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Regularization of the fractional Rayleigh–Stokes equation using a fractional Landweber method

Nguyen Hoang Luc[1], Le Nhat Huynh[2,3], Donal O'Regan[4] and Nguyen Huu Can[5*]

*Correspondence:
nguyenhuucan@tdtu.edu.vn
[5]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract

In this paper, we consider a time-fractional backward problem for the fractional Rayleigh–Stokes equation in a general bounded domain. We propose a fractional Landweber regularization method for solving this problem. Error estimates between the regularized solution and the sought solution are also obtained under some choice rules for both a-priori and a-posterior regularization parameters.

Keywords: Fractional Rayleigh–Stokes equation; Fractional Landweber regularization method; Regularization

1 Introduction

Fractional partial differential equations have applications in applied science and engineering, and these applications appear in fluid flow, heat conduction, image processing (filtering, denoising [30], restorations, segmentation, edge enhancement/detection), see [6, 26]. Nonlocal properties of the fractional operator, fractional partial differential equations are useful for simulating real super-diffusion and sub-diffusion phenomena. In this paper we discuss the Rayleigh–Stokes problem for a heated generalized second grade fluid with a fractional derivative. Equation (1) below arises in Newtonian fluids and magnetohydrodynamic flows in porous media [9] and initial value problems for fractional Rayleigh–Stokes was studied, for example, in [1–4, 20, 29].

In this paper, we consider a backward problem of the fractional Rayleigh–Stokes equation with variable coefficient in a bounded domain:

$$\begin{cases} \partial_t u - (1+\gamma\partial_t^{\alpha})(\mathcal{L}u)(t,\mathbf{x}) = \mathcal{F}(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,\mathcal{T}) \times \Omega, \\ u(t,\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \\ u(\mathcal{T},\mathbf{x}) = g, & \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(1 \le d \le 3)$ with sufficiently smooth boundary $\partial\Omega$ and $\mathcal{T} > 0$ is a given time. Here $\gamma > 0$ is a constant, g is the final data in $L^2(\Omega)$, $\partial_t = \partial/\partial t$, and

∂_t^α is the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ defined by [13, 22]:

$$\partial_t^\alpha u(t,\mathbf{x}) = \begin{cases} \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-\xi)u(\xi,\mathbf{x})\,d\xi, & \omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0<\alpha<1, \\ u_t(t,\mathbf{x}) = 0, & \alpha = 1. \end{cases} \tag{2}$$

Let $\mathcal{L}$ be

$$\mathcal{L}\mathfrak{v}(\mathbf{x}) = \sum_{i=1}^{d} \frac{\partial}{\partial \mathbf{x}_i}\left(\sum_{j=1}^{d} \mathcal{B}_{ij}(\mathbf{x})\frac{\partial}{\partial \mathbf{x}_j}\mathfrak{v}(\mathbf{x})\right) + \mathfrak{C}(\mathbf{x})\mathfrak{v}(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Here:

(1) $\mathcal{B}_{ij} = \mathcal{B}_{ji}$, $1 \le i,j \le d$, $\mathcal{B}_{ij} \in C^1(\overline{\Omega})$.

(2) There exists a constant χ such that

$$\chi\sum_{i=1}^{d}\xi_i^2 \le \sum_{i=1}^{d}\sum_{j=1}^{d}\mathcal{B}_{ij}(\mathbf{x})\xi_i\xi_j, \quad \mathbf{x} \in \overline{\Omega}, \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d,$$

(3) The function $\mathfrak{C} \in C(\overline{\Omega})$ satisfies $\mathfrak{C}(x) \le 0$, $\mathbf{x} \in \overline{\Omega}$.

Our goal is to construct the initial data $h(\mathbf{x}) = u(0,\mathbf{x})$ from the given data $(g,\mathcal{F})$. When we observe the data $(g,\mathcal{F})$, we get approximate data $(g^\delta,\mathcal{F}^\delta)$ such that

$$\|g - g^\delta\|_{L^2(\Omega)} \le \delta, \qquad \|\mathcal{F} - \mathcal{F}^\delta\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} \le \delta, \tag{3}$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm and $\delta > 0$ is the noise level.

The corresponding direct problem for (1) is stated as follows:

$$\begin{cases} \partial_t u - (1+\gamma\partial_t^\alpha)(\mathcal{L}u)(t,\mathbf{x}) = \mathcal{F}(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,\mathcal{T}) \times \Omega, \\ u(t,\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \\ u(0,\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{4}$$

The backward problem for the time fractional diffusion equation was studied by many authors; see, for example, [5, 18, 23, 24]. Such a problem is ill-posed in the sense of Hadamard. The solution (if it exists) does not depend continuously on the given data. Indeed, a small error of the given observation can result in that the solution may have a large error. This makes numerical computation troublesome. Hence a regularization is needed. There are very few results on the backward problem for the fractional Rayleigh–Stokes equation, and the first regularization result for such problems seems to be that of Tuan et al. [19] where they regularized a Rayleigh–Stokes problem with random noise.

In this paper, we do not follow the method in [19]. We will present another method called the fractional Landweber method to find a regularized solution. This method was introduced by Klann and Ramlau [15] to consider a linear ill-posed problem. The main idea of the fractional Landweber method is based on iterative sequences, which is similar to the classical iterative method [7, 11, 12, 16, 25]. Using this method, some authors developed and established a fractional Tikhonov method [10, 21, 27] and fractional Landweber method [28] for solving some linear ill-posed models.

The outline of the paper is as follows: Sect. 2 discusses mild solutions and the ill-posedness of the problem. In Sect. 3, we introduce the fractional Landweber regularization method and present a convergence estimate under an a-priori assumption on the exact solution. The a-posteriori parameter choice rule is also discussed.

2 The backward time of the fractional Rayleigh–Stokes equation

2.1 Preliminaries

In this section, we introduce some notation and preliminaries. Assume that $-\mathcal{L}$ has eigenvalues $\{\widetilde{\lambda}_n\}$ and corresponding eigenfunctions $\{\mathcal{X}_n\}$ with $\mathcal{X}_n \in H^2(\Omega) \cap H_0^1(\Omega)$.

Note that $\{\widetilde{\lambda}_n\}$ satisfies

$$0 < \widetilde{\lambda}_1 \le \widetilde{\lambda}_2 \le \widetilde{\lambda}_3 \le \cdots \le \widetilde{\lambda}_n \le \cdots$$

and $\widetilde{\lambda}_n \to \infty$ as $n \to \infty$. Moreover,

$$\begin{cases} \mathcal{L}\mathcal{X}_n(\mathbf{x}) = -\widetilde{\lambda}_n \mathcal{X}_n(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathcal{X}_n(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

For $p > 0$, we let

$$D\big((-\mathcal{L})^p\big) = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \widetilde{\lambda}_n^{2p} \big|\langle v, \mathcal{X}_n\rangle\big|^2 < +\infty \right\}, \tag{5}$$

with the following norm:

$$\|v\|_{D((-\mathcal{L})^p)} = \left(\sum_{n=1}^{\infty} \widetilde{\lambda}_n^{2p} \big|\langle v, \mathcal{X}_n\rangle\big|^2 \right)^{\frac{1}{2}}.$$

In the following, we present a mild solution of our direct problem (4). Indeed, suppose that problem (4) has a solution u. Then using the result in [1], we obtain

$$u_n(t) = \mathcal{Q}(t, n, \alpha) h_n + \int_0^t \mathcal{Q}(t - \varsigma, n, \alpha) \mathcal{F}_n(\varsigma)\, d\varsigma, \tag{6}$$

where $h_n = \langle h, \mathcal{X}_n\rangle$, $\mathcal{F}_n(t) = \langle \mathcal{F}(t, \cdot), \mathcal{X}_n\rangle$ and

$$\mathcal{Q}(t, n, \alpha) = \frac{\gamma}{\pi} \int_0^{\infty} \frac{\widetilde{\lambda}_n \sin(\alpha\pi) \rho^{\alpha} e^{-\rho t}}{(-\rho + \widetilde{\lambda}_n \gamma \rho^{\alpha} \cos(\alpha\pi) + \widetilde{\lambda}_n)^2 + (\widetilde{\lambda}_n \gamma \rho^{\alpha} \sin(\alpha\pi))^2}\, d\rho. \tag{7}$$

This implies that

$$u(t, \mathbf{x}) = \sum_{n=1}^{\infty} \left[\mathcal{Q}(t, n, \alpha) h_n + \int_0^t \mathcal{Q}(t - \varsigma, n, \alpha) \mathcal{F}_n(\varsigma)\, d\varsigma \right] \mathcal{X}_n(\mathbf{x}), \tag{8}$$

where $h_n = \langle h(\cdot), \mathcal{X}_n\rangle$. For the convenience of the reader, we repeat the relevant material from [19].

Lemma 2.1 *Let us assume that* $\alpha \in (0, 1)$. *The following estimates hold*:

- *There exists* $\mathbf{D}_1(\mathcal{T},\alpha) > 0$ *such that*

$$\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha) \geq \frac{1}{\widetilde{\lambda}_\mathfrak{n}}\mathbf{D}_1(\mathcal{T},\alpha). \tag{9}$$

- *There exists a constant* $\mathbf{D}_2(\alpha) > 0$ *such that*

$$\mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha) \leq \frac{\mathbf{D}_2(\alpha)}{\widetilde{\lambda}_\mathfrak{n}} \min\left(\frac{1}{\mathfrak{t}}, \frac{1}{\mathfrak{t}^{1-\alpha}}\right), \quad 0 < \mathfrak{t} \leq \mathcal{T}. \tag{10}$$

Hence

$$\int_0^{\mathcal{T}} \left|\mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha)\right| d\mathfrak{t} \leq \frac{\mathbf{D}_2(\alpha)}{\widetilde{\lambda}_\mathfrak{n}}\left(\frac{1}{\alpha} + \ln\left(\max\{\mathcal{T},1\}\right)\right), \tag{11}$$

where

$$\mathbf{D}_1(\mathcal{T},\alpha) = \frac{\gamma\sin(\alpha\pi)}{3\pi}\int_0^{+\infty} \frac{e^{-\rho\mathcal{T}}\rho^\alpha \, d\rho}{\gamma^2\rho^{2\alpha} + 1 + \frac{\rho^2}{\lambda_1^2}}. \tag{12}$$

Lemma 2.2 *Let* $\mathcal{F} \in L^\infty(0,\mathcal{T};L^2(\Omega))$. *If* $1 \leq d \leq 3$ *then there exists a positive constant* $\mathcal{G}(\alpha,d)$ *such that*

$$\sum_{\mathfrak{n}=1}^{\infty}\left|\int_0^{\mathfrak{t}} \mathcal{Q}(\mathfrak{t}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_\mathfrak{n}(\varsigma)\,d\varsigma\right|^2 \leq \mathcal{G}(\alpha,d)\|\mathcal{F}\|^2_{L^\infty(0,\mathcal{T};L^2(\Omega))}, \tag{13}$$

where

$$\mathcal{G}(\alpha,d) = \frac{\mathbf{D}_2^2(\alpha)(\frac{1}{\alpha} + \ln(\max\{\mathcal{T},1\}))^2}{\mathcal{C}^2}\sum_{\mathfrak{n}=1}^{\infty}\frac{1}{\mathfrak{n}^{\frac{4}{d}}},$$

with $\mathcal{C}$ *being a positive constant independent of* $\mathfrak{n}$.

Next we present a representation for the mild solution of problem (1). We assume that problem (1) has a unique solution u and then u satisfies (8). By letting $\mathfrak{t} = \mathcal{T}$, we have

$$g_\mathfrak{n} = \mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)h_\mathfrak{n} + \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_\mathfrak{n}(\varsigma)\,d\varsigma \tag{14}$$

where $\mathcal{G}_\mathfrak{n} = \langle\mathcal{G}(\cdot),\mathcal{X}_\mathfrak{n}\rangle$. It follows that

$$\begin{aligned} h_\mathfrak{n} &= \frac{1}{\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)}\left(g_\mathfrak{n} - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_\mathfrak{n}(\varsigma)\,d\varsigma\right) \\ &= \sum_{\mathfrak{n}=1}^{\infty}\frac{1}{\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)}\mathbf{H}_\mathfrak{n} \end{aligned} \tag{15}$$

where

$$\mathbf{H}(\mathbf{x}) = \sum_{\mathfrak{n}=1}^{\infty}\left|g_\mathfrak{n} - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_\mathfrak{n}(\varsigma)\,d\varsigma\right|\mathcal{X}_\mathfrak{n}(\mathbf{x}). \tag{16}$$

By substituting $h_{\mathfrak{n}}$ into (8), we obtain

$$u_{\mathfrak{n}}(\mathfrak{t}) = \frac{\mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha)}{\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)}\left(g_{\mathfrak{n}} - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma\right) + \int_0^{\mathfrak{t}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma. \tag{17}$$

Hence, we get

$$u(\mathfrak{t},\mathbf{x}) = \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha)}{\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)}\left(g_{\mathfrak{n}} - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma\right)\mathcal{X}_{\mathfrak{n}}(\mathbf{x}) + \sum_{\mathfrak{n}=1}^{\infty}\left[\int_0^{\mathfrak{t}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma\right]\mathcal{X}_{\mathfrak{n}}(\mathbf{x}). \tag{18}$$

From [1, Theorem 2.2], the functions $\mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha)$, $\mathfrak{n} = 1,2,\dots$, are completely monotone for $\mathfrak{t} \geq 0$, and we get

$$\begin{cases} 0 < \mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha) < 1, & \mathfrak{t} > 0, \\ \mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha) = 1, & \mathfrak{t} = 0. \end{cases}$$

Our main goal is to find the initial value $u(0,\mathbf{x}) = h(\mathbf{x})$ from given data $(g,\mathcal{F})$. To find $h(\mathbf{x})$, we need to solve the integral equation as follows:

$$\mathcal{K}h(\mathbf{x}) = \int_0^{\mathcal{T}} k(\mathbf{x},\zeta)h(\zeta)\,d\zeta = \mathbf{H}(\mathbf{x}), \tag{19}$$

where

$$k(\mathbf{x},\zeta) = \sum_{\mathfrak{n}=1}^{\infty} \mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)\mathcal{X}_{\mathfrak{n}}(\mathbf{x})\mathcal{X}_{\mathfrak{n}}(\zeta).$$

Since $k(\mathbf{x},\zeta) = k(\zeta,\mathbf{x})$, it is clear that the operator $\mathcal{K}$ is self-adjoint. Now we prove that the operator $\mathcal{K}$ is a compact operator. Let us consider the finite rank operator $\mathcal{K}_{\mathfrak{M}}$ defined by

$$\mathcal{K}_{\mathfrak{M}}f(\mathbf{x}) = \sum_{\mathfrak{n}=1}^{\mathfrak{M}} \mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)\langle h,\mathcal{X}_{\mathfrak{n}}\rangle\mathcal{X}_{\mathfrak{n}}(\mathbf{x}). \tag{20}$$

We have

$$\big\|\mathcal{K}_{\mathfrak{M}}h(\mathbf{x}) - \mathcal{K}h(\mathbf{x})\big\|^2_{L^2(\Omega)} = \sum_{\mathfrak{M}+1}^{\infty}\big|\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)\big|^2\big|\langle h,\mathcal{X}_{\mathfrak{n}}\rangle\big|^2 \tag{21}$$

$$\leq \left[\frac{\mathbf{D}_2(\alpha)}{\widetilde{\lambda}_{\mathfrak{M}}}\min\left(\frac{1}{\mathfrak{t}},\frac{1}{\mathfrak{t}^{1-\alpha}}\right)\right]^2\|h\|^2_{L^2(\Omega)}. \tag{22}$$

Therefore $\|\mathcal{K}_{\mathfrak{M}}h(\mathbf{x}) - \mathcal{K}h(\mathbf{x})\| \to 0$ when $\mathfrak{M} \to \infty$ in $\mathbb{L}(L^2(\Omega),L^2(\Omega))$. Hence, $\mathcal{K}$ is a compact operator and from a result by Kirsch [14], we know that the problem is ill-posed. Hence we introduce the fractional Landweber regularization method to recover it.

Let us denote by μ_n the singular values for the linear self-adjoint compact operator $\mathcal{K}$:

$$\mu_n = \mathcal{Q}(\mathcal{T}, n, \alpha), \quad n = 1, 2, 3, \ldots \tag{23}$$

2.2 The ill-posedness of a backward time-fractional problem

To illustrate an ill-posedness of the backward problem, we give an example. Let $(g, \mathcal{F}) = (0, 0)$ and $(\widehat{g}, \widehat{\mathcal{F}}) = (\frac{1}{\sqrt{\lambda_q}}\mathcal{X}_q, \frac{1}{\sqrt{\lambda_q}}\mathcal{X}_q)$. It is easy to see that

$$\|\widehat{g} - g\| = \frac{1}{\sqrt{\lambda_q}} \quad \text{and} \quad \|\widehat{\mathcal{F}} - \mathcal{F}\| = \frac{1}{\sqrt{\lambda_q}}.$$

Hence

$$\lim_{q\to\infty} \|\widehat{g} - g\| = 0 \quad \text{and} \quad \lim_{q\to\infty} \|\widehat{\mathcal{F}} - \mathcal{F}\| = 0, \tag{24}$$

so we know that $(\widehat{g}, \widehat{\mathcal{F}})$ is an approximation of $(g, \mathcal{F})$ when q is large enough. Using $(\widehat{g}, \widehat{\mathcal{F}})$, we get the corresponding initial data $\widehat{h}$ and the function $\widehat{\mathbf{H}}$ as follows:

$$\widehat{\mathbf{H}}(\mathbf{x}) = \sum_{n=1}^{\infty} \left| \widehat{g}_n - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, q, \alpha) \widehat{\mathcal{F}}_n(\varsigma)\, d\varsigma \right| \mathcal{X}_n(\mathbf{x}),$$

$$\widehat{h}(\mathbf{x}) = \frac{1}{\mathcal{Q}(\mathcal{T}, n, \alpha)} \sum_{n=1}^{\infty} \left| \widehat{g}_n - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, n, \alpha) \widehat{\mathcal{F}}_n(\varsigma)\, d\varsigma \right| \mathcal{X}_n(\mathbf{x}).$$

Using Lemma 2.1, we obtain

$$\begin{aligned}
\|\widehat{\mathbf{H}} - \mathbf{H}\|^2 &= \sum_{n=1}^{\infty} \left[\langle \widehat{g} - g, \mathcal{X}_n \rangle - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, n, \alpha) \langle \widehat{\mathcal{F}} - \mathcal{F}, \mathcal{X}_n \rangle\, d\varsigma \right]^2 \\
&= \left[\frac{1}{\sqrt{\lambda_q}} - \frac{1}{\sqrt{\lambda_q}} \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, q, \alpha)\, d\varsigma \right]^2 \\
&= \frac{1}{\lambda_q} \left[1 - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, q, \alpha)\, d\varsigma \right]^2.
\end{aligned}$$

This gives

$$\lim_{q\to\infty} \|\widehat{\mathbf{H}} - \mathbf{H}\| = 0. \tag{25}$$

On the other hand, we have

$$\begin{aligned}
\|\widehat{h} - h\|^2 &= \sum_{n=1}^{\infty} \frac{1}{\mathcal{Q}^2(\mathcal{T}, n, \alpha)} \left[\langle \widehat{g} - g, \mathcal{X}_n \rangle - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, n, \alpha) \langle \widehat{\mathcal{F}} - \mathcal{F}, \mathcal{X}_n \rangle\, d\varsigma \right]^2 \\
&\geq \frac{\lambda_q^2}{\mathbf{D}_2^2(\alpha) \min(\frac{1}{t}, \frac{1}{t^{1-\alpha}})} \left[\frac{1}{\sqrt{\lambda_q}} - \frac{1}{\sqrt{\lambda_q}} \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, q, \alpha)\, d\varsigma \right]^2 \\
&\geq \frac{\lambda_q^2}{\mathbf{D}_2^2(\alpha) \min(\frac{1}{t}, \frac{1}{t^{1-\alpha}})} \frac{1}{\lambda_q} \left[1 - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, q, \alpha)\, d\varsigma \right]^2
\end{aligned}$$

$$\geq \frac{1}{\mathbf{D}_2^2(\alpha)\min(\frac{1}{\mathfrak{t}},\frac{1}{\mathfrak{t}^{1-\alpha}})}\lambda_{\mathfrak{q}}\left[1-\int_0^{\mathcal{T}}\mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{q},\alpha)\,d\varsigma\right]^2. \tag{26}$$

Therefore

$$\lim_{\mathfrak{q}\to\infty}\|\widehat{h}-h\| = +\infty. \tag{27}$$

We conclude that the backward problem is ill-posed in the Hadamard sense. Hence a regularization method is necessary.

2.3 Conditional stability

We impose the following a priori bound condition on the initial value $u(0,\mathbf{x}) = h(\mathbf{x})$:

$$\|h\|_{D((-\mathcal{L})^{\frac{\mathfrak{m}}{2}})} = \left(\sum_{\mathfrak{n}=1}^{\infty}|\widetilde{\lambda}_{\mathfrak{n}}|^{\mathfrak{m}}|h|^2\right)^{\frac{1}{2}} \leq \mathcal{P}, \tag{28}$$

where $\mathcal{P}$ and $\mathfrak{m}$ are both positive constants. Now we construct a conditional stability estimate for this backward problem.

Theorem 2.1 *Suppose $h \in D((-\mathcal{L})^{\frac{\mathfrak{m}}{2}}) \subset \mathcal{H}^{\mathfrak{m}}(\Omega)$ satisfies $\|h\|_{D((-\mathcal{L})^{\frac{\mathfrak{m}}{2}})} \leq \mathcal{P}$. Then we get*

$$\|h\|_{L^2(\Omega)} \leq \mathbf{P}(\mathcal{T},\alpha)\left[\|g\|^2_{L^2(\Omega)} + \mathcal{G}(\alpha,d)\|\mathcal{F}\|^2_{L^\infty(0,\mathcal{T};L^2(\Omega))}\right]^{\frac{\mathfrak{m}}{2\mathfrak{m}+2}}\mathcal{P}^{\frac{1}{\mathfrak{m}+1}}, \tag{29}$$

where $\mathcal{P}$ is a positive constant and $\mathbf{P}(\mathcal{T},\alpha) = \frac{2^{\frac{\mathfrak{m}}{2\mathfrak{m}+2}}}{[\mathbf{D}_1(\mathcal{T},\alpha)]^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}}$.

Proof Using (15) and Hölder's inequality, we get

$$\begin{aligned}
&\|h\|^2_{L^2(\Omega)} \\
&\quad= \sum_{\mathfrak{n}=1}^{\infty}\frac{[g_{\mathfrak{n}}-\int_0^{\mathcal{T}}\mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma]^2}{|\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)|^2} \\
&\quad\leq \sum_{\mathfrak{n}=1}^{\infty}\frac{[g_{\mathfrak{n}}-\int_0^{\mathcal{T}}\mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma]^{\frac{2\mathfrak{m}}{\mathfrak{m}+1}}[g_{\mathfrak{n}}-\int_0^{\mathcal{T}}\mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma]^{\frac{2}{\mathfrak{m}+1}}}{|\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)|^2} \\
&\quad= \mathfrak{I}_1^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}\mathfrak{I}_2^{\frac{1}{\mathfrak{m}+1}},
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&\mathfrak{I}_1 = \sum_{\mathfrak{n}=1}^{\infty}\left[g_{\mathfrak{n}}-\int_0^{\mathcal{T}}\mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma\right]^2 \quad\text{and} \\
&\mathfrak{I}_2 = \sum_{\mathfrak{n}=1}^{\infty}\frac{[g_{\mathfrak{n}}-\int_0^{\mathcal{T}}\mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma]^2}{|\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)|^{2(\mathfrak{m}+1)}}.
\end{aligned} \tag{31}$$

From Lemma 2.2, we estimate $\mathfrak{I}_1$ as follows:

$$\begin{aligned}\mathfrak{I}_1 &\le 2\left[\sum_{\mathfrak{n}=1}^{\infty} g_{\mathfrak{n}}^2 + \sum_{\mathfrak{n}=1}^{\infty}\left|\int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma\right|^2\right] \\ &\le 2\big[\|g\|^2_{L^2(\Omega)} + \mathcal{G}(\alpha,d)\|\mathcal{F}\|^2_{L^\infty(0,\mathcal{T};L^2(\Omega))}\big]. \end{aligned} \tag{32}$$

For estimating $\mathfrak{I}_2$, we use Lemma 2.1 to obtain

$$\begin{aligned}\mathfrak{I}_2 &= \sum_{\mathfrak{n}=1}^{\infty} \frac{1}{|\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)|^{2\mathfrak{m}}} \frac{[g_{\mathfrak{n}} - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T}-\varsigma,\mathfrak{n},\alpha)\mathcal{F}_{\mathfrak{n}}(\varsigma)\,d\varsigma]^2}{|\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)|^2} \\ &\le \frac{1}{[\mathbf{D}_1(\mathcal{T},\alpha)]^{2\mathfrak{m}}} \sum_{\mathfrak{n}=1}^{\infty} \widetilde{\lambda}_{\mathfrak{n}}^{2\mathfrak{m}} \big|\langle h,\mathcal{X}_{\mathfrak{n}}\rangle\big|^2 \\ &= \frac{1}{[\mathbf{D}_1(\mathcal{T},\alpha)]^{2\mathfrak{m}}} \|h\|^2_{D((-\mathcal{L})^{\mathfrak{m}})(\Omega)}. \end{aligned} \tag{33}$$

Combining (30), (32) and (33), we get

$$\begin{aligned}\|h\|^2_{L^2(\Omega)} &\le \frac{2^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}}{[\mathbf{D}_1(\mathcal{T},\alpha)]^{\frac{2\mathfrak{m}}{\mathfrak{m}+1}}} \big[\|g\|^2_{L^2(\Omega)} + \mathcal{G}(\alpha,d)\|\mathcal{F}\|^2_{L^\infty(0,\mathcal{T};L^2(\Omega))}\big]^{\frac{\mathfrak{m}}{\mathfrak{m}+1}} \\ &\quad \times \|h\|^{\frac{2}{\mathfrak{m}+1}}_{D((-\mathcal{L})^{\mathfrak{m}})(\Omega)}. \end{aligned} \tag{34}$$

Thus

$$\|h\|_{L^2(\Omega)} \le \mathbf{P}(\mathcal{T},\alpha)\big[\|g\|^2_{L^2(\Omega)} + \mathcal{G}(\alpha,d)\|\mathcal{F}\|^2_{L^\infty(0,\mathcal{T};L^2(\Omega))}\big]^{\frac{\mathfrak{m}}{2\mathfrak{m}+2}} \mathcal{P}^{\frac{1}{\mathfrak{m}+1}}, \tag{35}$$

which completes the proof of the theorem. □

3 Regularization method and error estimate under two parameter choice rules

In this section, we introduce the fractional Landweber regularization method and also analyze the convergence properties of regularization methods under two parameter choice rules.

From [14], the operator equation $\mathcal{K}h = \mathbf{H}$ is equivalent to the following equation:

$$h = \big(I - \mathrm{a}\mathcal{K}^*\mathcal{K}\big)h + \mathrm{a}\mathcal{K}^*\mathbf{H}, \tag{36}$$

for any $\mathrm{a} > 0$. Here, $\mathcal{K}^*$ is the adjoint operator of $\mathcal{K}$, and $\mathrm{a} > 0$ satisfies $0 < \mathrm{a} < \frac{1}{\|\mathcal{K}\|^2}$. The iterative implementation of the fractional Landweber method was constructed in [15]. Let us denote the fractional Landweber regularization solution by

$$h_{\beta,\vartheta} = \sum_{\mathfrak{n}=1}^{\infty}\big[1 - \big(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\big)^{\beta}\big]^{\vartheta} \frac{1}{\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)} \langle \mathbf{H},\mathcal{X}_{\mathfrak{n}}\rangle\mathcal{X}_{\mathfrak{n}}, \tag{37}$$

and the fractional Landweber regularization solution with the noisy data by

$$h^{\delta}_{\beta,\vartheta} = \sum_{\mathfrak{n}=1}^{\infty}\big[1 - \big(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\big)^{\beta}\big]^{\vartheta} \frac{1}{\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)} \langle \mathbf{H}^{\delta},\mathcal{X}_{\mathfrak{n}}\rangle\mathcal{X}_{\mathfrak{n}}, \tag{38}$$

where $\vartheta \in (\frac{1}{2}, 1]$ is called the fractional parameter, and $\beta = 1, 2, 3, \dots$ is a regularization parameter. When $\vartheta = 1$, this is the classical Landweber method.

Lemma 3.1 *For $\widetilde{\lambda}_{\mathfrak{n}} > 0$, $\beta > 0$, $\vartheta \in (\frac{1}{2}, 1]$, and $0 < a\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha) < 1$, we get*

$$\sup_{\widetilde{\lambda}_{\mathfrak{n}} > 0} \bigl[1 - \bigl(1 - a\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\beta}\bigr]^{\vartheta} \frac{1}{\mathcal{Q}(\mathcal{T}, \mathfrak{n}, \alpha)} \le a^{\frac{1}{2}} \beta^{\frac{1}{2}}. \tag{39}$$

Proof Let us denote two functions with $\zeta^2 := \mathrm{a}\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)$:

$$\Psi(\zeta) = \mathrm{a}\zeta^{-2}\bigl[1 - \bigl(1 - \zeta^2\bigr)^{\beta}\bigr]^{2\vartheta} \tag{40}$$

and

$$\widetilde{\Psi}(\zeta) = \zeta^{-2}\bigl[1 - \bigl(1 - \zeta^2\bigr)^{\eta}\bigr]^{2\vartheta}. \tag{41}$$

Note that $\Psi(\zeta) = \mathrm{a}\widetilde{\Psi}(\zeta)$. These two functions are continuous in $[0, +\infty)$ when $\zeta \in (0, 1)$.

For $\vartheta \in (\frac{1}{2}, 1]$ and $\mathrm{a} < \frac{1}{\|\mathcal{K}\|^2}$, using Lemma 3.3 in [15], we have

$$\widetilde{\Psi}(\zeta) \le \beta. \tag{42}$$

Combining (40) and (42) gives

$$\sup_{\mu_{\mathfrak{n}} > 0} \Psi(\zeta) \le \mathrm{a}\beta. \tag{43}$$

Therefore

$$\sup_{\widetilde{\lambda}_{\mathfrak{n}} > 0} \bigl[1 - \bigl(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\beta}\bigr]^{\vartheta} \frac{1}{\mathcal{Q}(\mathcal{T}, \mathfrak{n}, \alpha)} \le \mathrm{a}^{\frac{1}{2}} \beta^{\frac{1}{2}}, \tag{44}$$

and this is precisely the assertion of the lemma. □

Lemma 3.2 *For $\widetilde{\lambda}_{\mathfrak{n}} > 0$, $\beta > 0$, and $0 < a\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha) < 1$, we have*

$$\sup_{\widetilde{\lambda}_{\mathfrak{n}} > 0} \bigl(1 - a\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\beta} \mathcal{Q}^{\mathfrak{m}}(\mathcal{T}, \mathfrak{n}, \alpha) \le \left(\frac{\mathfrak{m}}{2a}\right)^{\frac{\mathfrak{m}}{2}} \beta^{-\frac{\mathfrak{m}}{2}}. \tag{45}$$

Proof Consider the function $f(\mathbf{z}) = (1 - \mathrm{a}\mathbf{z})^{\beta} \mathbf{z}^{\frac{\mathfrak{m}}{2}}$, where $\mathbf{z} := \mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha) < \frac{1}{\mathrm{a}}$.

It is easy to see that there exists a unique $\mathbf{z}_0 = \frac{c}{\mathrm{a}(c+\beta)}$ with $c = \frac{\mathfrak{m}}{2}$ such that $f'(\mathbf{z}_0) = 0$. This implies that

$$\begin{aligned} f(\mathbf{z}) \le f(\mathbf{z}_0) &\le \left(1 - \frac{c}{c+\beta}\right)^{\beta} \left(\frac{c}{\mathrm{a}(c+\beta)}\right)^{c} \le \left(\frac{c}{\mathrm{a}}\right)^{c} \left(\frac{1}{c+\beta}\right)^{c} \\ &< \left(\frac{c}{\mathrm{a}}\right)^{c} \left(\frac{1}{\beta}\right)^{c} = \left(\frac{\mathfrak{m}}{2\mathrm{a}}\right)^{\frac{\mathfrak{m}}{2}} \left(\frac{1}{\beta}\right)^{\frac{\mathfrak{m}}{2}}, \end{aligned}$$

which completes the proof. □

3.1 The a-priori parameter choice

Theorem 3.1 *Let $h \in L^2(\Omega)$, given by (15), be the initial value of problem (1). Suppose the a priori bound condition (28) and (23) hold. Then the error estimate between the exact solution and its regularized solution with the exact data is as follows*:

$$\|h - h_{\beta,\vartheta}\|_{L^2(\Omega)} \le \frac{1}{\mathbf{D}_1(\mathcal{T},\alpha)}\left(\frac{\mathfrak{m}}{2a}\right)^{\frac{\mathfrak{m}}{2}} \beta^{-\frac{\mathfrak{m}}{2}}\mathcal{P}. \tag{46}$$

Proof Using Parseval's equality, we get

$$\begin{aligned}\|h - h_{\beta,\vartheta}\|^2_{L^2(\Omega)} &= \sum_{\mathfrak{n}=1}^{\infty}\Big[1 - \big[1 - \big(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\big)^{\beta}\big]^{\vartheta}\Big]^2 \frac{1}{\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)}\big|\langle \mathbf{H},\mathcal{X}_{\mathfrak{n}}\rangle\big|^2 \\ &\le \sum_{\mathfrak{n}=1}^{\infty}\big(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\big)^{2\beta}\big|\langle h,\mathcal{X}_{\mathfrak{n}}\rangle\big|^2 \\ &\le \sum_{\mathfrak{n}=1}^{\infty}\big(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\big)^{2\beta}\widetilde{\lambda}_{\mathfrak{n}}^{-2\mathfrak{m}} \sum_{n=1}^{\infty}\widetilde{\lambda}_{\mathfrak{n}}^{2\mathfrak{m}}\big|\langle h,\mathcal{X}_{\mathfrak{n}}\rangle\big|^2.\end{aligned} \tag{47}$$

From Lemma 2.1, we deduce that

$$\frac{1}{\widetilde{\lambda}_{\mathfrak{n}}} \le \frac{\mathcal{Q}(\mathcal{T},n,\alpha)}{\mathbf{D}_1(\mathcal{T},\alpha)}. \tag{48}$$

Applying Lemma 3.2, we have

$$\begin{aligned}&\|h - h_{\beta,\vartheta}\|^2_{L^2(\Omega)} \\ &\quad\le \frac{1}{\mathbf{D}_1^{2\mathfrak{m}}(\mathcal{T},\alpha)} \sup_{\widetilde{\lambda}_{\mathfrak{n}}>0}\big(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\big)^{2\beta}\mathcal{Q}^{2\mathfrak{m}}(\mathcal{T},\mathfrak{n},\alpha)\|h\|^2_{D((-\mathcal{L})^{\mathfrak{m}})(\Omega)} \\ &\quad\le \mathbf{D}_1^{-2\mathfrak{m}}(\mathcal{T},\alpha)\left(\frac{\mathfrak{m}}{2\mathrm{a}}\right)^{\mathfrak{m}}\beta^{-\mathfrak{m}}\mathcal{P}^2.\end{aligned} \tag{49}$$

Thus we get

$$\|h - h_{\beta,\vartheta}\|_{L^2(\Omega)} \le \mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha)\left(\frac{\mathfrak{m}}{2\mathrm{a}}\right)^{\frac{\mathfrak{m}}{2}}\beta^{-\frac{\mathfrak{m}}{2}}\mathcal{P}. \tag{50}$$

□

Theorem 3.2 *Let $h \in L^2(\Omega)$ and $\mathcal{F} \in L^\infty(0,\mathcal{T};L^2(\Omega))$. Assume the a priori bound condition (28) holds. If we choose the regularization parameter $\beta = [\Lambda]$ where*

$$\Lambda = \left(\frac{\mathcal{P}}{\delta}\right)^{\frac{2}{\mathfrak{m}+1}},$$

then we get the following error estimate between the exact solution and its regularization solution with the noisy data:

$$\big\|h - h^{\delta}_{\beta,\vartheta}\big\|_{L^2(\Omega)} \le \left(\mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha)\left(\frac{\mathfrak{m}}{2a}\right)^{\frac{\mathfrak{m}}{2}} + a^{\frac{1}{2}}\big(2 + 2\mathcal{G}(\alpha,d)\big)^{\frac{1}{2}}\right)\mathcal{P}^{\frac{1}{\mathfrak{m}+1}}\delta^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}, \tag{51}$$

where $[\Lambda]$ denotes the largest integer less than or equal to Λ.

Proof From the triangle inequality, we get

$$\left\| h - h^{\delta}_{\beta,\vartheta} \right\|_{L^2(\Omega)} \le \| h - h_{\beta,\vartheta} \|_{L^2(\Omega)} + \left\| h_{\beta,\vartheta} - h^{\delta}_{\beta,\vartheta} \right\|_{L^2(\Omega)}. \tag{52}$$

Using Parseval's equality, we obtain that

$$\begin{aligned} &\left\| h_{\beta,\vartheta} - h^{\delta}_{\beta,\vartheta} \right\|^2_{L^2(\Omega)} \\ &\quad = \sum_{\mathfrak{n}=1}^{\infty} \left[1 - \left(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta}\right]^{2\vartheta} \frac{1}{\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)} \left|\left\langle \mathbf{H} - \mathbf{H}^{\delta}, \mathcal{X}_{\mathfrak{n}} \right\rangle\right|^2 \\ &\quad = \sup_{\widetilde{\lambda}_{\mathfrak{n}}>0} \left[1 - \left(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta}\right]^{2\vartheta} \frac{1}{\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)} \sum_{\mathfrak{n}=1}^{\infty} \left|\left\langle \mathbf{H} - \mathbf{H}^{\delta}, \mathcal{X}_{\mathfrak{n}} \right\rangle\right|^2. \end{aligned} \tag{53}$$

Note that

$$\begin{aligned} &\sum_{\mathfrak{n}=1}^{\infty} \left|\left\langle \mathbf{H} - \mathbf{H}^{\delta}, \mathcal{X}_{\mathfrak{n}} \right\rangle\right|^2 \\ &\quad = \sum_{\mathfrak{n}=1}^{\infty} \left[\left\langle g - g^{\delta}, \mathcal{X}_{\mathfrak{n}} \right\rangle - \int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, \mathfrak{n}, \alpha) \left\langle \mathcal{F} - \mathcal{F}^{\delta}, \mathcal{X}_{\mathfrak{n}} \right\rangle d\varsigma \right]^2 \\ &\quad \le 2\left\| g - g^{\delta} \right\|^2_{L^2(\Omega)} + 2 \sum_{\mathfrak{n}=1}^{\infty} \left[\int_0^{\mathcal{T}} \mathcal{Q}(\mathcal{T} - \varsigma, \mathfrak{n}, \alpha) \left\langle \mathcal{F} - \mathcal{F}^{\delta}, \mathcal{X}_{\mathfrak{n}} \right\rangle d\varsigma \right]^2 \\ &\quad \le 2\left\| g - g^{\delta} \right\|^2_{L^2(\Omega)} + 2\mathcal{G}(\alpha,d) \left\| \mathcal{F} - \mathcal{F}^{\delta} \right\|^2_{L^{\infty}(0,\mathcal{T};L^2(\Omega))} \\ &\quad \le 2\delta^2\left(1 + \mathcal{G}(\alpha,d)\right). \end{aligned} \tag{54}$$

From (53) and (54), by Lemma 3.1, we get

$$\left\| h_{\beta,\vartheta} - h^{\delta}_{\beta,\vartheta} \right\|_{L^2(\Omega)} \le \mathrm{a}^{\frac{1}{2}} \beta^{\frac{1}{2}} \delta\left(2 + 2\mathcal{G}(\alpha,d)\right)^{\frac{1}{2}}. \tag{55}$$

Combining the above two inequalities (50) and (55), we obtain

$$\left\| h - h^{\delta}_{\beta,\vartheta} \right\|_{L^2(\Omega)} \le \mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha) \left(\frac{\mathfrak{m}}{2\mathrm{a}}\right)^{\frac{\mathfrak{m}}{2}} \beta^{-\frac{\mathfrak{m}}{2}} \mathcal{P} + \mathrm{a}^{\frac{1}{2}} \beta^{\frac{1}{2}} \delta\left(2 + 2\mathcal{G}(\alpha,d)\right)^{\frac{1}{2}}. \tag{56}$$

Choosing the regularization parameter β as

$$\beta = \left[\left(\frac{\mathcal{P}}{\delta}\right)^{\frac{2}{\mathfrak{m}+1}} \right],$$

we then obtain the following error estimate:

$$\left\| h - h^{\delta}_{\beta,\vartheta} \right\|_{L^2(\Omega)} \le \left(\mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha) \left(\frac{\mathfrak{m}}{2\mathrm{a}}\right)^{\frac{\mathfrak{m}}{2}} + \mathrm{a}^{\frac{1}{2}} \left(2 + 2\mathcal{G}(\alpha,d)\right)^{\frac{1}{2}} \right) \mathcal{P}^{\frac{1}{\mathfrak{m}+1}} \delta^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}. \tag{57}$$

□

3.2 A-posteriori parameter choice rule and convergence analysis

In this subsection, we give the convergence estimate between the regularized solution and the exact solution by using an a posteriori choice rule for the regularization parameter. From results in Morozov's discrepancy principal [8], the general a-posteriori rule can be formulated as follows:

$$\left\|\mathcal{K}h^{\delta}_{\beta,\vartheta} - \mathbf{H}^{\delta}\right\| \le \wp\delta, \tag{58}$$

where $\wp > 1$ is a constant independent of δ, $\beta > 0$ is the regularization parameter which makes (58) hold at the first iteration time.

Lemma 3.3 *Set* $\mathcal{M}(\beta) = \|\mathcal{K}h^{\delta}_{\beta,\vartheta} - \mathbf{H}^{\delta}\|$. *Then we have the following conclusions*:

(a) $\mathcal{M}(\beta)$ *is a continuous function*.
(b) $\mathcal{M}(\beta) \to 0$ *as* $\beta \to \infty$.
(c) $\mathcal{M}(\beta) \to \|\mathbf{H}^{\delta}\|_{L^2(\Omega)}$ *as* $\beta \to 0$.
(d) $\mathcal{M}(\beta)$ *is a strictly decreasing function*, *for any* $\beta \in (0, +\infty)$.

Proof From our results, we get

$$\begin{aligned}\mathcal{M}(\beta) &= \left\|\mathcal{K}h^{\delta}_{\beta,\vartheta} - \mathbf{H}^{\delta}\right\| \\ &= \left(\sum_{\mathfrak{n}=1}^{\infty}\left[1 - \left[1 - \left(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta}\right]^{\vartheta}\right]^2\left|\langle\mathbf{H}^{\delta},\mathcal{X}_{\mathfrak{n}}\rangle\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\lim_{\beta\to 0}\mathcal{M}(\beta) = \left(\sum_{\mathfrak{n}=1}^{\infty}\left|\langle\mathbf{H}^{\delta},\mathcal{X}_{\mathfrak{n}}\rangle\right|^2\right)^{\frac{1}{2}} = \left\|\mathbf{H}^{\delta}\right\|_{L^2(\Omega)},$$

and the conditions (a) through (d) hold. □

Remark 3.1 In this paper, without loss of generality we can assume that the noisy data $\|\mathbf{H}^{\delta}\|_{L^2(\Omega)}$ is large enough such that $0 < \wp\delta \le \|\mathbf{H}^{\delta}\|_{L^2(\Omega)}$. From Lemma 3.3, there exists a unique minimal solution for the inequality (58).

Lemma 3.4 *Let* β *satisfy* (58). *Then*, *we have the following inequality*:

$$\beta \le \frac{\mathfrak{m}+1}{2a}\left(\frac{\mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha)}{\wp - (2 + 2\mathcal{G}(\alpha,d))^{\frac{1}{2}}}\right)^{\frac{2}{\mathfrak{m}+1}}\left(\frac{\mathcal{P}}{\delta}\right)^{\frac{2}{\mathfrak{m}+1}}. \tag{59}$$

Proof From the definition of β, $\vartheta \in (\frac{1}{2}, 1]$, and $0 < \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha) < 1$, we have

$$\begin{aligned}&\left\|\mathcal{K}h^{\delta}_{\beta-1,\vartheta} - \mathbf{H}^{\delta}\right\| \\ &\quad= \left\|\sum_{\mathfrak{n}=1}^{\infty}\left[1 - \left[1 - \left(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta-1}\right]^{\vartheta}\right]\langle\mathbf{H}^{\delta},\mathcal{X}_{\mathfrak{n}}\rangle\mathcal{X}_{\mathfrak{n}}(\mathbf{x})\right\| \\ &\quad\le \left\|\sum_{\mathfrak{n}=1}^{\infty}\left[1 - \left[1 - \left(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta-1}\right]^{\vartheta}\right]\langle\mathbf{H}^{\delta} - \mathbf{H},\mathcal{X}_{\mathfrak{n}}\rangle\mathcal{X}_{\mathfrak{n}}(\mathbf{x})\right\|\end{aligned}$$

$$+\left\|\sum_{\mathfrak{n}=1}^{\infty}\left[1-\left[1-\left(1-\mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta-1}\right]^{\vartheta}\right]\langle\mathbf{H},\mathcal{X}_\mathfrak{n}\rangle\mathcal{X}_\mathfrak{n}(\mathbf{x})\right\|$$

$$\leq\left\|\mathbf{H}^\delta-\mathbf{H}\right\|+\left\|\sum_{\mathfrak{n}=1}^{\infty}\left(1-\mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta-1}\mathcal{Q}(\mathcal{T},\mathfrak{n},\alpha)\langle h,\mathcal{X}_\mathfrak{n}\rangle\mathcal{X}_\mathfrak{n}(\mathbf{x})\right\|. \tag{60}$$

By Lemma 2.1 and (54), we get

$$\left\|\mathcal{K}h^\delta_{\beta-1,\vartheta}-\mathbf{H}^\delta\right\|\leq\delta\left(2+2\mathcal{G}(\alpha,d)\right)^{\frac{1}{2}}$$
$$+\sup_{\widetilde{\lambda}_\mathfrak{n}>0}\left(1-\mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta-1}\mathcal{Q}^{\mathfrak{m}+1}(\mathcal{T},\mathfrak{n},\alpha)\mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha)\|h\|_{D((-\mathcal{L})^{\frac{\mathfrak{m}}{2}})}.$$

From [17] and [25], for $0<\kappa<1$, $m>0$, and $p\in\mathbb{N}$:

$$(1-\kappa)^p\kappa^m\leq m^m(p+1)^{-m}.$$

This implies that

$$\wp\delta\leq\delta\left(2+2\mathcal{G}(\alpha,d)\right)^{\frac{1}{2}}+\mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha)\mathcal{P}\left(\frac{\mathfrak{m}+1}{2\mathrm{a}}\right)^{\frac{\mathfrak{m}+1}{2}}\beta^{-\frac{\mathfrak{m}+1}{2}},$$

so

$$\beta\leq\frac{\mathfrak{m}+1}{2\mathrm{a}}\left(\frac{\mathbf{D}_1^{-\mathfrak{m}}(\mathcal{T},\alpha)}{\wp-(2+2\mathcal{G}(\alpha,d))^{\frac{1}{2}}}\right)^{\frac{2}{\mathfrak{m}+1}}\left(\frac{\mathcal{P}}{\delta}\right)^{\frac{2}{\mathfrak{m}+1}}.$$

□

Theorem 3.3 *If the a-priori condition* (28) *and the noise assumption* (3) *hold, then we have the following convergence estimate*;

$$\left\|h-h^\delta_{\beta,\vartheta}\right\|_{L^2(\Omega)}\leq[\mathcal{M}_1+\mathcal{M}_2]\mathcal{P}^{\frac{1}{\mathfrak{m}+1}}\delta^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}. \tag{61}$$

Proof From the triangle inequality, we get

$$\left\|h-h^\delta_{\beta,\vartheta}\right\|_{L^2(\Omega)}\leq\|h-h_{\beta,\vartheta}\|_{L^2(\Omega)}+\left\|h_{\beta,\vartheta}-h^\delta_{\beta,\vartheta}\right\|_{L^2(\Omega)}. \tag{62}$$

From Lemma 3.4 and (55), we see that

$$\left\|h_{\beta,\vartheta}-h^\delta_{\beta,\vartheta}\right\|_{L^2(\Omega)}\leq\mathrm{a}^{\frac{1}{2}}\beta^{\frac{1}{2}}\delta\left(2+2\mathcal{G}(\alpha,d)\right)^{\frac{1}{2}} \tag{63}$$

$$\leq\left(\frac{\mathrm{a}(1+\mathcal{G}(\alpha,d))(\mathfrak{m}+1)}{(\wp-(2+2\mathcal{G}(\alpha,d))^{\frac{1}{2}})^{\frac{2}{\mathfrak{m}+1}}\mathbf{D}_1^{\frac{2\mathfrak{m}}{\mathfrak{m}+1}}(\mathcal{T},\alpha)}\right)^{\frac{1}{2}}\mathcal{P}^{\frac{1}{\mathfrak{m}+1}}\delta^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}. \tag{64}$$

Using the triangle inequality, the a-priori bound condition (58), and $0<\mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)<1$, it follows that

$$\|h-h_{\beta,\vartheta}\|_{L^2(\Omega)}$$
$$\leq\left\|\sum_{\mathfrak{n}=1}^{\infty}\left[1-\left[1-\left(1-\mathrm{a}\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)\right)^{\beta}\right]^{\vartheta}\right]\frac{1}{\mathcal{Q}^2(\mathcal{T},\mathfrak{n},\alpha)}\langle\mathbf{H},\mathcal{X}_\mathfrak{n}\rangle\mathcal{X}_\mathfrak{n}(\mathbf{x})\right\|$$

$$
\begin{aligned}
&\leq \left\| \sum_{\mathfrak{n}=1}^{\infty} \langle h, \mathcal{X}_{\mathfrak{n}} \rangle \mathcal{X}_{\mathfrak{n}}(\mathbf{x}) \right\|^{\frac{1}{\mathfrak{m}+1}} \\
&\quad \times \left\| \sum_{\mathfrak{n}=1}^{\infty} \bigl(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\beta} \frac{1}{\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)} \langle \mathbf{H}, \mathcal{X}_{\mathfrak{n}} \rangle \mathcal{X}_{\mathfrak{n}}(\mathbf{x}) \right\|^{\frac{\mathfrak{m}}{\mathfrak{m}+1}} \\
&\leq \left\| \sum_{\mathfrak{n}=1}^{\infty} \widetilde{\lambda}_{\mathfrak{n}}^{-\mathfrak{m}} \widetilde{\lambda}_{\mathfrak{n}}^{\mathfrak{m}} \langle h, \mathcal{X}_{\mathfrak{n}} \rangle \mathcal{X}_{\mathfrak{n}}(\mathbf{x}) \right\|^{\frac{1}{\mathfrak{m}+1}} \\
&\quad \times \left(\left\| \sum_{\mathfrak{n}=1}^{\infty} \bigl(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\beta} \frac{1}{\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)} \langle \mathbf{H}^{\delta}, \mathcal{X}_{\mathfrak{n}} \rangle \mathcal{X}_{\mathfrak{n}}(\mathbf{x}) \right\| \right. \\
&\quad \left. + \left\| \sum_{\mathfrak{n}=1}^{\infty} \bigl(1 - \mathrm{a}\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\beta} \frac{1}{\mathcal{Q}^2(\mathcal{T}, \mathfrak{n}, \alpha)} \langle \mathbf{H} - \mathbf{H}^{\delta}, \mathcal{X}_{\mathfrak{n}} \rangle \mathcal{X}_{\mathfrak{n}}(\mathbf{x}) \right\| \right)^{\frac{\mathfrak{m}}{\mathfrak{m}+1}} \\
&\leq \sup_{\widetilde{\lambda}_{\mathfrak{n}} > 0} \bigl(\widetilde{\lambda}_{\mathfrak{n}}^{-1} \mathcal{Q}^{-1}(\mathcal{T}, \mathfrak{n}, \alpha)\bigr)^{\frac{\mathfrak{m}}{\mathfrak{m}+1}} \bigl(\bigl(2 + 2\mathcal{G}(\alpha, d)\bigr)^{\frac{1}{2}} \delta + \wp\delta\bigr)^{\frac{\mathfrak{m}}{\mathfrak{m}+1}} \|h\|_{D((-\mathcal{L})^{\mathfrak{m}})(\Omega)}^{\frac{1}{\mathfrak{m}+1}}.
\end{aligned}
$$

From Lemma 2.1, we have

$$
\|h - h_{\beta,\vartheta}\|_{L^2(\Omega)} \leq \left(\frac{(2 + 2\mathcal{G}(\alpha, d))^{\frac{1}{2}} + \wp}{\mathbf{D}_1(\mathcal{T}, \alpha)} \right)^{\frac{\mathfrak{m}}{\mathfrak{m}+1}} \mathcal{P}^{\frac{1}{\mathfrak{m}+1}} \delta^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}.
$$

Thus we have

$$
\bigl\|h - h_{\beta,\vartheta}^{\delta}\bigr\|_{L^2(\Omega)} \leq [\mathcal{M}_1 + \mathcal{M}_2] \mathcal{P}^{\frac{1}{\mathfrak{m}+1}} \delta^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}, \tag{65}
$$

where

$$
\begin{aligned}
\mathcal{M}_1 &= \left(\frac{\mathrm{a}(1 + \mathcal{G}(\alpha, d))(\mathfrak{m} + 1)}{(\wp - (2 + 2\mathcal{G}(\alpha, d))^{\frac{1}{2}})^{\frac{2}{\mathfrak{m}+1}} \mathbf{D}_1^{\frac{2\mathfrak{m}}{\mathfrak{m}+1}}(\mathcal{T}, \alpha)} \right)^{\frac{1}{2}}, \\
\mathcal{M}_2 &= \left(\frac{(2 + 2\mathcal{G}(\alpha, d))^{\frac{1}{2}} + \wp}{\mathbf{D}_1(\mathcal{T}, \alpha)} \right)^{\frac{\mathfrak{m}}{\mathfrak{m}+1}}.
\end{aligned}
$$

□

4 Simulation theory

4.1 Numerical example

The main objective of this subsection is to present an example to simulate the theory of this study in the case of an a-priori parameter choice. We consider the time-fractional backward problem of finding $u = u(\mathrm{t}, \mathbf{x})$, $(\mathrm{t}, \mathbf{x}) \in (0, \mathcal{T}) \times \Omega := (0, 1) \times (0, \pi)$ such that

$$
\begin{cases}
u_{\mathrm{t}} + \gamma \partial_{\mathrm{t}}^{\alpha} \Delta u + \Delta u = \sqrt{2/\pi}\, \mathrm{t} \sin(\mathbf{x}), & (\mathrm{t}, \mathbf{x}) \in (0, 1) \times (0, \pi), \\
u(\mathrm{t}, \mathbf{x}) = 0, & (\mathrm{t}, \mathbf{x}) \in (0, 1) \times \{0; \pi\}, \\
u(1, \mathbf{x}) = \sqrt{2/\pi}\, \sin(2\mathbf{x}), & \mathbf{x} \in (0, \pi).
\end{cases} \tag{66}
$$

We choose the Laplace operator $\mathcal{L} = -\Delta$ associated with the Dirichlet boundary condition. Then, it has the eigenvalues $\widetilde{\lambda}_{\mathfrak{n}} = \mathfrak{n}^2$, $\mathfrak{n} \geq 1$, and corresponding eigenfunctions $\mathcal{X}_{\mathfrak{n}}(\mathbf{x}) =$

$\sqrt{2/\pi}\sin(\mathfrak{n}\mathbf{x})$, $\mathfrak{n} \geq 1$. The solution of problem (66) is given by

$$u(\mathfrak{t},\mathbf{x}) = \sqrt{\frac{2}{\pi}}\sin(\mathbf{x})\left[\frac{2\mathcal{Q}(\mathfrak{t},2,\alpha)\cos(\mathbf{x})}{\mathcal{Q}(1,2,\alpha)} - \frac{\mathcal{Q}(\mathfrak{t},1,\alpha)}{\mathcal{Q}(1,1,\alpha)}\int_0^1 \mathcal{Q}(1-\varsigma,1,\alpha)\varsigma\,d\varsigma \right. \\ \left. + \int_0^{\mathfrak{t}} \mathcal{Q}(1-\varsigma,1,\alpha)\varsigma\,d\varsigma\right], \tag{67}$$

where we recall that

$$\mathcal{Q}(\mathfrak{t},\mathfrak{n},\alpha) = \frac{\gamma}{\pi}\int_0^\infty \frac{\mathfrak{n}^2\sin(\alpha\pi)\rho^\alpha e^{-\rho\mathfrak{t}}}{(-\rho+\mathfrak{n}^2\gamma\rho^\alpha\cos(\alpha\pi)+\mathfrak{n}^2)^2+\mathfrak{n}^4\gamma^2\rho^{2\alpha}\sin^2(\alpha\pi)}\,d\rho. \tag{68}$$

Next, we present the composite Simpson's rule to approximate the integral as follows: Suppose that the interval $[a,b]$ is split up into k subintervals, with k being an even number. Then, the composite Simpson's rule is given by

$$\int_a^b f(z)\,dz \approx \frac{\Delta z}{3}\left[f(z_0) + 2\sum_{i=1}^{k/2-1} f(z_{2i}) + 4\sum_{i=1}^{k/2} f(z_{2i-1}) + f(z_k)\right], \tag{69}$$

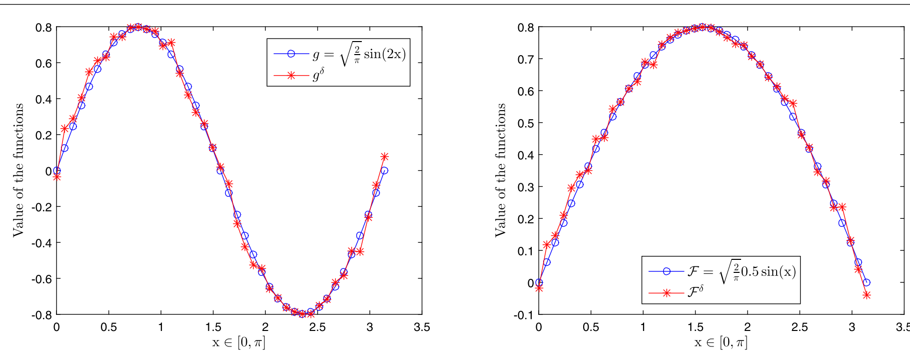
where $z_i = a + i\Delta z$ for $i = 0,1,\dots,k$ with $\Delta z = \frac{b-a}{k}$, and in particular, $z_0 = a$ and $z_n = b$. In the following simulation results, we will discretize the time and spatial variables as follows:

$$x_i = (i-1)\Delta x, \qquad t_j = (j-1)\Delta t,$$
$$\Delta x = \frac{\pi}{N_x}, \qquad \Delta t = \frac{1}{N_t}, \quad i = 1,\dots,N_x+1, j = 1,\dots,N_t+1,$$

where $N_x, N_t \in \mathbb{Z}^+ \setminus \{0\}$.

Instead of observing the exact data $(g,\mathcal{F})$, we get approximate data $(g^\delta,\mathcal{F}^\delta)$ such that

$$\|g-g^\delta\|_{L^2(\Omega)} \leq \delta, \qquad \|\mathcal{F}-\mathcal{F}^\delta\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} \leq \delta, \tag{70}$$



(a) The functions g and g^δ (b) The functions $\mathcal{F}$ and $\mathcal{F}^\delta$ at $t = 0.5$

Figure 1 The input data and its approximations

Table 1 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.2$ in case 1

δ	$\gamma = 0.7$, $\mathfrak{m} = 1/2$, $\vartheta = 3/4$, $\mathcal{P} = 30$, $N_x = 40$, $N_t = 40$		
	$\text{Error}_1(t = 0.1)$	$\text{Error}_1(t = 0.5)$	$\text{Error}_1(t = 0.9)$
0.1	2.370005530364097	1.839421744056028	1.114272879651872
0.01	0.266013210499944	0.179032675679955	0.119410328707072
0.001	0.027337033449183	0.016692692749124	0.012092996690034

Table 2 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.2$ in case 2

δ	$N(\delta) = \frac{1}{\max(T^{1-\alpha}, T)\delta}$, $\gamma = 0.7$		
	$\text{Error}_2(t = 0.1)$	$\text{Error}_2(t = 0.5)$	$\text{Error}_2(t = 0.9)$
0.1	3.067277122239126	2.331560024339361	0.586862037977597
0.01	0.282303322277925	0.693116629713161	0.046189744153395
0.001	0.035344214548148	0.201830859315601	0.040932510968230

Table 3 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.4$ in case 1

δ	$\gamma = 0.7$, $\mathfrak{m} = 1/2$, $\vartheta = 3/4$, $\mathcal{P} = 30$, $N_x = 40$, $N_t = 40$		
	$\text{Error}_1(t = 0.1)$	$\text{Error}_1(t = 0.5)$	$\text{Error}_1(t = 0.9)$
0.1	3.441083614522570	1.967441863324420	1.216004031281940
0.01	0.392794589419522	0.175387014637054	0.124872454990796
0.001	0.040783838085983	0.016522743057455	0.013353281312731

Table 4 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.4$ in case 2

δ	$N(\delta) = \frac{1}{\max(\mathcal{T}^{1-\alpha}, \mathcal{T})\delta}$, $\gamma = 0.7$		
	$\text{Error}_2(t = 0.1)$	$\text{Error}_2(t = 0.5)$	$\text{Error}_2(t = 0.9)$
0.1	4.538099120142149	2.098712218495121	1.474085282648231
0.01	0.459148594925824	0.281395186718633	0.173056572314173
0.001	0.057943611026572	0.025385813139180	0.014230357573040

Table 5 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.6$ in case 1

δ	$\gamma = 0.7$, $\mathfrak{m} = 1/2$, $\vartheta = 3/4$, $\mathcal{P} = 30$, $N_x = 40$, $N_t = 40$		
	$\text{Error}_1(t = 0.1)$	$\text{Error}_1(t = 0.5)$	$\text{Error}_1(t = 0.9)$
0.1	4.528918237889845	1.508794132566739	1.600628856280399
0.01	0.514335896636502	0.139127374227336	0.157363853188444
0.001	0.053388298785060	0.090959770500309	0.017066574494708

Table 6 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.6$ in case 2

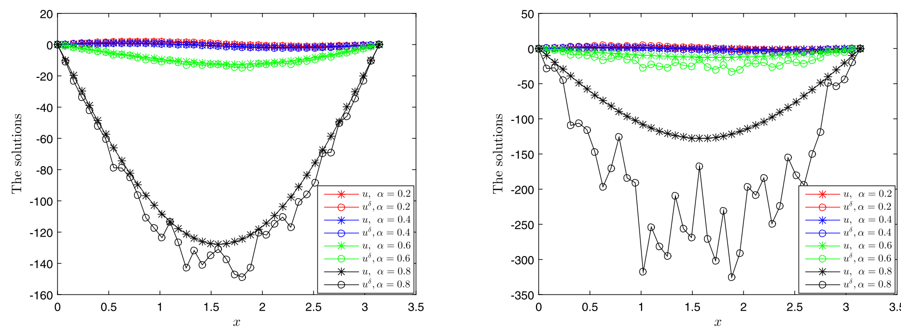
δ	$N(\delta) = \frac{1}{\max(\mathcal{T}^{1-\alpha}, \mathcal{T})\delta}$, $\gamma = 0.7$		
	$\text{Error}_2(t = 0.1)$	$\text{Error}_2(t = 0.5)$	$\text{Error}_2(t = 0.9)$
0.1	6.728343436454919	2.083436656341341	2.194194359940908
0.01	0.573679745877681	1.547353873964686	0.188447118306793
0.001	0.065624643204875	0.185606287444821	0.022053101663947

Table 7 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.8$ in case 1

δ	$\gamma = 0.7$, $\mathfrak{m} = 1/2$, $\vartheta = 3/4$, $\mathcal{P} = 30$, $N_x = 40$, $N_t = 40$		
	$\text{Error}_1(t = 0.1)$	$\text{Error}_1(t = 0.5)$	$\text{Error}_1(t = 0.9)$
0.1	4.242912131190551	1.549288857217745	1.398671745776383
0.01	0.479566420425554	0.144428484048293	0.138497851165066
0.001	0.049714128627384	0.085696070388215	0.015046030605746

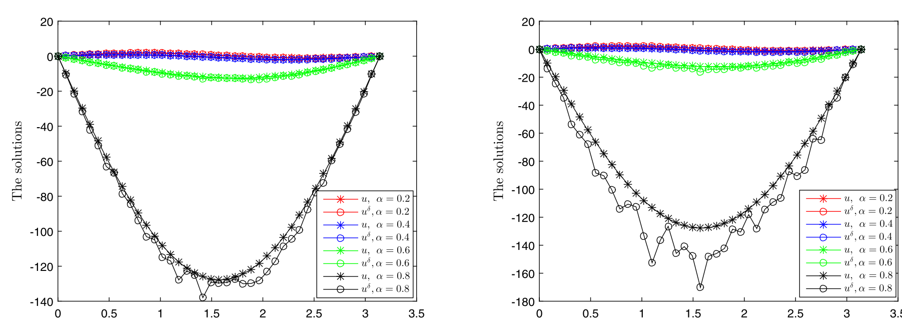
Table 8 The output errors for $t \in \{0.1; 0.5; 0.9\}$, $\delta \in \{0.1; 0.01; 0.001\}$, and $\alpha = 0.8$ in case 2

δ	$N(\delta) = \frac{1}{\max(\mathcal{T}^{1-\alpha}, \mathcal{T})\delta}$, $\gamma = 0.7$		
	$\text{Error}_2(t = 0.1)$	$\text{Error}_2(t = 0.5)$	$\text{Error}_2(t = 0.9)$
0.1	5.267407080684832	2.322503778497919	1.894375818559956
0.01	0.600229963944443	0.935180709782072	0.181090792426397
0.001	0.061597621175013	0.113528567084024	0.020271875129425



(a) The solutions in case 1 (b) The solutions in case 2

Figure 2 A comparison between the exact and regularized solutions of the proposed method (case 1) and according to [19] (case 2) at $t = 0.1$ for $\alpha \in \{0.2; 0.4; 0.6; 0.8\}$, $\delta = 0.1$, and $N_x = N_t = 40$



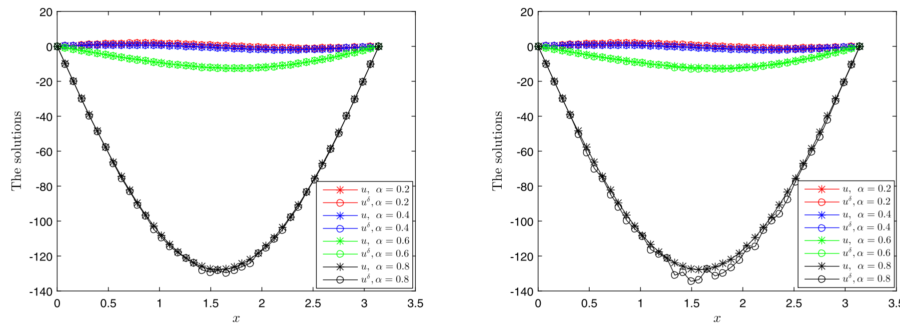
(a) The solutions in case 1 (b) The solutions in case 2

Figure 3 A comparison between the exact & regularized solutions of the proposed method (case 1) and according to [19] (case 2) at $t = 0.1$ for $\alpha \in \{0.2; 0.4; 0.6; 0.8\}$, $\delta = 0.01$, $N_x = N_t = 40$

where $\delta > 0$ is the noise level. Then the couple $(g^\delta, \mathcal{F}^\delta)$, which is determined below, plays the role of measured data with a random noise as follows (see Fig. 1):

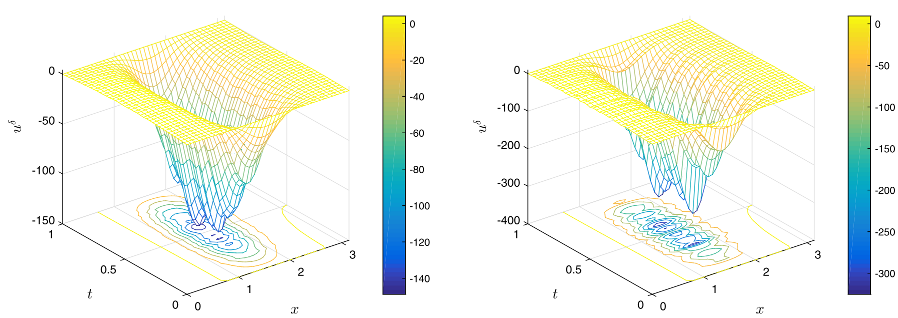
$$g^\delta(\cdot) = g(\cdot) + \delta\big(\text{rand}(\cdot) + 1\big), \qquad \mathcal{F}^\delta(\cdot) = \mathcal{F}(\cdot) + 2\delta\,\text{rand}(\cdot). \tag{71}$$

For the best of reader's comparison, we present some results between the result of this study and the result in [19] in two subsections as follows.



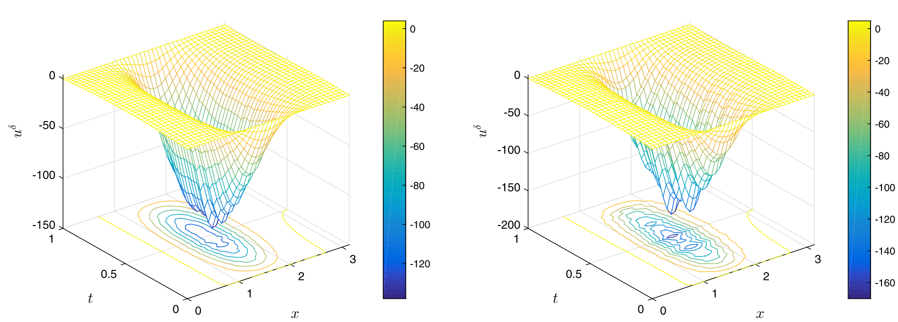
(a) The solutions in case 1 (b) The solutions in case 2

Figure 4 A comparison between the exact and regularized solutions of the proposed method (case 1) and according to [19] (case 2) at $t = 0.1$ for $\boldsymbol{\alpha} \in \{0.2; 0.4; 0.6; 0.8\}$, $\delta = 0.001$, and $N_x = N_t = 40$



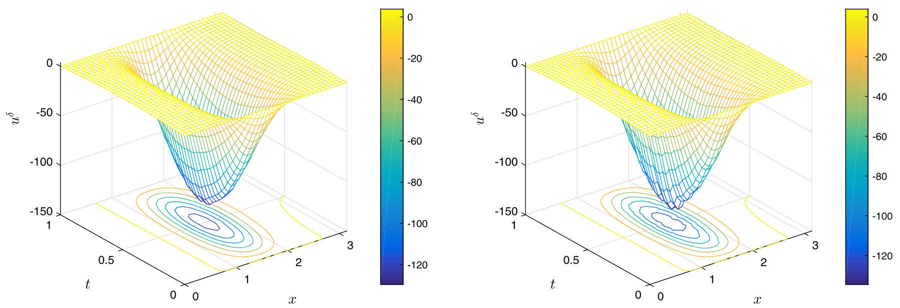
(a) The regularized solution in case 1 (b) The regularized solution in case 2

Figure 5 A comparison between the regularized solution in case 1 and case 2 for $\boldsymbol{\alpha} = 0.8$, $\delta = 0.1$, and $N_x = N_t = 40$ on $(\mathbf{t}, \mathbf{x}) \in [0, 1]) \times [0, \pi]$



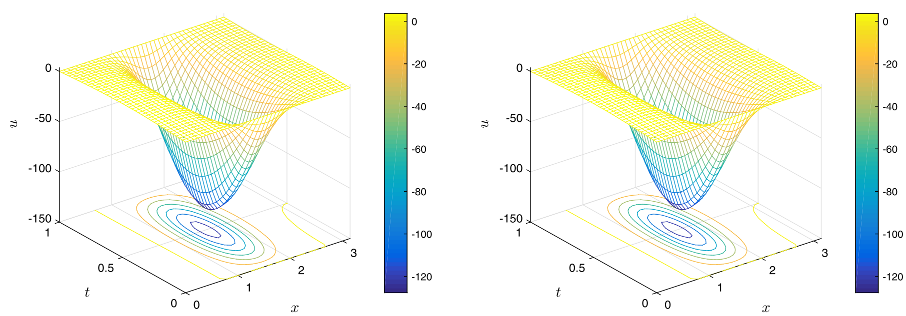
(a) The regularized solution in case 1 (b) The regularized solution in case 2

Figure 6 A comparison between the regularized solution in case 1 and case 2 for $\boldsymbol{\alpha} = 0.8$, $\delta = 0.01$, and $N_x = N_t = 40$ on $(\mathbf{t}, \mathbf{x}) \in [0, 1]) \times [0, \pi]$



(a) The regularized solution in case 1 (b) The regularized solution in case 2

Figure 7 A comparison between the regularized solution in case 1 and case 2 for $\alpha = 0.8$, $\delta = 0.001$, and $N_x = N_t = 40$ on $(\mathrm{t}, \mathbf{x}) \in [0, 1]) \times [0, \pi]$



(a) The exact solution in case 1 (b) The exact solution in case 2

Figure 8 A comparison between the exact solution in case 1 and case 2 for $\alpha = 0.8$ and $N_x = N_t = 40$ on $(\mathrm{t}, \mathbf{x}) \in [0, 1]) \times [0, \pi]$

4.2 Case 1: the Landweber method

We choose the regularization parameter $\beta = [(\frac{\mathcal{P}}{\delta})^{\frac{2}{\mathrm{m}+1}}]$, then we get the absolute error estimate between the exact solution and its regularization solution as follows:

$$\mathrm{Error}_1(\mathrm{t}) = \sqrt{\frac{1}{N_x}\sum_{i=1}^{N_x}\left|u(\mathrm{t}, \mathbf{x}_i) - u^\delta(\mathrm{t}, \mathbf{x}_i)\right|^2}, \tag{72}$$

where $u^\delta = u^\delta_{\beta,\vartheta}$ is defined by (38).

4.3 Case 2: the filter regularization method

In this case, we present the result which was shown in [19]. There the authors considered a general filter regularization method, then they gave the following regularized solution:

$$u^\delta(\mathrm{t}, \mathbf{x}) = \sum_{\mathrm{n}=1}^{\infty} \frac{\mathbf{R}_\mathrm{n}(\delta)}{\mathbf{P}_\mathrm{n}(\mathcal{T})}\left(g^\delta_\mathrm{n} - \int_0^{\mathcal{T}} \mathbf{P}_\mathrm{n}(\mathcal{T} - \varsigma)\mathcal{F}^\delta_\mathrm{n}(\varsigma)\,d\varsigma\right)\mathcal{X}_\mathrm{n}(\mathbf{x}), \tag{73}$$

where

$$\mathbf{R}_{\mathrm{n}}(\delta) = \frac{\mathbf{P}_{\mathrm{n}}(\mathcal{T})}{\delta + \mathbf{P}_{\mathrm{n}}(\mathcal{T})}, \quad \text{and} \quad \mathbf{P}_{\mathrm{n}}(\mathcal{T}) = \int_0^\infty \exp(-\xi t) M_n(\xi)\, d\xi,$$

$$M_n(\xi) = \frac{\gamma}{\pi} \frac{\widetilde{\lambda}_{\mathrm{n}} \sin(\alpha\pi)\xi^\alpha}{(-\xi + \widetilde{\lambda}_{\mathrm{n}}\gamma\xi^\alpha \cos(\alpha\pi) + \widetilde{\lambda}_{\mathrm{n}})^2 + (\widetilde{\lambda}_{\mathrm{n}}\gamma\xi^\alpha \sin(\alpha\pi))^2}.$$

The absolute error estimate between the exact and regularized solutions is given by

$$\mathrm{Error}_2(\mathrm{t}) = \sqrt{\frac{1}{N(\delta)} \sum_{i=1}^{N(\delta)} \left| u(\mathrm{t}, \mathbf{x}_i) - u^\delta(\mathrm{t}, \mathbf{x}_i) \right|^2}. \tag{74}$$

Take $t \in \{0.1; 0.5; 0.9\}$, $\alpha \in \{0.2; 0.6; 0.8\}$, and $\delta \in \{0.1; 0.01; 0.001\}$, respectively. The numerical results are included in Tables 1–8 and Figs. 2–8, i.e., we show the estimates of the exact and regularized solutions for $\alpha = 0.2$ in case 1 (Table 1) and in case 2 (Table 2), for $\alpha = 0.4$ in case 1 (Table 3) and in case 2 (Table 4), for $\alpha = 0.6$ in case 1 (Table 5) and in case 2 (Table 6), for $\alpha = 0.8$ in case 1 (Table 7) and in case 2 (Table 8), respectively. We also present the 2D graphs of the exact and regularized solutions of two cases at $t = 0.1$ for $\delta = 0.1$ (Fig. 2), $\delta = 0.01$ (Fig. 3) and $\delta = 0.001$ (Fig. 4). In addition, the 3D graphs of the solutions, for $\alpha = 0.8$ on the domain $(\mathrm{t}, \mathbf{x}) \in [0,1] \times [0,\pi]$, are shown in Figs. 5–8. From the above results, it is clear that the smaller input error, the smaller output error, when δ tends to zero, the regularized solution approaches the exact solution, the convergence results of case 1 are better compared to case 2. It is clear that the experiment convergence orders are consistent with theoretical analysis.

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam. [2]Department of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam. [3]Vietnam National University, Ho Chi Minh City, Vietnam. [4]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland. [5]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

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