Ali et al. *Advances in Difference Equations* (2020) 2020:470
https://doi.org/10.1186/s13662-020-02918-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Existence and uniqueness of solutions for coupled system of fractional differential equations involving proportional delay by means of topological degree theory

Anwar Ali[1], Muhammad Sarwar[1*], Mian Bahadur Zada[1] and Thabet Abdeljawad[2,3,4*]

*Correspondence:
sarwarswati@gmail.com;
tabdeljawad@psu.edu.sa
[1]Department of Mathematics, University of Malakand, Chakdara, Khyber Pakhtunkhwa, Pakistan
[2]Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this manuscript, we obtain sufficient conditions required for the existence of solution to the following coupled system of nonlinear fractional order differential equations:

$$D^{\gamma}\omega(\ell)=\mathcal{F}(\ell,\omega(\lambda\ell),\upsilon(\lambda\ell)),$$
$$D^{\delta}\upsilon(\ell)=\overline{\mathcal{F}}(\ell,\omega(\lambda\ell),\upsilon(\lambda\ell)),$$

with fractional integral boundary conditions

$$\mathfrak{a}_1\omega(0)-\mathfrak{b}_1\omega(\eta)-\mathfrak{c}_1\omega(1)=\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi(\rho,\omega(\rho))\,d\rho \quad \text{and}$$
$$\mathfrak{a}_2\upsilon(0)-\mathfrak{b}_2\upsilon(\xi)-\mathfrak{c}_2\upsilon(1)=\frac{1}{\Gamma(\delta)}\int_0^1(1-\rho)^{\delta-1}\psi(\rho,\upsilon(\rho))\,d\rho,$$

where $\ell\in\mathfrak{Z}=[0,1]$, $\gamma,\delta\in(0,1]$, $0<\lambda<1$, D denotes the Caputo fractional derivative (in short CFD), $\mathcal{F},\overline{\mathcal{F}}:\mathfrak{Z}\times\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ and $\phi,\psi:\mathfrak{Z}\times\mathfrak{R}\to\mathfrak{R}$ are continuous functions. The parameters η,ξ are such that $0<\eta,\xi<1$, and $\mathfrak{a}_i,\mathfrak{b}_i,\mathfrak{c}_i$ $(i=1,2)$ are real numbers with $\mathfrak{a}_i\neq\mathfrak{b}_i+\mathfrak{c}_i$ $(i=1,2)$. Using topological degree theory, sufficient results are constructed for the existence of at least one and unique solution to the concerned problem. For the validity of our result, an appropriate example is presented in the end.

MSC: 34A08; 35R11

Keywords: Fractional differential equations; Boundary value problems; Existence results; Topological degree theory

1 Introduction

It has been proved that fractional differential equations (in short FDEs) are a powerful tool for modeling various phenomena of physical and chemical as well as biological sciences.

Besides, it has also been proved that FDEs have numerous applications in various scientific and engineering disciplines such as chemistry, physics, biology, and optimization theory [1–5].

Many mathematicians give much attention to the existence theory of FDEs with multi-point boundary conditions, and there is rapidly growing area of research due to its wide range of applications in real world problems [6–10]. For the existence and uniqueness of solutions of FDEs, different methods are used like topological degree theory and fixed point theory. Here we use topological degree theory. After studying the present literature, we noticed that FDEs having fractional integral type boundary conditions are not well examined through topological degree theory. Very few articles used topological degree theory for simple initial and boundary value problems (BVPs) having CFD [11–15]. If viewed carefully, the existence of solutions to FDEs having integral boundary conditions has a wide range of applications in optimization theory, viscoelasticity, fluid mechanics, and quantitative theory which have been studied by many researchers [16–21]. Keeping in mind the applications of topological degree theory, Ali *et al.* [22] studied the existence of solutions to the following FDE:

$$
\begin{aligned}
&{}^{c}D^{\gamma}\omega(\ell) = \mathcal{F}\big(\ell, \omega(\ell)\big), \quad 1 < \gamma \leq 2, \ell \in \mathfrak{Z}, \\
&\mathfrak{a}_1\omega(0) + \mathfrak{b}_1\omega(1) = \overline{\mathcal{F}}_1(\omega), \\
&\mathfrak{a}_2\omega'(0) + \mathfrak{b}_2\omega'(1) = \overline{\mathcal{F}}_2(\omega),
\end{aligned}
$$

where $\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2 : C(\mathfrak{Z}, \mathfrak{R}) \to \mathfrak{R}$ and $\mathcal{F} : \mathfrak{Z} \times \mathfrak{R} \to \mathfrak{R}$ are continuous functions and $\mathfrak{a}_i$, $\mathfrak{b}_i$ are real numbers with $\mathfrak{a}_i + \mathfrak{b}_i \neq 0$, $i = 1, 2$. Using fixed point theory, Cabada *et al.* [23] discussed the following problem:

$$
\begin{aligned}
&{}^{c}D^{\gamma}\omega(\ell) + \mathcal{F}\big(\ell, \omega(\ell)\big) = 0, \quad \ell \in (0, 1), \\
&\omega(0) + \omega''(0) = 0, \qquad \omega(1) = \mathfrak{a}\int_0^1 \omega(\rho)\,d\rho,
\end{aligned}
$$

where $2 < \gamma < 3$, $0 < \mathfrak{a} < 2$, D is the CFD and $\mathcal{F} : \mathfrak{Z} \times [0, \infty) \to [0, \infty)$.

Motivated by [22] and [23], we examine the results for the existence of solution to the following nonlinear coupled system of FDEs through topological degree theory:

$$
\begin{cases}
D^{\gamma}\omega(\ell) = \mathcal{F}(\ell, \omega(\lambda\ell), \upsilon(\lambda\ell)), \\
D^{\delta}\upsilon(\ell) = \overline{\mathcal{F}}(\ell, \omega(\lambda\ell), \upsilon(\lambda\ell)), \\
\mathfrak{a}_1\omega(0) - \mathfrak{b}_1\omega(\eta) - \mathfrak{c}_1\omega(1) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-\rho)^{\gamma-1}\phi(\rho, \omega(\rho))\,d\rho, \\
\mathfrak{a}_2\upsilon(0) - \mathfrak{b}_2\upsilon(\xi) - \mathfrak{c}_2\upsilon(1) = \frac{1}{\Gamma(\delta)}\int_0^1 (1-\rho)^{\delta-1}\psi(\rho, \upsilon(\rho))\,d\rho,
\end{cases}
\tag{1.1}
$$

where $\ell \in \mathfrak{Z}$, $\gamma, \delta \in (0, 1]$, $0 < \lambda < 1$, D denotes the CFD. Further $\mathcal{F}, \overline{\mathcal{F}} : \mathfrak{Z} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ and $\phi, \psi : \mathfrak{Z} \times \mathfrak{R} \to \mathfrak{R}$ are continuous functions. The parameters η, ξ are such that $0 < \eta, \xi < 1$ and $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i$ $(i = 1, 2)$ are real numbers with $\mathfrak{a}_i \neq \mathfrak{b}_i + \mathfrak{c}_i$.

2 Preliminaries

In this section we recollect some facts, definitions, and results. Throughout this work $\mathcal{U} = C(\mathfrak{Z}, \mathfrak{R})$, $\mathcal{V} = C(\mathfrak{Z}, \mathfrak{R})$ represent the Banach spaces for all continuous function defined on $\mathfrak{Z}$ into $\mathfrak{R}$ with norm $\|\omega\| = \sup\{|\omega(\ell)| : 0 \leq \ell \leq 1\}$. The product space $\mathcal{U} \times \mathcal{V}$ is a Banach space with norm $\|(\omega, \upsilon)\| = \|\omega\| + \|\upsilon\|$.

Definition 2.1 ([24]) Let $\mathcal{H} : V \to \mathcal{U}$ be a continuous bounded map, where $V \subseteq \mathcal{U}$. Then $\mathcal{H}$ is

(1) σ-Lipschitz if there exists $\hbar \geq 0$ such that $\sigma(\mathcal{H}(S)) \leq \hbar\sigma(S)$ for all bounded subsets $S \subseteq V$;
(2) strict σ-contraction if there exists $0 \leq \hbar < 1$ with $\sigma(\mathcal{H}(S)) \leq \hbar\sigma(S)$ for all bounded subsets $S \subseteq V$;
(3) σ-condensing if $\sigma(\mathcal{H}(S)) < \sigma(S)$ for all bounded subsets $S \subseteq V$ having $\sigma(S) > 0$. In other words, $\sigma(\mathcal{H}(S)) \geq \sigma(S)$ implies $\sigma(S) = 0$.

Moreover, $\mathcal{H} : V \to \mathcal{U}$ is Lipschitz whenever there is $\hbar > 0$ such that

$$\left\|\mathcal{H}(\omega) - \mathcal{H}(\upsilon)\right\| \leq \hbar|\omega - \upsilon| \quad \text{for all } \omega, \upsilon \in V.$$

Further $\mathcal{H}$ will be a strict contraction if $\hbar < 1$.

Proposition 2.1 ([25]) *If $\mathcal{H}, G : V \to \mathcal{U}$ are σ-Lipschitz with constants $\hbar_1$ and $\hbar_2$ respectively, then $\mathcal{H} + G$ is σ-Lipschitz with constant $\hbar_1 + \hbar_2$.*

Proposition 2.2 ([25]) *If $\mathcal{H} : V \to \mathcal{U}$ is Lipschitz with constant $\hbar$, then $\mathcal{H}$ is σ-Lipschitz with the same constant $\hbar$.*

Proposition 2.3 ([25]) *If $\mathcal{H} : V \to \mathcal{U}$ is compact, then $\mathcal{H}$ is σ-Lipschitz with constant $\hbar = 0$.*

Theorem 2.1 ([25]) *Let $\mathcal{H} : \mathcal{U} \to \mathcal{U}$ be σ-condensing such that*

$$\Lambda = \{\omega \in \mathcal{U} : \textit{there exists } 0 \leq \vartheta \leq 1 \textit{ such that } \omega = \vartheta\mathcal{H}\omega\}.$$

If Λ is bounded in $\mathcal{U}$, so there exists $r > 0$ such that $\Lambda \subset S_r(0)$, then the degree

$$\mathcal{D}\bigl(I - \vartheta\mathcal{H}, S_r(0), 0\bigr) = 1 \quad \textit{for all } \vartheta \in [0, 1].$$

Consequently, $\mathcal{H}$ has at least one fixed point which lies in $S_r(0)$.

Definition 2.2 ([26]) The fractional order integral of a function $\mathcal{F} : \mathfrak{R}^+ \to \mathfrak{R}$ is defined by

$$I^\gamma\mathcal{F}(\ell) = \frac{1}{\Gamma(\gamma)}\int_0^\ell (\ell - \rho)^{\gamma-1}\mathcal{F}(\rho)\,d\rho. \tag{2.1}$$

Definition 2.3 ([26]) The CFD of order $\gamma > 0$ of a function $\mathcal{F} : \mathfrak{R}^+ \to \mathfrak{R}$ is defined by

$$D^\gamma\mathcal{F}(\ell) = \frac{1}{\Gamma(n - \gamma)}\int_0^\ell (\ell - \rho)^{n-\gamma-1}\mathcal{F}^{(n)}(\rho)\,d\rho. \tag{2.2}$$

Lemma 2.1 ([26]) *Let $\gamma > 0$, then*

$$I^{\gamma}\left[{}^{c}D^{\gamma}h(\ell)\right] = h(\ell) + c_0 + c_1\ell + c_2\ell^2 + \cdots + c_{n-1}\ell^{n-1}$$

for arbitrary $c_i \in \mathfrak{R}$, $i = 0,1,2,\ldots,n-1$.

3 Main results

In this section, we discuss the existence and uniqueness criteria for BVP (1.1). Before we start our main work, we need the following hypotheses.

(C_1) For arbitrary $\omega, \upsilon, \overline{\omega}, \overline{\upsilon} \in \mathfrak{R}$, there exist constants $k_\phi, k_\psi \in [0,1)$ such that

$$\begin{aligned}
&\left|\phi(\rho,\omega) - \phi(\rho,\overline{\omega})\right| \leq k_\phi \|\omega - \overline{\omega}\|, \\
&\left|\psi(\rho,\upsilon) - \psi(\rho,\overline{\upsilon})\right| \leq k_\psi \|\upsilon - \overline{\upsilon}\|.
\end{aligned}$$

(C_2) For arbitrary $\omega, \upsilon \in \mathfrak{R}$, there exist constants $c_\phi, c_\psi, M_\phi, M_\psi \geq 0$ and $q_1 \in [0,1)$ such that

$$\begin{aligned}
&\left|\phi(\rho,\omega)\right| \leq c_\phi \|\omega\|^{q_1} + M_\phi, \\
&\left|\psi(\rho,\upsilon)\right| \leq c_\psi \|\upsilon\|^{q_1} + M_\psi.
\end{aligned}$$

(C_3) For arbitrary $\omega, \upsilon \in \mathfrak{R}$, there exist constants c_i, d_i $(i = 1,2)$, $M_{\mathcal{F}}, M_{\overline{\mathcal{F}}}$ and $q_2 \in [0,1)$ such that

$$\begin{aligned}
&\left|\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\right| \leq c_1\|\omega\|^{q_2} + c_2\|\upsilon\|^{q_2} + M_{\mathcal{F}}, \\
&\left|\overline{\mathcal{F}}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\right| \leq d_1\|\omega\|^{q_2} + d_2\|\upsilon\|^{q_2} + M_{\overline{\mathcal{F}}}.
\end{aligned}$$

(C_4) For arbitrary $\omega, \upsilon, \overline{\omega}, \overline{\upsilon} \in \mathfrak{R}$, there exist constants $L_{\mathcal{F}}, L_{\overline{\mathcal{F}}} > 0$ such that

$$\begin{aligned}
&\left|\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big) - \mathcal{F}\big(\rho,\overline{\omega}(\lambda\rho),\overline{\upsilon}(\lambda\rho)\big)\right| \leq L_{\mathcal{F}}\big(\|\omega - \overline{\omega}\| + \|\upsilon - \overline{\upsilon}\|\big), \\
&\left|\overline{\mathcal{F}}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big) - \mathcal{F}\big(\rho,\overline{\omega}(\lambda\rho),\overline{\upsilon}(\lambda\rho)\big)\right| \leq L_{\overline{\mathcal{F}}}\big(\|\omega - \overline{\omega}\| + \|\upsilon - \overline{\upsilon}\|\big).
\end{aligned}$$

Lemma 3.1 *If $h : \mathfrak{Z} \to \mathfrak{R}$ is a γ times integrable function, then the FDE*

$$D^{\gamma}\omega(\ell) = h(\ell), \quad 0 < \gamma \leq 1, \ell \in \mathfrak{Z},$$

with integral type boundary conditions

$$\mathfrak{a}_1\omega(0) - \mathfrak{b}_1\omega(\eta) - \mathfrak{c}_1\omega(1) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-\rho)^{\gamma-1}\phi\big(\rho,\omega(\rho)\big)\,d\rho,$$

has a solution

$$\begin{aligned}
\omega(\ell) = {}& \frac{1}{\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1 (1-\rho)^{\gamma-1}\phi\big(\rho,\omega(\rho)\big)\,d\rho \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^{\ell} (\ell-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho
\end{aligned}$$

$$+\frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho$$
$$+\frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho.$$

Proof Applying fractional integrable operator I^γ to $D^\gamma\omega(\ell)=h(\ell)$ and using Lemma 2.1, we get

$$\omega(\ell)=c_0+I^\gamma h(\ell). \tag{3.1}$$

On applying boundary conditions to (3.1), we have

$$c_0(\mathfrak{a}_1-\mathfrak{b}_1-\mathfrak{c}_1)-\mathfrak{b}_1I^\gamma h(\eta)-\mathfrak{c}_1I^\gamma h(1)=\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi\big(\rho,\omega(\rho)\big)\,d\rho.$$

By rearranging, we get

$$\begin{aligned}c_0&=\frac{1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi\big(\rho,\omega(\rho)\big)\,d\rho\\&\quad+\frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\\&\quad+\frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho.\end{aligned}$$

□

By Lemma 3.1, the solution of system (1.1) is a solution of the following system of integral equations:

$$\begin{cases}\omega(\ell)=\frac{1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi(\rho,\omega(\rho))\,d\rho\\ \qquad+\frac{1}{\Gamma(\gamma)}\int_0^\ell(\ell-\rho)^{\gamma-1}\mathcal{F}(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho))\,d\rho\\ \qquad+\frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho))\,d\rho\\ \qquad+\frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho))\,d\rho,\\ \upsilon(\ell)=\frac{1}{\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)}\frac{1}{\Gamma(\delta)}\int_0^1(1-\rho)^{\delta-1}\psi(\rho,\upsilon(\rho))\,d\rho\\ \qquad+\frac{1}{\Gamma(\delta)}\int_0^\ell(\ell-\rho)^{\delta-1}\overline{\mathcal{F}}(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho))\,d\rho\\ \qquad+\frac{\mathfrak{c}_2}{\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)}\frac{1}{\Gamma(\delta)}\int_0^1(1-\rho)^{\delta-1}\overline{\mathcal{F}}(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho))\,d\rho\\ \qquad+\frac{\mathfrak{b}_2}{\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)}\frac{1}{\Gamma(\delta)}\int_0^\xi(\xi-\rho)^{\delta-1}\overline{\mathcal{F}}(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho))\,d\rho.\end{cases} \tag{3.2}$$

Define the operator $\mathcal{J}:\mathcal{U}\times\mathcal{V}\to\mathcal{U}\times\mathcal{V}$ by

$$\mathcal{J}(\omega,\upsilon)(\ell)=\big(\mathcal{J}_1\omega(\ell),\mathcal{J}_2\upsilon(\ell)\big),$$

where

$$\mathcal{J}_1\omega(\ell)=\frac{1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi\big(\rho,\omega(\rho)\big)\,d\rho$$

and

$$\mathcal{J}_2\upsilon(\ell)=\frac{1}{\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)}\frac{1}{\Gamma(\delta)}\int_0^1(1-\rho)^{\delta-1}\psi\big(\rho,\upsilon(\rho)\big)\,d\rho.$$

Also define the operator $\mathcal{G}:\mathcal{U}\times\mathcal{V}\to\mathcal{U}\times\mathcal{V}$ by

$$\mathcal{G}(\omega,\upsilon)(\ell)=\big(\mathcal{G}_1(\omega,\upsilon)(\ell),\mathcal{G}_2(\omega,\upsilon)(\ell)\big),$$

where

$$\begin{aligned}\mathcal{G}_1(\omega,\upsilon)(\ell)&=\frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\\&\quad+\frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\\&\quad+\frac{1}{\Gamma(\gamma)}\int_0^\ell(\ell-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_2(\omega,\upsilon)(\ell)&=\frac{\mathfrak{c}_2}{\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)}\frac{1}{\Gamma(\delta)}\int_0^1(1-\rho)^{\delta-1}\overline{\mathcal{F}}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\\&\quad+\frac{\mathfrak{b}_2}{\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)}\frac{1}{\Gamma(\delta)}\int_0^\xi(\xi-\rho)^{\delta-1}\overline{\mathcal{F}}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\\&\quad+\frac{1}{\Gamma(\delta)}\int_0^\ell(\ell-\rho)^{\delta-1}\overline{\mathcal{F}}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho.\end{aligned}$$

Further, we define $\mathcal{T}=\mathcal{J}+\mathcal{G}$. Then the system of integral equations (3.2) can be written as an operator form

$$(\omega,\upsilon)=\mathcal{T}(\omega,\upsilon)=\mathcal{J}(\omega,\upsilon)+\mathcal{G}(\omega,\upsilon),$$

which is the solution of system (1.1) in the operator form.

Lemma 3.2 *The operator $\mathcal{J}$ satisfies the Lipschitz condition*

$$\big\|\mathcal{J}(\omega,\upsilon)-\mathcal{J}(\overline{\omega},\overline{\upsilon})\big\|\le k\big\|(\omega,\upsilon)-(\overline{\omega},\overline{\upsilon})\big\|. \tag{3.3}$$

Proof For arbitrary $(\omega,\upsilon),(\overline{\omega},\overline{\upsilon})\in\mathcal{U}\times\mathcal{V}$, we have

$$\begin{aligned}|\mathcal{J}_1\omega-\mathcal{J}_1\overline{\omega}|&=\bigg|\frac{1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi\big(\rho,\omega(\rho)\big)\,d\rho\\&\quad-\frac{1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\phi\big(\rho,\overline{\omega}(\rho)\big)\,d\rho\bigg|\\&=\bigg|\frac{1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\big[\phi\big(\rho,\omega(\rho)\big)-\phi\big(\rho,\overline{\omega}(\rho)\big)\big]\,d\rho\bigg|,\end{aligned}$$

which implies that

$$\|\mathcal{J}_1\omega-\mathcal{J}_1\overline{\omega}\|\le\frac{k_\phi}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}\|\omega-\overline{\omega}\|. \tag{3.4}$$

Similarly,

$$\|\mathcal{J}_2\upsilon - \mathcal{J}_2\overline{\upsilon}\| \leq \frac{k_\psi}{|\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)|}\|\upsilon - \overline{\upsilon}\|. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}
\big\|\mathcal{J}(\omega,\upsilon) - \mathcal{J}(\overline{\omega},\overline{\upsilon})\big\| &= \big\|\mathcal{J}_1\omega(\ell) - \mathcal{J}_1\overline{\omega}(\ell) + \mathcal{J}_2\upsilon(\ell) - \mathcal{J}_2\overline{\upsilon}(\ell)\big\| \\
&\leq \frac{k_\phi}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|}\|\omega - \overline{\omega}\| + \frac{k_\psi}{|\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)|}\|\upsilon - \overline{\upsilon}\| \\
&\leq k\big\|(\omega,\upsilon) - (\overline{\omega},\overline{\upsilon})\big\|,
\end{aligned}$$

where $k = \max(\frac{k_\phi}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}, \frac{k_\psi}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|})$. Thus $\mathcal{J}$ is Lipschitz with constant k, and therefore by Proposition 2.2, $\mathcal{J}$ is σ-Lipschitz with constant k. □

Lemma 3.3 *The operator $\mathcal{G}:\mathcal{U}\times\mathcal{V}\to\mathcal{U}\times\mathcal{V}$ is continuous.*

Proof Consider a sequence $\{(\omega_n,\upsilon_n)\}_{n\in\mathbb{N}}$ in a bounded set

$$B_r = \big\{\big\|(\omega,\upsilon)\big\| \leq r : (\omega,\upsilon)\in\mathcal{U}\times\mathcal{V}\big\}$$

such that $(\omega_n,\upsilon_n)_{n\in\mathbb{N}} \to (\omega,\upsilon)$ as $n\to+\infty$ in B_r. To check that $\mathcal{G}$ is continuous, we have to prove that

$$\big\|\mathcal{G}(\omega_n,\upsilon_n)(\ell) - \mathcal{G}(\omega,\upsilon)(\ell)\big\| \to 0 \quad \text{as } n\to+\infty.$$

For this, we have

$$\begin{aligned}
&\big|\mathcal{G}_1(\omega_n,\upsilon_n)(\ell) - \mathcal{G}_1(\omega,\upsilon)(\ell)\big| \\
&\quad= \bigg|\frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega_n(\lambda\rho),\upsilon_n(\lambda\rho)\big)\,d\rho \\
&\qquad+ \frac{1}{\Gamma(\gamma)}\int_0^\ell(\ell-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega_n(\lambda\rho),\upsilon_n(\lambda\rho)\big)\,d\rho \\
&\qquad+ \frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega_n(\lambda\rho),\upsilon_n(\lambda\rho)\big)\,d\rho \\
&\qquad- \frac{1}{\Gamma(\gamma)}\int_0^\ell(\ell-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho \\
&\qquad- \frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho \\
&\qquad- \frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\,d\rho\bigg| \\
&\quad\leq \frac{|\mathfrak{c}_1|}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}\frac{1}{\Gamma(\gamma)} \\
&\qquad\times\int_0^1(1-\rho)^{\gamma-1}\big|\mathcal{F}\big(\rho,\omega_n(\lambda\rho),\upsilon_n(\lambda\rho)\big) - \mathcal{F}\big(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\big)\big|\,d\rho
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|\mathfrak{b}_1|}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} \frac{1}{\Gamma(\gamma)} \\
&\times \int_0^{\eta} (\eta - \rho)^{\gamma - 1} \big| \mathcal{F}\big(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho)\big) - \mathcal{F}\big(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\big) \big| \, d\rho \\
&+ \frac{1}{\Gamma(\gamma)} \int_0^{\ell} (\ell - \rho)^{\gamma - 1} \big| \mathcal{F}\big(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho)\big) - \mathcal{F}\big(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\big) \big| \, d\rho.
\end{aligned}
$$

From the continuity of $\mathcal{F}$, it follows that

$$\mathcal{F}\big(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho)\big) \to \mathcal{F}\big(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\big) \quad \text{as } n \to +\infty.$$

For every $\ell \in \mathfrak{Z}$ and by using (C_3), we get

$$\int_0^{\ell} \frac{(\ell - \rho)^{\gamma - 1}}{\Gamma(\gamma)} \big| \mathcal{F}\big(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho)\big) - \mathcal{F}\big(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\big) \big| \, d\rho \to 0 \quad \text{as } n \to +\infty.$$

Similarly other terms approach 0 as $n \to +\infty$. It follows that

$$\big\| \mathcal{G}_1(\omega_n, \upsilon_n)(\ell) - \mathcal{G}_1(\omega, \upsilon)(\ell) \big\| \to 0 \quad \text{as } n \to +\infty.$$

That is, $\mathcal{G}_1$ is continuous. Proceeding the same way as above, we can show that

$$\big\| \mathcal{G}_2(\omega_n, \upsilon_n)(\ell) - \mathcal{G}_2(\omega, \upsilon)(\ell) \big\| \to 0 \quad \text{as } n \to +\infty.$$

That is, $\mathcal{G}_2$ is continuous and hence $\mathcal{G}$ is continuous. □

Lemma 3.4 *The operators $\mathcal{J}$ and $\mathcal{G}$ satisfy the following growth conditions*:

$$\big\| \mathcal{J}(\omega, \upsilon) \big\| \le C \big\| (\omega, \upsilon) \big\|^{q_1} + M \quad \textit{for each } (\omega, \upsilon) \in \mathcal{U} \times \mathcal{V} \tag{3.6}$$

and

$$\big\| \mathcal{G}(\omega, \upsilon) \big\| \le \Delta\big(\big\| (\omega, \upsilon) \big\|^{q_2} + M^* \big) \quad \textit{for each } (\omega, \upsilon) \in \mathcal{U} \times \mathcal{V}, \tag{3.7}$$

respectively, where $c = \max(c_1, c_2)$, $d = \max(d_1, d_2)$, $C = \max(\frac{c_\phi}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|}, \frac{c_\psi}{|\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)|})$, $\Delta = \max(\frac{c[2|\mathfrak{c}_1| + 2|\mathfrak{b}_1| + |\mathfrak{a}_1|]}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|}, \frac{d[2|\mathfrak{c}_2| + 2|\mathfrak{b}_2| + |\mathfrak{a}_2|]}{|\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)|})$.

Proof For the growth condition on $\mathcal{J}$, consider

$$
\begin{aligned}
\big\| \mathcal{J}(\omega, \upsilon) \big\| &= \big\| \big(\mathcal{J}_1 \omega(\ell), \mathcal{J}_2 \upsilon(\ell) \big) \big\| \\
&= \big\| \mathcal{J}_1 \omega(\ell) \big\| + \big\| \mathcal{J}_2 \upsilon(\ell) \big\| \\
&= \bigg\| \frac{1}{\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} \phi\big(\rho, \omega(\rho)\big) \, d\rho \\
&\quad + \frac{1}{\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)} \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta - 1} \psi\big(\rho, \upsilon(\rho)\big) \, d\rho \bigg\| \\
&\le \frac{c_\phi \|\omega\|^{q_1}}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} + M_\phi + \frac{c_\psi \|\upsilon\|^{q_1}}{|\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)|} + M_\psi \\
&\le C \big\| (\omega, \upsilon) \big\|^{q_1} + M,
\end{aligned}
$$

where $M = \max(M_\phi, M_\psi)$, which is the growth condition for $\mathcal{J}$. Now, for the growth condition on $\mathcal{G}$, we have

$$\begin{aligned}
\left|\mathcal{G}_1(\omega,\upsilon)(\ell)\right| &= \left|\frac{\mathfrak{c}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^1(1-\rho)^{\gamma-1}\mathcal{F}\bigl(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\bigr)\,d\rho\right.\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^\ell(\ell-\rho)^{\gamma-1}\mathcal{F}\bigl(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\bigr)\,d\rho\\
&\quad\left.+\frac{\mathfrak{b}_1}{\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)}\frac{1}{\Gamma(\gamma)}\int_0^\eta(\eta-\rho)^{\gamma-1}\mathcal{F}\bigl(\rho,\omega(\lambda\rho),\upsilon(\lambda\rho)\bigr)\,d\rho\right|\\
&\le\frac{|\mathfrak{c}_1|}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}\bigl(c_1\|\omega\|^{q_2}+c_2\|\upsilon\|^{q_2}+M_{\mathcal{F}}\bigr)\\
&\quad+\frac{|\mathfrak{b}_1|\eta^\gamma}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}\bigl(c_1\|\omega\|^{q_2}+c_2\|\upsilon\|^{q_2}+M_{\mathcal{F}}\bigr)\\
&\quad+\ell^\gamma\bigl(c_1\|\omega\|^{q_2}+c_2\|\upsilon\|^{q_2}+M_{\mathcal{F}}\bigr),
\end{aligned}$$

which implies that

$$\left\|\mathcal{G}_1(\omega,\upsilon)(\ell)\right\|\le\left(\frac{2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}\right)c_1\|\omega\|^{q_2}+c_2\|\upsilon\|^{q_2}+M_{\mathcal{F}}. \tag{3.8}$$

Similarly,

$$\left\|\mathcal{G}_2(\omega,\upsilon)(\ell)\right\|\le\left(\frac{2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}\right)d_1\|\omega\|^{q_2}+d_2\|\upsilon\|^{q_2}+M_{\overline{\mathcal{F}}}. \tag{3.9}$$

Now, from (3.8) and (3.9), we have

$$\begin{aligned}
\left\|\mathcal{G}(\omega,\upsilon)(\ell)\right\| &= \left\|\mathcal{G}_1(\omega,\upsilon)(\ell)\right\|+\left\|\mathcal{G}_2(\omega,\upsilon)(\ell)\right\|\\
&\le\left(\frac{2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}\right)c_1\|\omega\|^{q_2}+c_2\|\upsilon\|^{q_2}+M_{\mathcal{F}}\\
&\quad+\left(\frac{2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}\right)d_1\|\omega\|^{q_2}+d_2\|\upsilon\|^{q_2}+M_{\overline{\mathcal{F}}}\\
&\le\Delta\bigl(\left\|(\omega,\upsilon)\right\|^{q_2}+M^*\bigr),
\end{aligned}$$

where $M^* = \max(M_{\mathcal{F}}, M_{\overline{\mathcal{F}}})$. Hence $\mathcal{G}$ satisfies the growth condition. □

Lemma 3.5 *The operator $\mathcal{G}:\mathcal{U}\times\mathcal{V}\to\mathcal{U}\times\mathcal{V}$ is compact.*

Proof Let $\mathcal{B}$ be a bounded subset of $B_r\subseteq\mathcal{U}\times\mathcal{V}$ and $\{(\omega_n,\upsilon_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}$, then by using the growth condition of $\mathcal{G}$, it is clear that $\mathcal{G}(\mathcal{B})$ is bounded in $\mathcal{U}\times\mathcal{V}$. Now, we need to show that $\mathcal{G}$ is equicontinuous. Let $0\le\ell\le\tau\le1$, then we have

$$\begin{aligned}
&\left|\mathcal{G}_1(\omega_n,\upsilon_n)(\ell)-\mathcal{G}_1(\omega_n,\upsilon_n)(\tau)\right|\\
&\quad=\left|\frac{1}{\Gamma(\gamma)}\int_0^\ell(\ell-\rho)^{\gamma-1}\mathcal{F}(\rho,\omega_n(\lambda\rho),\upsilon_n(\lambda\rho)\,d\rho\right.\\
&\qquad\left.-\frac{1}{\Gamma(\gamma)}\int_0^\tau(\tau-\rho)^{\gamma-1}\mathcal{F}(\rho,\omega_n(\lambda\rho),\upsilon_n(\lambda\rho)\,d\rho\right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{1}{\Gamma(\gamma)} \int_0^{\ell} \left[(\ell-\rho)^{\gamma-1} - (\tau-\rho)^{\gamma-1} \right] \mathcal{F}(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho)\, d\rho \right. \\
&\quad \left. - \frac{1}{\Gamma(\gamma)} \int_{\ell}^{\tau} (\tau-\rho)^{\gamma-1} \mathcal{F}(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho)\, d\rho \right| \\
&\le \frac{1}{\Gamma(\gamma)} \int_0^{\ell} \left[(\ell-\rho)^{\gamma-1} - (\tau-\rho)^{\gamma-1} \right] \left| \mathcal{F}(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho) \right| d\rho \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{\ell}^{\tau} (\tau-\rho)^{\gamma-1} \left| \mathcal{F}(\rho, \omega_n(\lambda\rho), \upsilon_n(\lambda\rho) \right| d\rho \\
&\le \frac{1}{\Gamma(\gamma+1)} \left[\ell^{\gamma} - \tau^{\gamma} - 2(\ell-\tau)^{\gamma} \right] c_1 \|\omega\|^{q_2} + c_2 \|\upsilon\|^{q_2} + M_{\mathcal{F}}.
\end{aligned}$$

Taking limit as $\ell \to \tau$, we get

$$\left\| \mathcal{G}_1(\omega_n, \upsilon_n)(\ell) - \mathcal{G}_1(\omega_n, \upsilon_n)(\tau) \right\| \to 0.$$

That is, there exists $\epsilon > 0$ such that

$$\left| \mathcal{G}_1(\omega_n, \upsilon_n)(\ell) - \mathcal{G}_1(\omega_n, \upsilon_n)(\tau) \right| < \frac{\epsilon}{2}. \tag{3.10}$$

Similarly,

$$\left| \mathcal{G}_2(\omega_n, \upsilon_n)(\ell) - \mathcal{G}_2(\omega_n, \upsilon_n)(\tau) \right| < \frac{\epsilon}{2}. \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$\left| \mathcal{G}(\omega_n, \upsilon_n)(\ell) - \mathcal{G}(\omega_n, \upsilon_n)(\tau) \right| < \epsilon. \tag{3.12}$$

Hence $\mathcal{G}$ is equicontinuous. Therefore $\mathcal{G}(\mathcal{B})$ is compact in $\mathcal{U} \times \mathcal{V}$ and hence by Proposition 2.1, $\mathcal{G}$ is σ-Lipschitz with constant zero. □

Theorem 3.1 *Under assumptions* (C_1)–(C_3), *BVP* (1.1) *has at least one solution* $(\omega, \upsilon) \in \mathcal{U} \times \mathcal{V}$. *Moreover, the solution set of* (1.1) *is bounded in* $\mathcal{U} \times \mathcal{V}$.

Proof From Lemma 3.2, $\mathcal{J}$ is Lipschitz with constant $k \in [0, 1)$, and from Lemma 3.5, $\mathcal{G}$ is Lipschitz with constant 0. It follows by Proposition 2.1 that $\mathcal{T}$ is a σ-contraction with constant k. Define

$$\mathfrak{B} = \left\{ (\omega, \upsilon) \in \mathcal{U} \times \mathcal{V} : \text{there exist } \varrho \in \mathfrak{Z}, (\omega, \upsilon) = \varrho \mathcal{T}(\omega, \upsilon) \right\}.$$

We have to show that $\mathfrak{B}$ is bounded in $\mathcal{U} \times \mathcal{V}$. Choose $(\omega, \upsilon) \in \mathfrak{B}$, then by using (3.6) and (3.7) we have

$$\begin{aligned}
\left\| (\omega, \upsilon) \right\| &= \left\| \varrho \mathcal{T}(\omega, \upsilon) \right\| \\
&= \varrho \left(\left\| \mathcal{J}(\omega, \upsilon) + \mathcal{G}(\omega, \upsilon) \right\| \right) \\
&\le \varrho \left(\left\| \mathcal{J}(\omega, \upsilon) \right\| + \left\| \mathcal{G}(\omega, \upsilon) \right\| \right) \\
&\le \varrho \left(C \left\| (\omega, \upsilon) \right\|^{q_1} + M + \Delta \left(\left\| (\omega, \upsilon) \right\|^{q_2} + M^* \right) \right) \\
&= \varrho \left(C \left\| (\omega, \upsilon) \right\|^{q_1} + \Delta \left\| (\omega, \upsilon) \right\|^{q_2} \right) + \varrho \left(M + \Delta M^* \right).
\end{aligned}$$

Thus $\mathfrak{B}$ is bounded in $\mathcal{U} \times \mathcal{V}$. Therefore Theorem 2.1 guarantees that $\mathcal{T}$ has at least one fixed point; consequently, BVP (1.1) has at least one solution. □

Theorem 3.2 *Under assumptions* (C_1)–(C_4), *assume that* $\mathcal{G}^* < 1$, *then BVP* (1.1) *has a unique solution, where*

$$\mathcal{G}^* = k + \frac{L_{\mathcal{F}}[2|\mathfrak{c}_1| + 2|\mathfrak{b}_1| + |\mathfrak{a}_1|]}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} + \frac{L_{\overline{\mathcal{F}}}[2|\mathfrak{c}_2| + 2|\mathfrak{b}_2| + |\mathfrak{a}_2|]}{|\mathfrak{a}_2 - (\mathfrak{c}_2 + \mathfrak{b}_2)|}.$$

Proof To find the unique solution of system (1.1), we use the Banach contraction theorem, that is, we have to show that $\mathcal{T}$ is a contraction. For this, let $(\omega, \upsilon), (\overline{\omega}, \overline{\upsilon}) \in \mathcal{U} \times \mathcal{V}$, then from (3.3) in Lemma 3.2, we showed that

$$\bigl|\mathcal{J}(\omega,\upsilon) - \mathcal{J}(\overline{\omega},\overline{\upsilon})\bigr| \le k\bigl\|(\omega,\upsilon) - (\overline{\omega},\overline{\upsilon})\bigr\|. \tag{3.13}$$

Next

$$\begin{aligned}
&\bigl|\mathcal{G}_1(\omega,\upsilon) - \mathcal{G}_1(\overline{\omega},\overline{\upsilon})\bigr| \\
&\quad= \biggl|\frac{\mathfrak{c}_1}{\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\rho)^{\gamma-1} \mathcal{F}\bigl(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\bigr)\,d\rho \\
&\qquad+ \frac{1}{\Gamma(\gamma)} \int_0^{\ell} (\ell-\rho)^{\gamma-1} \mathcal{F}\bigl(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\bigr)\,d\rho \\
&\qquad+ \frac{\mathfrak{b}_1}{\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)} \frac{1}{\Gamma(\gamma)} \int_0^{\eta} (\eta-\rho)^{\gamma-1} \mathcal{F}\bigl(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\bigr)\,d\rho \\
&\qquad- \frac{1}{\Gamma(\gamma)} \int_0^{\ell} (\ell-\rho)^{\gamma-1} \mathcal{F}\bigl(\rho, \overline{\omega}(\lambda\rho), \overline{\upsilon}(\lambda\rho)\bigr)\,d\rho \\
&\qquad- \frac{\mathfrak{c}_1}{\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\rho)^{\gamma-1} \mathcal{F}\bigl(\rho, \overline{\omega}(\lambda\rho), \overline{\upsilon}(\lambda\rho)\bigr)\,d\rho \\
&\qquad- \frac{\mathfrak{b}_1}{\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)} \frac{1}{\Gamma(\gamma)} \int_0^{\eta} (\eta-\rho)^{\gamma-1} \mathcal{F}\bigl(\rho, \overline{\omega}(\lambda\rho), \overline{\upsilon}(\lambda\rho)\bigr)\,d\rho\biggr| \\
&\quad\le \frac{|\mathfrak{c}_1|}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} \frac{1}{\Gamma(\gamma)} \\
&\qquad\times \int_0^1 (1-\rho)^{\gamma-1} \bigl|\mathcal{F}\bigl(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\bigr) - \mathcal{F}\bigl(\rho, \overline{\omega}(\lambda\rho), \overline{\upsilon}(\lambda\rho)\bigr)\bigr|\,d\rho \\
&\qquad+ \frac{|\mathfrak{b}_1|}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} \frac{1}{\Gamma(\gamma)} \\
&\qquad\times \int_0^{\eta} (\eta-\rho)^{\gamma-1} \bigl|\mathcal{F}\bigl(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\bigr) - \mathcal{F}\bigl(\rho, \overline{\omega}(\lambda\rho), \overline{\upsilon}(\lambda\rho)\bigr)\bigr|\,d\rho \\
&\qquad+ \frac{1}{\Gamma(\gamma)} \int_0^{\ell} (\ell-\rho)^{\gamma-1} \bigl|\mathcal{F}\bigl(\rho, \omega(\lambda\rho), \upsilon(\lambda\rho)\bigr) - \mathcal{F}\bigl(\rho, \overline{\omega}(\lambda\rho), \overline{\upsilon}(\lambda\rho)\bigr)\bigr|\,d\rho \\
&\quad\le \frac{|\mathfrak{c}_1|}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} L_{\mathcal{F}}\bigl(|\omega - \overline{\omega}| + |\upsilon - \overline{\upsilon}|\bigr) + \frac{|\mathfrak{b}_1|}{|\mathfrak{a}_1 - (\mathfrak{c}_1 + \mathfrak{b}_1)|} L_{\mathcal{F}}\bigl(|\omega - \overline{\omega}| + |\upsilon - \overline{\upsilon}|\bigr) \\
&\qquad+ L_{\mathcal{F}}\bigl(|\omega - \overline{\omega}| + |\upsilon - \overline{\upsilon}|\bigr),
\end{aligned}$$

which implies that

$$\left\|\mathcal{G}_1(\omega,\upsilon)-\mathcal{G}_1(\overline{\omega},\overline{\upsilon})\right\| \le \frac{2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}L_{\mathcal{F}}\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\|. \tag{3.14}$$

Similarly,

$$\left\|\mathcal{G}_2(\omega,\upsilon)-\mathcal{G}_2(\overline{\omega},\overline{\upsilon})\right\| \le \frac{2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}L_{\overline{\mathcal{F}}}\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\|. \tag{3.15}$$

From (3.14) and (3.15), it follows that

$$\begin{aligned}
\left\|\mathcal{G}(\omega,\upsilon)-\mathcal{G}(\overline{\omega},\overline{\upsilon})\right\| &= \left\|\mathcal{G}_1(\omega,\upsilon)-\mathcal{G}_1(\overline{\omega},\overline{\upsilon})\right\| + \left\|\mathcal{G}_2(\omega,\upsilon)-\mathcal{G}_2(\overline{\omega},\overline{\upsilon})\right\| \\
&\le \frac{2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}L_{\mathcal{F}}\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\| \\
&\quad + \frac{2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}L_{\overline{\mathcal{F}}}\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left\|\mathcal{G}(\omega,\upsilon)-\mathcal{G}(\overline{\omega},\overline{\upsilon})\right\| \\
&\quad\le \left(\frac{L_{\mathcal{F}}[2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|]}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}+\frac{L_{\overline{\mathcal{F}}}[2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|]}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}\right)\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\|.
\end{aligned} \tag{3.16}$$

Now, from (3.13) and (3.16), it follows that

$$\begin{aligned}
\left|\mathcal{T}(\omega,\upsilon)-\mathcal{T}(\overline{\omega},\overline{\upsilon})\right| &\le \left|\mathcal{J}(\omega,\upsilon)-\mathcal{J}(\overline{\omega},\overline{\upsilon})\right| + \left|\mathcal{G}(\omega,\upsilon)-\mathcal{G}(\overline{\omega},\overline{\upsilon})\right| \\
&\le k\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\| \\
&\quad + \left(\frac{L_{\mathcal{F}}[2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|]}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}+\frac{L_{\overline{\mathcal{F}}}[2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|]}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}\right) \\
&\quad \times \left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\| \\
&= \left(k+\frac{L_{\mathcal{F}}[2|\mathfrak{c}_1|+2|\mathfrak{b}_1|+|\mathfrak{a}_1|]}{|\mathfrak{a}_1-(\mathfrak{c}_1+\mathfrak{b}_1)|}+\frac{L_{\overline{\mathcal{F}}}[2|\mathfrak{c}_2|+2|\mathfrak{b}_2|+|\mathfrak{a}_2|]}{|\mathfrak{a}_2-(\mathfrak{c}_2+\mathfrak{b}_2)|}\right) \\
&\quad \times \left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\|,
\end{aligned}$$

which implies that

$$\left\|\mathcal{T}(\omega,\upsilon)-\mathcal{T}(\overline{\omega},\overline{\upsilon})\right\| \le \mathcal{G}^*\left\|(\omega,\upsilon)+(\overline{\omega},\overline{\upsilon})\right\|. \tag{3.17}$$

Thus $\mathcal{T}$ is a contraction and hence problem (1.1) has a unique solution. □

To illustrate our results, we provide the following example.

Example 3.1 Consider the following BVP:

$$\begin{cases} D^{2/3}\omega(\ell) = \frac{e^{-\pi\ell}}{10} + \frac{\sin|\omega(\frac{\ell}{2})|+\sin|\upsilon(\frac{\ell}{2})|}{51+\ell^2}, \quad \ell \in [0,1], \\ D^{3/4}\upsilon(\ell) = \frac{e^{-50\ell}}{20} + \frac{\sin|\omega(\frac{\ell}{2})|+\upsilon(\frac{\ell}{2})}{60+(\ell+1)^2}, \quad \ell \in [0,1], \\ \frac{1}{5}\omega(0) - \frac{1}{2}\omega(\frac{1}{2}) - 7\omega(1) = \frac{1}{\Gamma(\frac{2}{3})}\int_0^1 (1-\rho)^{\frac{-1}{2}} \frac{\cos\omega(\rho)}{2}\,d\rho, \\ \frac{1}{6}\upsilon(0) - \frac{1}{8}\upsilon(\frac{1}{2}) - 9\upsilon(1) = \frac{1}{\Gamma(\frac{3}{4})}\int_0^1 (1-\rho)^{\frac{-1}{2}} \frac{e^{-\upsilon(\rho)}}{3}\,d\rho. \end{cases} \tag{3.18}$$

Here, $\mathcal{F} = \frac{e^{-\pi\ell}}{10} + \frac{\sin|\omega(\frac{\ell}{2})|+\sin|\upsilon(\frac{\ell}{2})|}{51+\ell^2}$, $\overline{\mathcal{F}} = \frac{e^{-50\ell}}{20} + \frac{\sin|\omega(\frac{\ell}{2})|+\upsilon(\frac{\ell}{2})}{60+(\ell+1)^2}$, $\gamma = \frac{2}{3}$, $\delta = \frac{3}{4}$, $\mathfrak{a}_1 = \frac{1}{5}$, $\mathfrak{b}_1 = \frac{1}{2}$, $\mathfrak{c}_1 = 7$, $\mathfrak{a}_2 = \frac{1}{6}$, $\mathfrak{b}_2 = \frac{1}{8}$, $\mathfrak{c}_2 = 9$, $\eta = \xi = \frac{1}{2}$. Let $\varrho = \frac{1}{2}$, then by routine calculation we can easily find that $k_\phi = c_\phi = \frac{1}{2}$, $k_\psi = c_\psi = \frac{1}{3}$, $M_\phi = M_\psi = 0$, $c_1 = c_2 = L_{\mathcal{F}} = \frac{1}{51}$, $d_1 = d_2 = L_{\overline{\mathcal{F}}} = \frac{1}{61}$, $M_{\mathcal{F}} = \frac{1}{10}$, $M_{\overline{\mathcal{F}}} = \frac{1}{20}$, hence assumptions (C_1)–(C_4) are satisfied. Further

$$\begin{aligned} \bigl|\mathcal{J}(\omega,\upsilon)(\ell) - \mathcal{J}(\overline{\omega},\overline{\upsilon})(\ell)\bigr| &\le \frac{1}{17.890}\int_0^1 (1-\rho)^{\frac{-1}{2}}\bigl|\cos(\omega) - \cos(\overline{\omega})\bigr|\,d\rho \\ &\quad + \frac{1}{36.387}\int_0^1 (1-\rho)^{\frac{-1}{2}}\bigl|e^{-\upsilon(\rho)} - e^{-\overline{\upsilon}(\rho)}\bigr|\,d\rho \\ &\le \frac{2}{17.890}\|\omega - \overline{\omega}\| + \frac{2}{36.387}\|\upsilon - \overline{\upsilon}\| \\ &\le 0.112\bigl\|(\omega,\upsilon) - (\overline{\omega},\overline{\upsilon})\bigr\|, \end{aligned}$$

which means that $\mathcal{J}$ is σ-Lipschitz with constant 0.112 and $\mathcal{G}$ is σ-Lipschitz with constant zero, this implies that $\mathcal{T}$ is strict σ-Lipschitz with constant 0.112. Since

$$\mathfrak{B} = \bigl\{(\omega,\upsilon) \in \mathcal{U} \times \mathcal{V} : \text{there exists } \varrho \in \mathfrak{Z}, (\omega,\upsilon) = \varrho\mathcal{T}(\omega,\upsilon)\bigr\},$$

then, by routine calculation, we get

$$\bigl\|(\omega,\upsilon)\bigr\| \cong 0.0076 \le 1,$$

which implies that $\mathfrak{B}$ is bounded, and in the light of Theorem 3.1, BVP (3.18) has at least one solution. Moreover, $\mathcal{G}^* \cong 0.3348 < 1$. Hence the problem has a unique solution.

Acknowledgements
The authors are grateful to the editor and anonymous referees for their comments and remarks to improve this manuscript. The author Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this manuscript. All authors read and approved the final version.

Author details
[1]Department of Mathematics, University of Malakand, Chakdara, Khyber Pakhtunkhwa, Pakistan. [2]Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia. [3]Department of Medical Research, China Medical University, Taichung, Taiwan. [4]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 29 June 2020 Accepted: 20 August 2020 Published online: 04 September 2020

References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Math. Stud., vol. 204. Elsevier, Amsterdam (2006)
2. Lakshmikantham, V., Leela, S., Vasundhara, J.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
3. Ameen, R., Jarad, F., Abdeljawad, T.: Ulam stability for delay fractional differential equations with a generalized Caputo derivative. Filomat **32**(15), 5265–5274 (2018)
4. Todorčević, V.: Subharmonic behavior and quasiconformal mappings. Anal. Math. Phys. **9**(3), 1211–1225 (2019)
5. Sher, M., Shah, K., Rassias, J.: On qualitative theory of fractional order delay evolution equation via the prior estimate method. Math. Methods Appl. Sci. **43**, 6464–6475 (2020)
6. Wang, J., Xiang, H., Liu, Z.: Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations. Int. J. Differ. Equ. **2010**, Article ID 186928 (2010)
7. Agarwal, R.P., Ahmad, B., Alsaedi, A., Shahzad, N.: Existence and dimension of the set of mild solutions to semilinear fractional differential inclusions. Adv. Differ. Equ. **2012**, Article ID 74 (2012)
8. Ahmad, B., Nieto, J.J.: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**(9), 1838–1843 (2009)
9. Sarwar, M., Zada, M.B., Radenović, S.: Rational type inequality with applications to Volterra–Hammerstein nonlinear integral equations. Int. J. Nonlinear Sci. Numer. Simul. (2020). https://doi.org/10.1515/ijnsns-2018-0367
10. Shah, K., Khalil, H., Khan, R.A.: Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations. Chaos Solitons Fractals **77**, 240–246 (2015)
11. Todorčević, V.: Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics. Springer, Cham (2019)
12. Wang, J., Zhou, Y., Wei, W.: Study in fractional differential equations by means of topological degree methods. Numer. Funct. Anal. Optim. **33**(2), 216–238 (2012)
13. Shah, K., Hussain, W.: Investigating a class of nonlinear fractional differential equations and its Hyers–Ulam stability by means of topological degree theory. Numer. Funct. Anal. Optim. **40**(12), 1355–1372 (2019)
14. Shah, K., Ali, A., Khan, R.A.: Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems. Bound. Value Probl. **2016**(1), Article ID 43 (2016)
15. Yang, A., Weigao, G.: Positive solutions of multi-point boundary value problems of nonlinear fractional differential equation at resonance. J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **16**(2), 181–193 (2009)
16. Feng, M., Zhang, X., Weigao, G.: New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions. Bound. Value Probl. **2011**(1), Article ID 720702 (2011)
17. Ntouyas, S.K., Obaid, M.: A coupled system of fractional differential equations with nonlocal integral boundary conditions. Adv. Differ. Equ. **2012**(1), Article ID 97 (2012)
18. Ahmad, I., Shah, K., Rahman, G., Baleanu, D.: Stability analysis for a nonlinear coupled system of fractional hybrid delay differential equations. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6526
19. Sher, M., Shah, K., Feckan, M., Khan, R.A.: Qualitative analysis of multi-terms fractional order delay differential equations via the topological degree theory. Mathematics **8**(2), Article ID 218 (2020)
20. Yang, C., Zhai, C., Zhang, L.: Local uniqueness of positive solutions for a coupled system of fractional differential equations with integral boundary conditions. Adv. Differ. Equ. **2017**(1), Article ID 282 (2017)
21. Ali, A., Sarwar, M., Zada, M.B., Shah, K.: Degree theory and existence of positive solutions to coupled system involving proportional delay with fractional integral boundary conditions. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6311
22. Ali, A., Shah, K., Li, Y.: Topological degree theory and Ulam's stability analysis of a boundary value problem of fractional differential equations. In: Frontiers in Functional Equations and Analytic Inequalities, pp. 73–92 (2019)
23. Cabada, A., Wang, G.: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **398**(1), 403–411 (2012)
24. Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)
25. Isaia, F.: On a nonlinear integral equation without compactness. Acta Math. Univ. Comen. **75**(2), 233–240 (2006)
26. Shah, K., Khan, R.A.: Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions. Differ. Equ. Appl. **7**(2), 245–262 (2015)