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Dynamics of a stochastic Gilpin–Ayala population model with Markovian switching and impulsive perturbations

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Abstract

In this paper, we consider the dynamics of a stochastic Gilpin–Ayala model with regime switching and impulsive perturbations. The Gilpin–Ayala parameter is also allowed to switch. Sufficient conditions for extinction, nonpersistence in the mean, weak persistence, and stochastic permanence are provided. The critical number among the extinction, nonpersistence in the mean, and weak persistence is obtained. Our results demonstrate that the dynamics of the model have close relations with the impulses and the Markov switching.

Keywords: Gilpin–Ayala model; Markovian switching; Impulsive perturbations; Persistence

1 Introduction

Population ecology is an important branch of ecology; people often use some basic models in mathematical ecology and the mathematical analysis methods to study the living environment conditions of organisms. In many cases, mathematical models are used as a tool to study related problems in biology and ecology to make the internal laws of organisms more vividly displayed. One of the hot spots in the field of mathematical ecology today is the study of population size. The most basic mathematical model for a single species population growth is the logistic equation, which has been studied by many scholars. The classical logistic model is described by an ordinary differential equation:

$$\frac{dx(t)}{dt} = x(t)(r - kx(t)),$$

where $x(t)$ is the population size at time t , $r > 0$ stands for the intrinsic growth rate of species, $k > 0$ denotes the intraspecific competition coefficient, and $\frac{r}{k}$ is the carrying capacity. However, the logistic model assumes that the exponential growth of species in an ecosystem is a linear. This simplifies the model, but is very different from reality. Therefore, in order to better fit the actual data, Gilpin–Ayala [1] considered the following model:

$$\frac{dx(t)}{dt} = x(t)(r - kx^\theta(t)),$$

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where θ is a positive parameter to modify the classical logistic model. Many important research works about various forms of Gilpin–Ayala model have appeared in the literature, see [2–4] and the references therein.

In the real world, any individual organism in nature is inevitably affected by its own changes in quantity and environmental noises, which are important components of the ecosystem [5]. The effects of demographic stochasticity can be neglected when the population sizes are large enough. However, the environmental noise affects the population growth rate, carrying capacity, competition coefficient, and so on. Then it follows that a deterministic system cannot reflect the authenticity of population growth, and also ignores the impact of random fluctuations of various internal and external factors on population growth. Random external perturbations need to be considered. In recent years, stochastic Gilpin–Ayala models have been studied extensively [6–10].

Generally speaking, there are two types of environmental noise, namely, white noise and colored noise. White noise, which describes the small perturbations around the equilibrium state, is common in the real world. Considering the white noise in the growth rate in Gilpin–Ayala model, Mao et al. [11] presented the following stochastic Gilpin–Ayala model:

$$dx(t) = x(t)(r - kx^\theta(t))dt + \sigma x(t)dB(t).$$

The authors showed that this equation is stochastically permanent and the solution is globally attractive under some basic assumptions.

The colored noise can be explained as a switching between two or more regimes of environment [12–16]. It should be pointed out that colored noise has a great influence on the properties of the system. The switching among different environments is memoryless, and the waiting time for the next switch is exponentially distributed [17]. So the regime switching can be modeled by a right-continuous Markov chain $(\gamma(t))_{t \geq 0}$ taking values in a finite state space $S = \{1, 2, \dots, m\}$. It has been noted [18] that population models may experience random changes in their structure and parameters, by factors such as nutrition or as rain falls. These random changes cannot be described only by the white noise [13, 19]. Inspired by this, Liu et al. [17, 20] considered the following stochastic Gilpin–Ayala model under regime switching:

$$dx(t) = x(t)(r(\gamma(t)) - a(\gamma(t))x^\lambda(t))dt + \sigma_1(\gamma(t))x(t)dB_1(t) + \sigma_2(\gamma(t))x^{1+\theta}(t)dB_2(t).$$

They studied the existence of global positive solution, extinction, nonpersistence, and weak persistence of the species, and obtained the stochastic permanence of the species under the conditions that $0 < \theta \leq 1$ and $0 < \lambda \leq 1 + \theta$. Settati et al. [21] considered a Gilpin–Ayala model with the Gilpin–Ayala parameter θ under regime switching:

$$dx(t) = x(t)(r(\gamma(t)) - k(\gamma(t))x^{\theta(\gamma(t))}(t))dt + \alpha(\gamma(t))x(t)dB(t). \quad (1.1)$$

They presented global stability of the trivial solution and sufficient conditions for the extinction, persistence, and existence of stationary distribution. Later, in [22], they presented a stochastic Gilpin–Ayala model under regime switching on patches, studied the effect of environmental noises in the asymptotic properties of the model on patches under regime

switching, and established some sufficient conditions for extinction and persistence of species.

It is important to point out that the Markov chain may have significant impacts on the properties of the population dynamics. Takeuchi et al. [23] have investigated a predator–prey deterministic system described by Lotka–Volterra equations in a random environment and revealed a very interesting and surprising result of Markovian switching on the population system: both its subsystems develop periodically, but switching makes them become neither permanent nor dissipative.

Now let us take a further step. There is no independent living organism in nature, and any organism is subject to a variety of transient effects that cause sudden changes in system variables or growth patterns. The common transient effects include external factors, internal laws, and human factors. Impulsive differential or difference equation describes the rapid changes or jumps of some moving states at fixed or nonfixed moments. Hence, impulsive equation models are more realistic responses to some natural phenomena and biological processes, and have more abundant properties and contents than differential equations [24–27].

However, because of the jump in the state of the system, the solution of the continuous system is discontinuous, which makes the study of the impulse equation more difficult. In recent years, the stability of impulsive differential equations and the existence and stability of periodic solutions of periodic systems have been studied systematically. Wu [28] considered the Gilpin–Ayala model under regime switching with impulsive effects:

$$\begin{cases} dx(t) = x(t)[b(r(t)) - a(r(t))x^\lambda(t)] dt + \sigma_1(r(t))x(t) dB_1(t) \\ \quad + \sigma_2(r(t))x^{1+\theta}(t) dB_2(t), \quad t \neq t_k, k \in \mathbb{N}, \\ x(t_k^+) = x(t_k) + b_k x(t_k), \quad k \in \mathbb{N}. \end{cases} \quad (1.2)$$

He focused on exploring the effects of Markovian switching and impulse on the extinction, the threshold value of persistence and extinction. The author also established sufficient criteria for stochastic permanence by using the theory of M -matrices.

Motivated by the above discussion, this paper considers the following general stochastic hybrid system with Markovian switching and impulsive effects:

$$\begin{cases} dx(t) = x(t)r(\xi(t))[1 - \frac{x^{\theta(\xi(t))}(t)}{K(\xi(t))}] dt + \sigma(\xi(t))x(t) dB(t), \quad t \neq t_k, k \in \mathbb{N}, \\ x(t_k^+) = x(t_k) + h_k x(t_k), \quad h_k \in (-1, \infty), k \in \mathbb{N}, \end{cases} \quad (1.3)$$

where $(\xi(t))_{t \geq 0}$ is a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \dots, m\}$; $\theta(i) > 0$ for any $i \in S$. In regime $i \in S$, the system obeys

$$\begin{cases} dx(t) = x(t)r(i)[1 - \frac{x^{\theta(i)}(t)}{K(i)}] dt + \sigma(i)x(t) dB(t), \quad t \neq t_k, k \in \mathbb{N}, \\ x(t_k^+) = x(t_k) + h_k x(t_k), \quad h_k \in (-1, \infty), k \in \mathbb{N}. \end{cases} \quad (1.4)$$

The model is presented for the following two improvements:

- (i) Compared with model (1.1), the presented model (1.3) takes the impulsive effects into consideration. Moreover, it points out the range of h_k , where $h_k \in (-1, 0)$ means harvesting and $h_k \in (0, \infty)$ denotes planting.

- (ii) Compared with model (1.2), model (1.3) allows Gilpin–Ayala parameter θ to switch among different regimes.

The rest of the paper is arranged as follows. In Sect. 2, we show that model (1.3) has a unique positive global solution. Then we establish the critical number between persistence and extinction in Sect. 3. Afterwards, we provide the sufficient condition for the stochastic permanence of the system in Sect. 4.

2 Global positivity

Throughout this paper, we suppose that there is a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, in which the one-dimensional Brownian motion $B(t)$ is defined. Let $(\xi(t))$ be generated by $Q = (q_{ij})_{m \times m}$, that is,

$$\mathbb{P}\{\xi(t + \Delta t) = j | \xi(t) = i\} = \begin{cases} q_{ij} \Delta t + o(\Delta t), & \text{if } j \neq i, \\ 1 + q_{ii} \Delta t + o(\Delta t), & \text{if } j = i, \end{cases}$$

where $\Delta t > 0$, $q_{ij} \geq 0$ for all $i \neq j$ is the transition rate from i to j while $\sum_{j=1}^m q_{ij} = 0$. As a standing hypothesis, we assume in this paper that the Markov chain $(\xi(t))_{t \geq 0}$ is irreducible. Under this assumption, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \dots, \pi_m)$ satisfying $\pi Q = 0$ subject to $\sum_{i=1}^m \pi_i = 1$ and $\pi_i > 0, \forall i \in S$ [29].

Firstly, we show the global positivity of system (1.3), which is a necessary condition among biological models. For convenience and simplicity, we define

$$\hat{f} = \min_{i \in S} f(i), \quad \check{f} = \max_{i \in S} f(i), \quad [f] = t^{-1} \int_0^t f(s) ds.$$

Throughout the paper, we always assume that

$$\min_{i \in S} \frac{r(i)}{K(i)} > 0. \quad (\text{H1})$$

Theorem 2.1 *For any initial condition $x(0) = x_0 \in (0, \infty)$, there is a unique positive solution $x(t)$ of (1.3) for $t \geq 0$ almost surely.*

Proof Consider the following Itô equation without impulse:

$$dy(t) = y(t) \left[r(\xi) - \frac{r(\xi)}{K(\xi)} \prod_{0 < t_k < t} (1 + h_k)^{\theta(\xi)} y^{\theta(\xi)}(t) \right] dt + \sigma(\xi) y(t) dB(t) \quad (2.1)$$

with initial value $y(0) = x(0)$. Here for simplicity, we drop t from $r(\xi)$ and $\frac{r(\xi)}{K(\xi)}$, etc. By a similar proof as that of [17, Theorem 1], (2.1) has a unique global positive solution $y(t)$ for $t \geq 0$. Letting $x(t) = \prod_{0 < t_k < t} (1 + h_k) y(t)$, we claim that $x(t)$ is the solution of (1.3) with initial value $x(0)$. It can be seen that $x(t)$ is continuous on each interval $(t_k, t_{k+1}) \subset [0, \infty)$, $k \in \mathbb{N}$.

And for $t \neq t_k$,

$$\begin{aligned} dx(t) &= \prod_{0 < t_k < t} (1 + h_k) dy(t) \\ &= \prod_{0 < t_k < t} (1 + h_k) y(t) \left[r(\xi) - \frac{r(\xi)}{K(\xi)} \prod_{0 < t_k < t} (1 + h_k)^{\theta(\xi)} y^{\theta(\xi)}(t) \right] dt \\ &\quad + \sigma(\xi) \prod_{0 < t_k < t} (1 + h_k) y(t) dB(t) \\ &= x(t) \left[r(\xi) - \frac{r(\xi)}{K(\xi)} x^{\theta(\xi)}(t) \right] dt + \sigma(\xi) x(t) dB(t). \end{aligned}$$

Further, for every $t_k \in \mathbb{N}$,

$$\begin{aligned} x(t_k^+) &= \lim_{t \rightarrow t_k^+} x(t) = \lim_{t \rightarrow t_k^+} \prod_{0 < t_j < t} (1 + h_j) y(t) = \prod_{0 < t_j \leq t_k} (1 + h_j) y(t_k^+) \\ &= (1 + h_k) \prod_{0 < t_j < t_k} (1 + h_j) y(t_k) = (1 + h_k) x(t_k) \end{aligned}$$

and

$$\begin{aligned} x(t_k^-) &= \lim_{t \rightarrow t_k^-} x(t) = \lim_{t \rightarrow t_k^-} \prod_{0 < t_j < t} (1 + h_j) y(t) = \prod_{0 < t_j < t_k} (1 + h_j) y(t_k^-) \\ &= \prod_{0 < t_j < t_k} (1 + h_j) y(t_k) = x(t_k). \end{aligned}$$

This completes the proof. $\square$

3 Persistence and extinction

The persistence or extinction, as a measure of population size in the long run, is of importance in the study of a population model. This section will study the persistence and extinction of the system. Firstly, we introduce some concepts.

Definition 3.1 ([17]) Let $x(t)$ be the solution of (1.3) and θ be a positive constant. Then

- (i) $x(t)$ is said to be extinctive if $\lim_{t \rightarrow +\infty} x(t) = 0$,
- (ii) $x(t)$ is said to be nonpersistent in the mean if $\lim_{t \rightarrow +\infty} [x^\theta] = 0$,
- (iii) $x(t)$ is said to be weakly persistent in the mean if $\limsup_{t \rightarrow \infty} x(t) > 0$.

Now, we are ready to give our main results of this section.

Theorem 3.1 For any initial value $x(0) > 0$, if $\bar{b} < 0$, then species x will go extinct, where

$$\bar{b} = \sum_{\xi=1}^m \pi_\xi b(\xi) + \limsup_{t \rightarrow \infty} \frac{\sum_{0 < t_k < t} \ln(1 + h_k)}{t}, \quad b(\xi) = r(\xi) - \frac{1}{2} \sigma^2(\xi).$$

Proof It follows from the ergodicity of $\xi(t)$ that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t b(\xi(s)) ds = \sum_{\xi=1}^m \pi_\xi b(\xi). \quad (3.1)$$

Applying generalized Itô formula to Eq. (2.1), one can see that

$$\begin{aligned} d \ln y(t) &= \frac{dy(t)}{y(t)} - \frac{(dy(t))^2}{(y(t))^2} \\ &= \left[r(\xi) - \frac{r(\xi)}{K(\xi)} \prod_{0 < t_k < t} (1 + h_k)^{\theta(\xi)} y^{\theta(\xi)}(t) - \frac{1}{2} \sigma^2(\xi) \right] dt + \sigma(\xi) dB(t). \end{aligned}$$

Then we have

$$\begin{aligned} \ln y(t) &= \ln y(0) + \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) \right. \\ &\quad \left. - \frac{r(\xi(s))}{K(\xi(s))} \prod_{0 < t_k < t} (1 + h_k)^{\theta(\xi(s))} y^{\theta(\xi(s))}(s) \right] ds + M(t), \end{aligned}$$

where $M(t) = \int_0^t \sigma(\xi(s)) dB(s)$. The quadratic variation of $M(t)$ is

$$\langle M(t), M(t) \rangle = \int_0^t \sigma^2(\xi(s)) ds \leq \check{\sigma}^2 t.$$

Making use of the strong law of large numbers for martingales results in

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0. \quad (3.2)$$

Recalling that $x(t) = \prod_{0 < t_k < t} (1 + h_k) y(t)$, we have

$$\begin{aligned} \ln x(t) &= \ln x(0) + \sum_{0 < t_k < t} \ln(1 + h_k) + \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) \right. \\ &\quad \left. - \frac{r(\xi(s))}{K(\xi(s))} x^{\theta(\xi(s))}(s) \right] ds + M(t) \end{aligned} \quad (3.3)$$

for all $t > 0$. Hence, we have

$$\frac{\ln x(t)}{t} \leq \frac{\ln x(0)}{t} + \frac{1}{t} \sum_{0 < t_k < t} \ln(1 + h_k) + \frac{1}{t} \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) \right] ds + \frac{M(t)}{t}.$$

Taking limit superior of both sides and then making use of (3.1) and (3.2), we can conclude $\limsup_{t \rightarrow \infty} t^{-1} \ln x(t) \leq \bar{b}$. That is, if $\bar{b} < 0$, then $\lim_{t \rightarrow \infty} x(t) = 0$. The proof is completed. $\square$

Lemma 3.1 ([30]) Suppose $Y \in ([0, \infty) \times \Omega, (0, \infty))$ and $Z \in ([0, \infty) \times \Omega, R)$ is such that

$$\lim_{t \rightarrow \infty} \frac{Z(t)}{t} = 0.$$

If for all $t > 0$,

$$\ln Y(t) \geq v_0 t - v \int_0^t Y(s) ds + Z(t),$$

then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y(s) ds \geq \frac{v_0}{v}, \quad (3.4)$$

and if

$$\ln Y(t) \leq v_0 t - v \int_0^t Y(s) ds + Z(t),$$

then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y(s) ds \leq \frac{v_0}{v}, \quad (3.5)$$

where $v_0 \geq 0$ and $v > 0$ are two real numbers.

Theorem 3.2 *If $\bar{b} = 0$, then species x is nonpersistent in the mean.*

Proof For arbitrarily fixed $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$\frac{\ln x(0)}{t} \leq \frac{\varepsilon}{3}, \quad \frac{1}{t} \left[\sum_{0 < t_k < t} \ln(1 + h_k) + \int_0^t b(\xi(s)) ds \right] \leq \bar{b} + \frac{\varepsilon}{3} \quad \text{and} \quad \frac{M(t)}{t} \leq \frac{\varepsilon}{3}$$

for all $t \geq T$. By the above inequalities, (3.3) becomes

$$\ln x(t) \leq (\bar{b} + \varepsilon)t - \frac{\hat{r}}{\check{K}} \int_0^t x^{\hat{\theta}}(s) ds.$$

Using Lemma 3.1, we obtain

$$\limsup_{t \rightarrow +\infty} t^{-1} \int_0^t x^{\hat{\theta}}(s) ds \leq \frac{(\bar{b} + \varepsilon)\check{K}}{\hat{r}} \quad \text{a.s.}$$

Hence, condition $\bar{b} = 0$ and the arbitrariness of ε will lead to

$$\lim_{t \rightarrow +\infty} [x^{\hat{\theta}}(s)] = 0.$$

This completes the proof. $\square$

Theorem 3.3 *If $\bar{b} > 0$, then species x is weakly persistent in the mean.*

Proof To prove the result, we need to show $\limsup_{t \rightarrow \infty} x(t) > 0$ a.s. If the assertion is not true, then $\mathbb{P}(U) > 0$, where $U = \{\omega : \limsup_{t \rightarrow +\infty} x(t, \omega) = 0\}$. It follows from Eq. (3.3) that

$$\begin{aligned} \frac{\ln x(t)}{t} &= \frac{\ln x(0)}{t} + \frac{1}{t} \sum_{0 < t_k < t} \ln(1 + h_k) + \frac{1}{t} \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) \right] ds \\ &\quad - \frac{1}{t} \int_0^t \frac{r(\xi(s))}{K(\xi(s))} x^{\theta(\xi(s))}(s) ds + \frac{M(t)}{t}. \end{aligned} \quad (3.6)$$

If for all $\omega \in \mathcal{U}$, $\limsup_{t \rightarrow +\infty} x(t, \omega) = 0$, then

$$\limsup_{t \rightarrow \infty} \frac{\ln x(t, \omega)}{t} \leq 0, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{r(\xi(s))}{K(\xi(s))} x^{\theta(\xi(s))}(s, \omega) ds = 0, \quad \lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0.$$

Substituting the above into (3.6) results in

$$0 \geq \limsup_{t \rightarrow \infty} \frac{\ln x(t, \omega)}{t} = \bar{b} > 0,$$

which is a contradiction. This completes the proof. $\square$

Remark 3.1 Theorems 3.1–3.3 have a clear and interesting biological interpretation. Observe that the extinction and persistence of species $x(t)$ depend only on the expression

$$\bar{b} = \sum_{\xi=1}^m \pi_{\xi} \left(r(\xi) - \frac{1}{2} \sigma^2(\xi) \right) + \limsup_{t \rightarrow \infty} \frac{\sum_{0 < t_k < t} \ln(1 + h_k)}{t}.$$

If $\bar{b} > 0$, the population $x(t)$ is weakly persistent. If $\bar{b} < 0$, the population $x(t)$ becomes extinct. These also imply that the long time behaviors of system (1.3) have close relations with impulse, white noise, and stationary distribution of the Markovian switching.

Remark 3.2 Let us look at the effects of impulse on the extinction and persistence of species. The result in no-impulse system [17, Theorem 2] has shown that condition $\sum_{\xi=1}^m \pi_{\xi} (r(\xi) - \frac{1}{2} \sigma^2(\xi)) < 0$ leads to the species extinction. However, Theorem 3.3 tells us that if the impulsive effects h_k ($k \in \mathbb{N}$) are large enough such that $\limsup_{t \rightarrow \infty} t^{-1} \times \sum_{0 < t_k < t} \ln(1 + h_k) > |\sum_{\xi=1}^m \pi_{\xi} (r(\xi) - \frac{1}{2} \sigma^2(\xi))|$, i.e., $\bar{b} > 0$, then species will be weakly persistent. This means that a positive impulse is always advantageous for the existence of the species, e.g., planting. Furthermore, we can see that the positive impulse can resist the impact of the white noise, which in accord with reality and will not happen in the model without impulses.

Remark 3.3 Let us consider the effects of white noise on the species. From the expression of the critical value $\bar{b}$, we can see that the white noise $\sigma(\xi(t))$ imposed on the intrinsic growth rate $r(\xi(t))$ leads to the extinction of the species and is disadvantageous to the survival of the species.

Remark 3.4 Let us turn to the impact of the Markovian switching on the system. For subsystem (1.4), we can easily prove that (see [31]) if $\bar{b}(i) = r(i) - 0.5\sigma^2(i) + \limsup_{t \rightarrow \infty} t^{-1} \times \sum_{0 < t_k < t} \ln(1 + h_k) < 0$, then species $x(t)$ of (1.4) will go extinct. And if $\bar{b}(i) > 0$, the species $x(t)$ will be weakly persistent. Now, let us consider a case that the impulsive effect is non-positive. In this case, if $r(i) - 0.5\sigma^2(i) < 0$ for every $i \in S$, then every subsystem of (1.4) will become extinct. From Theorem 3.1 we can see that the ultimate behavior of (1.3) is also extinction even if Markovian switching exists. However, if there is a subset $S_0 \subset S$ such that $r(i) - 0.5\sigma^2(i) > 0$ for all $i \in S_0$, the irreducibility of the Markov chain will guarantee that if the species spends enough time in these “good” states (i.e., the states with $r(i) - 0.5\sigma^2(i) > 0$, $i \in S_0$ can contribute to the result $\bar{b} > 0$), the species can still be persistent under regime switching.

Theorem 3.4 *If*

$$\bar{b} - \sum_{\xi=1}^m \pi_{\xi} \frac{r(\xi)}{K(\xi)} \left(1 - \frac{\theta(\xi)}{\hat{\theta}} \right) \geq 0, \quad (3.7)$$

then the solution of system (1.3) obeys

$$\limsup_{t \rightarrow \infty} [x^{\hat{\theta}}] \leq \frac{\hat{\theta}}{\hat{\omega}} \left(\bar{b} - \sum_{\xi=1}^m \pi_{\xi} \frac{r(\xi)}{K(\xi)} \left(1 - \frac{\theta(\xi)}{\hat{\theta}} \right) \right),$$

where $\hat{\omega} = \min\{\omega(\xi), \xi \in S\}$ *and* $\omega(\xi) = r(\xi)\theta(\xi)/K(\xi)$.

Proof Applying the inequality (see [21])

$$z^{\rho} \geq 1 + \rho(z - 1) \quad \text{for } z = x^{\hat{\theta}}(t) \geq 0 \text{ and } \rho = \frac{\theta(\xi(t))}{\hat{\theta}} \geq 1$$

to (3.3), we have

$$\begin{aligned} \ln x(t) &\leq \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) - \frac{r(\xi(s))}{K(\xi(s))} \left(1 + \frac{\theta(\xi(s))}{\hat{\theta}} (x^{\hat{\theta}}(s) - 1) \right) \right] ds \\ &\quad + \sum_{0 < t_k < t} \ln(1 + h_k) + M(t) + \ln x(0) \\ &= \sum_{0 < t_k < t} \ln(1 + h_k) + \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) - \frac{r(\xi(s))}{K(\xi(s))} \left(1 - \frac{\theta(\xi(s))}{\hat{\theta}} \right) \right] ds \\ &\quad - \int_0^t \frac{r(\xi(s))}{K(\xi(s))} \frac{\theta(\xi(s))}{\hat{\theta}} x^{\hat{\theta}}(s) ds + M(t) + \ln x(0). \end{aligned} \quad (3.8)$$

By the ergodic theory of Markov chains, we have, for any $\varepsilon > 0$, there is a $T > 0$ such that

$$\begin{aligned} &t^{-1} \left[\sum_{0 < t_k < t} \ln(1 + h_k) + \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) - \frac{r(\xi(s))}{K(\xi(s))} \left(1 - \frac{\theta(\xi(s))}{\hat{\theta}} \right) \right] ds \right] \\ &\leq \bar{b} - \sum_{\xi=1}^m \pi_{\xi} \frac{r(\xi)}{K(\xi)} \left(1 - \frac{\theta(\xi)}{\hat{\theta}} \right) + \varepsilon \end{aligned}$$

for all $t \geq T$. Substituting this inequality into (3.8), we deduce that

$$\ln x(t) \leq \left(\bar{b} - \sum_{\xi=1}^m \pi_{\xi} \frac{r(\xi)}{K(\xi)} \left(1 - \frac{\theta(\xi)}{\hat{\theta}} \right) + \varepsilon \right) t - \frac{\hat{\omega}}{\hat{\theta}} \int_0^t x^{\hat{\theta}}(s) ds + M(t) + \ln x(0)$$

for all $t \geq T$. From (3.7) and using Lemma 3.1, we get

$$\limsup_{t \rightarrow \infty} [x^{\hat{\theta}}] \leq \frac{\hat{\theta}}{\hat{\omega}} \left(\bar{b} - \sum_{\xi=1}^m \pi_{\xi} \frac{r(\xi)}{K(\xi)} \left(1 - \frac{\theta(\xi)}{\hat{\theta}} \right) + \varepsilon \right).$$

Letting $\varepsilon \rightarrow 0$, we finally obtain the desired result. $\square$

Theorem 3.5 *If $\bar{d} > 0$, then*

$$\liminf_{t \rightarrow \infty} x(t) \leq \Lambda \leq \limsup_{t \rightarrow \infty} x(t) \quad a.s., \quad (3.9)$$

where

$$\bar{d} = \sum_{i=1}^m \pi_i \left(r(i) - \frac{1}{2} \sigma^2(i) \right) + \liminf_{t \rightarrow \infty} t^{-1} \sum_{0 < t_k < t} \ln(1 + h_k)$$

and Λ is the unique positive solution of the equation

$$\bar{d} - \sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} \Lambda^{\theta(i)} = 0. \quad (3.10)$$

Proof Consider the function

$$f(z) = \bar{d} - \sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} z^{\theta(i)}.$$

Obviously, $f(z)$ is a continuous, strictly decreasing function on $(0, \infty)$ and

$$f(0^+) = \bar{d} > 0, \quad f(+\infty) = -\infty.$$

Hence the existence, uniqueness, and positivity of Λ can be ensured by the intermediate value theorem. First we prove the right side of (3.9). If the assertion is not true, we suppose that

$$\mathbb{P}(\omega \in \Omega, \limsup_{t \rightarrow \infty} x(t, \omega) < \Lambda) > 0.$$

Then there is $A \in (\frac{1}{2}, 1) > 0$ such that $\mathbb{P}(\Omega_1) > 0$, where

$$\Omega_1 = \left\{ \omega \in \Omega, \limsup_{t \rightarrow \infty} x(t, \omega) \leq (2A - 1)\Lambda \right\}.$$

So, for every $\omega \in \Omega_1$, there is a $T(\omega) > 0$ such that for all $t \geq T(\omega)$,

$$x(t) \leq (2A - 1)\Lambda + (1 - A)\Lambda = A\Lambda. \quad (3.11)$$

From (3.3) and using (3.11), we deduce that

$$\begin{aligned} \ln x(t) &\geq \ln x(0) + \sum_{0 < t_k < t} \ln(1 + h_k) + \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) \right] ds \\ &\quad - \int_0^T \frac{r(\xi(s))}{K(\xi(s))} x^{\theta(\xi(s))}(s) ds - \int_T^t \frac{r(\xi(s))}{K(\xi(s))} (A\Lambda)^{\theta(\xi(s))} ds + \int_0^t \sigma(\xi(s)) dB(s). \end{aligned}$$

Since $A < 1$, we have

$$\begin{aligned} \ln x(t) &\geq \ln x(0) + \sum_{0 < t_k < t} \ln(1 + h_k) + \int_0^t \left[r(\xi(s)) - \frac{1}{2} \sigma^2(\xi(s)) \right] ds \\ &\quad - \int_0^T \frac{r(\xi(s))}{K(\xi(s))} x^{\theta(\xi(s))}(s) ds - A^{\hat{\theta}} \int_T^t \frac{r(\xi(s))}{K(\xi(s))} (\Lambda)^{\theta(\xi(s))} ds \\ &\quad + \int_0^t \sigma(\xi(s)) dB(s). \end{aligned} \quad (3.12)$$

Based on the ergodic theory of Markov chains and (3.2), we can easily show from (3.12) that there exists $\Omega'_1 \subset \Omega_1$ such that $\mathbb{P}(\Omega'_1) = 1$ and, for every $\omega \in \Omega'_1$, we get

$$\begin{aligned} \liminf_t \frac{1}{t} \ln x(t) &\geq \sum_{i=1}^m \pi_i \left(r(i) - \frac{1}{2} \sigma^2(i) \right) + \liminf_{t \rightarrow \infty} t^{-1} \sum_{0 < t_k < t} \ln(1 + h_k) \\ &\quad - A^{\hat{\theta}} \sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} \Lambda^{\theta(i)}. \end{aligned}$$

It follows from the Λ -equation (3.10) and $A^{\hat{\theta}} < 1$ that

$$\liminf_t \frac{1}{t} \ln x(t) \geq (1 - A^{\hat{\theta}}) \bar{d} > 0.$$

Therefore $\lim_{t \rightarrow \infty} x(t) = \infty$. This contradicts (3.11). The required assertion must therefore hold. Similarly, we can prove the left side of (3.9). $\square$

Theorem 3.5 tells us that solution of system (1.3) oscillates infinitely often about Λ with probability one. Therefore, it is helpful to know more information about Λ . For this, we have the following corollary.

Corollary 3.1 *Assume that $\bar{d} > 0$, then*

(i) *If*

$$\bar{d} - \sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} = 0,$$

one has

$$\liminf_{t \rightarrow \infty} x(t) \leq 1 \leq \limsup_{t \rightarrow \infty} x(t) \quad a.s.;$$

(ii) *If*

$$\bar{d} - \sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} > 0,$$

one has

$$\liminf_{t \rightarrow \infty} x(t) \leq \Lambda_2 \quad \text{and} \quad \limsup_{t \rightarrow \infty} x(t) \geq \Lambda_1 \quad a.s.,$$

(iii) Otherwise, one has

$$\liminf_{t \rightarrow \infty} x(t) \leq \Lambda_1 \quad \text{and} \quad \limsup_{t \rightarrow \infty} x(t) \geq \Lambda_2 \quad \text{a.s.,}$$

where

$$\Lambda_1 = \left[\bar{d} \left(\sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} \right)^{-1} \right]^{1/\check{\theta}} \quad \text{and} \quad \Lambda_2 = \left[\bar{d} \left(\sum_{i=1}^m \pi_i \frac{r(i)}{K(i)} \right)^{-1} \right]^{1/\hat{\theta}}.$$

Proof Denote the left-hand side of Eq. (3.10) by $f(\Lambda)$. The condition of claim (i) indicates that $f(1) = 0$, which means that $\Lambda = 1$. Hence, the result of (i) holds.

The condition of (ii) indicates that $f(1) > 0$. Applying the intermediate value theorem, we have $\Lambda > 1$. It is easy to calculate that for all $i \in S$, $\Lambda^{\hat{\theta}} \leq \Lambda^{\theta(i)} \leq \Lambda^{\check{\theta}}$. Then, from (3.8) we obtain that $\Lambda_1 \leq \Lambda \leq \Lambda_2$, which implies that $x(t)$ lies in the interval $[\Lambda_1, \Lambda_2]$ infinitely often with probability one.

Similarly, the condition of (iii) indicates that $f(1) < 0$. Then $\Lambda < 1$ and so $\Lambda_2 \leq \Lambda \leq \Lambda_1$. $\square$

4 Stochastic permanence

Stochastic permanence describes the persistence of a population in the presence of random disturbances. In the following, we will further study the stochastic permanence of the system. We first give the definition of stochastic permanence.

Definition 4.1 ([6]) Model (1.3) is said to be stochastically permanent if for any $\epsilon \in (0, 1)$, there exists a pair of positive constants $\beta_1 = \beta_1(\epsilon)$ and $\beta_2 = \beta_2(\epsilon)$ such that for any initial data $x_0 \in (0, \infty)$, the solution $x(t)$ of model (1.3) has the property that

$$\liminf_{t \rightarrow \infty} \mathbb{P}\{|x(t)| \leq \beta_1\} \geq 1 - \epsilon,$$

$$\liminf_{t \rightarrow \infty} \mathbb{P}\{|x(t)| \geq \beta_2\} \geq 1 - \epsilon.$$

Assumption 4.1 There exist two positive constants n and N such that $n \leq \prod_{0 < t_k < t} (1 + h_k) \leq N$ for all $t > 0$.

Assumption 4.2 For some $u \in S$, $q_{iu} > 0$, $\forall i \neq u$.

Lemma 4.1 ([13]) Under Assumption 4.1, for any $p \in (0, 1)$, there exists a constant K_0 such that any solution of (1.3) has the property

$$\limsup_{t \rightarrow \infty} \mathbb{E}[x^p(t)] \leq K_0.$$

Lemma 4.2 ([20]) Let Assumption 4.2 hold. If $b^* = \sum_{k=1}^m \pi_k b(k) > 0$, then for any constant $\lambda > 0$, there exists a constant $\alpha > 0$ such that the matrix

$$A(\alpha) := \text{diag}(\xi_1(\alpha), \xi_2(\alpha), \dots, \xi_m(\alpha)) - Q$$

is a nonsingular M -matrix, where $\xi_k(\alpha) = \alpha(1 + \lambda)b(k) - 0.5\alpha^2(1 + \lambda)^2\sigma^2(k)$.

Now we present our main result of this section. In the following, we write $V(y(t), r(t))$ with $V(y, k)$ for simplicity.

Theorem 4.1 *Under Assumptions 4.1 and 4.2, if $\bar{b} > 0$, then the population x is stochastically permanent.*

Proof First let us demonstrate that for arbitrary given $\varepsilon > 0$, there exists a constant $\beta_2 > 0$ such that $\liminf_{t \rightarrow \infty} \mathbb{P}\{|x(t)| \geq \beta_2\} \geq 1 - \varepsilon$. Define

$$V_1(y) = y^{-(1+\lambda)},$$

where λ is any positive number. Applying generalized Itô formula to Eq. (2.1), one can see that

$$\begin{aligned} dV_1(y(t)) &= -(1+\lambda)V_1 \left[r(\xi(t)) - \frac{r(\xi(t))}{K(\xi(t))} \prod_{0 < t_k < t} (1+h_k)^{\theta(\xi(t))} y^{\theta(\xi(t))} \right] dt \\ &\quad - (1+\lambda)V_1 \sigma(\xi(t)) dB(t) + 0.5(1+\lambda)(2+\lambda)V_1 \sigma^2(\xi(t)) dt \\ &= (1+\lambda)V_1 \left[\frac{r(\xi(t))}{K(\xi(t))} \prod_{0 < t_k < t} (1+h_k)^{\theta(\xi(t))} y^{\theta(\xi(t))} - r(\xi(t)) \right] dt \\ &\quad + 0.5(1+\lambda)(2+\lambda)V_1 \sigma^2(\xi(t)) dt - (1+\lambda)V_1 \sigma(\xi(t)) dB(t). \end{aligned}$$

For α given in Lemma 4.2, there exists a vector $\vec{p} = (p_1, p_2, \dots, p_m)^T \gg 0$ (where $\vec{p} \gg 0$ means $p_i > 0$ for every $i = 1, 2, \dots, m$) such that $A(\alpha)\vec{p} \gg 0$, which is equivalent to

$$\alpha p_k(1+\lambda)(r(k) - 0.5\sigma^2(k)) - 0.5\alpha^2 p_k(1+\lambda)^2 \sigma_1^2(k) - \sum_{j=1}^m q_{kj} p_j > 0$$

for $1 \leq k \leq m$. Define function $V_2: \mathbb{R}_+ \times S \rightarrow \mathbb{R}_+$ by $V_2(y, k) = p_k(1 + V_1)^\alpha$. Making use of Itô formula leads to

$$dV_2(y, k) = LV_2(y, k) dt - \alpha p_k(1+\lambda)\sigma(k)(1+V_1)^{\alpha-1} V_1 dB(t),$$

where

$$\begin{aligned} dV_2(y, k) &= \alpha p_k(1+V_1)^{\alpha-2} \left\{ (1+\lambda)(1+V_1)V_1 \left[\frac{r(k)}{K(k)} \prod_{0 < t_k < t} (1+h_k)^{\theta(k)} y^{\theta(k)} - r(k) \right] \right. \\ &\quad \left. + 0.5(1+\lambda)(2+\lambda)(1+V_1)V_1 \sigma^2(k) + 0.5(\alpha-1)(1+\lambda)^2 V_1^2 \sigma^2(k) \right\} \\ &\quad + (1+V_1)^\alpha \sum_{j=1}^m q_{kj} p_j \\ &= (1+V_1)^{\alpha-2} \\ &\quad \times \left\{ -V_1^2 \left[\alpha p_k(1+\lambda)(r(k) - 0.5\sigma^2(k)) - 0.5\alpha^2 p_k(1+\lambda)^2 \sigma^2(k) - \sum_{j=1}^m q_{kj} p_j \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + V_1 \left[-\alpha p_k(1+\lambda)r(k) + 0.5\alpha p_k(1+\lambda)(2+\lambda)\sigma^2(k) + 2 \sum_{j=1}^m q_{kj}p_j \right] \\
& + \sum_{j=1}^m q_{kj}p_j + \alpha p_k(1+\lambda) \frac{r(k)}{K(k)} \prod_{0 < t_k < t} (1+h_k)^{\theta(k)} V_1 y^{\theta(k)-1-\lambda} \\
& + \alpha p_k(1+\lambda) \frac{r(k)}{K(k)} \prod_{0 < t_k < t} (1+h_k)^{\theta(k)} y^{\theta(k)-1-\lambda} \Big\}.
\end{aligned}$$

Now, choose $\eta > 0$ sufficiently small to satisfy

$$\alpha p_k(1+\lambda)(r(k) - 0.5\sigma^2(k)) - 0.5\alpha^2 p_k(1+\lambda)^2 \sigma^2(k) - \sum_{j=1}^m q_{kj}p_j - \eta p_k > 0.$$

Define $V_3(y, k) = e^{\eta t} V_2(y, k)$ and let $\frac{r}{K} = \psi$. By virtue of the generalized Itô formula,

$$dV_3(y, k) = \eta e^{\eta t} V_2(y, k) dt + e^{\eta t} dV_2(y, k).$$

Integrating this inequality and then taking expectations of both sides, we can see that

$$\mathbb{E}[V_3(y(t), r(t))] = V_2(y(0), r(0)) + \mathbb{E} \int_0^t e^{\eta s} [LV_2(y(s), r(s)) + \eta V_2(y(s), r(s))] ds,$$

where

$$\begin{aligned}
& LV_2(y, k) + \eta V_2(y, k) \\
& = (1 + V_1)^{\alpha-2} \left\{ -V_1^2 \left[\alpha p_k(1+\lambda)(r(k) - 0.5\sigma^2(k)) - 0.5\alpha^2 p_k(1+\lambda)^2 \sigma^2(k) - \sum_{j=1}^m q_{kj}p_j \right. \right. \\
& \quad \left. \left. - \eta p_k \right] + V_1 \left[-\alpha p_k(1+\lambda)r(k) + 0.5\alpha p_k(1+\lambda)(2+\lambda)\sigma^2(k) + 2 \sum_{j=1}^m q_{kj}p_j + 2\eta p_k \right] \right. \\
& \quad \left. + \eta p_k + \sum_{j=1}^m q_{kj}p_j + \alpha p_k(1+\lambda) \frac{r(k)}{K(k)} \prod_{0 < t_k < t} (1+h_k)^{\theta(k)} V_1 y^{\theta(k)-1-\lambda} \right. \\
& \quad \left. + \alpha p_k(1+\lambda) \frac{r(k)}{K(k)} \prod_{0 < t_k < t} (1+h_k)^{\theta(k)} y^{\theta(k)-1-\lambda} \right\} \\
& \leq (1 + V_1)^{\alpha-2} \left\{ -V_1^2 \left[\alpha p_k(1+\lambda)(r(k) - 0.5\sigma^2(k)) - 0.5\alpha^2 p_k(1+\lambda)^2 \sigma^2(k) - \sum_{j=1}^m q_{kj}p_j \right. \right. \\
& \quad \left. \left. - \eta p_k \right] + V_1 \left[0.5\alpha p_k(1+\lambda)(2+\lambda)\sigma^2 + 2 \sum_{j=1}^m q_{kj}p_j + 2\eta p_k \right] \right. \\
& \quad \left. + \eta p_k + \sum_{j=1}^m q_{kj}p_j + \alpha p_k(1+\lambda) \check{\psi} N^{\check{\theta}} V_1 y^{\theta(k)-1-\lambda} + \alpha p_k(1+\lambda) \check{\psi} N^{\check{\theta}} y^{\theta(k)-1-\lambda} \right\} \\
& := L(y, k).
\end{aligned}$$

We claim that $L(y, k)$ is upper-bounded. Without loss of generality:

(1) If $y \geq 1$, since $\lambda > 0$ is arbitrarily, we can choose λ to satisfy $\check{\theta} \leq 1 + \lambda$. Then by the definition of V_1 , we can check that $L(y, k)$ is upper-bounded. That is to say, there exists a positive number L_{1k} such that $\sup_{y \geq 1} L(y, k) < L_{1k}$.

(2) If $y < 1$, for the above λ , for any $k \in S$ and $0 < \theta(k) \leq \check{\theta} \leq 1 + \lambda$, we can see that there exists a constant $\tau(k) \in (0, 1]$ such that $\theta(k) = \tau(k)(1 + \lambda)$ and $y^{\theta(k)-1-\lambda} = y^{\tau(k)(1+\lambda)} y^{-(1+\lambda)} = V_1^{1-\tau(k)}$. Then we can obtain

$$\begin{aligned} L(y, k) = (1 + V_1)^{\alpha-2} & \left\{ -V_1^2 \left[\alpha p_k(1 + \lambda)(r(k) - 0.5\sigma^2(k)) - 0.5\alpha^2 p_k(1 + \lambda)^2 \sigma^2(k) \right. \right. \\ & \left. \left. - \sum_{j=1}^m q_{kj} p_j - \eta p_k \right] + V_1 \left[0.5\alpha p_k(1 + \lambda)(2 + \lambda)\check{\sigma}^2 + 2 \sum_{j=1}^m q_{kj} p_j + 2\eta p_k \right] \right. \\ & \left. + \eta p_k + \sum_{j=1}^m q_{kj} p_j + \alpha p_k(1 + \lambda)\check{\psi} N^{\check{\theta}} V_1^{2-\tau(k)} + \alpha p_k(1 + \lambda)\check{\psi} N^{\check{\theta}} V_1^{1-\tau(k)} \right\}. \end{aligned}$$

Now, we can conclude that if $y < 1$, there exists a positive number L_{2k} such that $\sup_{y < 1} L(y, k) < L_{2k}$. Choose $L = \max_{k \in S} \{L_{1k}, L_{2k}\}$, then $\sup_{y \in \mathbb{R}_+, k \in S} L(y, k) < L < \infty$. Hence

$$\begin{aligned} & p_k e^{\eta t} \mathbb{E}[(1 + V_1(y(t)))^\alpha] \\ & = p_k (1 + V_1(y(0)))^\alpha + \mathbb{E} \int_0^t e^{\eta s} [LV_2(y(s), r(s)) + \eta V_2(y(s), r(s))] ds \\ & \leq p_k (1 + V_1(y(0)))^\alpha + \frac{1}{\eta} L(e^{\eta t} - 1). \end{aligned}$$

Further,

$$\limsup_{t \rightarrow \infty} \mathbb{E}[y^{-\alpha(1+\lambda)}(t)] = \limsup_{t \rightarrow \infty} \mathbb{E}[V_1^\alpha(y(t))] \leq \limsup_{t \rightarrow \infty} \mathbb{E}[(1 + V_1)^\alpha y(t)] \leq \frac{L}{\eta \hat{p}}.$$

Then,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{E}[x(t)^{-\alpha(1+\lambda)}] & = \limsup_{t \rightarrow \infty} \mathbb{E} \left[\prod_{0 < t_k < t} (1 + h_k)^{-\alpha(1+\lambda)} y^{-\alpha(1+\lambda)}(t) \right] \\ & \leq n^{-\alpha(1+\lambda)} \limsup_{t \rightarrow \infty} \mathbb{E}[y^{-\alpha(1+\lambda)}(t)] \\ & \leq n^{-\alpha(1+\lambda)} \frac{L}{\eta \hat{p}} := \tilde{L}. \end{aligned}$$

Thus for any given $\varepsilon > 0$, letting $\beta_2 = (\varepsilon/\tilde{L})^{-\alpha(1+\lambda)}$, by virtue of Chebyshev's inequality, we derive

$$\begin{aligned} \mathbb{P}\{x(t) \leq \beta_2\} & = \mathbb{P}\{x^{-\alpha(1+\lambda)}(t) \geq \beta_2^{-\alpha(1+\lambda)}\} \leq \mathbb{E}[x^{-\alpha(1+\lambda)}(t)/\beta_2^{-\alpha(1+\lambda)}] \\ & = \beta_2^{\alpha(1+\lambda)} \mathbb{E}[x^{-\alpha(1+\lambda)}(t)]. \end{aligned}$$

That is to say, $\limsup_{t \rightarrow \infty} \mathbb{P}\{x(t) \leq \beta_2\} \leq (\varepsilon/\tilde{L})\tilde{L} = \varepsilon$. Consequently, $\limsup_{t \rightarrow \infty} \mathbb{P}\{x(t) \geq \beta_2\} \geq 1 - \varepsilon$. As an application of Chebyshev's inequality and Lemma 4.1, we get $\limsup_{t \rightarrow \infty} \mathbb{P}\{x(t) \leq \beta_1\} \geq 1 - \varepsilon$. This completes the proof. $\square$

Remark 4.1 The stochastic permanence of the system means that the population will remain in a positive and bounded range with probability one. From the conditions of Theorem 4.1, we can see that the stochastic permanence has close relations with the impulse and stationary distribution of the Markov chain. In order to make the population survive forever, the impulse must be in a certain range, which is consistent with reality.

5 Conclusions and further remarks

This paper is concerned with the dynamics of a stochastic Gilpin–Ayala model with regime switching and impulsive perturbations. The Gilpin–Ayala parameter is also allowed to switch. We established some criteria on the permanence and extinction of the system. Further, we obtained some sufficient conditions on the extinction and stochastic permanence. The critical value among the extinction, nonpersistence in the mean and weak persistence has been given. Our results demonstrate that the dynamics of the model have close relations with the impulse and the Markov switching. Some interesting topics deserve further investigating. It is an interesting question to investigate the ergodicity and stationary distribution of the system. It is also interesting to study the asymptotic properties of stochastic Lotka–Volterra systems under regime switching.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The corresponding author ZL provided the basic idea of this work. The first author YJ wrote the draft of the manuscript, and the other authors, JY and YT, revised the manuscript. All authors read and approved the final manuscript.

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