{\mathcal{B}} \} \}. \end{split}$$

Now, using the power rule (2.11) and definitions (3.9) and (3.10), we conclude that

$$\|P(u,v)\|_{X} \le \Lambda_{1} \max\{ \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right|, \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| \},$$
(3.15)

where

$$\Lambda_1 := \frac{\max\{(b-a)^{n-(n-\alpha)(1-\beta)-1}, \frac{\Gamma(n-(n-\alpha)(1-\beta))}{(i-1)!}(b-a)^i, i=1,2,\ldots,n-1\}}{\Gamma(n-(n-\alpha)(1-\beta))}.$$

Similarly, we have

$$\left\|K_{P}(u,v)\right\|_{X} \leq \Lambda_{2} \max\left\{\|u\|_{\mathcal{B}}, \|v\|_{\mathcal{B}}\right\},\tag{3.16}$$

where

$$\Lambda_2 := \max\left\{\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, \frac{(b-a)^{i-\beta(n-\alpha)}}{\Gamma(i-\beta(n-\alpha)+1)}, i=1,2,\ldots,n-1\right\}.$$

Turning to Theorem 2.10, we have just been proved that the operators  $\mathcal{L}$  defined by (3.1) and  $\mathcal{N}$  defined by (3.4) are a Fredholm operator of index 0 and an *L*-compact operator, respectively. In what follows, we are going to identify conditions (i)–(iii) in this theorem. To this aim, we first consider the following hypotheses:

(*H*<sub>1</sub>) There exist positive real constants  $d_i$ ,  $i = 1, 2, b_k, c_k, \theta_k$ , and  $\lambda_k$  with  $\theta_k, \lambda_k \in [0, 1]$  for k = 1, 2, ..., n such that for each  $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ ,

$$\left|\Phi(t,x_1,x_2,\ldots,x_n)\right| \le d_1 + \sum_{k=1}^n b_k |x_k|^{\theta_k}, \quad t \in [a,b],$$
 (3.17)

$$\left|\Psi(t,x_1,x_2,\ldots,x_n)\right| \le d_2 + \sum_{k=1}^n c_k |x_k|^{\lambda_k}, \quad t \in [a,b].$$
 (3.18)

(*H*<sub>2</sub>) For a positive real constant *B*, such that for each  $x_i, y_i \in \mathbb{R}$ , i = 1, 2, ..., n, if  $\min\{|x_n|, |y_n|\} > B$ , then we have either

$$x_n \cdot \Phi(t, y_1, y_2, \dots, y_n) > 0$$
, or  $x_n \cdot \Phi(t, y_1, y_2, \dots, y_n) < 0$ ,  $t \in [a, b]$ , (3.19)

or

$$y_n \cdot \Phi(t, x_1, x_2, \dots, x_n) > 0$$
, or  $y_n \cdot \Phi(t, x_1, x_2, \dots, x_n) < 0$ ,  $t \in [a, b]$ . (3.20)

 $(H_3)$ 

$$(\Lambda_1 \Theta + \Lambda_2) \sum_{k=1}^{n} \xi_k < 1, \qquad \xi := b, c,$$
 (3.21)

$$\Lambda_1 \Theta \sum_{k=1}^n b_k + \Lambda_2 \sum_{k=1}^n c_k < 1,$$
(3.22)

$$\Lambda_1 \Theta \sum_{k=1}^n c_k + \Lambda_2 \sum_{k=1}^n b_k < 1,$$
(3.23)

where  $\Theta$  is given by (3.28).

Here we define

$$\Omega_1 = \left\{ (u, v) \in \operatorname{dom} \mathcal{L} \setminus \ker \mathcal{L} | \mathcal{L}(u, v) = \lambda \mathcal{N}(u, v), \lambda \in [0, 1] \right\},$$
(3.24a)

$$\Omega_2 = \{(u, v) \in \ker \mathcal{L} | \mathcal{N}(u, v) \in \operatorname{Im} \mathcal{L} \},$$
(3.24b)

$$\Omega_3 = \{(u, v) \in \ker \mathcal{L} | \lambda(u, v) + (1 - \lambda) Q \mathcal{N}(u, v) = (0, 0), \lambda \in [0, 1] \},$$
(3.24c)

$$\Omega'_{3} = \{(u, v) \in \ker \mathcal{L} | -\lambda(u, v) + (1 - \lambda)Q\mathcal{N}(u, v) = (0, 0), \lambda \in [0, 1]\}.$$
(3.24d)

Indeed, if we prove that the subsets  $\overline{\Omega_i} \subset X$ , i = 1, 2, 3, and  $\overline{\Omega'_3} \subset X$  are bounded, then relying on Theorem 2.10, we can directly conclude the existence of at least one solution for the Hilfer fractional resonant system (1.1)–(1.2). So, we start to prove the boundedness of these sets.

**Lemma 3.4**  $\overline{\Omega_1}$  defined by (3.24a) is bounded subset of X.

*Proof* For each  $(u, v) \in \overline{\Omega_1}$ ,  $(u, v) \notin \ker \mathcal{L}$ . So,  $\lambda \neq 0$ . Also, for each  $(u, v) \in \overline{\Omega_1}$ , we have  $\mathcal{L}(u, v) = \lambda \mathcal{N}(u, v) \in \ker Q$ , that is,

$$\begin{aligned} &\frac{\lambda(1-\beta(n-\alpha))}{(b-a)^{1-\beta(n-\alpha)}} \\ &\times \int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \Phi\left(s, \nu, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}\nu, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu\right) ds = 0, \\ &\frac{\lambda(1-\beta(n-\alpha))}{(b-a)^{1-\beta(n-\alpha)}} \\ &\times \int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \Psi\left(s, u, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u\right) ds = 0. \end{aligned}$$

Thus there are  $t_0, t_1 \in [a, b]$  such that

$$\Phi(t_{1}, \nu, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}\nu, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}\nu, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu) = 0,$$

$$\Psi(t_{0}, u, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u) = 0.$$
(3.25)

In viewpoint of hypothesis  $(H_2)$ , equalities (3.25) imply that

$$D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta}u(t_0) \le B, \qquad D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta}v(t_1) \le B.$$

For  $(u, v) \in \overline{\Omega_1}$ , we have that  $\mathcal{L}(u, v) = \lambda \mathcal{N}(u, v)$ , that is,

$$D_{a^+}^{\alpha,\beta}u(t) = \lambda \mathcal{N}_1 v(t), \qquad D_{a^+}^{\alpha,\beta}v(t) = \lambda \mathcal{N}_2 u(t).$$

So, we have

$$\begin{split} u(t) &= \lambda I_{a^+}^{\alpha} \mathcal{N}_1 v(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} D_{a^+}^{k-(n-\alpha)(1-\beta)} u(a), \\ v(t) &= \lambda I_{a^+}^{\alpha} \mathcal{N}_2 u(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} D_{a^+}^{k-(n-\alpha)(1-\beta)} v(a). \end{split}$$

Now taking  $D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta}$  on both sides of these equalities, we get

$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(t) = \lambda I_{a^{+}}^{1-\beta(n-\alpha)}\mathcal{N}_{1}v(t) + D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(a),$$
  
$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(t) = \lambda I_{a^{+}}^{1-\beta(n-\alpha)}\mathcal{N}_{2}u(t) + D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(a).$$

Setting  $t = t_0$  in the first equality and  $t = t_1$  in the second equality gives us

$$\begin{split} \left| D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| &\leq B + \lambda \frac{(b-a)^{1-\beta(n-\alpha)}}{\Gamma(2-\beta(n-\alpha))} \|\Phi\|, \\ \left| D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| &\leq B + \lambda \frac{(b-a)^{1-\beta(n-\alpha)}}{\Gamma(2-\beta(n-\alpha))} \|\Psi\|. \end{split}$$

Applying hypothesis  $(H_1)$  to these inequalities, we get

$$\left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| \le B + \lambda \Theta \left( d_1 + b_1 \|v\|^{\theta_1} + \sum_{k=2}^n b_k \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1,\beta} v\|^{\theta_k} \right), \quad (3.26)$$

$$\left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} \nu(a) \right| \le B + \lambda \Theta \left( d_2 + c_1 \|u\|^{\lambda_1} + \sum_{k=2}^n c_k \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1,\beta} u\|^{\lambda_k} \right), \quad (3.27)$$

where

$$\Theta := \frac{(b-a)^{1-\beta(n-\alpha)}}{\Gamma(2-\beta(n-\alpha))}.$$
(3.28)

Let  $(u, v) \in \Omega_1$ . Then  $(u, v) \in \text{dom } \mathcal{L} \setminus \text{ker } \mathcal{L}$ . Since  $P^2 = P$ ,  $(I - P)(u, v) \in \text{dom } \mathcal{L} \cap \text{ker } P$  and  $\mathcal{L}P(u, v) = (0, 0)$ . So, by inequality (3.16) it follows that

$$\| (I - P)(u, v) \|_{X} = \| K_{P} \mathcal{L} (I - P)(u, v) \|_{X} = \| K_{P} (\mathcal{L}_{1} u, \mathcal{L}_{2} v) \|_{X}$$
  
$$= \| (I_{a^{+}}^{\alpha} \mathcal{L}_{1} u, I_{a^{+}}^{\alpha} \mathcal{L}_{2} v) \|_{X} \le \lambda \Lambda_{2} \max \{ \| \mathcal{N}_{1} v \|_{\mathcal{B}}, \| \mathcal{N}_{2} u \|_{\mathcal{B}} \}$$
  
$$\le \Lambda_{2} \max \{ \| \mathcal{N}_{1} v \|_{\mathcal{B}}, \| \mathcal{N}_{2} u \|_{\mathcal{B}} \}.$$
(3.29)

Accordingly, based on Remark 3.3 and (3.29), we have

$$\begin{split} \left\| (u,v) \right\|_{X} &= \left\| P(u,v) + (I-P)(u,v) \right\|_{X} \leq \left\| P(u,v) \right\|_{X} + \left\| (I-P)(u,v) \right\|_{X} \\ &\leq \max \left\{ \left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{1} v \right\|_{\mathcal{B}} \right\}, \\ &\left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{2} u \right\|_{\mathcal{B}} \right\}, \\ &\left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{2} u \right\|_{\mathcal{B}} \right\}, \\ &\left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{1} v \right\|_{\mathcal{B}} \right\} \right\}. \end{split}$$
(3.30)

To complete the proof, since the right-hand side of inequality (3.30) depends on four cases, we will divide our boundedness estimation into four cases as follows:

Case i.

$$\|(u,v)\|_{X} \le \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| + \Lambda_{2} \|\mathcal{N}_{1}v\|_{\mathcal{B}}.$$
(3.31)

So, properties (3.17) and (3.18) in hypothesis  $(H_1)$  and (3.26)–(3.28) yield

$$\|(u,v)\|_{X} \leq \Lambda_{1}B + (\Lambda_{1}\Theta + \Lambda_{2}) \left( d_{1} + b_{1} \|v\|_{\mathcal{B}}^{\theta_{1}} + \sum_{k=2}^{n} b_{k} \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1}v\|_{\mathcal{B}}^{\theta_{k}} \right).$$
(3.32)

Case ii.

$$\|(u,v)\|_{X} \leq \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| + \Lambda_{2} \|\mathcal{N}_{2}u\|_{\mathcal{B}}.$$
(3.33)

The resources of case i imply that

$$\|(u,v)\|_{\chi} \le \Lambda_1 B + (\Lambda_1 \Theta + \Lambda_2) \left( d_2 + c_1 \|u\|_{\mathcal{B}}^{\lambda_1} + \sum_{k=2}^n c_k \|D_{a^+}^{k-(n-\alpha)(1-\beta)-1}u\|_{\mathcal{B}}^{\lambda_k} \right).$$
(3.34)

Case iii.

$$\|(u,v)\|_{X} \leq \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| + \Lambda_{2} \|\mathcal{N}_{2}u\|_{\mathcal{B}}.$$
(3.35)

We have

$$\|(u,v)\|_{X} \leq \Lambda_{1} \Theta \left\{ B + \left( d_{1} + b_{1} \|v\|_{\mathcal{B}}^{\theta_{1}} + \sum_{k=2}^{n} b_{k} \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1}v\|_{\mathcal{B}}^{\theta_{k}} \right) \right\} + \Lambda_{2} \left( d_{2} + c_{1} \|u\|_{\mathcal{B}}^{\lambda_{1}} + \sum_{k=2}^{n} c_{k} \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1}u\|_{\mathcal{B}}^{\lambda_{k}} \right).$$
(3.36)

Case iv.

$$\|(u,v)\|_{X} \leq \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| + \Lambda_{2} \|\mathcal{N}_{1}v\|_{\mathcal{B}}.$$
(3.37)

We get that

$$\|(u,v)\|_{X} \leq \Lambda_{1} \Theta \left\{ B + \left( d_{2} + c_{1} \|u\|_{\mathcal{B}}^{\lambda_{1}} + \sum_{k=2}^{n} c_{k} \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1}u\|_{\mathcal{B}}^{\lambda_{k}} \right) \right\} + \Lambda_{2} \left( d_{1} + b_{1} \|v\|_{\mathcal{B}}^{\theta_{1}} + \sum_{k=2}^{n} b_{k} \|D_{a^{+}}^{k-(n-\alpha)(1-\beta)-1}v\|_{\mathcal{B}}^{\theta_{k}} \right).$$
(3.38)

Some simple manipulations on inequalities (3.32), (3.34), (3.36), and (3.38), help us to reach the following results.

(i)

$$\left\| (u,v) \right\|_{X} \leq \frac{\Lambda_{1}B + d_{1}(\Lambda_{1}\Theta + \Lambda_{2})}{1 - (\Lambda_{1}\Theta + \Lambda_{2})\sum_{k=1}^{n} b_{k}}.$$
(3.39)

(ii)

$$\left\| (u,v) \right\|_{X} \le \frac{\Lambda_{1}B + d_{2}(\Lambda_{1}\Theta + \Lambda_{2})}{1 - (\Lambda_{1}\Theta + \Lambda_{2})\sum_{k=1}^{n} c_{k}}.$$
(3.40)

(iii)

$$\|(u,v)\|_{X} \leq \frac{\Lambda_{1}B + (\Lambda_{1}\Theta d_{1} + \Lambda_{2}d_{2})}{1 - \{\Lambda_{1}\Theta \sum_{k=1}^{n} b_{k} + \Lambda_{2} \sum_{i=k}^{n} c_{k}\}}.$$
(3.41)

(iv)

$$\|(u,v)\|_{X} \leq \frac{\Lambda_{1}B + (\Lambda_{1}\Theta d_{2} + \Lambda_{2}d_{1})}{1 - \{\Lambda_{1}\Theta \sum_{k=1}^{n} c_{k} + \Lambda_{2} \sum_{k=1}^{n} b_{k}\}}.$$
(3.42)

Inequalities (3.39)–(3.42) ensure that  $\overline{\Omega_1} \subset X$  is bounded.

**Lemma 3.5**  $\overline{\Omega_2}$  defined by (3.24b) is a bounded subset of *X*.

*Proof* Let  $(u, v) \in \overline{\Omega_2}$ . Then  $u(t) := c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}$  and  $v(t) := d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}$ . On the other hand,

$$\mathcal{N}(u, v) = (\mathcal{N}_1 v, \mathcal{N}_2 u) \in \operatorname{Im} \mathcal{L} = \ker Q.$$

So, we have

$$\int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \times \Phi\left(s, c_{n-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}, c_{n-1}D_{a^{+}}^{1-(n-\alpha)(1-\beta)}(s-a)^{n-(n-\alpha)(1-\beta)-1}, \dots, c_{n-1}D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}\right) ds = 0$$
(3.43)

and

$$\int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \times \Psi(s, d_{n-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}, d_{n-1}D_{a^{+}}^{1-(n-\alpha)(1-\beta)}(s-a)^{n-(n-\alpha)(1-\beta)-1}, \dots, d_{n-1}D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}) \, ds = 0.$$
(3.44)

Now, according to hypothesis  $(H_2)$ , we come to the conclusion that

$$|c_{n-1}|, |d_{n-1}| \le \frac{B}{\Gamma(n-(n-\alpha)(1-\beta))},$$

that is,  $\overline{\Omega_2}$  is a bounded subset of *X*.

The next case deals with the boundedness of the subset  $\overline{\Omega_3}$ .

# **Lemma 3.6** $\overline{\Omega_3}$ defined by (3.24c) is a bounded subset of *X*.

*Proof* Our proof basically depends on hypothesis  $(H_2)$ . Let  $(u, v) \in \overline{\Omega_3}$ . Thus  $(u, v) \in \ker \mathcal{L}$ , that is,

$$(u,v) = (c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}, d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1}).$$

On the other hand, since ker  $\mathcal{L} = \text{Im } Q$ , from  $\lambda(u, v) - (1 - \lambda)Q\mathcal{N}(u, v) = (0, 0)$  we deduce that

$$\lambda c_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1} + (1-\lambda) \int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \\ \times \Phi\left(s, c_{n-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}, c_{n-1}D_{a^{+}}^{1-(n-\alpha)(1-\beta)}(s-a)^{n-(n-\alpha)(1-\beta)-1}, \\ \dots, c_{n-1}D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}\right) ds = 0$$
(3.45)

and

$$\lambda d_{n-1}(t-a)^{n-(n-\alpha)(1-\beta)-1} + (1-\lambda) \int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \\ \times \Psi \left( s, d_{n-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}, d_{n-1} D_{a^{+}}^{1-(n-\alpha)(1-\beta)}(s-a)^{n-(n-\alpha)(1-\beta)-1}, \\ \dots, d_{n-1} D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1}(s-a)^{n-(n-\alpha)(1-\beta)-1} \right) ds = 0.$$
(3.46)

If in equalities (3.45) and (3.46), we take  $\lambda = 0$ , then the proof reduces to the proof of Lemma 3.5. So, let  $\lambda \in (0, 1]$ . Similarly to the previous case, this case gives us the desired outcome. The assumption  $|c_{n-1}|, |d_{n-1}| > B$ , together with the hypothesis ( $H_2$ ), leads us to

$$\begin{split} \lambda c_{n-1}^2 (t-a)^{2(n-(n-\alpha)(1-\beta)-1)} + (1-\lambda)c_{n-1} \int_a^b (b-s)^{-\beta(n-\alpha)} (s-a)^{n-(n-\alpha)(1-\beta)-1} \\ &\times \Phi \left( s, c_{n-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}, c_{n-1} D_{a^+}^{1-(n-\alpha)(1-\beta)} (s-a)^{n-(n-\alpha)(1-\beta)-1} \right) \\ &\dots, c_{n-1} D_{a^+}^{n-(n-\alpha)(1-\beta)-1} (s-a)^{n-(n-\alpha)(1-\beta)-1} \right) ds > 0 \end{split}$$

and

$$\begin{split} \lambda d_{n-1}^2 (t-a)^{2(n-(n-\alpha)(1-\beta)-1)} + (1-\lambda)d_{n-1} \int_a^b (b-s)^{-\beta(n-\alpha)} (s-a)^{n-(n-\alpha)(1-\beta)-1} \\ & \times \Psi \big( s, d_{n-1}(s-a)^{n-(n-\alpha)(1-\beta)-1}, d_{n-1} D_{a^+}^{1-(n-\alpha)(1-\beta)} (s-a)^{n-(n-\alpha)(1-\beta)-1}, \\ & \dots, d_{n-1} D_{a^+}^{n-(n-\alpha)(1-\beta)-1} (s-a)^{n-(n-\alpha)(1-\beta)-1} \big) \, ds > 0, \end{split}$$

which contradicts equalities (3.45) and (3.46). This completes the proof.

Concentrating on the second parts of hypothesis ( $H_2$ ), that is, the right-hand side inequalities in (3.19) and (3.20), helps us to prove boundedness of  $\overline{\Omega'_3} \subset X$ . So, we present this result without proof in the next lemma.

# **Lemma 3.7** $\overline{\Omega'_3}$ defined by (3.24d) is a bounded subset of *X*.

Since all conditions of Theorem 2.10 are satisfied in the frame of Lemmas 3.1, 3.2, and (3.4)-(3.7), we are ready to prove the existence of at least one solution of the Hilfer fractional resonant system (1.1)-(1.2) as follows.

**Theorem 3.8** Assume that hypotheses  $(H_1)-(H_3)$  are satisfied. Then the Hilfer fractional resonant system (1.1)–(1.2) has at least one solution in X.

*Proof* Consider  $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_{i}} \cup \{0\}$  (or  $\Omega \supset \bigcup_{i=1}^{2} \overline{\Omega_{i}} \cup \Omega'_{3} \cup \{0\}$ ) as a bounded open subset of *X*. According to the Lemma 3.2, we know that  $\mathcal{N}$  is an *L*-compact operator on  $\Omega$ . Besides, via Lemmas 3.4–3.7, we get the following:

- $\mathcal{L}(u, v) = \lambda \mathcal{N}(u, v)$  for every  $((u, v), \lambda) \in [\operatorname{dom} \mathcal{L} \setminus \ker \mathcal{L} \cap \partial \Omega] \times (0, 1)$ .
- ►  $\mathcal{N}(u, v) \notin \operatorname{Im} \mathcal{L}$  for every  $(u, v) \in \ker \mathcal{L} \cap \partial \Omega$ .

So, it just remains to prove that

$$\deg(JQ\mathcal{N}|_{\ker \mathcal{L}}, \Omega \cap \ker \mathcal{L}, 0) \neq 0.$$

To prove this, we define the homotopy

$$H((u, v), \lambda) = \pm \lambda \operatorname{Id}(u, v) + (1 - \lambda)JQ\mathcal{N}(u, v).$$

According to the degree property of invariance under a homotopy, if  $u \in \ker \mathcal{L} \cap \partial \Omega$ , then

$$deg(JQ\mathcal{N}|_{\ker\mathcal{L}}, \Omega \cap \ker\mathcal{L}, 0)$$
  
= deg(H(\cdot, 0), \Omega \cap \ker\mathcal{L}, 0)  
= deg(H(\cdot, 1), \Omega \cap \ker\mathcal{L}, 0)  
= deg(\pm Id, \Omega \cap \ker\mathcal{L}, 0) \neq 0.

Therefore, since assumption (iii) in Theorem 2.10 is satisfied, we conclude that the Hilfer fractional resonant system (1.1)-(1.2) has at least one solution in *X*.

As explained before, our investigation is divided into the couple of stages including the existence and uniqueness of solutions for the Hilfer fractional resonant system (1.1)-(1.2). So far, the existence of at least one solution is proved. So, in what follows, we state and prove a uniqueness criterion for the solutions of the coupled system (1.1)-(1.2). To do this, we first need the following hypotheses.

 $(H'_1)$  There exist positive constants  $(\mu_i, \nu_i) \in \mathbb{R}^2$ , i = 1, ..., n, and  $(\eta_i, \zeta_i) \in [0, 1]^2$ , i = 1, 2, ..., n, such that for all  $((x_i)_1^n, (y_i)_1^n) \in \mathbb{R}^n \times \mathbb{R}^n$ ,

$$\left|\Phi(t, x_1, x_2, \dots, x_n) - \Phi(t, y_1, y_2, \dots, y_n)\right| \le \sum_{i=1}^n \mu_i |x_i - y_i|^{\eta_i}$$
 (3.47)

and

$$|\Psi(t, x_1, x_2, \dots, x_n) - \Psi(t, y_1, y_2, \dots, y_n)| \le \sum_{i=1}^n \nu_i |x_i - y_i|^{\zeta_i}.$$
 (3.48)

(*H*<sub>2</sub>) There exist positive constants  $(k_i, l_i) \in \mathbb{R}^2$ , i = 1, ..., n, such that for all  $((x_i)_1^n, (y_i)_1^n) \in \mathbb{R}^n \times \mathbb{R}^n$ ,

$$\left|\Phi(t, x_1, x_2, \dots, x_n) - \Phi(t, y_1, y_2, \dots, y_n)\right| \ge k_n |x_n - y_n| - \sum_{i=1}^{n-1} k_i |x_i - y_i|$$
 (3.49)

and

$$\left|\Psi(t,x_1,x_2,\ldots,x_n) - \Psi(t,y_1,y_2,\ldots,y_n)\right| \ge l_n |x_n - y_n| - \sum_{i=1}^{n-1} l_i |x_i - y_i|.$$
 (3.50)

 $(H'_{3})$ 

$$\Lambda_1 \left\{ \sum_{i=1}^{n-1} \frac{l_i}{l_n} + \Theta \sum_{i=1}^n \mu_i \right\} + \Lambda_2 \sum_{i=1}^n b_i > 1,$$
(3.51)

$$\Lambda_1 \left\{ \sum_{i=1}^{n-1} \frac{k_i}{k_n} + \Theta \sum_{i=1}^n \nu_i \right\} + \Lambda_2 \sum_{i=1}^n c_i > 1,$$
(3.52)

$$\Lambda_1 \left\{ \sum_{i=1}^{n-1} \frac{l_i}{l_n} + \Theta \sum_{i=1}^n \mu_i \right\} + \Lambda_2 \sum_{i=1}^n c_i > 1,$$
(3.53)

$$\Lambda_1 \left\{ \sum_{i=1}^{n-1} \frac{k_i}{k_n} + \Theta \sum_{i=1}^n \nu_i \right\} + \Lambda_2 \sum_{i=1}^n b_i > 1.$$
(3.54)

**Theorem 3.9** Assume that hypotheses  $(H_2)-(H_3)$  and  $(H'_1)-(H'_3)$  are satisfied. Then the Hilfer fractional resonant system (1.1)-(1.2) has a unique solution on X.

*Proof* To prove the uniqueness of solutions, we first will prove the existence of solutions. Since hypotheses  $(H_2)-(H_3)$  are satisfied, we just need to check hypothesis  $(H_1)$ . To do this, for each i = 1, 2, ..., n, assume that  $y_i = 0$ . Also suppose that

$$d_1 := \Phi(t, \underbrace{0, 0, \dots, 0}_{n-1 \text{ tupled}}), \qquad d_2 := \Psi(t, \underbrace{0, 0, \dots, 0}_{n-1 \text{ tupled}}).$$
(3.55)

In this case, hypothesis  $(H'_1)$  reduces to hypothesis  $(H_1)$ . Having the hypotheses  $(H_1)-(H_3)$  in hand, Theorem 3.8 guarantees the existence of at least one solution for the Hilfer fractional resonant system (1.1)–(1.2). Now it is time to prove the uniqueness of solutions. In the traditional way, we consider the solutions  $(u_k, v_k) \in X$ , k = 1, 2, of the coupled system (1.1)–(1.2) and will prove that  $u_1 = u_2$  and  $v_1 = v_2$ . As supposed, for k = 1, 2,

$$D_{a^{+}}^{\alpha,\beta}u_{k}(t) = \Phi(t, v_{k}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}v_{k}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}v_{k}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v_{k}),$$
  
$$D_{a^{+}}^{\alpha,\beta}v_{k}(t) = \Psi(t, u_{k}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{k}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{k}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{k}).$$

Let us consider  $u := u_1 - u_2$  and  $v := v_1 - v_2$ . It follows that

$$D_{a^{+}}^{\alpha,\beta}u(t) = \Phi(t,\nu_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}\nu_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}\nu_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu_{1}) - \Phi(t,\nu_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}\nu_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}\nu_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu_{2})$$
(3.56)

and

$$D_{a^{+}}^{\alpha,\beta}\nu(t) = \Psi\left(t, u_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{1}\right) - \Psi\left(t, u_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{2}\right).$$
(3.57)

Since  $\operatorname{Im} \mathcal{L} = \ker Q$ , we get that

$$\int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \times \left\{ \Phi\left(s, v_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta} v_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta} v_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v_{1} \right) - \Phi\left(s, v_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta} v_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta} v_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v_{2} \right) \right\} ds = 0$$
(3.58)

and

$$\int_{a}^{b} (b-s)^{-\beta(n-\alpha)} \times \left\{ \Psi(s, u_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta} u_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta} u_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u_{1}) - \Psi(s, u_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta} u_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta} u_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u_{2}) \right\} ds = 0.$$
(3.59)

Equalities (3.58) and (3.59) imply that there are  $t_2, t_3 \in [a, b]$  such that

$$\Phi(t_{2}, \nu_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}\nu_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}\nu_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu_{1})$$

$$= \Phi(t_{2}, \nu_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}\nu_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}\nu_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu_{2})$$
(3.60)

and

$$\Psi(t_{3}, u_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{1})$$
  
=  $\Psi(t_{3}, u_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{2}).$  (3.61)

So, in view of hypothesis  $(H'_2)$ , we get the following inequality:

$$0 = \left| \Phi\left(t_{2}, v_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta} v_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta} v_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v_{1}\right) - \Phi\left(t_{2}, v_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta} v_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta} v_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v_{2}\right) \right|$$
  
$$\geq k_{n} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(t_{2}) \right| - k_{1} \left| v(t_{2}) \right| - \sum_{i=2}^{n-1} k_{i} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta} v(t_{2}) \right|.$$

Therefore this inequality gives us

$$\left| D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} v(t_2) \right| \le \frac{k_1}{k_n} \left| v(t_2) \right| + \sum_{i=2}^{n-1} \frac{k_i}{k_n} \left| D_{a^+}^{n-(n-\alpha)(1-\beta)-i,\beta} v(t_2) \right|,$$

that is,

$$\left|D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}\nu(t_{2})\right| \leq \frac{k_{1}}{k_{n}}\left\|\nu(t_{2})\right\| + \sum_{i=2}^{n-1}\frac{k_{i}}{k_{n}}\left\|D_{a^{+}}^{n-(n-\alpha)(1-\beta)-i,\beta}\nu(t_{2})\right\|.$$

From this inequality we immediately derive that

$$\left| D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} \nu(t_2) \right| \le \sum_{i=1}^{n-1} \frac{k_i}{k_n} \|\nu\|_{\mathcal{B}}.$$
(3.62)

Similarly, we can obtain

$$\left| D_{a^+}^{n-(n-\alpha)(1-\beta)-1,\beta} u(t_3) \right| \le \sum_{i=1}^{n-1} \frac{l_i}{l_n} \| u \|_{\mathcal{B}}.$$
(3.63)

Let us reconsider equalities (3.56) and (3.57). Accordingly, we have

$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(t) = I_{a^{+}}^{1-\beta(n-\alpha)} \left\{ \Phi(t_{2},v_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}v_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}v_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v_{1}) - \Phi(t_{2},v_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}v_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}v_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v_{2}) \right\} + D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u(a)$$
(3.64)

and

$$D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(t) = I_{a^{+}}^{1-\beta(n-\alpha)} \left\{ \Psi(t_{3}, u_{1}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{1}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{1}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{1}) - \Psi(t_{3}, u_{2}, D_{a^{+}}^{1-(n-\alpha)(1-\beta),\beta}u_{2}, D_{a^{+}}^{2-(n-\alpha)(1-\beta),\beta}u_{2}, \dots, D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}u_{2}) \right\} + D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta}v(a).$$
(3.65)

If we take  $t = t_3$  in (3.64) and  $t = t_2$  in (3.65), then (3.28), together with inequalities (3.62) and (3.63), yields

$$\left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| \le \sum_{i=1}^{n-1} \frac{l_i}{l_n} \| u \|_{\mathcal{B}} + \Theta \sum_{i=1}^n \mu_i \| v \|_{\mathcal{B}},$$
(3.66)

$$\left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} \nu(a) \right| \le \sum_{i=1}^{n-1} \frac{k_i}{k_n} \|\nu\|_{\mathcal{B}} + \Theta \sum_{i=1}^n \nu_i \|u\|_{\mathcal{B}}.$$
(3.67)

Here we recall once again inequality (3.30):

$$\begin{split} \left\| (u,v) \right\|_{X} &= \left\| P(u,v) + (I-P)(u,v) \right\|_{X} \le \left\| P(u,v) \right\|_{X} + \left\| (I-P)(u,v) \right\|_{X} \\ &\le \max \left\{ \left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{1} v \right\|_{\mathcal{B}} \right\}, \\ &\left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{2} u \right\|_{\mathcal{B}} \right\}, \\ &\left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} u(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{2} u \right\|_{\mathcal{B}} \right\}, \\ &\left\{ \Lambda_{1} \left| D_{a^{+}}^{n-(n-\alpha)(1-\beta)-1,\beta} v(a) \right| + \Lambda_{2} \left\| \mathcal{N}_{1} v \right\|_{\mathcal{B}} \right\} \right\}. \end{split}$$

Similarly to case i, (3.32), case ii, (3.34), case iii, (3.36), and case iv, (3.38), we conclude that

$$\|(u,v)\|_{X} \leq \frac{\Lambda_{2}d_{1}}{1 - [\Lambda_{1}\{\sum_{i=1}^{n-1}\frac{l_{i}}{l_{n}} + \Theta\sum_{i=1}^{n}\mu_{i}\} + \Lambda_{2}\sum_{i=1}^{n}b_{i}]},$$
(3.68)

$$\|(u,v)\|_{X} \leq \frac{\Lambda_{2}d_{2}}{1 - [\Lambda_{1}\{\sum_{i=1}^{n-1}\frac{k_{i}}{k_{n}} + \Theta\sum_{i=1}^{n}v_{i}\} + \Lambda_{2}\sum_{i=1}^{n}c_{i}]},$$
(3.69)

$$\|(u,v)\|_{X} \leq \frac{\Lambda_{2}d_{2}}{1 - [\Lambda_{1}\{\sum_{i=1}^{n-1}\frac{l_{i}}{l_{n}} + \Theta\sum_{i=1}^{n}\mu_{i}\} + \Lambda_{2}\sum_{i=1}^{n}c_{i}]},$$
(3.70)

and

$$\|(u,v)\|_{X} \leq \frac{\Lambda_{2}d_{1}}{1 - [\Lambda_{1}\{\sum_{i=1}^{n-1}\frac{k_{i}}{k_{n}} + \Theta\sum_{i=1}^{n}v_{i}\} + \Lambda_{2}\sum_{i=1}^{n}b_{i}]}.$$
(3.71)

Finally, if we impose conditions (3.51)-(3.54) into inequalities (3.68)-(3.71), respectively, then it follows that u = v = 0. More precisely, we have proven that  $u_1 = u_2$  and  $v_1 = v_2$ . This completes the proof of uniqueness of the solutions of the Hilfer fractional resonant system (1.1)-(1.2).

## 4 An application

In this section, we present an application to illustrate the obtained theoretical existence and uniqueness criteria in the frame of Theorems 3.8 and 3.9.

*Example* 4.1 Let us consider the Hilfer fractional resonant system

$$\begin{cases} D_{0^+}^{\frac{5}{2},\frac{1}{2}} u(t) = \Phi(t, v, D_{0^+}^{\frac{1}{2},\frac{1}{2}} v, D_{0^+}^{\frac{3}{2},\frac{1}{2}} v), & 0 < t < 1, \\ D_{0^+}^{\frac{5}{2},\frac{1}{2}} v(t) = \Psi(t, u, D_{0^+}^{\frac{1}{2},\frac{1}{2}} u, D_{0^+}^{\frac{3}{2},\frac{1}{2}} u), & 0 < t < 1, \end{cases}$$

$$\tag{4.1}$$

with nonlinearities

$$\Phi(t, x_1, x_2, x_3) := \frac{t + |x_1| + |x_2| + |x_3|}{10},$$
  

$$\Psi(t, x_1, x_2, x_3) := \frac{t + |\sin(x_1)| + |\sin(x_2)| + |\sin(x_3)|}{10},$$
(4.2)

subject to the boundary conditions

$$\begin{cases} I_{0^+}^{0.25,0.5}u(0) = 0, & D_{0^+}^{0.75,0.5}u(0) = 0, & D_{0^+}^{1.75,0.5}u(0) = D_{0^+}^{1.75,0.5}u(1), \\ I_{0^+}^{0.25,0.5}v(0) = 0, & D_{0^+}^{0.75,0.5}v(0) = 0, & D_{0^+}^{1.75,0.5}v(0) = D_{0^+}^{1.75,0.5}v(1). \end{cases}$$
(4.3)

Indeed, in the Hilfer fractional resonant system (1.1)-(1.2), we impose the following setting to reach the fractional resonant system (4.1)-(4.3):

$$\alpha := 2.5, \qquad \beta := 0.5, \qquad n = 3, \qquad a := 0, \qquad b := 1$$

By a direct calculation we get that

$$\Lambda_1 := 1, \qquad \Lambda_2 = \Theta := 0.100946$$

Finally, taking B := 1,  $\eta_i = \zeta_i := 1$ , i = 1, 2, 3, and  $k_i = l_i := 0.1$ , i = 1, 2, 3, it is easy to check that the hypotheses  $(H_2)-(H_3)$  and  $(H'_1)-(H'_3)$  are satisfied. So, the Hilfer fractional resonant system (4.1)-(4.3) has a unique solution on X.

## 5 Discussion and concluding remarks

In this paper, we studied the higher-order Hilfer fractional resonant system (1.1)-(1.2). Our aim in this investigation was to apply the coincidence degree theory to obtain at least one solution for the resonant system (1.1)-(1.2). Besides, having certain conditions on the nonlinearities, we presented a uniqueness criterion. One of the advantages of this investigation is that, to the best of our knowledge, this is the first time in the literature that the Hilfer fractional differential equations have been considered to establish the fractional resonant problems. At this position, we discuss other advantages of the Hilfer fractional derivatives. As mentioned in Definition 2.4, taking  $\beta = 0$  gives us the Riemann–Liouville fractional resonant system

$$\begin{cases} D_{a^{+}}^{\alpha} u(t) = \Phi(t, v, D_{a^{+}}^{\alpha-n+1}v, D_{a^{+}}^{\alpha-n+2}v, \dots, D_{a^{+}}^{\alpha-1}v), & a < t < b, \\ D_{a^{+}}^{\alpha} v(t) = \Psi(t, u, D_{a^{+}}^{\alpha-n+1}u, D_{a^{+}}^{\alpha-n+2}u, \dots, D_{a^{+}}^{\alpha-1}u), & a < t < b, \end{cases}$$
(5.1)

subject to the boundary conditions

$$\begin{cases} I_{a^+}^{n-\alpha}u(a) = 0, & D_{a^+}^{\alpha-i}u(a) = 0, \\ D_{a^+}^{\alpha-1}u(a) = D_{a^+}^{\alpha-1}u(b), & i = 2, 3, \dots, n-1, \\ I_{a^+}^{n-\alpha}v(a) = 0, & D_{a^+}^{\alpha-i}v(a) = 0, \\ D_{a^+}^{\alpha-1}v(a) = D_{a^+}^{\alpha-1}v(b), & i = 2, 3, \dots, n-1, \end{cases}$$
(5.2)

which generalizes all the Riemann–Liouville-based fractional resonant problems with boundary conditions of the form (5.2).

Also, taking  $\beta := 1$ , we get the following Caputo fractional resonant system:

$${}^{c}D_{a^{+}}^{\alpha}u(t) = \Phi(t, v, v', v'', \dots, v^{(n-1)}), \quad a < t < b,$$

$${}^{c}D_{a^{+}}^{\alpha}v(t) = \Psi(t, u, u', u'', \dots, u^{(n-1)}), \quad a < t < b,$$

$$(5.3)$$

subject to the boundary conditions

$$\begin{cases} u(a) = u'(a) = u''(a) = \dots = u^{(n-2)}(a) = 0, & u^{(n-1)}(a) = u^{(n-1)}(b), \\ v(a) = v'(a) = v''(a) = \dots = v^{(n-2)}(a) = 0, & v^{(n-1)}(a) = v^{(n-1)}(b), \end{cases}$$
(5.4)

which coincides with the main problem in [19], that is, (1.7)-(1.8), and it generalizes [20] and [39].

We believe that due to the unifying characteristics of the Hilfer fractional derivatives, differential equations equipped with this generalized derivatives have much more potential to reach new results both in the theory and in the applications of the fractional-order problems. For instance, as a future research works, one may consider the half-linear Hilfer fractional differential equations and try to extract the corresponding Lyapunov-type inequalities to describe the qualitative dynamics of these problems involving stability, disconjugacy, nonexistence, algebraic properties of nontrivial solutions, and so on. Furthermore, applicability of the half-linear dynamical systems in the porous medium opens a new research line to find more applications of the Hilfer fractional dynamical systems.

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#### **Competing interests**

The author declares that he has no competing interests regarding the publication of this paper.

## Authors' contributions

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