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Certain new weighted estimates proposing generalized proportional fractional operator in another sense

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Abstract

The present work investigates the applicability and effectiveness of generalized proportional fractional integral ($\mathcal{GPFI}$) operator in another sense. We aim to derive novel weighted generalizations involving a family of positive functions n ($n \in \mathbb{N}$) for this recently proposed operator. As applications of this operator, we can generate notable outcomes for Riemann–Liouville ($\mathcal{RL}$) fractional, generalized $\mathcal{RL}$-fractional operator, conformable fractional operator, Katugampola fractional integral operator, and Hadamard fractional integral operator by changing the domain. The proposed strategy is vivid, explicit, and it can be used to derive new solutions for various fractional differential equations applied in mathematical physics. Certain remarkable consequences of the main theorems are also figured.

MSC: 26D15; 26A33; 26A51

Keywords: Integral inequality; Generalized proportional fractional integral with respect to another function; Increasing and decreasing functions

1 Introduction

Fractional calculus [1–20] is genuinely viewed to be the real-world framework, and it has wide applications in mathematics, physics, biology, medicine, and many other natural and social sciences [21–35], for instance, a correspondence structure that contains indulgent interfacing, dependent parts that are used to accomplish a bound together with a goal of transmitting and getting signals, can be depicted using complex framework models [36–43]. This structure is considered as a stunning framework, and the units that make the entire system are seen as the centers of the complex framework. An attracting particularization of this field is that there are various fractional operators, and this allows the scientists to pick out the most suitable operator for the purpose of displaying the issue under scrutiny [44–47]. Moreover, due to its effortlessness in applications, analysts have paid more noteworthy enthusiasm to fractional operators without singular kernels [48–50], after which numerous articles considering these sorts of fractional operators have as of late become visible. These methods had been created by various mathematicians with a scarcely explicit definition, as in the case of $\mathcal{RL}$, Weyl, Erdelyi–Kober, Hadamard integral, Liouville and Katugampola fractional operator [44–46, 48–52].

Recently, Jarad et al. [53] explored the idea of $\mathcal{GPFI}$ operators which have been applied to characterize certain probability density functions and have fertile applications in statistics. In [54, 55], Rashid et al. proposed a different novel fractional approach having an exponential function in its kernel which comes into existence in the theory of fractional calculus, which is known as $\mathcal{GPFI}$ operators in another sense. The significant characterization of the $\mathcal{GPFI}$ operator in another sense is that it can discover the bulk of complex problems in one direction, and then again the generalized proportional fractional derivative in the sense of another function can catch various sorts of complexities, hence assembling these two ideas can help us to comprehend the complexities of existing nature in a vastly improved manner. The $\mathcal{GPFI}$ operators have captivated the interest of many researchers from several areas of science. This novel concept provides an avenue for interested readers towards various scientific fields of research, including control theory, engineering, fluid dynamics, meteorology, analysis, aerodynamics, and many more.

Inequalities are an important part of the whole field of mathematical research [56–69], fractional calculus can be applied on many equalities and inequalities that have been explored by many authors, such as the Hardy, Ostrowski, Gagliardo–Nirenberg, Olsen, and trapezoidal-type inequalities, which are utilized in imperative significant systems among scientists and amass prolific utilitarian applications in different regions of science [70–72]. Fractional integral inequalities have potential applications in several areas of science, such as technology, mathematics, chemistry, plasma physics, and so on. Particularly, we bring up the initial value problem, the stability of linear transformation, integrodifferential equations, and transform equations [44–46]. Such utilizations of fractional integral operators constrained us to show the speculation by utilizing a group of n positive functions including $\mathcal{GPFI}$ operators in another sense.

In the present study, we introduce new weighted versions of several generalizations and enunciate a new generalized fractional proportional integrals, which we name $\mathcal{GPFI}$ operator in another sense. To be more precise, we establish a new version for a class of family of n ($n \in \mathbb{N}$) continuous positive decreasing functions in the frame of the $\mathcal{GPFI}$ operator in another sense and also provide some of its trendy splendor consequences observing Remark 2.2. New findings are introduced and new theorems which relate to $\mathcal{GPFI}$ and $\mathcal{RL}$ are derived that correlate with the earlier results.

2 Prelude

This section consists of some useful preliminaries from fractional calculus used in our subsequent discussion. The major subtleties are given in the monograph by Kilbas et al. [73].

Now, we demonstrate a novel fractional operator which is known as the $\mathcal{GPFI}$ operator of a function in another sense proposed by Jarad et al. [54] and Rashid et al. [55], independently.

Definition 2.1 (see [54, 55]) Let $\zeta > 0$, (μ_1, μ_2) $(-\infty \leq \mu_1 < \mu_2 \leq \infty)$ be a finite or infinite real interval, and $\varphi(y_1)$ an increasing and positive monotone function on $(\mu_1, \mu_2]$. Then the left- and right-sided $\mathcal{GPFI}$ operators of a function $\mathcal{P}$ with respect to another function

φ of order $\zeta > 0$ are defined by (2.1) and (2.2) as follows:

$$ {}^{\varphi}\mathcal{K}_{\mu_1}^{\zeta,\sigma}\mathcal{P}(\varsigma) = \frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\int_{\mu_1}^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{P}(y_1)\,dy_1 \quad (\mu_1 < \varsigma), \tag{2.1}$$

$$ {}^{\varphi}\mathcal{K}_{\mu_2}^{\zeta,\sigma}\mathcal{P}(\varsigma) = \frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\int_{\varsigma}^{\mu_2}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(y_1)-\varphi(\varsigma))]}{(\varphi(y_1)-\varphi(\varsigma))^{1-\zeta}}\varphi'(y_1)\mathcal{P}(y_1)\,dy_1 \quad (\varsigma < \mu_2), \tag{2.2}$$

where the proportionality index $\sigma \in (0,1]$, $\varsigma \in \mathcal{C}$ with $\Re(\varsigma) > 0$ and $\Gamma(\varsigma) = \int_0^{\infty} y_1^{\varsigma-1}e^{-y_1}\,dy_1$ is the Gamma function.

Remark 2.2 From Definition 2.1 we clearly see that

(1) If $\varphi(y_1) = y_1$, then the left- and right-sided $\mathcal{GPFI}$ operator reduces to the operator given in [53].
(2) If $\sigma = 1$, then the left- and right-sided generalized $\mathcal{RL}$-fractional integral operator reduces to the operator defined in [73].
(3) If $\sigma = 1$ and $\varphi(y_1) = y_1$, then the left- and right-sided $\mathcal{RL}$-fractional integral operator reduces to the operator given in [73].
(4) If $\varphi(y_1) = \ln y_1$, then the left- and right-sided generalized proportional Hadamard fractional integral operator reduces to the operator given in [74].
(5) If $\varphi(y_1) = \ln y_1$ and $\sigma = 1$, then the left- and right-sided Hadamard fractional integral operator reduces to the operator defined in [75].

Next, we present the definition of the one-sided $\mathcal{GPFI}$ operator with respect to another function φ.

Definition 2.3 Let (μ_1,μ_2) $(-\infty \le \mu_1 < \mu_2 \le \infty)$ be a finite or infinite real interval, $\zeta > 0$, and $\varphi(y_1)$ an increasing and positive monotone function on $(\mu_1,\mu_2]$. Then the one-sided $\mathcal{GPFI}$ operator of a function $\mathcal{P}$ with respect to another function φ of order $\zeta > 0$ is defined by

$$ {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\mathcal{P}(\varsigma) = \frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\int_{0}^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{P}(y_1)\,dy_1 \quad (\varsigma > 0). \tag{2.3}$$

3 Main results

In what follows, we suppose that $\varphi(y_1)$ a continuous, increasing, and positive function on $[0,\infty)$ with $\varphi(0) = 0$.

Throughout this paper, we suppose that $\varphi(y_1)$ is an increasing, positive monotone function on $[0,\infty)$, and also $\varphi(y_1)$ is continuous on $[0,\infty)$ with $\varphi(0) = 0$.

Next, we provide certain inequalities for a class of family of continuous, positive and decreasing functions via $\mathcal{GPFI}$ defined in (2.3).

Theorem 3.1 *Assume that there are two positive and continuous functions $\mathcal{P}$ and $\mathcal{Q}$ defined on $[0,\infty)$ with the assumption that $\mathcal{R} : [0,\infty) \to \mathbb{R}$ is positive and continuous. Then for all $\varsigma > 0$, the following inequality holds:*

$$\begin{aligned} &{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta+\rho}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big] \\ &\quad \ge {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big], \end{aligned} \tag{3.1}$$

where $\beta \ge \eta > 0$, $\rho > 0$, $\sigma \in (0,1]$, and $\zeta \in \mathcal{C}$ $(\Re(\zeta) > 0)$.

Proof Since $\mathcal{P}$ and $\mathcal{Q}$ are positive and continuous functions defined on $[0,\infty)$, therefore for all $y_1, z_1 \in (0,\varsigma)$, $\varsigma > 0$ and for any $\rho > 0$, $\beta \geq \eta > 0$, we have

$$\big(\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\rho}(y_1) - \mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\rho}(z_1)\big)\big(\mathcal{P}^{\beta-\eta}(y_1) - \mathcal{P}^{\beta-\eta}(z_1)\big) \geq 0.$$

It follows that

$$\begin{aligned} &\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\rho+\beta-\eta}(y_1) + \mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\rho+\beta-\eta}(z_1) \\ &\quad \geq \mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\rho}(y_1)\mathcal{P}^{\beta-\eta}(z_1) + \mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(y_1). \end{aligned} \tag{3.2}$$

Multiplying both sides of (3.2) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)$ and then integrating the obtained inequality with respect to y_1 over $(0,\varsigma)$, we get

$$\begin{aligned} &\frac{\mathcal{Q}^{\rho}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{P}^{\rho+\beta-\eta}(y_1)\,dy_1 \\ &\qquad + \frac{\mathcal{P}^{\rho+\beta-\eta}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{Q}^{\rho}(y_1)\,dy_1 \\ &\quad \geq \frac{\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{P}^{\rho}(y_1)\,dy_1 \\ &\qquad + \frac{\mathcal{P}^{\rho}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\beta-\eta}(y_1)\,dy_1. \end{aligned}$$

In view of Definition 2.3, we have

$$\begin{aligned} &\mathcal{Q}^{\rho}(z_1){}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta+\rho}(\varsigma)\big] + \mathcal{P}^{\rho+\beta-\eta}(z_1){}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big] \\ &\quad \geq \mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1){}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(y_1)\big] + \mathcal{P}^{\rho}(z_1)\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]. \end{aligned} \tag{3.3}$$

Again, if we multiply both sides of (3.3) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\zeta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)$ and integrate the obtained inequality with respect to z_1 over $(0,\varsigma)$, then, by using (2.3), we obtain

$$\begin{aligned} &{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)\big] \\ &\qquad + {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big] \\ &\quad \geq {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big] \\ &\qquad + {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big], \end{aligned}$$

which gives the desired inequality (3.1). □

Theorem 3.2 *Assume that there are two positive and continuous functions $\mathcal{P}$ and $\mathcal{Q}$ defined on $[0,\infty)$ with the assumption that $\mathcal{R}:[0,\infty)\to\mathbb{R}$ is positive and continuous. Then for all $\varsigma > 0$, we have the following inequality*:

$$\begin{aligned} &{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big] \\ &\qquad + {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)\big] \end{aligned}$$

$$\begin{aligned}&\geq {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\\&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\zeta)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big],\end{aligned}\tag{3.4}$$

where $\beta\geq\eta>0$, $\rho>0$, $\sigma\in(0,1]$, $\zeta\in\mathcal{C}$, $\Re(\zeta)>0$.

Proof Multiplying both sides of (3.3) by $\frac{1}{\sigma^{\vartheta}\Gamma(\vartheta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)$, where $\vartheta,\eta>0$ and $z_1\in(0,\varsigma)$, $\varsigma>0$, and then integrating the obtained inequality with respect to z_1 over $(0,\varsigma)$, we have

$$\begin{aligned}&\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\,dz_1\\&\quad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{P}^{\rho+\beta-\eta}(z_1)\,dz_1\\&\geq\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1)\,dz_1\\&\quad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{P}^{\rho}(z_1)\,dz_1.\end{aligned}$$

It follows from Definition 2.3 that

$$\begin{aligned}&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\beta}(\varsigma)\big]\\&\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\\&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(\varsigma)\big],\end{aligned}$$

which is the desired inequality (3.4). □

Remark 3.3 Applying Theorem 3.2 for $\zeta=\vartheta$, we get Theorem 3.1.

Theorem 3.4 *Assume that there are two positive and continuous functions* $\mathcal{P}$ *and* $\mathcal{Q}$ *defined on* $[0,\infty)$ *with the assumption that* $\mathcal{R}:[0,\infty)\to\mathbb{R}$ *is positive and continuous. Then for all* $\varsigma>0$, *we have the inequality*

$$\begin{aligned}&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\&\quad\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big],\end{aligned}\tag{3.5}$$

where $\beta\geq\eta>0$, $\rho>0$, $\sigma\in(0,1]$, $\zeta\in\mathcal{C}$, $\Re(\zeta)>0$.

Proof Since $\mathcal{P}$ and $\mathcal{Q}$ are positive and continuous functions defined on $[0,\infty)$ such that a decreasing function $\mathcal{P}$ and an increasing function $\mathcal{Q}$ defined on $[0,\infty)$, hence for all $\rho > 0$, $\beta \geq \eta > 0$, $y_1, z_1 \in (0,\varsigma)$, $\varsigma > 0$, we have

$$\big(\mathcal{Q}^{\rho}(z_1) - \mathcal{Q}^{\rho}(y_1)\big)\big(\mathcal{P}^{\beta-\eta}(y_1) - \mathcal{P}^{\beta-\eta}(z_1)\big) \geq 0. \tag{3.6}$$

Inequality (3.6) leads to

$$\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(y_1) + \mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\beta-\eta}(z_1) \geq \mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1) + \mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(y_1). \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)$ and then integrating the obtained inequality with respect to y_1 over $(0,\varsigma)$, we get

$$\begin{aligned}
&\frac{\mathcal{Q}^{\rho}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{P}^{\beta-\eta}(y_1)\,dy_1\\
&\quad+\frac{\mathcal{P}^{\beta-\eta}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{Q}^{\rho}\,dy_1\\
&\quad\geq\frac{\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)(y_1)\,dy_1\\
&\quad+\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\mathcal{P}^{\eta}(y_1)\mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\beta-\eta}(y_1)\,dy_1.
\end{aligned}$$

From Definition 2.3 we clearly see that

$$\begin{aligned}
&\mathcal{Q}^{\rho}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big] + \mathcal{P}^{\beta-\eta}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\
&\quad\geq \mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big] + {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big].
\end{aligned} \tag{3.8}$$

Further, if we multiply both sides of (3.8) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\zeta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)$ and integrate the obtained inequality with respect to z_1 over $(0,\varsigma)$, and then employ (2.3), we obtain

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\\
&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\
&\quad\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\
&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big],
\end{aligned}$$

which gives the desired inequality (3.5). □

Theorem 3.5 *Assume that there are two positive and continuous functions $\mathcal{P}$ and $\mathcal{Q}$ defined on $[0,\infty)$, function $\mathcal{P}$ is decreasing and function $\mathcal{Q}$ is an increasing $[0,\infty)$, with the assumption that the function $\mathcal{R}:[0,\infty)\to\mathbb{R}$ is positive and continuous. Then for all $\varsigma > 0$, one has*

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\,{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\
&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]
\end{aligned}$$

$$\begin{aligned}&\geq {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\\&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big],\end{aligned}\tag{3.9}$$

where $\beta\geq\eta>0$, $\rho>0$, $\sigma\in(0,1]$, $\zeta\in\mathcal{C}$, $\Re(\zeta)>0$.

Proof Multiplying both sides of (3.8) by $\frac{1}{\sigma^{\vartheta}\Gamma(\vartheta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)$, where $\vartheta,\eta>0$ and $z_1\in(0,\varsigma)$, $\varsigma>0$, and then integrating the previous inequality with respect to z_1 over $(0,\varsigma)$, we have

$$\begin{aligned}&\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\,dz_1\\&\quad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{P}^{\beta-\eta}(z_1)\,dz_1\\&\geq\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}^{\rho}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\mathcal{P}^{\beta-\eta}(z_1)\,dz_1\\&\quad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\mathcal{P}^{\eta}(z_1)\,dz_1.\end{aligned}$$

From (2.3) we clearly see that the above inequality can be rewritten as

$$\begin{aligned}&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]\\&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\\&\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]\\&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\big]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\big[\mathcal{R}(\varsigma)\mathcal{P}^{\eta}(\varsigma)\big],\end{aligned}$$

which is the desired inequality (3.9). □

Remark 3.6 Applying Theorem 3.5 for $\zeta=\vartheta$, we get Theorem 3.4.

Theorem 3.7 *Assume that there are two functions* $\mathcal{P}_s$ $(s=1,2,\dots,n)$ *and* $\mathcal{Q}$ *which are positive and continuous, defined on* $[0,\infty)$, *with the assumption that the function* $\mathcal{R}:[0,\infty)\to\mathbb{R}$ *is positive and continuous. Then for all* $\varsigma>0$, *we have*

$$\begin{aligned}&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta+\rho}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg]\\&\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{U}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg],\end{aligned}\tag{3.10}$$

where $\beta\geq\eta_p>0$, $\rho>0$ $(p=1,2,\dots,n)$, $\sigma\in(0,1]$, $\zeta\in\mathcal{C}$, $\Re(\zeta)>0$.

Proof Since $\mathcal{P}_s$ and $\mathcal{Q}$ are positive and continuous functions on $[0,\infty)$, for all $y_1, z_1 \in (0,\varsigma)$, $\varsigma > 0$ and for any $\rho > 0$, $\beta \leq \eta_p > 0$, we have

$$\big(\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\rho}(y_1) - \mathcal{Q}^{\rho}(y_1)\mathcal{P}^{\rho}(z_1)\big)\big(\mathcal{P}_p^{\beta-\eta_p}(y_1) - \mathcal{P}_p^{\beta-\eta_p}(z_1)\big) \geq 0. \tag{3.11}$$

It follows from (3.11) that

$$\begin{aligned}&\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\rho+\beta-\eta_p}(y_1) + \mathcal{Q}^{\rho}(y_1)\mathcal{P}_p^{\rho+\beta-\eta_p}(z_1)\\&\quad\geq \mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\rho}(y_1)\mathcal{P}_p^{\beta-\eta_p}(z_1) + \mathcal{Q}^{\rho}(y_1)\mathcal{P}_p^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(y_1).\end{aligned} \tag{3.12}$$

Multiplying both sides of (3.12) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)$ ($\eta_s > 0$, $s = 1,2,\ldots,n$ and $y_1 \in (0,\varsigma)$) and then integrating the obtained inequality with respect to y_1 over $(0,\varsigma)$, we have

$$\begin{aligned}&\frac{\mathcal{Q}^{\rho}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{P}_p^{\rho+\beta-\eta_p}(y_1)\,dy_1\\
&\quad+\frac{\mathcal{P}_p^{\rho+\beta-\eta_p}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{Q}^{\rho}(y_1)\,dy_1\\
&\geq\frac{\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{P}_p^{\rho}(y_1)\,dy_1\\
&\quad+\frac{\mathcal{P}_p^{\rho}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\\
&\quad\times\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{Q}^{\rho}(y_1)\mathcal{P}_p^{\beta-\eta_p}(y_1)\,dy_1.\end{aligned}$$

In view of Definition 2.3, we have

$$\begin{aligned}&\mathcal{Q}^{\rho}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta+\rho}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\quad+\mathcal{P}_p^{\rho+\beta-\eta}(z_1)\mathcal{H}_{1,\varsigma}^{\zeta,\rho}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\eta_p}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\geq\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}^{\rho+\eta}(y_1)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\quad+\mathcal{P}_p^{\rho}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg].\end{aligned} \tag{3.13}$$

Again, if we multiply both sides of (3.13) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\zeta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(z_1)$ ($\eta_s > 0$, $s = 1,2,\ldots,n$ and $z_1 \in (0,\varsigma)$) and integrate the obtained inequality with respect to z_1 over $(0,\varsigma)$, then, by using (2.3), we obtain

$${}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]$$

$$+ {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]$$

$$\geq {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]$$

$$+ {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right],$$

which is the desired inequality (3.10). □

Theorem 3.8 *Assume that there are two functions $\mathcal{P}_s$ ($s = 1, 2, \ldots, n$) and $\mathcal{Q}$ which are positive and continuous, defined on $[0,\infty)$, with the assumption that the function $\mathcal{R} : [0,\infty) \to \mathbb{R}$ is positive and continuous. Then for all $\varsigma > 0$, we have the inequality*

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\right]\\
&\quad+ {}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\right]\\
&\quad\geq {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]\\
&\quad+ {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right],
\end{aligned}\tag{3.14}$$

where $\beta \geq \eta_p > 0$, $\rho > 0$ ($p = 1, 2, \ldots, n$), $\sigma \in (0, 1]$, $\zeta \in \mathcal{C}$, $\Re(\zeta) > 0$.

Proof Multiplying both sides of (3.13) by $\frac{1}{\sigma^{\vartheta}\Gamma(\vartheta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1) \times \prod_{s=1}^{n}\mathcal{P}^{\eta}(\varsigma)(z_1)$, where $\vartheta, \eta_s > 0$ ($s = 1, 2, \ldots, n$), $z_1 \in (0, \varsigma)$, $\varsigma > 0$ and integrating the obtained inequality with respect to z_1 over $(0, \varsigma)$, we have

$$\begin{aligned}
&\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\
&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\,dz_1\\
&\quad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\
&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{P}_p^{\rho+\beta-\eta_p}(z_1)\,dz_1\\
&\geq\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}\\
&\quad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\,dz_1
\end{aligned}$$

$$+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)}$$
$$\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{P}_p^{\rho}(z_1)\,dz_1.$$

It follows from Definition 2.3 that

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\Bigg]\\
&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho+\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\Bigg]\\
&\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_p^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg],
\end{aligned}$$

which is the required inequality (3.14). □

Remark 3.9 Applying Theorem 3.8 for $\zeta=\vartheta$, we get Theorem 3.7.

Theorem 3.10 *Assume that there are two functions $\mathcal{P}_s$ $(s=1,2,\dots,n)$ and $\mathcal{Q}$ which are positive and continuous, defined on $[0,\infty)$, with the assumption that the function $\mathcal{R}:[0,\infty)\rightarrow\mathbb{R}$ is positive and continuous, function $\mathcal{Q}$ is increasing and function $\mathcal{P}_s$ $(s=1,2,\dots,n)$ is decreasing on $[0,\infty)$. Then for all $\varsigma>0$, the following inequality holds*:

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg]\\
&\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg],
\end{aligned}\tag{3.15}$$

where $\beta\geq\eta_p>0$, $\rho>0$ $(p=1,2,\dots,n)$, $\sigma\in(0,1]$, $\zeta\in\mathcal{C}$, $\Re(\zeta)>0$.

Proof Utilizing hypothesis mentioned in Theorem 3.10, one obtains

$$\big(\mathcal{Q}^{\rho}(z_1)-\mathcal{Q}^{\rho}(y_1)\big)\big(\mathcal{P}_p^{\beta-\eta_p}(y_1)-\mathcal{P}_p^{\beta-\eta_p}(z_1)\big)\geq 0,\tag{3.16}$$

for any $y_1,z_1\in(0,\varsigma)$, $\varsigma>0$, $\rho>0$, $\beta\geq\eta>0$, and $p=1,2,\dots,n$.

It follows from (3.16) that

$$\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(y_1)+\mathcal{Q}^{\rho}(y_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\geq\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)+\mathcal{Q}^{\rho}(y_1)\mathcal{P}_p^{\beta-\eta_p}(z_1).\tag{3.17}$$

Multiplying both sides of (3.17) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)$ and integrating the obtained inequality with respect to y_1 over $(0,\varsigma)$, we have

$$\begin{aligned}
&\frac{\mathcal{Q}^{\rho}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{P}_p^{\beta-\eta_p}(y_1)\,dy_1\\
&\quad+\frac{\mathcal{P}_p^{\beta-\eta_p}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{Q}^{\rho}(y_1)\,dy_1\\
&\quad\geq\frac{\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\,dy_1\\
&\quad+\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(y_1))]}{(\varphi(\varsigma)-\varphi(y_1))^{1-\zeta}}\varphi'(y_1)\mathcal{R}(y_1)\\
&\quad\times\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(y_1)\mathcal{Q}^{\rho}(y_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\,dy_1.
\end{aligned}$$

Now, in view of Definition 2.3, we get

$$\begin{aligned}
&\mathcal{Q}^{\rho}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\quad+\mathcal{P}_p^{\beta-\eta_p}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\geq\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\,{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]\\
&\quad+{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{U}_p^{\beta}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg].
\end{aligned}\tag{3.18}$$

Again, multiplying both sides of (3.18) by $\frac{1}{\sigma^{\zeta}\Gamma(\zeta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\zeta}}\varphi'(z_1)\mathcal{R}(y_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(z_1)$, integrating the obtained inequality with respect to z_1 over $(0,\varsigma)$, and then using (2.3), we obtain

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg]\\
&\geq{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg],
\end{aligned}$$

which is the desired inequality (3.15). □

Theorem 3.11 *Assume that two functions $\mathcal{P}_s$ $(s=1,2,\dots,n)$ and $\mathcal{Q}$ defined on $[0,\infty)$ are positive and continuous with the assumption that the function $\mathcal{R}:[0,\infty)\to\mathbb{R}$ is positive and continuous, function $\mathcal{Q}$ is increasing and function $\mathcal{P}_s$ $(s=1,2,\dots,n)$ is decreasing on*

$[0,\infty)$. *Then for all* $\varsigma > 0$, *the following inequality holds*:

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg] \\
&\quad + {}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg] \\
&\geq {}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg] \\
&\quad + {}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg],
\end{aligned}\tag{3.19}$$

where $\beta \geq \eta_p > 0$, $\rho > 0$, $p = 1,2,\ldots,n$ *and* $\sigma \in (0,1]$, $\zeta \in \mathcal{C}$, $\Re(\zeta) > 0$.

Proof To obtain the desired assertion (3.19), we multiply (3.18) by $\frac{1}{\sigma^{\vartheta}\Gamma(\vartheta)}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\times \varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(z_1)$, where $\vartheta, \eta_s > 0$ $(s = 1,2,\ldots,n)$, $z_1 \in (0,\varsigma)$, $\varsigma > 0$, and then integrate the obtained inequality with respect to z_1 over $(0,\varsigma)$ to get

$$\begin{aligned}
&\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)} \\
&\qquad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\,dz_1 \\
&\qquad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)} \\
&\qquad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\,dz_1 \\
&\quad\geq\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)} \\
&\qquad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\mathcal{Q}^{\rho}(z_1)\mathcal{P}_p^{\beta-\eta_p}(z_1)\,dz_1 \\
&\qquad+\frac{{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)]}{\sigma^{\vartheta}\Gamma(\vartheta)} \\
&\qquad\times\int_0^{\varsigma}\frac{\exp[\frac{\sigma-1}{\sigma}(\varphi(\varsigma)-\varphi(z_1))]}{(\varphi(\varsigma)-\varphi(z_1))^{1-\vartheta}}\varphi'(z_1)\mathcal{R}(z_1)\prod_{s=1}^{n}\mathcal{P}_s^{\eta}(z_1)\,dz_1.
\end{aligned}$$

It follows from Definition 2.3 that

$$\begin{aligned}
&{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg] \\
&\quad + {}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\Bigg]{}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\Bigg[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\prod_{s=1}^{n}\mathcal{P}^{\eta_s}(\varsigma)\Bigg]
\end{aligned}$$

$$\geq {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\left[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]$$
$$+ {}^{\varphi}\mathcal{K}_{0^+}^{\zeta,\sigma}\left[\mathcal{R}(\varsigma)\mathcal{Q}^{\rho}(\varsigma)\mathcal{P}_p^{\beta}(\varsigma)\prod_{s\neq p}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right]{}^{\varphi}\mathcal{K}_{0^+}^{\vartheta,\sigma}\left[\mathcal{R}(\varsigma)\prod_{s=1}^{n}\mathcal{P}_s^{\eta_s}(\varsigma)\right],$$

which completes the proof of Theorem 3.11. □

Remark 3.12 Applying Theorem 3.11 for $\zeta = \vartheta$, we get Theorem 3.10.

4 Conclusions

In the article, we have derived certain variants by the use of the newly defined $\mathcal{GPFI}$ with respect to another function φ related to a class of n positive continuous and decreasing functions defined on $[\mu_1, \mu_2]$. In [76], Liu et al. investigated thought-provoking variants for continuous functions. Recently, Dahmani [77] has presented more generalizations of the work in [76] by utilizing the $\mathcal{RL}$-fractional integral operators. Therefore our findings in the present article are generalizations of integral inequalities involving the $\mathcal{RL}$-fractional integral operators. If we take into account $\mathcal{R}(\varsigma) = \mu = 1$ and $\varphi(x) = x$, then our findings derived in the present paper will become variants associated with the $\mathcal{RL}$-fractional integral operators introduced by Dahmani [77]. Particular cases of our consequences could be observed in [76]. The consequences acquired in this paper deliver some contributions to the direction of the idea of integral inequalities, fractional calculus, and anticipated results in some applications for establishing the uniqueness of solutions of integrodifferential equations and for finding the analytical solutions of some space-time fractional differential equations.

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