(2020) 2020:480

# RESEARCH

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# Uniqueness of meromorphic functions concerning their difference operators and derivatives



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# Abstract

Let f and g be two nonconstant meromorphic functions. Shared value problems related to f and g are investigated in this paper. We give sufficient conditions in terms of weighted value sharing which imply that f is a linear transformation or inversion transformation of g. We also investigate the uniqueness problem of meromorphic functions with their difference operators and derivatives sharing some values.

MSC: 30D35; 39A05

**Keywords:** Meromorphic function; Difference operator; Shared values; Periodic function; Finite order

# 1 Introduction and main results

Throughout this paper, a meromorphic function is assumed to be meromorphic in the whole complex plane. The basic notations and results of Nevanlinna value distribution theory of meromorphic function are assumed to be known to the reader; see, e.g., [10, 21]. For a meromorphic function f(z), we define the order and the lower order of f(z) by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r},$$

while

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r} \quad \text{and} \quad \delta(a, f) = \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}$$

stand for the exponents of convergence of zero sequence of f and the deficiency of f at the point a, respectively. For a nonconstant meromorphic function h, we denote by T(r, h) the Nevanlinna characteristic of h and by S(r, h) any quantity satisfying S(r, h) = o(T(r, h)), as r runs to infinity outside of a set  $E \subset (0, +\infty)$  of finite linear measure. We say that h is a small function of f if T(r, h) = S(r, f). In the following, we denote by I a set of infinite linear measure not necessarily the same in all its occurrences.

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We say that f and g share the value a IM (ignoring multiplicities), if f and g have the same a point. If f and g have the same a point with the same multiplicities, then we say f, g share the value a CM (counting multiplicities). Let k be a positive integer, we denote by  $N_{k}(r, \frac{1}{f-a})$  the counting function of a points of f with multiplicity  $\leq k$  and by  $N_{(k+1}(r, \frac{1}{f-a})$  the counting function of a points of f with multiplicity > k, where each a point is counted according to its multiplicity. Similarly we define  $\overline{N}_{k}(r, \frac{1}{f-a})$  and  $\overline{N}_{(k+1}(r, \frac{1}{f-a})$  where in counting the a points of f we ignore the multiplicities.

**Definition 1** ([12]) Let *k* be a nonnegative integer or infinity. For  $a \in \mathbb{C} \cup \{\infty\}$ , we denote by  $E_k(a; f)$  the set of all *a* points of f(z) where an *a* point of multiplicity *m* is counted *m* times if  $m \le k$  and k + 1 times if m > k. If  $E_k(a; f) = E_k(a; g)$ , then we say that *f*, *g* share the value *a* with weight *k*.

We write f and g share (a, k) to mean that f, g share the value a with weight k.

**Definition 2** ([12]) If *s* is a positive integer, then we denote by  $N(r, \frac{1}{f-a}| = s)$  the counting function of those *a* points of *f* whose multiplicity is *s*, where each *a* point is counted according to its multiplicity. For a positive integer *m*, denote by  $N(r, a; f| \ge m)$  the counting function of those *a* points of *f* whose multiplicities are not less than *m* where each *a* point is counted according to its multiplicity.

**Definition 3** ([13]) Denote by  $N_2(r, \frac{1}{f-a})$  the sum of  $\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-a}) \ge 2$ .

The classical four point and five point theorems of Nevanlinna [15] show f is a Möbius transformation of g if two meromorphic functions f and g share four distinct values CM, and f = g if f and g share five distinct values IM. The assumption 4CM of the four-point theorem and 5IM of the five-point theorem have been improved to 2CM+2IM and 3CM+1IM, while 1CM+3IM remains an open problem.

Some researchers also considered whether the conditions of shared values can be replaced by other conditions. Ozawa [16] obtained the following.

**Theorem A** Let f and g be nonconstant entire functions of finite order such that f and g share 0 and 1 CM. If  $\delta(0, f) > \frac{1}{2}$ , then  $f \cdot g = 1$  or f = g.

Removing the order restriction Ueda [19] and Yi [22] obtained some improvements of Theorem A. Especially, Yi [23] obtained the following.

**Theorem B** ([23]) Suppose that f(z) and g(z) are nonconstant meromorphic functions. If f, g share  $0, 1, \infty$  CM and  $\overline{N}(r, \frac{1}{f}) + \overline{N}(r, f) < (d + o(1))T(r, f)$  for  $r \in I$  and  $r \in \infty$ , where d is a positive number satisfying  $0 < d < \frac{1}{2}$ , while  $I \subset (0, +\infty)$  is a subset of infinite linear measure, then f = g or  $f \cdot g = 1$ .

For the sake of relaxing the nature of sharing of values and improving Theorem B, Lahiri in [12] obtained the following result in terms of the weighted value sharing.

**Theorem C** ([12]) Suppose that f(z) and g(z) are nonconstant meromorphic functions. Let f and g share  $(0, 1), (\infty, 0), (1, \infty)$ . If

$$N\left(r,\frac{1}{f-a}\Big|=1\right)+4\overline{N}(r,f)<\left(d+o(1)\right)T(r,f),$$

then either f = g or  $f \cdot g = 1$ .

Our first result shall follow this direction and investigate the uniqueness of meromorphic functions in terms of weighted value sharing.

**Theorem 1.1** Let f and g be meromorphic functions with finite order, and let  $c \in \mathbb{C} \setminus \{0\}$ . Suppose that  $f^n$  and  $g^n$  share (R(z), l), where R(z) is a rational function and l, n are integer. If one of the following cases holds:

(1)  $l = 0, n \ge 15;$ (2)  $l = 1, n \ge 10;$ (3)  $l \ge 2, n \ge 9,$ then  $f = tg \text{ or } f \cdot g = t\alpha$ , where  $t^n = 1, \alpha^n = R^2$ .

Rubel and Yang [18] in 1977 initiated the study of entire functions sharing values with their derivatives instead of studying the problem of sharing value of two meromorphic functions f and g.

**Theorem D** Let f be a nonconstant entire function. If f shares two distinct finite values CM with f', then  $f \equiv f'$ .

More results on the uniqueness of f' with its nth derivative  $f^{(n)}$  were obtained by several authors; see [1, 8, 20]. In view of the progress on the difference analogues of classical Nevanlinna theory of meromorphic functions [4, 9], it is quite natural to investigate the uniqueness problems of meromorphic functions and their difference operators; see [3, 6, 11, 25]. So a natural question arises, that is, how about the uniqueness of the derivatives and the difference operators of f(z)?

In 2018, Qi et al. [17] obtained some results in the case that f'(z) shares values with  $\Delta f$  or f(z + c).

**Theorem E** Let f be a meromorphic function of finite order. Suppose that f' and  $\Delta f$  share  $a_1, a_2, a_3, a_4$  IM, where  $a_1, a_2, a_3, a_4$  are four distinct finite values. Then  $f' = \Delta f$ .

**Theorem F** Let f be a transcendental entire function of finite order, and a be a nonzero finite value. If f'(z) and f(z + c) share 0, a CM, then f'(z) = f(z + c) for all  $z \in C$ .

Regarding Theorems E and F, one may ask the following question:

**Question 1** What can be said when the difference operators of f(z) shares some values with its derivatives?

In this direction, we will prove the following results.

**Theorem 1.2** Let f be a nonconstant entire function of finite order with periodic  $\eta \neq 0$ such that  $\mu(f) > 1$ , where  $\eta$  is a finite nonzero value, and let  $a_1$  and  $a_2$  be two distinct finite values, and k be a positive integer. If  $\Delta f$  and  $f^{(k)}$  share  $a_1$  CM,  $\Delta f$  and  $f^{(k)}$  share  $a_2$  IM, then  $\Delta f = f^{(k)}$ .

If we replace  $\Delta f(z)$  by the shift f(z + c), we can obtain the following.

**Theorem 1.3** Let f be a nonconstant meromorphic function of finite order, and let c be a finite nonzero value, let k be a positive integer satisfying  $k \ge 2$ , and let  $a_1, a_2, a_3$  be three finite values such that  $a_1 \ne 0$ ,  $a_2 \ne 0$  and  $N(r, \frac{1}{f-a_3}) = S(r, f)$ . If f(z + c) and  $f^{(k)}(z)$  share  $a_1$  *CM* and  $a_2$  *IM*, then  $f(z + c) = f^{(k)}(z)$  for all  $z \in C$ .

*Remark* The results obtained in this manuscript have many applications in the qualitative analysis of differential equations with piecewise constant arguments and porous medium problems; see [5, 14]. The following example shows that the assumption  $\mu(f) > 1$  of Theorem 1.2 is the best possible.

*Example* 1 Let  $f(z) = e^{Az}$ , where  $A \neq 0$  is a constant. Then  $f^{(k)} = A^k e^{Az}$  and  $\Delta(f) = f(z + 1) - f(z) = (e^A - 1)e^{Az}$ . Clearly,  $\Delta(f)$  and  $f^{(k)}$  share 0 CM and  $\infty$  IM, and that  $\mu(f) = 1$ . We can choose A such that  $e^A - 1 \neq A^k$ , and so  $f' \neq \Delta(f)$ .

### 2 Preliminaries

First we denote by *H* the following function:

$$H = \left(\frac{F^{\prime\prime}}{F^{\prime}} - \frac{2F^{\prime}}{F-1}\right) - \left(\frac{G^{\prime\prime}}{G^{\prime}} - \frac{2G^{\prime}}{G-1}\right),$$

where and in what follows, F and G are some two distinct nonconstant meromorphic functions.

Next we introduce some lemmas needed in the proofs of the main results of this paper.

**Lemma 2.1** ([2]) Let F and G be two nonconstant meromorphic functions sharing (1,0) and  $H \neq 0$ . Then

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right)$$
$$+ \overline{N}\left(r,\frac{1}{G}\right) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G)$$

and the same inequality holds for T(r, G).

**Lemma 2.2** ([2]) Let F and G be two nonconstant meromorphic functions sharing (1,1) and  $H \neq 0$ . Then

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G)$$

and the same inequality holds for T(r, G).

**Lemma 2.3** ([12]) *Let f and g be two nonconstant meromorphic functions sharing* (1,2). *Then one of the following cases holds:* 

- (i)  $T(r) \le N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r, f) + N_2(r, g) + S(r),$ (ii) f = g,
- (iii) fg = 1, where  $T(r) = \max\{T(r, f), T(r, g)\}$  and  $S(r) = o\{T(r)\}$ , as  $r \notin E$ , where  $E \subset (0, +\infty)$  is a subset of finite linear measure.

**Lemma 2.4** ([7]) Let f and g be two meromorphic functions, and let k be a positive integer. If  $E_k(1;f) = E_k(1;g)$ , then one of the following cases must occur:

$$T(r,f) + T(r,g) \le N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) - N_{11}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(k+1)}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(k+1)}\left(r,\frac{1}{g-1}\right) + S(r,f) + S(r,g);$$

(ii)  $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$ , where  $a \neq 0$ , b are two constants.

**Lemma 2.5** ([21]) Let f be a meromorphic function. If

$$g = \frac{af + b}{cf + d},$$

where  $a, b, c, d \in S(f)$  and  $ad - bc \neq 0$ , then T(r,g) = T(r,f) + S(r,f).

**Lemma 2.6** ([4]) Let f be a nonconstant meromorphic function, let  $\varepsilon > 0$  and let  $c \in \mathbb{C}$ . If f is of finite order, then there exists a set  $E = E(f, \varepsilon) \subset (0, +\infty)$  satisfying

$$\limsup_{r \to \infty} \frac{\int_{E \cap [1,r)} \frac{dt}{t}}{\log r} \le \varepsilon$$

i.e., of logarithmic density at most  $\varepsilon$ , such that

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = O\left(\frac{\log r}{r}T(r,f(z))\right)$$

for all out of the set *E*. If  $\rho_2 = \rho_2 < 1$  and  $\varepsilon > 0$ , then

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f(z))}{r^{1-\rho_2-\varepsilon}}\right)$$

for all  $r \in (0, +\infty)$  outside of a set of finite logarithmic measure.

**Lemma 2.7** ([1, Lemma 2.3]) Let k be a positive integer, and let f be a nonconstant meromorphic function such that  $f^{(k+1)} \neq 0$ . If  $\overline{N}(r, \frac{1}{f}) = S(r, f)$ , then

$$kN_{1}(r,f(z)) \leq \overline{N}_{(2}(r,f) + N_{1}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f).$$

**Lemma 2.8** ([24]) Let f be a meromorphic function such that  $f^{(k)}$  is not constant. Then

$$T(r,f) \le N\left(r,\frac{1}{f}\right) + N_{1}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}(r,f) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f).$$

**Lemma 2.9** ([24]) Let f be a transcendental meromorphic function. Then, for each positive real number  $\varepsilon$  and for each positive integer n,

$$(n-1)\overline{N}(r,f) \le (1+\varepsilon)N\left(r,\frac{1}{f^{(n)}}\right) + (1+\varepsilon)\left(N(r,f) - \overline{N}(r,f)\right) + S(r,f).$$

**Lemma 2.10** ([4]) *Let* f *be a nonconstant meromorphic function of finite order and*  $c \in \mathbb{C}$ *. Then* 

$$\begin{split} T(r,f(z+c) &= T(r,f) + S(r,f), \\ N(r,f(z+c) &= N(r,f) + S(r,f), \\ N\left(r,\frac{1}{f(z+c)}\right) &= N\left(r,\frac{1}{f(z)}\right) + S(r,f) \end{split}$$

and

$$\overline{N}(r,f(z+c)=\overline{N}(r,f)+S(r,f),\qquad \overline{N}\left(r,\frac{1}{f(z+c)}\right)=\overline{N}\left(r,\frac{1}{f(z)}\right)+S(r,f).$$

# 3 Proof of Theorem 1.1

Set  $F = \frac{f^n}{R}$ ,  $G = \frac{g^n}{R}$ . Clearly, F and G share (1, l). Write  $T(r) = \max\{T(r, f(z)), T(r, g(z))\}$  and  $S(r) = o\{T(r)\}$  as  $r \notin E$  and  $r \to \infty$ , where  $E \subset (0, +\infty)$  is a subset of finite linear measure. *Case 1.* l = 0 and  $n \ge 15$ .

Assume that  $H \neq 0$ . By Lemma 2.1, we have

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right)$$
$$+ \overline{N}\left(r,\frac{1}{G}\right) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G).$$

Clearly,

$$nT(r,f) \leq 4\overline{N}\left(r,\frac{1}{F}\right) + 3\overline{N}\left(r,\frac{1}{G}\right) + 4\overline{N}(r,F) + 3\overline{N}(r,G) + S(r,F) + S(r,G)$$

$$\leq 4\overline{N}\left(r,\frac{1}{f}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + 4\overline{N}(r,f) + 3\overline{N}(r,g) + S(r,f) + S(r,g)$$

$$\leq 8T(r,f) + 6T(r,g) + S(r,f) + S(r,g). \tag{1}$$

Similarly, we have

$$nT(r,g) \le 8T(r,g) + 6T(r,f) + S(r,f) + S(r,g).$$
(2)

Now (1) and (2) yield

$$(n-14)\big(T(r,f)+T(r,g)\big) \leq S(r,f)+S(r,g).$$

Hence

$$(n-14)T(r) \le S(r),$$

which contradicts  $n \ge 15$ .

Therefore,  $H \equiv 0$  and we have

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$
(3)

It follows from (3) that

$$\frac{1}{F-1} = \frac{A}{G-1} + B,$$
(4)

where  $A \neq 0$ , *B* are constants.

Subcase 1.1. If B = 0, then (4) leads to  $F = \frac{G-1+A}{A}$  and G = AF - (A - 1). Suppose that A = 1. Clearly, we have F = G, and thus f = tg, where  $t^n = 1$ . If  $A \neq 1$ , then we have  $\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F-\frac{A-1}{A}})$  and  $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G+(A-1)})$ . By Nevanlinna's second fundamental theorem,

$$T(r,F) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right) + \overline{N}(r,F) + S(r,F)$$
$$\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + S(r,F).$$

Thus,

$$nT(r,f) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,g) + \overline{N}(r,f) + S(r,f)$$
  
$$\le 2T(r,f) + T(r,g) + S(r,f).$$
(5)

Similarly, we have

$$nT(r,g) \le 2T(r,g) + T(r,f) + S(r,g).$$
 (6)

Combining (7) and (8), we obtain

$$(n-3)\big(T(r,f)+T(r,g)\big) \leq S(r,f)+S(r,g).$$

Clearly, it is a contradiction by  $n \ge 15$ .

Subcase 1.2  $B \neq 0$ .

If  $A \neq B$ , then we have  $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ . Applying Nevanlinna's second fundamental theorem to F with considering 0 point, 1 point, and  $\infty$  point, we can also get a contradiction by similar discussion as in Subcase 1.1.

*Case 2.* l = 1 and  $n \ge 10$ .

Assume that  $H \neq 0$ . By Lemma 2.2., we have

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F)$$

$$+S(r,F) + S(r,G)$$

$$\leq \frac{5}{2}\overline{N}\left(r,\frac{1}{f(z)}\right) + 2\overline{N}\left(r,\frac{1}{g(z)}\right) + \frac{5}{2}\overline{N}(r,f) + 2\overline{N}(r,g) + S(r,f) + S(r,g).$$

Thus,

$$nT(r,f) \le 5T(r,f) + 4T(r,g) + S(r,f) + S(r,g).$$
(7)

Similarly, we have

$$nT(r,g) \le 5T(r,g) + 4T(r,f) + S(r,f) + S(r,g).$$
(8)

Combining (5) and (6) yields

$$(n-9)\big(T(r,f)+T(r,g)\big) \le S(r,f)+S(r,g),$$

which contradicts  $n \ge 10$ .

Therefore,  $H \equiv 0$ . We can deduce the same conclusion by similar discussion as in Case 1. *Case 3.*  $l \ge 2$  and  $n \ge 9$ . *Subcase 3.1.* l = 2.

From Lemma 2.3, if (i) holds, then we deduce that

$$\max\{T(r,F), T(r,G)\} \le N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G).$$
(9)

That is,

$$nT(r) = n \max\left\{T(r,F), T(r,G)\right\} \le 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) + 2\overline{N}(r,f) + 2\overline{N}(r,g) + S(r)$$
$$\le 8T(r) + S(r).$$

Therefore,  $(n - 8)T(r) \le S(r)$ , which contradicts  $n \ge 9$ . Thus we have F = G or FG = 1. If F = G, then  $f^n = g^n$ , which yields f = tg, where  $t^n = 1$ . If FG = 1, then  $(fg)^n = (R)^2$ , which yields  $fg = t\alpha$ , where  $t^n = 1$ ,  $\alpha^n = R^2$ .

*Subcase 3.2.*  $l \ge 3$ *.* 

By Lemma 2.4, either (i) or (ii) holds. If (i) holds, then we obtain

$$T(r,F) + T(r,G) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_{11}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(k+1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(k+1)}\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G) \leq N_2(r,F) + N_2(r,G) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F-1}\right) + \frac{1}{2}\overline{N}\left(r,\frac{1}{G-1}\right)$$

+ 
$$S(r,F)$$
 +  $S(r,G)$   
 $\leq N_2(r,F) + N_2(r,G) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + T(r,F) + T(r,G) + S(r,F) + S(r,G).$ 

Therefore, we get

$$\frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) \le 2\overline{N}(r,f) + 2\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g).$$
(10)

Consequently,

$$\frac{1}{2}n\{T(r,f)+T(r,g)\} \le 4\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which leads to

$$\left(\frac{1}{2}n-4\right)\left\{T(r,f)+T(r,g)\right\} \le S(r,f)+S(r,g)$$

This is a contradiction since  $n \ge 9$ .

Hence, (ii) holds, which means

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)},$$

where  $a \neq 0$ , *b* are constants.

Suppose that b = 0. Then we have F = G when a - 1 = 0, that is, f = tg, where  $t^n = 1$ . If  $a - 1 \neq 0$ , then we obtain  $F = \frac{G+a-1}{a}$  and  $G = a(F + \frac{1-a}{a})$ , and so  $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G+a-1})$ ,  $\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F+\frac{1-a}{2}})$ . By Nevanlinna's second fundamental theorem we get

$$T(r,G) \leq \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G+a-1}\right) + \overline{N}(r,G) + S(r,G).$$

This yields

$$nT(r,g) \le 2T(r,g) + T(r,f) + S(r,f).$$

Similarly, we have

$$nT(r,f) \le 2T(r,f) + T(r,g) + S(r,f).$$

Thus we obtain  $(n - 3)(T(r, f) + T(r, g)) \le S(r, f) + S(r, g)$ , which is a contradiction with  $n \ge 9$ . Suppose that b = -1. If a + 1 = 0, then  $F \cdot G \equiv 1$ . Hence  $f \cdot g = t\alpha$ , where  $t^n = 1$ ,  $\alpha^n = R^2$ . If  $a + 1 \ne 0$ , similarly to above, then we can obtain a contradiction. Suppose  $b \ne 0$  and  $b \ne -1$ . By a similar reasoning to the case b = 0, we can also obtain a contradiction.

This completes the proof of Theorem 1.1.

# 4 Proof of Theorem 1.2

Without loss of any generality, we can assume  $a_1 = 0$ ,  $a_2 = 1$ . Since  $\Delta f(z)$ ,  $f^{(k)}$  share 0 CM, we have

$$\frac{\Delta f}{f^{(k)}} = e^Q,\tag{11}$$

where Q is a polynomial.

Since the period of *f* is  $c \in \mathbb{C} \setminus \{0\}$ , we have  $e^{Q(z)} = e^{Q(z+\eta)}$ .

Consequently,  $e^{Q(z+\eta)-Q(z)} = 1$ , which leads to  $Q'(z+\eta) = Q'(z)$ . Then Q'(z) has a period  $\eta$  and Q'(z) must be a constant.

Now write

$$Q(z) = az + b,$$

where *a*, *b* are constants.

Since  $\Delta f$ ,  $f^{(k)}$  share 1 *IM*, we get

$$\frac{\Delta f - 1}{f^{(k)} - 1} = \alpha(z), \tag{12}$$

where  $\alpha$  is a meromorphic function.

By (11), (12) and Q(z) = az + b, we deduce

$$\alpha = \frac{f^{(k)}e^{az+b} - 1}{f^{(k)} - 1}.$$
(13)

By Lemma 2.5, we obtain

$$T(r,\alpha) = T(r,f^{(k)}) + S(r,f) = (k+1)T(r,f) + S(r,f).$$
(14)

Now we estimate the number of zeros, poles of  $\alpha$ . From the assumption that  $\mu(f) > 1$ , we know that  $T(r, e^Q) = S(r, f)$ .

Since  $\Delta f$ ,  $f^{(k)}$  share 1 *IM*, it follows from (13) that the zero of  $\Delta f(z) - 1$  and  $f^{(k)} - 1$  must be the zero of  $e^{az+b} - 1$ . Noting that  $f^{(k)} - 1$  have the same poles with  $f^{(k)}e^{az+b} - 1$ , then, by (12), we have

$$\overline{N}(r,\alpha) = \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) \le \overline{N}\left(r,\frac{1}{e^{az+b}-1}\right) = S(r,f)$$

and

$$\overline{N}\left(r,\frac{1}{\alpha}\right) = \overline{N}\left(r,\frac{1}{\Delta f - 1}\right) \le \overline{N}\left(r,\frac{1}{e^{az+b} - 1}\right) = S(r,f).$$

Therefore, from the Nevanlinna second fundamental theorem, we obtain

$$T(r,\alpha) \le \overline{N}(r,\alpha) + \overline{N}\left(r,\frac{1}{\alpha}\right) + \overline{N}\left(r,\frac{1}{\alpha - e^Q}\right) + S(r,\alpha).$$
(15)

Combining (11) and (12), we may write  $\alpha - e^Q = \frac{\Delta f - f^{(k)}}{(f^{(k)} - 1)f^{(k)}} = \frac{e^Q - 1}{f^{(k)} - 1}$ . Then, by (15), we conclude that

$$T(r,\alpha) \le 3\overline{N}\left(r,\frac{1}{e^{az+b}-1}\right) + S(r,f) = S(r,f).$$
(16)

It contradicts (14).

Therefore,  $\Delta f = f^{(k)}$ . This completes the proof of Theorem 1.2.

## 5 Proof of Theorem 1.3

Some ideas of our proof come from [1, 26]. Without loss of generality, we assume that f(z + c) and  $f^{(k)}(z)$  share 1 CM and  $\infty$  IM, and  $N(r, \frac{1}{f}) = S(r, f)$ . For the general case, we take the transformation  $T(z) = \frac{z-a_3}{z-a_1} \frac{a_2-a_1}{a_2-a_3}$ , and so  $T(a_1) = \infty$ ,  $T(a_2) = 1$ ,  $T(a_3) = 0$ . Suppose that  $f(z + c) \neq f^{(k)}(z)$ . Set

$$G(z) = \frac{1}{f(z+c)} \left( \frac{f^{(k+1)}(z)}{f^{(k)}(z) - 1} - \frac{f'(z+c)}{f(z+c) - 1} \right)$$
(17)

$$=\frac{f^{(k)}(z)}{f(z+c)}\left(\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}-\frac{f^{(k+1)}(z)}{f^{(k)}(z)}\right)-\left(\frac{f'(z+c)}{f(z+c)-1}-\frac{f'(z+c)}{f(z+c)}\right).$$
(18)

It follows from the lemma of the logarithmic derivative, Lemma 2.6 and (18) that m(r, G) = S(r, f).

By (17), we see that the possible poles of *G* can occur at the zeros of f(z + c), the 1 points of f(z + c) and  $f^{(k)}(z)$ , and the poles of f(z + c) and  $f^{(k)}(z)$ . If  $z_0$  is a 1 point of f(z + c), then by a short calculation with Laurent series and (17) we see that G(z) is analytic at  $z_0$ . Since f(z + c) and  $f^{(k)}(z)$  share 1 CM, we know the 1 points of f(z + c) and  $f^{(k)}(z)$  are not the poles of G(z). If f(z + c) has a pole  $z_0$  with multiplicity  $p (\ge 1)$ , we need to consider two cases: (i)  $z_0$  is also a pole of  $f^{(k)}(z)$ , then by (17)  $G(z) = O((z - z_0)^{p-1})$ ; (ii)  $z_0$  is not a pole of  $f^{(k)}(z)$ , and hence  $z_0$  is not a pole of  $f^{(k+1)}(z)$ . Then we also have  $G(z) = O((z - z_0)^{p-1})$ . Similarly, the poles of  $f^{(k)}(z)$  are not also the poles of G(z). Therefore, the poles of F can only occur at the zeros of f(z + c). By Lemma 2.7 and the hypothesis of Theorem 1.3, it follows that  $N(r, \frac{1}{f(z+c)}) = N(r, \frac{1}{f}) + S(r, f) = S(r, f)$ , and so we have N(r, G) = S(r, f). Thus,

$$T(r,G) = S(r,f). \tag{19}$$

If  $G \equiv 0$ , then, by (17), we find that  $f^{(k)}(z) - 1 = t(f(z+c)-1)$ , with  $t \neq 0$  constant. Thus,  $(1-t)m(r, \frac{1}{f}) \leq m(r, \frac{f^{(k)}(z)}{f(z)}) + m(r, \frac{f(z+c)}{f(z)}) = S(r, f)$ . Since  $N(r, \frac{1}{f}) = S(r, f)$ , we have  $T(r, \frac{1}{f}) = S(r, f)$ . It is a contradiction. Then  $G \not\equiv 0$ . And so we deduce from (17) and (19) that

$$m(r,f(z+c)) \le m\left(r,\frac{1}{G}\right) + m\left(r,\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} - \frac{f'(z+c)}{f(z+c)-1}\right) \le T(r,G) + S(r,f)$$
  
= S(r,f). (20)

If  $z_0$  is a pole of f(z + c) of multiplicity  $p \ge 2$ , then by (17) we know that  $z_0$  is possible a zero of *G* with multiplicity p - 1. Consequently, it follows from (19) that

$$N_{(2}(r, f(z+c)) \le 2N\left(r, \frac{1}{G}\right) \le 2T(r, G) + O(1) = S(r, f).$$
(21)

Let  $z_0$  be a simple pole of f(z + c). Set

$$H(z) = \frac{f^{(k+1)}(z)(f(z+c)-1)}{f'(z+c)(f^{(k)}(z)-1)}.$$
(22)

By a short calculation with Laurent series, it follows that  $H(z_0) = k + 1$ . If  $H(z) \equiv k + 1$ , then we have  $f^{(k)}(z) - 1 = t(f(z + c) - 1)^{k+1}$  with  $t \neq 0$  constant. This a contradiction, since  $f^{(k)}(z)$  and f(z + c) share 1 CM. Thus  $H \neq k + 1$ , and so

$$N_{1}(r, f(z+c)) \le N\left(r, \frac{1}{H - (k+1)}\right) \le T(r, H) + O(1).$$
(23)

We now estimate the poles of *H*. Clearly, the poles of *H* can only occur at the 1 points of  $f^{(k)}(z)$ , the zeros of f'(z + c), and the poles of f(z + c) and  $f^{(k+1)}(z)$ . Since  $f^{(k)}(z)$ , f(z + c) share 1 CM and  $\infty$  IM, *H* is holomorphic at the 1 points of  $f^{(k)}(z)$  and the poles of f(z + c) and  $f^{(k+1)}(z)$ . Thus

$$N(r,H) \le N_0\left(r,\frac{1}{f'(z+c)}\right) + S(r,f),\tag{24}$$

where  $N_0(r, \frac{1}{f'(z+c)})$  denotes the zeros of f'(z+c) which are not zeros of f(z+c) - 1. Again by (22), we see that

$$m(r,H) = S(r,f),$$
(25)

from this, (23) and (24) we find that

$$N_{1}(r, f(z+c)) \le N_0\left(r, \frac{1}{f'(z+c)}\right) + S(r, f).$$
(26)

Combining this, Nevanlinna's second fundamental theorem (see [10, Theorem 3.2])for f(z + c), (21) and the hypothesis  $N(r, \frac{1}{f(z)}) = S(r, f)$  we have

$$T(r,f(z+c)) \leq N\left(r,\frac{1}{f(z+c)}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-1}\right) + \overline{N}\left(r,f(z+c)\right)$$
$$-N_0\left(r,\frac{1}{f'(z+c)}\right) + S\left(r,f(z+c)\right)$$
$$\leq \overline{N}\left(r,\frac{1}{f(z+c)-1}\right) + S(r,f)$$
$$\leq N_{10}\left(r,\frac{1}{f(z+c)-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{f(z+c)-1}\right) + S(r,f).$$
(27)

From this and Nevanlinna's first fundamental theorem, it is easy to deduce that

$$m\left(r,\frac{1}{f(z+c)-1}\right) + N_{(2}\left(r,\frac{1}{f(z+c)-1}\right) \le \overline{N}_{(2}\left(r,\frac{1}{f(z+c)-1}\right) + S(r,f),\tag{28}$$

which implies

$$m\left(r,\frac{1}{f(z+c)-1}\right) + N_{(2}\left(r,\frac{1}{f(z+c)-1}\right) = S(r,f).$$
(29)

From (29), we see that

$$T(r, f(z+c)) = m\left(r, \frac{1}{f(z+c)-1}\right) + N_{10}\left(r, \frac{1}{f(z+c)-1}\right) + N_{(2}\left(r, \frac{1}{f(z+c)-1}\right) + S(r, f)$$
$$= N_{10}\left(r, \frac{1}{f(z+c)-1}\right) + S(r, f).$$
(30)

Since f(z + c) and  $f^{(k)}(z)$  share 1 CM, it follows from Lemma 2.7 and (21) that

$$(k-1)N_{1}(r,f(z+c)) \le \overline{N}\left(r,\frac{1}{f^{(k+1)}(z)}\right) + S(r,f).$$
(31)

By Lemma 2.8, (29) (30) and (21), we have

$$N\left(r,\frac{1}{f^{(k+1)}(z+c)}\right) \le N\left(r,\frac{1}{f(z+c)}\right) + N_{11}\left(r,f(z+c)\right) + N\left(r,\frac{1}{f(z+c)-1}\right) - T\left(r,f(z+c)\right) + S\left(r,f(z+c)\right) \\ \le N_{11}\left(r,f(z+c)\right) + S(r,f).$$
(32)

It follows from Lemma 2.10 that

$$N\left(r,\frac{1}{f^{(k+1)}(z)}\right) = N\left(r,\frac{1}{f^{(k+1)}(z+c)}\right) + S(r,f).$$

From this, (31) and (32), we see that  $(k - 2)N_{1}(r, f(z + c)) = S(r, f)$ . If  $k \ge 3$ , then  $N_{1}(r, f(z + c)) = S(r, f)$ . Combining this, (20) and (21), we have T(r, f(z + c)) = S(r, f), which is a contradiction.

Let k = 2.

*Case 1.* If f(z) is transcendental, then, by Lemma 2.9, for a positive constant  $\varepsilon < 1$  we have

$$2N_{1}(r,f(z+c)) \le (1+\varepsilon)N\left(r,\frac{1}{f^{(k+1)}(z+c)}\right) + (1+\varepsilon)\left[N\left(r,f(z+c)\right) - \overline{N}\left(r,f(z+c)\right)\right] + S(r,f).$$
(33)

From this and (21) we have

$$(1-\varepsilon)N_{1}(r,f(z+c)) = S(r,f).$$

Combining this, (20) and (21), we have T(r, f(z + c)) = S(r, f). This is a contradiction.

*Case 2.* If f(z) is rational, then by  $N(r, \frac{1}{f}) = S(r, f)$  we know that f has no zeros, and hence we can write  $f(z + c) = \frac{1}{P(z)}$ , where P(z) is a nonconstant polynomial. Set  $\Phi = \frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} - \frac{f'(z+c)}{f(z+c)-1} - 2\frac{f'(z+c)}{f(z+c)}$ . Clearly  $T(r, \Phi) = S(r, f)$ . Combining this and (17), we have

$$2f'(z+c) = Gf^2 - \Phi f.$$
 (34)

Substituting  $f(z + c) = \frac{1}{P(z)}$  into (34) we obtain  $-2P' = G - \Phi P$ . This shows that T(r, P') = T(r, P) + S(r, f) and so T(r, P) = S(r, f). This is a contradiction.

Therefore,  $f(z + c) = f^{(k)}(z)$ . This completes the proof of Theorem 1.3.

#### Acknowledgements

The authors are grateful to the referees for their valuable suggestions and comments which have improved the clarity of the paper.

#### Funding

The work was supported by NNSF of China (No.11971344).

#### Availability of data and materials

Not applicable.

#### **Competing interests**

The authors declare that they have no competing interests.

#### Authors' contributions

Each of the authors contributed to each part of this study equally and approved the final version of this manuscript.

#### **Publisher's Note**

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#### Received: 26 May 2020 Accepted: 31 August 2020 Published online: 09 September 2020

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