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Extension of generalized Fox's H-function operator to certain set of generalized integrable functions

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Abstract

In this article, we investigate the so-called Inayat integral operator $T^{m,n}_{p,q}$, $p,q,m,n \in \mathbb{Z}$, $1 \leq m \leq q$, $0 \leq n \leq p$, on classes of generalized integrable functions. We make use of the Mellin-type convolution product and produce a concurrent convolution product, which, taken together, establishes the Inayat integral convolution theorem. The Inayat convolution theorem and a class of delta sequences were derived and employed for constructing sequence spaces of Boehmians. Moreover, by the aid of the concept of quotients of sequences, we present a generalization of the Inayat integral operator in the context of Boehmians. Various results related to the generalized integral operator and its inversion formula are also derived.

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1 Introduction

Special functions are a generalization of the more familiar elementary functions and include, among many others, gamma functions, zeta functions, Bessel functions, Legendre functions, Laguerre functions, hypergeometric functions, and Hermite polynomials. The special functions of mathematical physics and chemistry are mostly obtained in the solution of differential equations, which were already met in some elementary analysis, series solutions of the harmonic oscillator and atomic one-electron problems.

If $p,q,m,n \in \mathbb{Z}$, $1 \leq m \leq q$; $0 \leq n \leq p$, and A_j $(j = 1,\dots,p)$, B_j $(j = 1,\dots,q)$ are complex numbers, then, in terms of Mellin–Barnes type contour integrals, the Inayat $\bar{H}^{m,n}_{p,q}$-function is defined as [1]

$$\bar{H}^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (\alpha_j, A_j, a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j, b_j)_{1,q} \end{matrix}\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \theta(s) z^s \, ds, \tag{1}$$

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where

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j s) \prod_{j=1}^{n} (\Gamma(1-\alpha_j + A_j s))^{a_i}}{\prod_{j=m+1}^{q} (\Gamma(1-\beta_j + B_j s))^{b_j} \prod_{j=n+1}^{p} (\Gamma(\alpha_j + A_j s))}$$

is a function of fractional powers of sums of Γ functions and z is a nonzero real or complex number. The exponents a_j ($j = 1,\dots,n$) and b_j ($j = m+1,\dots,q$) take non-integer values, whereas the exponents α_j and β_j take integer values ($\alpha_j, \beta_j = 1, \forall i, j$) and the empty product is interpreted as a unity. The $\bar{H}_{p,q}^{m,n}$ or the $\bar{H}$-function generalizes the Fox H-function (see, e.g., [2–4])

$$H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (\alpha_j, A_j)_{1,p} \\ (\beta_j, B_j)_{1,q} \end{matrix}\right) = \frac{1}{2\pi i} \int_L X(s) z^{-s}\, ds, \tag{2}$$

provided

$$X(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j s) \prod_{j=1}^{n} \Gamma(1-\alpha_j + A_j s)}{\prod_{j=m+1}^{q} \Gamma(1-\beta_j + B_j s) \prod_{j=n+1}^{p} (\Gamma(\alpha_j + A_j s))},$$

$m, n, p, q \in \mathbb{N}_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_j, B_j \in \mathbb{R}_+$, $\alpha_j, \beta_j \in \mathbb{R}$ or $\mathbb{C}$ and L is a suitable contour that separates the poles of $\Gamma(\beta_j - B_j s)$ ($j = 1,\dots,m$) from the poles of $\Gamma(1-\alpha_j + A_j s)$ ($j = 1,\dots,n$). The $\bar{H}_{p,q}^{m,n}$-function satisfies the order property (see, e.g., [5, Equation 1.3])

$$\bar{H}_{p,q}^{m,n}(z) = o\big(|z|^g\big), g = \min_{1\le j\le m} \operatorname{Re} \frac{\beta_j}{B_j}, \tag{3}$$

for small values of z where, for large values z, it satisfies (see, e.g., [5, Equation 1.4])

$$\bar{H}_{p,q}^{m,n}(z) = O\big(|z|^h\big), h = \max_{1\le j\le n} \operatorname{Re}\left(a_j\left(\frac{\alpha_j - 1}{A_j}\right)\right). \tag{4}$$

The sufficient condition for the absolute convergence of the contour integral and the region of the absolute convergence of Eq. (1) are, respectively, given by

$$0 < \Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{q} |A_j| \quad \text{and} \quad \big|\arg(z)\big| < \frac{1}{2}\Omega\pi. \tag{5}$$

Owing to the following interpretations, the $\bar{H}$-function makes sense and defines an analytic function of z in the following two cases (see, e.g., [6]):

(i) $\mu > 0$ and $0 < |z| < \infty$ holds, where μ is defined by

$$\mu = \sum_{j=1}^{m} |B_j| + \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=1}^{n} |A_j \alpha_j| - \sum_{j=n+1}^{p} |A_j|. \tag{6}$$

(ii) $\mu = 0$ and $0 < |z| < \tau^{-1}$ holds, where τ is defined by

$$\tau = \prod_{j=1}^{m} (B_j)^{-B_j} \prod_{j=1}^{n} (A_j)^{A_j \alpha_j} \prod_{j=n+1}^{p} (A_j)^{A_j} \prod_{j=m+1}^{q} (B_j)^{-B_j \beta_j}. \tag{7}$$

In addition to the general properties of the $\bar{H}$-function, the fractional calculus theory of the $\bar{H}$-function was derived in [5] by involving Appell functions on a basis of generalized fractional integration and differentiation of arbitrary complex orders. However, due to the most general character of the $\bar{H}$-function, it is very interesting to mention here that various special cases of the $\bar{H}$-function which are associated with numerous transcendental functions such as Mittag-Leffler functions, Bessel functions, hypergeomatric ${}_pF_q$ functions and Meiger's G-functions can be deduced from using special cases of the H-function. On the other hand, there are various special functions of practical importance which are $\bar{H}$-functions but are not H-functions (see, e.g., [7]). The Fox–Wright ψ-function is written in terms of the $\bar{H}$-function as (see, e.g., [7, Eq. (3.3)])

$$ {}_p\psi_q\left(\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\middle| -z\right) = \bar{H}^{1,p}_{p,q+1}\left(z\middle|\begin{matrix}(1-a_p, A_p; 1)\\(0,1),(1-b_q, B_q; 1)\end{matrix}\right), $$

whereas the Mittag-Leffler function $E_{\alpha,\beta}$ was duly expressed in terms of the $\bar{H}$-function as follows (see, e.g., [7, Remark 3.3]):

$$ E_{\alpha,\beta}(z) = \bar{H}^{1,1}_{1,2}\left(-z\middle|\begin{matrix}(0,1;1)\\(0,1,(1-\beta,\alpha;1))\end{matrix}\right), $$

where α and β are complex numbers such that $\min(\mathrm{Re}(\alpha), \mathrm{Re}(\beta)) > 0$.

We state the following definition.

Definition 1 Let $g(t) = 0$, $-\infty < t < T$, be a locally integrable function and let there be a real number u such that for every positive number w the function

$$ \exp(-wut)\bar{H}^{m,n}_{p,q}\left(ut\middle|\begin{matrix}(a_j, A_j; a_j)_{1,n}\\(\beta_j, B_j)_{1,m}\end{matrix}\right) \tag{8} $$

is absolutely integrable over $-\infty < t < \infty$. Then, under the assumption that only sectors having $\bar{H}^{m,n}_{p,q}(st) = O(e^{st})$ at infinity with $w > 0$ and $s > 0$ and the sectors having the asymptotic expansion of the $\bar{H}$-function of algebraic order at infinity with $ws > 0$ should be considered, the Inayat integral operator $T^{m,n}_{p,q}$ is defined by [1]

$$ T^{m,n}_{p,q}(g)(s) = \int_{-\infty}^{\infty}\exp(-wst)\bar{H}^{m,n}_{p,q}\left(st\middle|\begin{matrix}(a_j, A_j; a_j)_{1,n}\\(\beta_j, B_j)_{1,m}\end{matrix}\right)g(t)\,dt. \tag{9} $$

The fact that $g(t) = 0$ for $-\infty < t < T$ permits us to write the $T^{m,n}_{p,q}$ integral operator in the form

$$ T^{m,n}_{p,q}(g)(s) = \int_{0}^{\infty}\exp(-wst)\bar{H}^{m,n}_{p,q}\left(st\middle|\begin{matrix}(a_j, A_j; a_j)_{1,n}\\(\beta_j, B_j)_{1,m}\end{matrix}\right)g(t)\,dt. \tag{10} $$

Therefore, we shall refer to Eq. (10) as the right-sided $\bar{H}$-function operator in order to indicate that the lower limit on the integral is a finite number.

By L^A we denote the space of all absolutely integrable functions on I; $I =: (0,\infty)$, and by D we denote the class of all smooth functions of compact supports over I. By Δ we denote the set of delta sequences $\{\delta_n\}$ from D such that the following properties hold:

$$\begin{aligned} &\int_0^\infty \delta_n = 1, \qquad |\delta_n| < M, \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \\ &s(\delta_n) \subseteq (a_n, c_n), \qquad a_n, c_n \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{11}$$

where M is a certain real number and $s(\delta_n)$ is the support of $\{\delta_n\}$. By $\times$ we denote the Mellin-type convolution product (see, e.g., [8])

$$(g \times \psi)(s) = \int_0^\infty x^{-1} g\bigl(x^{-1}s\bigr)\psi(x)\,dx, \tag{12}$$

when the integral part exists for every $s > 0$. The promising and very mutual convolution product we need here is defined by the integral

$$(\phi \otimes \psi)(s) = \int_0^\infty \phi(sx)\psi(x)\,dx \tag{13}$$

when the integral exists for every $s > 0$.

Therefore, on the bases of Eq. (12) and Eq. (13), the convolution theorem of the Inayat integral operator can be presented as follows.

Theorem 1 *Let $g \in L^A$ and $s > 0$. Then we have*

$$T_{p,q}^{m,n}(g \times \psi)(s) = \bigl(T_{p,q}^{m,n} g \otimes \psi\bigr)(s) \quad \textit{for all } \psi \in D.$$

Proof Let $g \in L^A$ and $\psi \in D$ be given. Then, by using Eq. (12), we have

$$T_{p,q}^{m,n}(g \times \psi)(s) = \int_0^\infty \psi(x)x^{-1} \int_0^\infty \exp(-wst)\bar{H}_{p,q}^{m,n}\left(st \left| \begin{matrix} (a_j, A_j; a_j)_{1,n} \\ (\beta_j, B_j)_{1,m} \end{matrix} \right.\right) g\bigl(x^{-1}t\bigr)\,dx\,dt.$$

By making a change of variables, $t = yx$, the preceding equation can be given as

$$T_{p,q}^{m,n}(g \times \psi)(s) = \int_0^\infty \psi(x) \int_0^\infty \exp(-wsyx)\bar{H}_{p,q}^{m,n}\left(syx \left| \begin{matrix} (a_j, A_j; a_j)_{1,n} \\ (\beta_j, B_j)_{1,m} \end{matrix} \right.\right) g(y)\,dy\,dx.$$

Therefore, by taking into account Eq. (10), the above equation leads to

$$T_{p,q}^{m,n}(g \times \psi)(s) = \int_0^\infty \psi(x) T_{p,q}^{m,n} g(sx)\,dx.$$

Hence, the proof of the convolution theorem is completed. □

In the sequel, we make use of the convolution theorem and the delta sequences, we already obtained, to derive the deterministic axioms necessary for defining the spaces of extension of the Inayat integral operator. The generalized spaces will be furnished with certain convergence and topology. However, our results spread over three sections. In Sect. 2, two spaces are determined and their topology is defined. In Sect. 3, the generalized operator is obtained and certain properties are introduced. The conclusion is given in Sect. 4 as a closing section.

2 Generalized function spaces

The field of generalized functions has been developed along the requirements of its applications in linear and nonlinear partial differential equations, geometry, mathematical physics, stochastic analysis as well as in harmonic analysis, both in theoretical and numerical aspects. The recent space of generalized functions or Boehmians is obtained by abstract algebra similar to that of field of quotients (see, e.g., [4, 9–18]). In this section, we investigate the Boehmian spaces where the Inayat integral operator is well defined. We commence our investigation by constructing the Boehmian space β_{L^A} with the set $(L^A, \otimes)$, the subset $(D, \times)$ and the set Δ of delta sequences. We prove the necessary axioms (Theorem 2–Theorem 4) as follows.

Theorem 2 *Let $g, \{g_n\} \in L^A$, $\alpha \in \mathbb{C}$, $\psi_1, \psi_2 \in D$ and $g_n \to g$ as $n \to \infty$. Then the following hold in L^A:*

$$g \otimes (\psi_1 + \psi_2) = g \otimes \psi_1 + g \otimes \psi_2, \qquad \alpha(g \otimes \psi_1) = (\alpha g) \otimes \psi_1 \quad \textit{and} \quad g_n \otimes \psi_1 \to g \otimes \psi_1$$

for large values of n.

Proof The proof of this theorem is trivial and follows from simple integration. Hence, we omit all details. □

For a permissible use, in the sequel, we claim that

$$g \otimes \psi \in L^A \quad \text{for every } g \in L^A \text{ and } \psi \in \Delta. \tag{14}$$

To prove such a claim, we consider an interval $[a, b]$ such that $s(\psi) \subseteq [a, b]$, $a > 0$ and $b > 0$ are real numbers. Then, by virtue of Eq. (13), we write

$$\int_0^\infty \left|(g \otimes \psi)(y)\right| dy \leq \int_a^b \left|\psi(x)\right| \int_0^\infty \left|g(yx)\right| dy\, dx. \tag{15}$$

Therefore, the hypothesis that $g \in L^A$ implies

$$\int_0^\infty \left|(g \otimes \psi)(y)\right| dy \leq M \int_a^b \left|\psi(x)\right| dx,$$

where M is a positive constant. Hence, by the fact that $\psi \in D$, we have $\int_a^b |\psi(x)|\, dx < \infty$. Thus, our requirement has been fulfilled.

Theorem 3 *Let $g \in L^A$ and $\{\delta_n\} \in \Delta$. Then we have $g \otimes \delta_n \to g$ as $n \to \infty$.*

Proof Under the assumption that $g \in L^A$ and $\{\delta_n\} \in \Delta$ we, by the aid of Eq. (13) and Eq. (11), write

$$\int_0^\infty \left|\left((g \otimes \delta_n) - g\right)(y)\right| dy \leq \int_0^\infty \left|\delta_n(x)\right| \int_0^\infty \left|g(yx) - g(y)\right| dy\, dx. \tag{16}$$

The mapping $\phi(x) = g(yx) - g(y)$, indeed, belongs to L^A for every $y > 0$. Hence, from (16), we get

$$\int_0^\infty \left|\left((g \otimes \delta_n) - g\right)(y)\right| dy \le M_1 \int_0^\infty \left|\delta_n(x)\right| dx, \tag{17}$$

where M_1 is a positive constant. Also, from Eq. (11), we derive that

$$\left|\delta_n(x)\right| < M \quad (\forall n \in \mathbb{N}). \tag{18}$$

Also, from Eq. (11), we have

$$s(\delta_n) \subseteq (a_n, c_n), \qquad a_n, c_n \to 0 \quad \text{as } n \to \infty. \tag{19}$$

Hence, by invoking Eq. (18) and Eq. (19) in Eq. (17), we get

$$\int_0^\infty \left|\left((g \otimes \delta_n) - g\right)(y)\right| dy \le M_1 M(b_n - a_n) \to 0 \quad \text{as } n \to \infty.$$

Therefore, by the linearity of the integral, we write

$$\int_0^\infty \left|(g \otimes \delta_n)(y)\right| dy \to \int_0^\infty \left|g(y)\right| dy \quad \text{as } n \to \infty.$$

This finishes the proof of this theorem. □

Theorem 4 *Let $\{\delta_n\}$ and $\{\epsilon_n\}$ be two sequences in Δ. Then the sequence $\{\delta_n \times \epsilon_n\}$ is also in Δ.*

The proof of this theorem follows from the property $\phi \times \psi = \psi \times \phi$ (see, e.g., [8]). Hence, we omit the details.

The space β_{L^A} is, therefore, generated. The pair $(T_{p,q}^{m,n} g_n, \delta_n)$ of sequences is said to be a quotient of sequences if $T_{p,q}^{m,n} g_n \otimes \delta_m = T_{p,q}^{m,n} g_m \otimes \delta_n$ $(\forall m, n \in \mathbb{N})$. The quotients $(T_{p,q}^{m,n} g_n, \delta_n)$ and $(T_{p,q}^{m,n} \theta_n, t_n)$ of sequences in β_{L^A} are equivalent, $(T_{p,q}^{m,n} \theta_n, t_n) \sim (T_{p,q}^{m,n} \theta_n, t_n)$, if $T_{p,q}^{m,n} g_n \otimes t_m = T_{p,q}^{m,n} g_m \otimes \delta_n$ $(\forall m, n \in \mathbb{N})$. Every equivalence class in β_{L^A} is the Boehmian $\frac{T_{p,q}^{m,n} g_n}{\delta_n}$. An embedding from the space $L^A(\mathbb{R})$ into the space β_{L^A} can be expressed as

$$x \to x \otimes \frac{\delta_n}{\delta_n}$$

for every $n \in \mathbb{N}$. If $\frac{T_{p,q}^{m,n} g_n}{\delta_n} \in \beta_{L^A}$ and $z \in \beta_{L^A}$, then it follows that

$$\frac{T_{p,q}^{m,n} g_n}{\delta_n} \otimes z = \frac{T_{p,q}^{m,n} g_n \otimes z}{\delta_n}.$$

Let $\mu \in \mathbb{C}$ and $\omega \in D$. Then the sum of two Boehmians, multiplication by a scalar α, $\otimes$ and differentiation are, respectively, defined by

$$\frac{T_{p,q}^{m,n}\theta_n}{\omega_n} + \frac{T_{p,q}^{m,n}g_n}{\psi_n} = \frac{T_{p,q}^{m,n}\theta_n \otimes \psi_n + T_{p,q}^{m,n}g_n \otimes \omega_n}{\omega_n \times \psi_n}, \qquad \mu\frac{T_{p,q}^{m,n}\theta_n}{\omega_n} = \frac{\mu T_{p,q}^{m,n}\theta_n}{\omega_n},$$
$$\frac{T_{p,q}^{m,n}\theta_n}{\omega_n} \otimes \frac{T_{p,q}^{m,n}g_n}{\delta_n} = \frac{T_{p,q}^{m,n}\theta_n \otimes T_{p,q}^{m,n}g_n}{\omega_n \times \delta_n} \quad \text{and} \quad D\frac{T_{p,q}^{m,n}\theta_n}{\omega_n} = \frac{D(T_{p,q}^{m,n}\theta_n)}{\omega_n}.$$

A sequence of Boehmians (y_n) in β_{L^A} is said to be δ-convergent to the Boehmian y, $y_n \xrightarrow{\delta} y$, if there exists a delta sequence (ω_n) such that

$$y_n \otimes \omega_n, \qquad y \otimes \omega_n \in L^A, \quad \forall k, n \in \mathbb{N},$$

and

$$y_n \otimes \omega_k \to y \otimes \omega_k \quad \text{as } n \to \infty, \text{in } L^A, \text{ for every } k \in \mathbb{N}.$$

The equivalent statement for the δ-convergence is that $y_n \xrightarrow{\delta} y$ $(n \to \infty)$ if and only if there is $\theta_{n,k}, \theta_k \in L^A$ and $(\omega_k) \in \Delta$ such that $\beta_n = \frac{T_{p,q}^{m,n}\theta_{n,k}}{\omega_k}$, $y = \frac{T_{p,q}^{m,n}\theta_k}{\omega_k}$ and for each $k \in \mathbb{N}$, $T_{p,q}^{m,n}\theta_{n,k} \to T_{p,q}^{m,n}\theta_k$ as $n \to \infty$ in L^A.

A sequence of Boehmians (y_n) in β_{L^A} is said to be Δ-convergent to a Boehmian y in β_{L^A}, $\beta_n \xrightarrow{\Delta} y$, if there exists a $(\omega_n) \in \Delta$ such that $(y_n - \beta) \otimes T_{p,q}^{m,n}\omega_n \in \beta_{L^A}$, $\forall n \in \mathbb{N}$, and $(y_n - y) \otimes T_{p,q}^{m,n}\omega_n \to 0$ as $n \to \infty$ in β_{L^A}.

The space $\beta_{L^A}^*$ with the sets $(L^A, \times)$, $(D, \times)$ and Δ follows from a similar technique. Indeed, it follows from the commutativity $\phi \times \psi = \psi \times \phi$ and the associativity $\phi \times (\psi_1 \times \psi_2) = (\phi \times \psi_1) \times \psi_2$ of the product $\times$; see [8]. Every Boehmian in $\beta_{L^A}^*$ is denoted

$$\frac{g_n}{\delta_n},$$

where $g_n \in L^A$ and $\delta_n \in \Delta$, $n \in \mathbb{N}$. An embedding between $L^A(\mathbb{R})$ and $\beta_{L^A}^*$ can be expressed as $w \to w \times \frac{\delta_n}{\delta_n}$, $n \in \mathbb{N}$. Likewise, if $\frac{g_n}{\delta_n} \in \beta_{L^A}^*$, $\alpha \in \mathbb{C}$ and $\omega \in D$, the sum of two Boehmians, multiplication of a Boehmian by α, the convolution $\times$ and the differentiation of a Boehmian are, respectively, defined by

$$\frac{\theta_n}{\omega_n} + \frac{g_n}{\psi_n} = \frac{\theta_n \times \psi_n + g_n \times \omega_n}{\omega_n \times \psi_n}, \qquad \alpha\frac{\theta_n}{\omega_n} = \alpha\frac{\theta_n}{\omega_n} = \frac{\alpha\theta_n}{\omega_n},$$
$$\frac{\theta_n}{\omega_n} \times \frac{g_n}{\delta_n} = \frac{\theta_n \times g_n}{\omega_n \times \delta_n}, \qquad D\frac{\theta_n}{\omega_n} = \frac{D\theta_n}{\omega_n}.$$

A sequence of Boehmians (x_n) in $\beta_{L^A}^*$ is said to be δ-convergent to a Boehmian x in $\beta_{L^A}^*$, $x_n \xrightarrow{\delta} x$, if there exists a delta sequence (ω_n) such that

$$x_n \times \omega_n, \qquad x \times \omega_n \in L^A, \quad \forall k, n \in \mathbb{N},$$

and

$$x_n \times \omega_k \to x \times \omega_k \quad \text{as } n \to \infty, \text{in } L^A, \text{ for every } k \in \mathbb{N}.$$

This can alternatively be written as $x_n \xrightarrow{\delta} x$ $(n \to \infty)$ in $\beta^*_{L^A}$ if and only if there is $\theta_{n,k}, \theta_k \in L^A$ and $(\omega_k) \in \Delta$ such that $x_n = \frac{\theta_{n,k}}{\omega_k}$, $x = \frac{\theta_k}{\omega_k}$ and for each $k \in \mathbb{N}$, $\theta_{n,k} \to \theta_k$ as $n \to \infty$ in L^A.

A sequence of Boehmians (x_n) in $\beta^*_{L^A}$ is said to be Δ-convergent to a Boehmian x in $\beta^*_{L^A}$, $x_n \xrightarrow{\Delta} x$, if there exists a $(\omega_n) \in \Delta$ such that $(x_n - x) \times \omega_n \in \beta^*_{L^A}$, $\forall n \in \mathbb{N}$, and $(x_n - x) \times \omega_n \to 0$ as $n \to \infty$ in $\beta^*_{L^A}$.

3 Generalized and inverse generalized Inayat operator

This section derives new definitions and properties of the generalized Inayat integral operator and its inversion formula in the class of Boehmians. It asserts that the spaces β_{L^A} and $\beta^*_{L^A}$ are isomorphic and the generalized Inayat integral operator is linear and injective. It, further, establishes a generalized convolution theorem.

Let $\frac{g_n}{\delta_n} \in \beta^*_{L^A}$, $g_n \in L^A$ and $\delta_n \in \Delta$ for every $n \in \mathbb{N}$. Then we define the generalized T integral operator of $\frac{g_n}{\delta_n}$ as a member of β_{L^A} given as

$$T_\beta \frac{g_n}{\delta_n} = \frac{T^{m,n}_{p,q} g_n}{\delta_n}. \tag{20}$$

Indeed, T_β belongs to β_{L^A} by the fact that $T^{m,n}_{p,q} g_n \in L^A$ which is justified by Theorem 1. Clearly, $T_\beta : \beta^*_{L^A} \to \beta_{L^A}$ is well defined and linear (see, e.g., [14, 16]).

Theorem 5 *The generalized operator $T_\beta : \beta^*_{L^A} \to \beta_{L^A}$ coincides with the operator $T : L^A \to L^A$.*

Proof Let $g \in L^A$ and $\frac{g \times \delta_n}{\delta_n}$ be the representation of g in $\beta^*_{L^A}$ and $\{\delta_n\} \in \Delta$ $(\forall n \in \mathbb{N})$. Then it is clear that $\{\delta_n\}$ is independent from the representation for all $n \in \mathbb{N}$. Hence, by Eq. (20) and the convolution theorem, we get

$$T_\beta \frac{g \times \delta_n}{\delta_n} = \frac{T^{m,n}_{p,q}(g \times \delta_n)}{\delta_n} = \frac{T^{m,n}_{p,q} g \otimes \delta_n}{\delta_n}.$$

This, indeed, shows that $\frac{T^{m,n}_{p,q} g \otimes \delta_n}{\delta_n}$ is the representation of $T^{m,n}_{p,q} g$ in L^A.

The proof is therefore finished. □

Theorem 6 *The operator $T_\beta : \beta^*_{L^A} \to \beta_{L^A}$ is continuous with respect to the convergence of type δ and Δ.*

A similar proof for this theorem can be easily deduced from [14–16]. Hence it has been omitted.

By aid of the aforesaid analysis, we introduce the generalized inverse integral operator of T_β as follows.

Definition 2 Let $\frac{T^{m,n}_{p,q} g_n}{\delta_n}$ be a Boehmian in β_{L^A}. Then we define the inverse T_β operator of the Boehmian $\frac{T^{m,n}_{p,q} g_n}{\delta_n}$ as

$$(T_\beta)^{-1} \frac{T^{m,n}_{p,q} g_n}{\delta_n} = \frac{g_n}{\delta_n},$$

for each $\{\delta_n\} \in \Delta$.

Theorem 7 *The generalized inverse T_β operator $(T_\beta)^{-1} : \beta_{L^A} \to \beta^*_{L^A}$ is linear.*

Proof Consider two arbitrary Boehmians $\frac{T_{p,q}^{m,n}\psi_n}{\delta_n}$ and $\frac{T_{p,q}^{m,n}g_n}{\delta_n}$ in β_{L^A}. Then the notion of addition of Boehmians reveals that

$$\frac{T_{p,q}^{m,n}\psi_n}{\delta_n} + \frac{T_{p,q}^{m,n}g_n}{\epsilon_n} = \frac{T_{p,q}^{m,n}\psi_n \times \epsilon_n + T_{p,q}^{m,n}g_n \times \epsilon_n}{\delta_n \times \epsilon_n},$$

for all $n \in \mathbb{N}$. By applying Definition 2 and the Inayat convolution theorem, the above equation becomes

$$(T_\beta)^{-1}\left(\frac{T_{p,q}^{m,n}\psi_n}{\delta_n} + \frac{T_{p,q}^{m,n}g_n}{\epsilon_n}\right) = \frac{(T_{p,q}^{m,n})^{-1}(T_{p,q}^{m,n}\psi_n \otimes \epsilon_n + T_{p,q}^{m,n}g_n \otimes \epsilon_n)}{\delta_n \times \epsilon_n}.$$

Once again, by taking into account the Inayat convolution theorem we obtain

$$(T_\beta)^{-1}\left(\frac{T_{p,q}^{m,n}\psi_n}{\delta_n} + \frac{T_{p,q}^{m,n}g_n}{\epsilon_n}\right) = \frac{(T_{p,q}^{m,n})^{-1}(T_{p,q}^{m,n}(\psi_n \times \epsilon_n) + T_{p,q}^{m,n}(g_n \times \delta_n))}{\delta_n \times \epsilon_n}.$$

Therefore, allowing the inverse operator to act on the right-hand side of the above equation gives

$$(T_\beta)^{-1}\left(\frac{T_{p,q}^{m,n}\psi_n}{\delta_n} + \frac{T_{p,q}^{m,n}g_n}{\epsilon_n}\right) = \frac{(\psi_n \times \epsilon_n) + (g_n \times \delta_n)}{\delta_n \times \epsilon_n}.$$

Hence, the notion of addition in $\beta^*_{L^A}$ implies

$$(T_\beta)^{-1}\left(\frac{T_{p,q}^{m,n}\psi_n}{\delta_n} + \frac{T_{p,q}^{m,n}g_n}{\epsilon_n}\right) = \frac{\psi_n}{\delta_n} + \frac{g_n}{\epsilon_n} \quad (\forall n \in \mathbb{N}).$$

Now, for all $\eta \in \mathbb{C}$ and all $n \in \mathbb{N}$, it is very natural to write

$$(T_\beta)^{-1}\left(\eta \frac{T_{p,q}^{m,n}\psi_n}{\delta_n}\right) = \eta (T_\beta)^{-1} \frac{T_{p,q}^{m,n}\psi_n}{\delta_n}.$$

This finishes the proof of the theorem. □

Theorem 8 *Let $\frac{T_{p,q}^{m,n}g_n}{\delta_n} \in \beta_{L^A}$. Then we have $(T_\beta)^{-1}(\frac{T_{p,q}^{m,n}g_n}{\delta_n} \otimes g) = \frac{g_n}{\delta_n} \times g$ for all $g \in D$.*

Proof Assume $\frac{T_{p,q}^{m,n}g_n}{\delta_n} \in \beta_{L^A}$. Then, for every $g \in D$, we have

$$(T_\beta)^{-1}\left(\frac{T_{p,q}^{m,n}g_n}{\delta_n} \otimes g\right) = (T_\beta)^{-1} \frac{T_{p,q}^{m,n}g_n \otimes g}{\delta_n}.$$

By using the convolution theorem, Definition 2 and splitting out the operation $\times$ we obtain

$$(T_\beta)^{-1}\left(\frac{T_{p,q}^{m,n}g_n}{\delta_n} \otimes g\right) = \frac{(T_{p,q}^{m,n})^{-1}(T_{p,q}^{m,n}g_n \otimes g)}{\delta_n} = \frac{g_n \times g}{\delta_n} = \frac{g_n}{\delta_n} \times g.$$

This finishes the proof of the theorem. □

By a similar technique, the reader can easily check the convolution formula of the generalized Inayat operator,

$$T_\beta\left(\frac{g_n}{\delta_n}\times g\right)=\frac{T^{m,n}_{p,q}g_n}{\delta_n}\otimes g,$$

where $g\in L^A$ and $g_n\in\beta^*_{L^A}$ for all $n\in\mathbb{N}$.

Theorem 9 *The generalized Inayat operator* $T_\beta:\beta^*_{L^A}\to\beta_{L^A}$ *is a bijection.*

Proof To show that the mapping is an injective mapping, assume $T_\beta\frac{\psi_n}{\delta_n}=T_\beta\frac{g_n}{\epsilon_n}$ $(\forall n\in\mathbb{N})$. Then it follows by Eq. (20) that

$$T^{m,n}_{p,q}\psi_n\otimes\epsilon_m=T^{m,n}_{p,q}g_m\otimes\delta_n\quad\text{for all }n\in\mathbb{N}.$$

Therefore, the Inayat convolution theorem leads to

$$T^{m,n}_{p,q}(\psi_n\times\epsilon_m)=T^{m,n}_{p,q}(g_m\times\delta_n)\quad\text{for all }m,n\in\mathbb{N}.$$

Hence, by applying the inverse operator we obtain

$$\psi_n\times\epsilon_m=g_m\times\delta_n\quad\text{for all }m,n\in\mathbb{N}.$$

Therefore, by the notion of addition, it happens that

$$\frac{\psi_n}{\delta_n}=\frac{g_n}{\epsilon_n}\quad\text{for all }n\in\mathbb{N}.$$

This finishes the proof of the theorem. □

4 Conclusion

In this paper a generalization of the Inayat integral operator is obtained and extended to a set of generalized functions. Two pairs of convolution products and a certain convolution theorem are derived and applied in the construction of the desired spaces of integrable Boehmians. In addition to an inversion formula, several properties of the generalized integral are also obtained in the sense of generalized functions.

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