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Gamma operators reproducing exponential functions

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Abstract

The present paper deals with reconstruction of Gamma operators preserving some exponential functions and studies their approximation properties: uniform convergence, rate of convergence, asymptotic formula and saturation. The effectiveness of new operators compared to classical ones is presented in certain senses as well. The last section is devoted to numerical results which compare the effectiveness of new constructions of Gamma operators.

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1 Introduction and preliminaries

The increasingly rapid developments in the field of approximation theory have been seen for the past 70 years. A considerable amount of literature has been published to find the answer to the question of how to find the best convergence to a given function. These studies give a number of approaches to the construction of approximating functions. In the meantime, Korovkin and Bohman introduced one of the most significant theorems in approximation theory, known as Bohman–Korovkins's approximation theorem in the literature, which presents a method to check whether a sequence of positive linear operators $(L_n)_{n\geq 1}$ converges to the identity operator with regard to the uniform norm of the space $C[a,b]$, that is, whether it represents or not an approximation process. This theorem provides insight into the studies on linear positive operators, and several new constructions of approximation operators have been introduced in the literature. One of these studies is due to King who provides new sequences of operators preserving the test functions $e_k(x) = x^k$, $k = 0, 2$, and their linear combinations, conditionally [17]. In other respects, Acar et al. [2] introduced a new family of linear positive operators, that reproduce the functions e^{2ax}, $a > 0$, instead of the usual polynomials. Another modification of such a construction was also considered in [3]. Further modifications and their approximation results showed the effectiveness of operators preserving some exponential functions in certain senses, hence many researchers have focused on the introduction and investigation of exponential-type operators. Among others, we refer the readers to [4–10, 20] and the references therein.

On the other hand, Gamma operators, introduced by Lupas and Müller [18], are one of the most widely used groups in approximation theory and have been extensively used for finding a better approximation to the target function. The classical Gamma operators are defined by

$$G_n(f;x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-xu}u^n f\left(\frac{n}{u}\right)du, \quad x \in \mathbb{R}^+ := (0,\infty), n \in \mathbb{N}.$$

In the literature, a number of approaches have appeared addressing to refine Gamma operators, which showed that new operators have similar approximation properties as their classical counterparts; see [1, 13, 15, 18, 19, 21] and the references therein. It is observed that the modified Gamma operators reproduce only constant and linear functions.

Motivated by the above-mentioned papers, the main aim of this paper is to provide a conceptual theoretical framework based on reconstruction of Gamma operators that reproduce the functions $e^{k\omega x}$, $\omega > 0$, $k = 1, 2$, instead of the usual polynomials, and we formulate a sufficient condition under which the new operators perform better than the classical ones.

The paper is organized as follows. In Sect. 2, the new construction of the Gamma operators is introduced, along with the preservation of the exponential test functions $e^{k\omega x}$, $\omega > 0$, $k = 1, 2$. In Sect. 3, the uniform convergence of the operators is investigated, while in Sect. 4 the rate of convergence is presented. Section 5 is devoted to weighted approximation and a Voronovskaya-type result, while comparisons, graphical and numerical examples are discussed in Sect. 6.

2 Reconstruction of gamma operators

A new construction of Gamma operators for the function $f \in C(\mathbb{R}^+)$, such that the generalized integral is convergent, can be defined as

$$\Gamma_n^\omega(f;x) = \frac{e^{\omega x}\gamma_n^{n+1}(x)}{n!}\int_0^\infty e^{-\gamma_n(x)u}u^n e^{-\omega\frac{x^2u}{n}} f\left(\frac{x^2u}{n}\right)du, \quad x \in \mathbb{R}^+, \tag{2.1}$$

where

$$\gamma_n(x) := \frac{\frac{\omega x^2}{n}e^{\frac{\omega x}{n+1}}}{e^{\frac{\omega x}{n+1}} - 1}, \tag{2.2}$$

$\omega > 0$ and $n \in \mathbb{N}$, such that the conditions

$$\Gamma_n^\omega\left(e^{\omega t};x\right) = e^{\omega x},$$

$$\Gamma_n^\omega\left(e^{2\omega t};x\right) = e^{2\omega x},$$

are satisfied for all $x \in \mathbb{R}^+$ and all $n \in \mathbb{N}$. Here we take attention of readers to the fact that the limit $\gamma_n(x) \to x$ as $n \to \infty$. The operators Γ_n^ω are linear, positive, and preserve the constant functions in the limit.

We shall introduce the moments, including central, of the operators (2.1) which will be necessary to prove our main theorems.

Integrating by parts, we easily get

$$\begin{aligned}
\Gamma_n^{\omega}\left(e^{\omega t};x\right) &= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{n!}\int_0^{\infty} e^{-\gamma_n(x)u}u^n e^{-\omega\frac{x^2u}{n}} e^{\frac{\omega x^2 u}{n}}\,du \\
&= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{n!}\int_0^{\infty} e^{-\gamma_n(x)u}u^n\,du \\
&= e^{\omega x}
\end{aligned}$$

and by a similar consideration

$$\begin{aligned}
\Gamma_n^{\omega}\left(e^{2\omega t};x\right) &= \frac{e^{\omega x}\gamma^{n+1}(x)}{n!}\int_0^{\infty} e^{-(\gamma(x)-\omega\frac{x^2}{n})u}u^n\,du \\
&= \frac{e^{\omega x}\gamma^{n+1}(x)}{n!(\gamma(x)-\omega\frac{x^2}{n})^{n+1}}\int_0^{\infty} e^{-t}t^n\,dt \\
&= \frac{e^{\omega x}\gamma^{n+1}(x)}{(\gamma(x)-\omega\frac{x^2}{n})^{n+1}} = e^{2\omega x}.
\end{aligned}$$

Let us now evaluate the constant functions under the operators $\Gamma_n^{\omega}(\cdot;x)$:

$$\begin{aligned}
\Gamma_n^{\omega}(1;x) &= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{n!}\int_0^{\infty} e^{-(\gamma_n(x)+\omega\frac{x^2}{n})u}u^n\,du \\
&= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{(\gamma_n(x)+\omega\frac{x^2}{n})^{n+1}}.
\end{aligned}$$

Thus, using the above expression, we immediately reach a corollary:

Corollary 1 *Let $\omega > 0$, $x \in \mathbb{R}^+$, $n \in \mathbb{N}$. Then*

$$\begin{aligned}
\Gamma_n^{\omega}(1;x) &= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{(\gamma_n(x)+\omega\frac{x^2}{n})^{n+1}}, \\
\Gamma_n^{\omega}\left(e^{\omega t};x\right) &= e^{\omega x}, \\
\Gamma_n^{\omega}\left(e^{2\omega t};x\right) &= e^{2\omega x},
\end{aligned} \tag{2.3}$$

where $\gamma_n(x)$ is as in (2.2).

Lemma 1 *Let $\omega > 0$, $x \in \mathbb{R}^+$ and $e_i(x) = x^i$, $i \in \mathbb{N}$. Then*

$$\Gamma_n^{\omega}(t;x) = \frac{(n+1)x^2 e^{\omega x}\gamma_n^{n+1}(x)}{n(\gamma_n(x)+\omega\frac{x^2}{n})^{n+2}}, \tag{2.4}$$

$$\Gamma_n^{\omega}\left(t^2;x\right) = \frac{(n+1)(n+2)x^4 e^{\omega x}\gamma_n^{n+1}(x)}{n^2(\gamma_n(x)+\omega\frac{x^2}{n})^{n+3}}. \tag{2.5}$$

Besides, let us consider the central moment operator by $m_n^s(x) := \Gamma_n^{\omega}((t-x)^s;x)$, $s \in \mathbb{N}$. Then we get

$$\lim_{n\to\infty} n m_n^1(x) = -\frac{3}{2}x^2\omega, \tag{2.6}$$

$$\lim_{n\to\infty} nm_n^2(x) = x^2. \tag{2.7}$$

Proof Let us first calculate $e_1(x)$:

$$\begin{aligned}
\Gamma_n^\omega(t;x) &= \frac{x^2 e^{\omega x}\gamma_n^{n+1}(x)}{n!n}\int_0^\infty e^{-\gamma_n(x)u}u^{n+1}e^{-\omega\frac{x^2u}{n}}\,du \\
&= \frac{x^2 e^{\omega x}\gamma_n^{n+1}(x)}{n!n(\gamma_n(x)+\omega\frac{x^2}{n})^{n+2}}\int_0^\infty e^{-t}t^{n+1}\,dt \\
&= \frac{x^2 e^{\omega x}\gamma_n^{n+1}(x)\Gamma(n+2)}{n!n(\gamma_n(x)+\omega\frac{x^2}{n})^{n+2}} \\
&= \frac{(n+1)x^2 e^{\omega x}\gamma_n^{n+1}(x)}{n(\gamma_n(x)+\omega\frac{x^2}{n})^{n+2}}.
\end{aligned}$$

So, we get

$$\begin{aligned}
\Gamma_n^\omega\bigl(t^2;x\bigr) &= \frac{x^4 e^{\omega x}\gamma_n^{n+1}(x)}{n!n^2}\int_0^\infty e^{-\gamma_n(x)u}u^{n+2}e^{-\omega\frac{x^2u}{n}}\,du \\
&= \frac{x^4 e^{\omega x}\gamma_n^{n+1}(x)\Gamma(n+3)}{n!n^2(\gamma_n(x)+\omega\frac{x^2}{n})^{n+3}} \\
&= \frac{(n+1)(n+2)x^4 e^{\omega x}\gamma_n^{n+1}(x)}{n^2(\gamma_n(x)+\omega\frac{x^2}{n})^{n+3}}.
\end{aligned}$$

Equalities (2.6) and (2.7) can be obtained as a result of (2.4) and (2.5). □

Remark 1 Here we note that

$$\begin{aligned}
\Gamma_n^\omega(1;x) &= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{(\gamma_n(x)+\omega\frac{x^2}{n})^{n+1}} \to 1, \\
\Gamma_n^\omega(t;x) &= \frac{x^2 e^{\omega x}\gamma_n^{n+1}(x)}{n(\gamma_n(x)+\omega\frac{x^2}{n})^{n+2}} \to x, \\
\Gamma_n^\omega\bigl(t^2;x\bigr) &= \frac{(n+1)(n+2)x^4 e^{\omega x}\gamma_n^{n+1}(x)}{n^2(\gamma_n(x)+\omega\frac{x^2}{n})^{n+3}} \to x^2
\end{aligned}$$

as $n\to\infty$.

From Lemma 1, one can see that the operators $\Gamma_n^\omega(t;x)$ and $\Gamma_n^\omega(t^2;x)$ do not reproduce the Korovkin test functions. However, Remark 1 shows that, based on the Bohman–Korovkin theorem, the values of the limits of the moments guarantee that $\Gamma_n^\omega(t;x)$ and $\Gamma_n^\omega(t^2;x)$ are an approximation process on any compact $K\subset\mathbb{R}^+$.

Now, let us consider the exponential functions under the proposed operators.

Lemma 2 *Let* $f_\lambda(t) := e^{-\lambda t}$, $t\in\mathbb{R}^+$, $\lambda\in\mathbb{R}$. *Then for all* $\lambda\in\mathbb{R}$, $\omega>0$, $x\in\mathbb{R}^+$,

$$\lim_{n\to\infty}\Gamma_n^\omega\bigl(e^{-\lambda t};x\bigr) = e^{-\lambda x}$$

holds.

Proof Indeed, considering the definition of the operators Γ_n^ω, we immediately have

$$\begin{aligned} \Gamma_n^\omega\left(e^{-\lambda t};x\right) &= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{n!}\int_0^\infty e^{-\gamma_n(x)u}u^n e^{-\omega\frac{x^2u}{n}}e^{-\lambda\frac{x^2u}{n}}\,du \\ &= \frac{e^{\omega x}\gamma_n^{n+1}(x)}{(\gamma_n(x)+(\omega+\lambda)\frac{x^2}{n})^{n+1}} \end{aligned} \tag{2.8}$$

which holds for all $\lambda\in\mathbb{R}$. Since $\gamma_n(x)\to x$ as $n\to\infty$, equality (2.8) yields

$$\lim_{n\to\infty}\Gamma_n^\omega\left(e^{-\lambda t};x\right) = e^{\omega x}\lim_{n\to\infty}\left(\frac{\gamma_n(x)}{\gamma_n(x)+(\omega+\lambda)\frac{x^2}{n}}\right)^{n+1} = e^{\omega x}e^{-(\lambda+\omega)x} = e^{-\lambda x}. \qquad \square$$

3 Uniform convergence of (Γ_n^ω)

To investigate the uniform convergence behavior of the sequences (Γ_n^ω), we shall use the following theorem due to Boyanov and Veselinov [11], in which spaces of functions,

$$C[0,\infty) \text{ and } C_*[0,\infty) = \left\{f\in C[0,\infty): \lim_{x\to\infty} f(x) \text{ exists}\right\},$$

are considered endowed with the uniform norm $\|\cdot\|_\infty$.

Theorem 1 ([11]) *A sequence $A_n: C_*[0,\infty)\to C_*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\big(f_\lambda(t);x\big) = f_\lambda(x), \quad \lambda = 0,1,2,$$

uniformly in $[0,\infty)$ if and only if

$$\lim_{n\to\infty} A_n(f;x) = f(x) \tag{3.1}$$

uniformly in $[0,\infty)$ for all $f\in C_*[0,\infty)$.

Let us show that (Γ_n^ω) is a sequence of positive linear operators acting on $C_*(\mathbb{R}^+)$.

Proposition 1 *The sequence (Γ_n^ω) is a sequence of positive linear operators acting on $C_*[0,\infty)$.*

Proof Let $\omega>0$ be fixed, n be a natural number, $u,x\in[0,\infty)$, and

$$K(x,u) := \frac{e^{\omega x}\gamma_n^{n+1}(x)}{n!}e^{-\gamma_n(x)u}u^n e^{-\omega\frac{x^2u}{n}}.$$

Then

$$\Gamma_n^\omega(f;x) = \int_0^\infty K(x,u)f\left(\frac{x^2u}{n}\right)du, \qquad \Gamma_n^\omega(\mathbf{1};x) = \frac{e^{\omega x}\gamma_n^{n+1}(x)}{(\gamma_n(x)+\omega\frac{x^2}{n})^{n+1}}.$$

Also $\gamma_n^{n+1}(x)$ vanishes for $x=0$. It has polynomial growth at infinity; u^n has the same property, too. Also, since $\Gamma_n^\omega(\mathbf{1};x)\to 1$, $\Gamma_n^\omega(\mathbf{1};x)<C_1$. When we consider both functions and

$e^{-\gamma_n(x)u}$, we can say that $K(x,u)$ is bounded, i.e., there is $C = C(n,\omega)$ such that $K(x,u) \le C$, $u, x \ge 0$. Let $f \in C[0,\infty)$, $f \neq \mathbf{0}$, $\lim_{x\to\infty} f(x) = 0$. Fix $\varepsilon > 0$. Then there is $M > 0$ such that $|f(x)| \le \frac{\varepsilon}{2}$, $\forall x \ge M$. Let $a := \frac{\varepsilon}{2C_1\|f\|_\infty}$ and $x \ge (\frac{nM}{a})^{1/2}$. We have

$$\int_a^\infty K(x,u)\,du \le \int_0^\infty K(x,u)\,du < C_1,$$

and $\frac{x^2u}{n} \ge \frac{x^2a}{n} \ge M$, hence $|f(\frac{x^2u}{n})| \le \frac{\varepsilon}{2}$, $\forall u \ge a$. Now,

$$\begin{aligned}
\left|\Gamma_n^\omega(f;x)\right| &\le \int_0^\infty K(x,u)\left|f\left(\frac{x^2u}{n}\right)\right|du \\
&\le \int_0^a K(x,u)\|f\|_\infty\,du + \int_a^\infty K(x,u)\frac{\varepsilon}{2}\,du \\
&\le aC_1\|f\|_\infty + \frac{\varepsilon}{2}\int_a^\infty K(x,u)\,du \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

that is, $|\Gamma_n^\omega(f;x)| \le \varepsilon$, $\forall x \ge (\frac{nM}{a})^{1/2}$, i.e., $\lim_{x\to\infty} \Gamma_n^\omega(f;x) = 0$, hence $\Gamma_n^\omega(f) \in C^*[0,\infty)$, $\Gamma_n^\omega : C^*[0,\infty) \to C^*[0,\infty)$. $\square$

Theorem 2 *Let $\omega > 0$. Then, the sequence of operators $\Gamma_n^\omega : C_*(\mathbb{R}^+) \to C_*(\mathbb{R}^+)$ converges to $f(x)$ as $n \to \infty$ uniformly in $\mathbb{R}^+$, for all $f \in C_*(\mathbb{R}^+)$.*

Proof Due to Theorem 1, let us check $\sup_{x\in\mathbb{R}^+} |\Gamma_n^\omega(f_\lambda(t);x) - f_\lambda(x)|$ for $\lambda = 0, 1, 2$. Since $\Gamma_n^\omega(f_0(t);x) = \Gamma_n^\omega(e_0(t);x)$, using equality (2.3), we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^+}\left|\Gamma_n^\omega\big(f_0(t);x\big) - f_0(x)\right| \\
&\quad= \frac{\omega^2x^2}{n} + \frac{\omega^4x^4 - 2\omega^3x^3 - 2\omega^2x^2}{2n^2} + O\left(\frac{1}{n^3}\right) \\
&\quad= \frac{\omega^2}{n}\left[4e^{-2} + 27e^{-3} + 128e^{-4} + \frac{3125}{6}e^{-5} + O\big(e^{-6}\big)\right] \\
&\qquad+ \frac{\omega^4}{2n^2}\left[256e^{-4} + 3125e^{-5} + 23328e^{-6} + \frac{823543}{6}e^{-7} + O\big(e^{-8}\big)\right] \\
&\qquad- \frac{\omega^3}{n^2}\left[27e^{-3} + 256e^{-4} + \frac{3125}{3}e^{-5} + 7776e^{-6} + O\big(e^{-7}\big)\right] \\
&\qquad- \frac{\omega^2}{n^2}\left[4e^{-2} + 27e^{-3} + 128e^{-4} + \frac{3125}{6}e^{-5} + O\big(e^{-6}\big)\right] + O\left(\frac{1}{n^3}\right) \\
&\quad= \alpha_n.
\end{aligned}$$

Also, using Lemma 2, we can write the expansion

$$\begin{aligned}
\left\|\Gamma_n^\omega(f_1) - f_1\right\| \le{}& \frac{2e^{-2}}{n}(\omega+1)(2\omega+1) + \frac{32e^{-4}}{3n^2}(\omega+1)\big(12\omega^3 + 24\omega^2 + 15\omega + 3\big) \\
&- \frac{9e^{-3}}{8n^2}(\omega+1)\big(24\omega^2 + 28\omega + 8\big) - \frac{e^{-2}}{6n^2}(\omega+1)(24\omega+12) + O\left(\frac{1}{n^3}\right)
\end{aligned}$$

$$:= \beta_n,$$

which is indeed $\mathcal{O}(n^{-1})$. Finally, we also get in a similar manner that

$$\begin{aligned}\left\|\Gamma_n^{\omega}(f_2)-f_2\right\| &\leq \frac{e^{-2}}{n}(\omega+2)(\omega+1)+\frac{8e^{-4}}{3n^2}(\omega+2)\big(3\omega^3+12\omega^2+15\omega+6\big)\\ &\quad -\frac{9}{16n^2}(\omega+2)\big(6\omega^2+14\omega+8\big)-\frac{e^{-2}}{6n^2}(\omega+2)(6\omega+6)+O\left(\frac{1}{n^3}\right)\\ &:= \sigma_n.\end{aligned}$$

Here note that $\sigma_n = \mathcal{O}(n^{-1})$. Also note that the sequences (β_n) and (σ_n) tend to zero as $n \to \infty$ uniformly in $\mathbb{R}^+$, which establishes the desired result. □

4 Rate of convergence

An estimate with appropriate modulus of continuity for a sequence of operators satisfying the conditions of Theorem 1 was presented by Holhos [14]. The modulus of continuity $\omega^*(f;\delta)$ considered there is defined by

$$\omega^*(f;\delta)=\sup_{x,t\geq 0,|e^{-x}-e^{-t}|\leq\delta}\left|f(x)-f(t)\right|$$

for $\delta > 0$ and $f \in C_*[0,\infty)$. The modulus of continuity satisfies

$$\omega^*(f;\delta)=\omega\big(\Phi(f);\delta\big),$$

where $\omega(\cdot;\delta)$ is the usual modulus of continuity and $\Phi : C_*[0,\infty) \to C[0,1]$ is an isometric isomorphism given by

$$\Phi(f)(t)=\begin{cases} f(-\ln t), & 0<t\leq 1,\\ \lim_{x\to\infty} f(x), & t=0,\end{cases}\qquad \text{for every } f\in C_*[0,\infty).$$

Hence, the above-mentioned quantitative result is given in [14] as:

Theorem 3 *Let $A_n : C_*[0,\infty) \to C_*[0,\infty)$ be a sequence of positive linear operators and set*

$$\begin{aligned}\left\|A_n(f_0)-f_0\right\|_\infty &= \alpha_n,\\ \left\|A_n(f_1)-f_1\right\|_\infty &= \beta_n,\\ \left\|A_n(f_2)-f_2\right\|_\infty &= \sigma_n.\end{aligned}$$

Under the hypothesis that all sequences (α_n), (β_n), (σ_n) vanish at infinity, the following estimate holds: for every $f \in C_*[0,\infty)$,

$$\left\|A_n(f)-f\right\|_\infty \leq \|f\|_\infty\alpha_n+(2+\alpha_n)\omega^*(f;\sqrt{\alpha_n+2\beta_n+\sigma_n}).$$

As a consequence of Theorem 3, we have the following corollary.

Corollary 2 *For* $f \in C_*(\mathbb{R}^+)$, *the following inequality holds*:

$$\left\| \Gamma_n^{\omega}(f) - f \right\|_{\infty} \leq 2\omega^*\left(f; \sqrt{\alpha_n + 2\beta_n + \sigma_n}\right), \tag{4.1}$$

where (α_n), (β_n), *and* (σ_n) *are as in the proof of Theorem* 2.

Note that inequality (4.1) presents uniform convergence $\Gamma_n^{\omega}(f) \to f$. Furthermore, since both (β_n) and (σ_n) are $\mathcal{O}(n^{-1})$, the rate of uniform convergence (4.1) is $1/\sqrt{n}$.

5 Weighted approximation

Let $\varphi(x) = 1 + e^{2\omega x}$, $x \in \mathbb{R}^+, \omega > 2$. Then we consider spaces of functions:

$$\begin{aligned} B_{\varphi}(\mathbb{R}^+) &= \left\{ f : \mathbb{R}^+ \to \mathbb{R} \mid \left| f(x) \right| \leq M_f \varphi(x), x \geq 0 \right\}, \\ C_{\varphi}(\mathbb{R}^+) &= C(\mathbb{R}^+) \cap B_{\varphi}(\mathbb{R}^+), \\ C_{\varphi}^{k}(\mathbb{R}^+) &= \left\{ f \in C_{\varphi}(\mathbb{R}^+) : \lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = k_f \text{ exists} \right\}, \end{aligned}$$

where M_f and k_f are constants depending only on f. The spaces of functions are also normed spaces with

$$\|f\|_{\varphi} = \sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

Remark 2 Let $f \in C_{\varphi}(\mathbb{R}^+)$. Then the operators Γ_n^{ω} satisfy

$$\left\| \Gamma_n^{\omega}(f) \right\|_{\varphi} \leq C \|f\|_{\varphi}.$$

Here we note that the fundamental concepts of weighted approximation of a sequence of linear positive operators can be found in [12]. Our first result on weighted approximation of the sequence (Γ_n^{ω}) is as follows.

Theorem 4 *Let* $f \in C_{\varphi}^{k}(\mathbb{R}^+)$. *Then*

$$\lim_{n \to \infty} \left\| \Gamma_n^{\omega}(f) - f \right\|_{\varphi} = 0$$

holds.

Proof According to the method presented in [12], assuming the conditions

$$\lim_{n \to \infty} \left\| \Gamma_n^{\omega}\left(e^{k\omega \cdot}\right) - e^{k\omega \cdot} \right\|_{\varphi} = 0, \quad k = 0, 1, 2$$

is sufficient to prove the uniform convergence of the corresponding operators. Since $\Gamma_n^{\omega}(e^{\omega t}; x) = e^{\omega x}$ and $\Gamma_n^{\omega}(e^{2\omega t}; x) = e^{2\omega x}$, the condition follows immediately for $k = 1$ and $k = 2$. To complete the proof, it is enough to check that for $k = 0$,

$$\lim_{n \to \infty} \left\| \Gamma_n^{\omega}(e_0) - e_0 \right\|_{\varphi} = 0,$$

which completes the proof. □

6 Pointwise convergence

Theorem 5 *Let $f \in C_{\varphi}(\mathbb{R}^{+})$ be twice differentiable at any $x \in \mathbb{R}^{+}$ such that f'' is continuous at $x \in \mathbb{R}^{+}$, then we have*

$$\lim_{n\to\infty} 2n\left[\Gamma_n^{\omega}(f;x) - f(x)\right] = -3x^2\omega f'(x) + 2x^2 f''(x). \tag{6.1}$$

Proof By the Taylor's formula, there exists ξ lying between x and t such that

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)(t-x)^2}{2} + h(t,x)(t-x)^2, \tag{6.2}$$

where

$$h(t,x) = \frac{f''(\xi) - f''(x)}{2},$$

which is a continuous function vanishing as $t \to x$. When we apply the operators G_n^{ω} to both sides of equality (6.2), we have

$$\Gamma_n^{\omega}(f;x) - f(x) = f'(x)m_n^1(x) + \frac{f''(x)}{2}m_n^2(x) + \Gamma_n^{\omega}\left(h(t,x)(t-x)^2;x\right)$$

and

$$n\left[\Gamma_n^{\omega}(f;x) - f(x)\right] = f'(x)nm_n^1(x) + \frac{f''(x)}{2}nm_n^2(x) + n\Gamma_n^{\omega}\left(h(t,x)(t-x)^2;x\right).$$

On the other hand, by Lemma 1, we get

$$\lim_{n\to\infty} nm_n^1(x) = -\frac{3}{2}x^2\omega,$$
$$\lim_{n\to\infty} nm_n^2(x) = x^2.$$

We know from [3, p. 1402] that

$$\left|h(t,x)(t-x)^2\right| \le \varepsilon m_n^2(x) + \frac{M}{\delta^2}m_n^4(x).$$

Since $m_n^4(x) = \mathcal{O}(n^{-2})$, we conclude that

$$\lim_{n\to\infty} n\Gamma_n^{\omega}\left(h(t,x)(t-x)^2;x\right) = 0.$$

Hence we have the desired result. □

7 Comparisons

In this section, we compare the operators $\Gamma_n^{\omega}(f;x)$ with classical Gamma operators. The consequences obtained in this part reveal that the newly introduced operators provide a better approximation.

Theorem 6 *Let $f \in C^2(\mathbb{R}^+)$. Assume that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \leq \Gamma_n^{\omega}(f;x) \leq G_n(f;x) \quad \text{for all } n \geq n_0, x \in \mathbb{R}^+, \omega > 0. \tag{7.1}$$

Then

$$0 \leq 3\omega f'(x) \leq f''(x). \tag{7.2}$$

Conversely, if inequality (7.2) holds at a given point $x \in \mathbb{R}^+$, then there exists $n_0 \in \mathbb{N}$ such that

$$f(x) \leq \Gamma_n^{\omega}(f;x) \leq G_n(f;x) \quad \text{for all } n \geq n_0.$$

Proof Rempulska et al. [19] obtained Voronovskaya-type theorem for the operators $G_n(f;\cdot)$ as $2n(G_n(f;x) - f(x)) \to x^2 f''(x)$.

By inequality (7.1), we have

$$0 \leq 2n\big(\Gamma_n^{\omega}(f;x) - f(x)\big) \leq 2n\big(G_n(f;x) - f(x)\big).$$

Hence we get

$$0 \leq -3\omega f'(x) + f''(x) \leq f''(x),$$

which allows us to write

$$0 \leq 3\omega f'(x) \leq f''(x).$$

Thus, we immediately have inequality (7.2).

Conversely, if inequality (7.2) holds with strict inequalities for a given $x \in \mathbb{R}^+$, then directly

$$0 \leq -3\omega f'(x) + f''(x) \leq f''(x),$$

which directly implies inequality (7.1). □

8 Numerical and graphical examples

In this section, we present a series of numerical experiments highlighting the good performance of the proposed new construction of Gamma operators. All the implementations of these operators are performed in MATLAB.

Example 1 We shall now illustrate the convergence of the new Gamma operator based on its classical counterparts. The new construction of Gamma operator and its standard-version algorithm is applied to the test function $f(x) : [5,6] \to \mathbb{R}$, with

$$f(x) = x^{\log(1+x)}.$$

Table 1 Root-mean-square errors for classical Gamma operators and newly constructed Gamma operators with $\mu = 0.25$ for different values of n, test function $f(x) = x^{\log(1+x)}$, on an equally spaced 10000 evaluation grid

n	RMS Error for $G_n(f;x)$	RMS Error for $\Gamma_n^{\mu}(f;x)$
1	6.402506e–01	2.202781e–02
5	2.846635e–01	5.771027e–03
10	9.328952e–02	2.756072e–03
20	6.356742e–02	1.305979e–03
30	2.464479e–02	8.495264e–04
40	1.747821e–02	6.284498e–04

Here *RMS Error for* $G_n(f;x)$ and *RMS Error for* $\Gamma_n^{\omega}(f;x)$ are root-mean-square (L-norm) errors for $G_n(f;x)$ and $\Gamma_n^{\omega}(f;x)$, respectively. We consider the problem with a fixed dimension and investigate the error behavior for the different values of n and $\omega = 0.25$. We see in Table 1 that, as expected, the new construction of Gamma operators achieves better convergence when compared with classical Gamma operators with the same level n. The error is tested on a 1000 uniform grid.

In Fig. 1(a), we graph the results of standard Gamma operators, new construction of Gamma operators, and target function. Obviously, the proposed operator shows better convergence behavior than its classic counterparts to the target functions.

Figure 1(b) shows convergence plots for the proposed method for $n = 1, 5, 10, 20$, and 40 with $\omega = 0.25$ for all levels. Clearly, the observed increase in convergence rate of approximations of the introduced operators could be attributed to increasing n as might be expected.

Example 2 Next, we consider another test function $g(x) : [1,6] \to \mathbb{R}$, given by

$$g(x) = x^4 \log\bigl(1 + x^4\bigr),$$

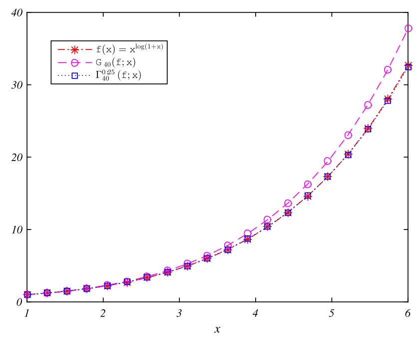
with $\omega = 0.3$. Similarly, Figur 2(a) has also confirmed that the introduced operators are generally superior to the standard Gamma operators in terms of convergence rate. On the other hand, Fig. 2(b), as expected, confirms that the bigger n gives better convergence.

9 Conclusions

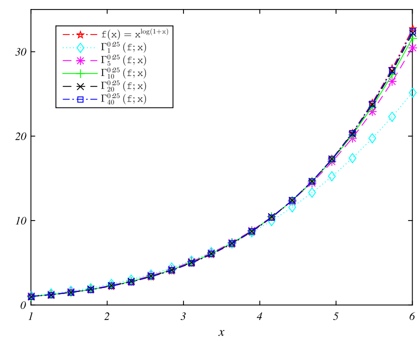
The new construction of Gamma operators which preserve the functions $e^{\omega t}$ and $e^{2\omega t}$, $\omega > 0$ is proposed and tested. One of the most significant positive aspects of the newly proposed algorithm is that it can yield better approximation behavior in comparison to its standard counterparts for large classes of functions. Numerical experiments also suggest that the introduced technique provides better approximation accuracy. The findings of this research might have a number of important implications for future practice.

There is a direction for further research if we define the degenerate Gamma operators by using the definition of the degenerate Gamma function. Recently, the concept of the degenerate gamma function and degenerate Laplace transformation was introduced by Kim–Kim [16]. For each $\lambda \in (0, \infty)$, the degenerate gamma function for the complex variable s with $0 < \mathrm{Re}(s) < 1/\lambda$ as follows:

$$\Gamma_\lambda(s) = \int_0^\infty (1 + \lambda t)^{-\frac{1}{\lambda}} t^{s-1}\, dt = \lambda^{-s} \beta\left(s, \frac{1}{\lambda} - s\right),$$

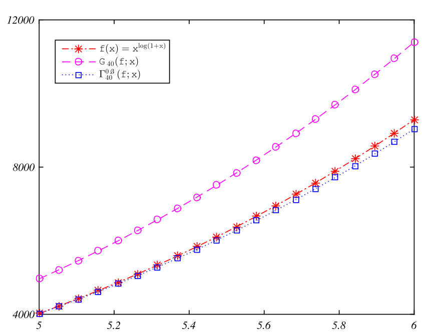


(a) Comparison of classical Gamma operators and newly constructed Gamma operators with exact function with $\mu = 0.25$ and $n = 40$; exact function (red-star), classical Gamma operators (purple-circle) and newly constructed Gamma operators (blue-square).

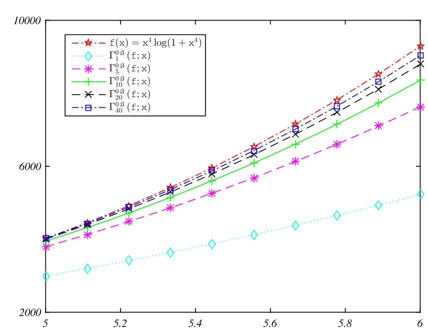


(b) Comparison of newly constructed Gamma operators with exact function for the value of $n = 1, 5, 10, 20,$ and 40 with $\mu = 0.25$; $\Gamma_1^{0.25}(f;x)$ (cyan-diamond), $\Gamma_5^{0.25}(f;x)$ (magenta-star), $\Gamma_{10}^{0.25}(f;x)$ (green-plus), $\Gamma_{20}^{0.25}(f;x)$ (black-cross), $\Gamma_{40}^{0.25}(f;x)$ (blue-square), and $f(x) = x^{\log(1+x)}$ (red-five-pointed star).

Figure 1 Approximation of target function $f(x)$ on a equally spaced evaluation grid of $[1,6]$



(a) Comparison of classical Gamma operators and newly constructed Gamma operators with exact function with $\mu = 0.3$ and $n = 40$; exact function (red-star), classical Gamma operators (purple-circle), and newly constructed Gamma operators (blue-square).



(b) Comparison of newly constructed Gamma operators with exact function for the value of $n = 1, 5, 10, 20,$ and 40 with $\mu = 0.3$; $\Gamma_1^{0.3}(g;x)$ (cyan-diamond), $\Gamma_5^{0.3}(g;x)$ (magenta-star), $\Gamma_{10}^{0.3}(g;x)$ (green-plus), $\Gamma_{20}^{0.3}(g;x)$ (black-cross), $\Gamma_{40}^{0.3}(g;x)$ (blue-square), and $g(x) = x^4 \log(1 + x^4)$ (red-five-pointed star).

Figure 2 Approximation of target function $g(x)$ on an equally spaced evaluation grid of $[5,6]$

where $\beta(x,y)$ is the Beta function. They studied some properties of the degenerate gamma and degenerate Laplace transformation and obtained their properties. The degenerate gamma and degenerate Laplace transformation can be included into engineer's mathematical toolbox to solve linear ODEs and related initial value problems.

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