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Hopf bifurcation in a diffusive predator–prey model with Smith growth rate and herd behavior

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Abstract

This paper mainly aims to consider the dynamical behaviors of a diffusive delayed predator–prey system with Smith growth and herd behavior subject to the homogeneous Neumann boundary condition. For the analysis of the predator–prey model, we have studied the existence of Hopf bifurcation by analyzing the distribution of the roots of associated characteristic equation. Then we have proved the stability of the periodic solution by calculating the normal form on the center of manifold which is associated to the Hopf bifurcation points. Some numerical simulations are also carried out in order to validate our analysis findings. The implications of our analytical and numerical findings are discussed critically.

MSC: 34A34

Keywords: Diffusion; Hopf bifurcation; Predator–prey model; Smith growth rate; Delay; Herd behavior

1 Introduction

Predator–prey models are basic differential equation models for describing the interactions between two species, and are of great interest to many researchers in mathematics and ecology. There are many factors which can affect dynamical properties of biological and mathematical models, such as functional response and harvesting.

Recently, herd behavior has been concluded as a kind of behavior, it is a behavior that the prey population uses to defend itself against the predator population. It makes a group which is called “group defense”, such that when the predators make contact with the prey, they can’t really reach the inside the space of the prey group which means that the predators just hunt on the boundary of the prey population. A new scientific study mainly aimed at making interaction to be more elaborated socially is running; the study suggests that a mix of two species shows a much more individualistic behavior [1–5]. The authors who mainly work in this field have already discovered a new kind of predator–prey model which is coordinated with the fact that the predator–prey interactions usually occur through the perimeter of the herd [2], and the new kind of model can be vividly and in detail described

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by the following ordinary differential equations:

$$\begin{cases} \frac{dX(t)}{dt} = r\left(1 - \frac{X(t)}{K}\right)X(t) - \alpha\sqrt{X(t)}Y(t), \\ \frac{dY(t)}{dt} = -sY(t) + c\alpha\sqrt{X(t)}Y(t), \end{cases}$$

Here $X(t)$ represents the prey density and $Y(t)$ the predator density, parameter r is the growth rate of the prey, parameter K is its carrying capacity, parameter s denotes the death rate of the predator in the absence of prey, parameter α is the search efficiency of $Y(t)$ for $X(t)$, and parameter c is viewed as the biomass conversion or consumption rate. This kind of model is also known to us as the predator–prey model with herd behavior, and the existence of the possibility of sustained limit cycles is real; what's more, the solution behavior near the origin shows to be more subtle and interesting.

As far as the growth of the prey is concerned, many researchers have considered logistic growth function to be a logically acceptable function

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right),$$

so that the average growth rate of $\frac{N'(t)}{N(t)}$ is a nonlinear function of the density function $N(t)$. This assumption is not realistic for a food-limited population which is under the effect of environmental toxicants. Therefore, the population dynamics with limited growth should be based on the proportion of unused available resources,

$$\frac{dN(t)}{dt} = rN(t)\frac{K - N(t)}{K + cN(t)},$$

where $\frac{K}{c}$ is viewed as the mass in the population at K , both environmental and food chain effects of toxicants' stress [6–9] should be taken into consideration.

In the real world, the prey and predator populations are always moving, so, in order to study the dynamics of this model, we should consider the condition that the two populations are spatially disperse. The spatial dynamics of the predator–prey models which include spatial diffusion have already been widely studied in the literature. We will assume the condition that both populations are in an isolated patch, which means that the immigration is neglected using the Neumann boundary conditions.

Based on the above discussions, we do rigorously consider the predator–prey system along with Smith growth rate and herb behavior as follows:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{u(x,t)(1-u(x,t))}{1+cu(x,t)} - \sqrt{u(x,t)}v(x,t) + d_1\Delta u(x,t), \\ \frac{\partial v(x,t)}{\partial t} = \gamma v(x,t)(-\beta + \sqrt{u(x,t)}) + d_2\Delta v(x,t), \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, \quad t \geq 0, \\ u(x,t) = \phi(x,t), v(x,t) = \psi(x,t) \geq 0, \quad x \in [0,\pi], \end{cases} \quad (1.1)$$

where $u(t)$ is the prey density and $v(t)$ predator density at time t ; $\beta\gamma$ is the death rate of the predator in the absence of prey, the conversion or the consumption rate of prey to predator is represented by the parameter γ .

In the real-world systems, delay exist almost everywhere. So in order to reflect the dynamical behavior of models that depend on the past history, a more realistic and better

approach is to incorporate the factor time delays into the models. Many interesting and surprising conclusions have been obtained, since incorporating time delays has a great impact and complicated effect on the dynamical behaviors of the systems. In a survey paper [10], Ruan made a conclusion that the delay differential equations show more complex dynamics than ordinary differential equations. Recently, many authors paid more attention to a partial differential system in the field of delay effects; this new kind of diffusion was taken into consideration in [3, 11–13]. Some authors put their focus on the study of the delay effects of the reaction–diffusion system, and they seriously investigated the stability/instability of the coexistence equilibrium and related with Hopf bifurcation [4, 5, 14, 15].

Hence, we will try to continue studying the dynamics of the following system:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{u(x,t)(1-u(x,t))}{1+cu(x,t)} - \sqrt{u(x,t)}v(x,t) + d_1 \Delta u(x,t), \\ \frac{\partial v(x,t)}{\partial t} = \gamma v(x,t)(-\beta + \sqrt{u(x,t-\tau)}) + d_2 \Delta v(x,t), \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, \quad t \geq 0, \\ u(x,t) = \phi(x,t), v(x,t) = \psi(x,t) \geq 0, \quad x \in [0, \pi] \times [-\tau, 0], \end{cases} \quad (1.2)$$

where the delay effect is represented by a nonnegative parameter τ .

In this paper, the main aim is to consider the delay-induced Hopf bifurcation for the predator–prey system (1.2) with the proper use of the normal form and the use of center manifold theory. The results of this paper could be summarized as follows. In Sect. 2, we first seriously consider the Hopf bifurcation of the local system (1.1). Secondly, we investigate the existence of the delay-induced Hopf bifurcation for the predator–prey model with strong diffusion. In Sect. 3, we carefully calculate the normal form on the center manifold to further discuss the dynamical behavior around the delay-induced Hopf bifurcation value. In Sect. 5, we present some accurate and vivid numerical simulations in order to precisely illustrate and better expand our theoretical results. In Sect. 8, this paper is ended with some discussions.

2 Stability and bifurcation analysis

2.1 Stability and bifurcation analysis for system (1.1) without diffusion

In order to further study the very complex dynamics of system (1.1), firstly, we discuss the dynamics of system (1.1) with no diffusion as follows:

$$\begin{cases} \frac{du(t)}{dt} = \frac{u(x,t)(1-u(x,t))}{1+cu(x,t)} - \sqrt{u(t)}v(t), \\ \frac{dv(t)}{dt} = \gamma v(t)(-\beta + \sqrt{u(t)}), \end{cases} \quad (2.1)$$

As for the system (1.2), when $0 < \beta < 1$, the unique positive equilibrium point $E^*(u^*, v^*)$ exists, where

$$u^* = \beta^2, \quad v^* = \frac{\beta(1-\beta^2)}{1+c\beta^2}.$$

The linearization of (2.1) at the unique positive equilibrium point $E^*(u^*, v^*)$ is

$$\begin{pmatrix} \frac{du(t)}{dt} \\ \frac{dv(t)}{dt} \end{pmatrix} = A \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad (2.2)$$

where

$$a_{11} = \frac{1 - 3\beta^2 - c\beta^2(1 + \beta^2)}{2(1 + c\beta^2)^2}, \quad a_{12} = -\beta, \quad a_{21} = \frac{\gamma(1 - \beta^2)}{2(1 + c\beta^2)}, \quad a_{22} = 0. \quad (2.3)$$

The characteristic equation is

$$\lambda^2 + T_0\lambda + D_0 = 0, \quad (2.4)$$

where

$$T_0 = -(a_{11} + a_{22}) = -\frac{1 - 3\beta^2 - c\beta^2(1 + \beta^2)}{2(1 + c\beta^2)^2},$$

$$D_0 = a_{11}a_{22} - a_{12}a_{21} = \frac{\gamma\beta(1 - \beta^2)}{2(1 + c\beta^2)}.$$

To study the stability of the positive equilibrium E^* for system (2.1) more accurately, the mathematical relation between c and β which appear in the preceding equations is needed. Denote

$$c_0(\beta) = \frac{1 - 3\beta^2}{\beta^2(1 + \beta^2)},$$

then $T_0(\beta, c_0(\beta)) = 0$.

In the following, we choose parameter c to be the bifurcation parameter and analyze the existence of Hopf bifurcation at the interior equilibrium E^* . In fact, parameter c can also be regarded as the Smith growth rate of prey, which plays a very important role in determining the stability of the interior equilibrium and also influences the existence of Hopf bifurcation.

If we choose to consider parameter c as a bifurcation parameter, then (2.4) will have a pair of opposite purely imaginary eigenvalues $\omega = \pm\sqrt{D_0}$ when the value of the parameter c is $c = c_0$. System (2.1) should have a small amplitude nonconstant periodic solution bifurcated from the positive E^* , when the parameter c crosses through c_0 if the transversality condition is satisfied.

Let $\lambda(c) = \alpha(c) + i\omega(c)$ be the root of (2.4), then

$$\alpha(c) = -\frac{1}{2}T_0(c), \quad \omega(c) = \frac{1}{2}\sqrt{4D_0(c) - T_0^2(c)}.$$

Hence, $\alpha(c_0) = 0$ and

$$\alpha'(c_0) = \frac{-\beta^2(1 + \beta^2)^3}{16(1 - \beta^2)^2} < 0 \quad (0 < \beta < 1). \quad (2.5)$$

From the above discussions, we can imply that system (2.1) will undergo Hopf bifurcation at E^* as c crosses through c_0 if the transversality condition (2.5) is satisfied.

Proposition *If $0 < \beta < 1$, and $\gamma > 0$ and $c > 0$, then*

- (a) *The positive equilibrium point $E^*(u^*, v^*)$ of system (2.1) is locally asymptotically stable when $c > c_0$, and $E^*(u^*, v^*)$ is actually unstable when $c < c_0$;*

- (b) System (2.1) will undergo the Hopf bifurcation at the positive equilibrium $E^*(u^*, v^*)$ when $c = c_0$.

For system (2.1), we can obtain the Hopf bifurcation line $H_0 : c_0 = \frac{1-3\beta^2}{\beta^2(1+\beta^2)}$, and the stability region is $D = \{(\beta, c) | c_0 < c\}$ of the positive equilibrium $E^*(u^*, v^*)$.

Next, we will continue to study the delay-induced Hopf bifurcation for the predator–prey model with diffusion.

2.2 Spatial-temporal dynamics for the diffusive predator–prey model

Let

$$\begin{aligned} f^{(1)}(u, v) &= \frac{u(x, t)(1 - u(x, t))}{1 + cu(x, t)} - \sqrt{u(x, t)}v(x, t), \\ f^{(2)}(u, v) &= \gamma v(x, t)(-\beta + \sqrt{u(x, t - \tau)}). \end{aligned}$$

The linearization of (1.2) at the positive equilibrium E^* is

$$\begin{pmatrix} \frac{\partial u(x, t)}{\partial t} \\ \frac{\partial v(x, t)}{\partial t} \end{pmatrix} = D\Delta \begin{pmatrix} u(x, t) \\ v(x, t) \end{pmatrix} + A_0 \begin{pmatrix} u(x, t) \\ v(x, t) \end{pmatrix} + A_1 \begin{pmatrix} u(x, t - \tau) \\ v(x, t - \tau) \end{pmatrix}, \quad (2.6)$$

with

$$D\Delta = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix}, \quad A_0 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix},$$

where a_{11} , a_{12} , a_{21} , a_{22} are already given in (2.3).

Hence, the characteristic equation of (2.6) is

$$\det(\lambda I - M_k - A_0 - A_1 e^{-\lambda\tau}) = 0, \quad (2.7)$$

where I is the 2×2 identity matrix and $M_k = -k^2 \operatorname{diag}\{d_1, d_2\}$, $k \in N_0$, from which we can conclude that

$$\begin{aligned} \lambda^2 + [(d_1 + d_2)k^2 - (a_{11} + a_{22})]\lambda \\ + [d_1 d_2 k^4 - (a_{11} d_2 + a_{22} d_1)k^2 + (a_{11} a_{22} - a_{12} a_{21} e^{-\lambda\tau})] = 0. \end{aligned} \quad (2.8)$$

When $\tau = 0$,

$$\lambda^2 + T_k \lambda + D_k = 0, \quad (2.9)$$

where

$$\begin{aligned} T_k &= (d_1 + d_2)k^2 - (a_{11} + a_{22}), \\ D_k &= d_1 d_2 k^4 - (a_{11} d_2 + a_{22} d_1)k^2 + (a_{11} a_{22} - a_{12} a_{21}). \end{aligned} \quad (2.10)$$

According to the result of [15], we always assume that the positive equilibrium E^* of system (1.2) without delay is asymptotically stable, which is equivalent to the condition $T_k > 0$, $D_k > 0$ for any $k \in N_0$,

Hence, we can obtain a long series of Hopf bifurcation lines H_k given by

$$c_k = \frac{-B + \sqrt{B^2 - 4C}}{2A},$$

with

$$\begin{aligned} A &= (d_1 + d_2)k^2\beta^4, & B &= 4(d_1 + d_2)k^2\beta^2 + \beta^2(1 + \beta^2), \\ C &= 2(d_1 + d_2)k^2 - (1 - 3\beta^2) < 0, & k &= 1, 2, 3, \dots \end{aligned}$$

When $\tau \neq 0$, we assume that $\lambda = i\omega$, and substitute $i\omega$ into (2.8), to obtain

$$\begin{aligned} &-\omega^2 + i[(d_1 + d_2)k^2 - (a_{11} + a_{22})]\omega \\ &+ [d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 + (a_{11}a_{22} - a_{12}a_{21}e^{-i\omega\tau})] = 0. \end{aligned} \quad (2.11)$$

Separating the real and imaginary parts, we have

$$\begin{cases} -\omega^2 + d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 + a_{11}a_{22} - a_{12}a_{21}\cos\omega\tau = 0, \\ [(d_1 + d_2)k^2 - (a_{11} + a_{22})]\omega + a_{12}a_{21}\sin\omega\tau = 0, \end{cases} \quad (2.12)$$

which is equivalent to

$$\omega^4 + P_k\omega^2 + Q_k = 0, \quad (2.13)$$

where

$$\begin{aligned} P_k &= [d_1k^2 - a_{11}]^2 + [d_2k^2 - a_{22}]^2, \\ Q_k &= D_k[d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 + (a_{11} + a_{22})]. \end{aligned}$$

For $0 < k < N_1$, there is a unique positive root ω_k of (2.13),

$$\omega_k = \sqrt{\frac{-P_k + \sqrt{P_k^2 - 4Q_k}}{2}}. \quad (2.14)$$

From (2.14), we can obtain that

$$\tau_k^j = \tau_k^0 + \frac{2\pi j}{\omega_k}, \quad \tau_k^0 = \frac{1}{\omega_k} \arccos \frac{-\omega^2 + d_1d_2k^4 - (a_{11}d_2 + a_{22}d_1)k^2 + a_{11}a_{22}}{a_{12}a_{21}}, \quad (2.15)$$

for $k \in \{0, 1, 2, \dots, N_1\}$.

Lemma 2.1 Assume that $0 < \beta < 1$ holds, then

$$\tau_{N_1}^j \geq \tau_{k+1}^j \geq \tau_k^j \geq \dots \geq \tau_1^j \geq \tau_0^j,$$

for $j \in N_0$.

Lemma 2.2 *If the condition $0 < \beta < 1$ holds, $T_k > 0$, $D_k > 0$ for any $k \in N_0$, then (2.8) has a pair of purely imaginary roots $i\omega_k$ for each $k \in \{0, 1, 2, \dots, N_1\}$ and (2.8) will have no purely imaginary roots for any $k \geq N_1 + 1$.*

Let $\lambda(\tau) = \alpha(\tau) + i\delta(\tau)$ be the roots of (2.8) near $\tau = \tau_k^j$ which satisfy $\alpha(\tau_k^j) = 0$, $\delta(\tau_k^j) = \omega_k$. Then, we can have the following transversality condition.

Lemma 2.3 *For $k \in \{0, 1, 2, \dots, N_1\}$ and $j \in N_0$, $\frac{d\operatorname{Re}(\lambda)}{d\tau}|_{\tau=\tau_k^j} > 0$.*

Proof Differentiating the two sides of (2.8), we get

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} = \operatorname{Re}\left[\frac{(2\lambda + T_k)e^{\lambda\tau}}{-a_{12}a_{21}\lambda} - \frac{\tau}{\lambda}\right].$$

Thus, by (2.12) and (2.13), we have

$$\begin{aligned} \operatorname{Re}\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)\Big|_{\tau=\tau_k^j} &= \operatorname{Re}\left[\frac{(2\lambda + T_k)e^{\lambda\tau}}{-a_{12}a_{21}\lambda} - \frac{\tau}{\lambda}\right]\Big|_{\tau=\tau_k^j} = \operatorname{Re}\left[\frac{(2i\omega_k + T_k)e^{i\omega_k\tau_k^j}}{-ia_{12}a_{21}\omega_k} - \frac{\tau_k^j}{i\omega_k}\right] \\ &= \frac{T_k \sin \omega_k \tau_k^j - 2\omega_k \cos \omega_k \tau_k^j}{a_{12}a_{21}\omega_k} = \frac{\omega_k^2 + P_k}{(a_{12}a_{21})^2} > 0. \end{aligned}$$

Clearly, $\tau_k^0 = \min_{j \in N_0} \{\tau_k^j\}$, $k \in \{0, 1, 2, \dots, N_1\}$, and from the above lemmas, we already know that $\tau_0^0 = \min\{\tau_k^j : 0 \leq k \leq N_1, j \in N_0\}$. Denote the parameter $\tau_* = \tau_0^0$. Let $\lambda(\tau) = \alpha(\tau) + i\delta(\tau)$ be the pair of roots of Eq. (2.8) near $\tau = \tau_k^j$ satisfying $\alpha(\tau_k^j) = 0$ and $\delta(\tau_k^j) = \omega_k$. Then, we can present the following results. $\square$

Theorem 2.1 *Assume that the condition $0 < \beta < 1$ holds, $T_k > 0$, $D_k > 0$ for any $k \in N_0$; ω_k and τ_k^j is defined by (2.14) and (2.15), respectively. And denote the minimum value of the critical values of delay by $\tau_* = \min_{k,j} \{\tau_k^j\}$.*

- The positive equilibrium $E^*(u^*, v^*)$ of system (1.2) is asymptotically stable for the parameter $\tau \in (0, \tau_*)$ and unstable for any $(\tau_*, +\infty)$;*
- System (1.2) undergoes Hopf bifurcations near the positive equilibrium $E^*(u^*, v^*)$ at τ_k^j for any $k \in \{0, 1, 2, \dots, N_1\}$ and any $j \in N_0$.*

3 Direction of Hopf bifurcation and stability of bifurcating periodic solution

3.1 Normal form of the Hopf bifurcation for a diffusive model

For $U_1 = (u_1, v_1)^T$, $U_2 = (u_2, v_2)^T \in X$, define the inner product

$$[U_1, U_2] = \int_0^\pi (u_1 u_2 + v_1 v_2) dx,$$

where $X = \{(u, v) \in W^{2,2}(0, \pi) \mid \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = 0, \pi\}$.

We denote $c^* = c_j$, and then introduce a new parameter $\varepsilon \in \mathbb{R}$ by setting the parameter $c = c^* + \varepsilon$ such that $\varepsilon = 0$ obviously becomes the bifurcation value. Then, we rewrite the positive equilibrium just as a parameter-dependent form $E_\varepsilon^*(u^*(\varepsilon), v^*(\varepsilon))$ with

$$u^*(\varepsilon) = \beta^2, v^*(\varepsilon) = \frac{\beta(1 - \beta^2)}{1 + (c^* + \varepsilon)\beta^2}.$$

Setting $\tilde{u}(\cdot, t) = u(\cdot, t) - u^*(\varepsilon)$, $\tilde{v}(\cdot, t) = v(\cdot, t) - v^*(\varepsilon)$, $\tilde{U}(t) = (\tilde{u}(\cdot, t), \tilde{v}(\cdot, t))$ and then dropping the tildes for the simplification of notation, system (1.1) can be written as the following equation:

$$\frac{dU(t)}{dt} = D\Delta U + L_0(U) + f(U, \varepsilon), \quad (3.1)$$

where

$$D\Delta U = \begin{pmatrix} d_1 \Delta u \\ d_2 \Delta v \end{pmatrix}, \quad L_0(U) = \begin{pmatrix} a_{11}u & a_{12}v \\ a_{21}u & a_{22}v \end{pmatrix},$$

$$f(U, \varepsilon) = \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl} u^i v^j \varepsilon^l, \quad f_{ijl} = (f_{ijl}^{(1)}, f_{ijl}^{(2)})^T,$$

with $f_{ijl}^{(n)} = \frac{\partial^{i+j+l} \tilde{f}^{(n)}(0,0,0)}{\partial u^i \partial v^j \partial \varepsilon^l}$, $n = 1, 2$, and

$$\tilde{f}^{(1)}(u, v, \varepsilon) = \frac{(u + u^*(\varepsilon))(1 - (u + u^*(\varepsilon)))}{(1 + (c^* + \varepsilon)u^*(\varepsilon))} - \sqrt{u + u^*(\varepsilon)}(v + v^*(\varepsilon)),$$

$$\tilde{f}^{(2)}(u, v, \varepsilon) = \gamma(v + v^*(\varepsilon))(-\beta + \sqrt{u + u^*(\varepsilon)}).$$

By a direct computation, we obtain that $f_{020} = f_{120} = f_{030} = 0$.

Assuming that there does exist a parameter $k \in N_0$ such that $\Delta_k = 0$ with $c = c^*$ which has a pair of purely imaginary roots $\pm i\omega_k$, the remaining roots of the characteristic Eq. (3.1) will actually have nonzero real parts, where

$$\omega_k = \sqrt{d_1 d_2 k^4 - d_2 \left(\frac{1 - 3\beta^2 - \beta^2(1 + \beta^2)}{2(1 + c^* \beta^2)^2} \right) k^2 + \frac{\gamma \beta(1 - \beta^2)}{2(1 + c^* \beta^2)}}.$$

In term of $\mathcal{M}_k p_k = i\omega_k p_k$ and $\mathcal{M}_k^T q_k = i\omega_k q_k$, we choose p_k and q_k such that $\langle q_k^T, p_k \rangle = 1$, where

$$p_k = \begin{pmatrix} 1 \\ \frac{i\omega_k + d_1 k^2 - a_{11}}{a_{12}} \end{pmatrix}, \quad q_k = D \begin{pmatrix} 1 \\ \frac{i\omega_k + d_1 k^2 - a_{11}}{a_{21}} \end{pmatrix},$$

with $D = [1 + \frac{(i\omega_k + d_1 k^2 - a_{11})^2}{a_{12}a_{21}}]^{-1}$.

By (3.1) and a very direct computation, we get

$$\frac{1}{2}f_2(U, \varepsilon) = f_{101}u\varepsilon + f_{011}v\varepsilon + \frac{1}{2}f_{020}u^2 + f_{110}uv + \frac{1}{2}f_{020}v^2,$$

then

$$\begin{aligned} \frac{1}{2}f_2(z, 0, \varepsilon) &= \frac{1}{2}f_2(\Phi_k z \gamma_k(x), 0) = f_{101}(p_{k1}z_1\varepsilon + \bar{p}_{k1}z_2\varepsilon)\gamma_k(x) \\ &\quad + f_{011}(p_{k2}z_1\varepsilon + \bar{p}_{k2}z_2\varepsilon)\gamma_k(x) + \frac{1}{2}(A_{k20}z_1^2 + A_{k11}z_1z_2 + A_{k02}z_2^2)\gamma_k^2(x), \end{aligned}$$

where

$$A_{k20} = f_{200}p_{k1}^2 + 2f_{110}p_{k1}p_{k2}, \quad A_{k02} = \bar{A}_{k20}, \quad A_{k11} = 2f_{200}|p_{k1}|^2 + 4f_{110}\operatorname{Re}\{p_{k1}\bar{p}_{k2}\}.$$

Thus, we obtain

$$\frac{1}{2}g_2^1(z, 0, \varepsilon) = \frac{1}{2}\text{Proj}_{\text{KerM}_2^1}f_2^1(z, 0, \varepsilon) = \begin{pmatrix} B_{k1}z_1\varepsilon \\ \bar{B}_{k1}z_2\varepsilon \end{pmatrix},$$

where $B_{k1} = q_k^T(f_{101}p_{k1} + f_{011}p_{k2})$.

The calculation of $\text{Proj}_S f_3^1(z, 0, 0)$.

$$\int_0^\pi \gamma_k^4(x) dx = \begin{cases} \frac{1}{\pi}, & k = 0, \\ \frac{3}{2\pi}, & k \neq 0, \end{cases}$$

It is easy to verify that

$$\frac{1}{3!}\text{Proj}_S f_3^1(z, 0, 0) = \begin{pmatrix} B_{k21}z_1^2z_2 \\ \bar{B}_{k21}z_1z_2^2 \end{pmatrix},$$

where

$$B_{k21} = \begin{cases} \frac{1}{2\pi}b_{k21}, & k = 0, \\ \frac{3}{4\pi}b_{k21}, & k \neq 0, \end{cases}$$

with $b_{k21} = q_k^T(f_{300}p_{k1}|p_{k1}|^2 + f_{210}(p_{k1}^2\bar{p}_{k2} + 2p_{k2}|p_{k1}|^2))$.

The calculation of $\text{Proj}_S[(D_z f_2^1)(z, 0, 0)U_2^1(z, 0)]$ requires

$$f_2^1(z, 0, 0) = \Psi_k(A_{k20}z_1^2 + A_{k11}z_1z_2 + A_{k02}z_2^2) \int_0^\pi \gamma_k^3(x) dx,$$

where there is a straightforward calculation which shows that

$$U_2^1(z, 0) = (M_2^1)^{-1}f_2^1(z, 0, 0) = \frac{\int_0^\pi \gamma_k^3(x) dx}{i\omega_k} \begin{pmatrix} q_k^T(A_{k20}z_1^2 - A_{k11}z_1z_2 - \frac{1}{3}A_{k02}z_2^2) \\ \bar{q}_k^T(\frac{1}{3}A_{k20}z_1^2 + A_{k11}z_1z_2 - A_{k02}z_2^2) \end{pmatrix},$$

and then

$$\frac{1}{3!}\text{Proj}_S[(D_z f_2^1)U_2^1](z, 0, 0) = \begin{pmatrix} C_{k21}z_1^2z_2 \\ \bar{C}_{k21}z_1z_2^2 \end{pmatrix},$$

with

$$C_{k21} = \begin{cases} \frac{1}{6\pi}c_{k21}, & k = 0, \\ 0, & k \neq 0, \end{cases}$$

where

$$c_{k21} = \frac{i}{\omega_k} \left((q_k^T A_{k20})(q_k^T A_{k11}) - |q_k^T A_{k11}|^2 - \frac{2}{3}|q_k^T A_{k02}|^2 \right).$$

The calculation of $\text{Proj}_S[(D_w f_2^1)(z, 0, 0)U_2^2(z, 0)]$ is as follows:

$$\frac{1}{3!}\text{Proj}_S(D_w f_2^1)(z, 0, 0)(h) = \begin{pmatrix} D_{k21}z_1^2z_2 \\ \bar{D}_{k21}z_1z_2^2 \end{pmatrix},$$

with

$$C_{k21} = \begin{cases} \frac{1}{3\sqrt{\pi}}E_{(0,0)}, & k = 0, \\ \frac{1}{3\sqrt{\pi}}E_{(k,0)} + \frac{1}{3\sqrt{2\pi}}E_{(k,2k)}, & k \neq 0, \end{cases}$$

where

$$E_{(k,j)} = q_k^T((f_{200}p_{k1} + f_{110}p_{k2})h_{kj11}^{(1)} + (f_{110}p_{k1} + f_{020}p_{k2})h_{kj11}^{(2)} \\ + (f_{200}\bar{p}_{k1} + f_{110}\bar{p}_{k2})h_{kj20}^{(1)} + (f_{110}\bar{p}_{k1} + f_{020}\bar{p}_{k2})h_{kj20}^{(2)}), \quad j = 0, 2k.$$

In order to obtain D_{k21} , we compute h_{kj20} and h_{kj11} as follows:

$$\begin{cases} h_{0020} = \frac{1}{\sqrt{\pi}}(2i\omega I_2 - \mathcal{M}_0)^{-1}(A_{020} - q_0^T A_{020}p_0 - \bar{q}_0^T A_{020}\bar{p}_0), \\ h_{0020} = -\frac{1}{\sqrt{\pi}}(A_{011} - q_0^T A_{011}p_0 - \bar{q}_0^T A_{011}\bar{p}_0), \quad k = 0, j = 0, \end{cases}$$

and

$$\begin{cases} h_{kj20} = c_{kj}(2i\omega I_2 - \mathcal{M}_0)^{-1}A_{020}, \\ h_{0020} = -c_{kj}A_{011}, \quad k \neq 0, j = 0, 2k. \end{cases}$$

Thus, the normal form of Hopf bifurcations on the center manifold is

$$\dot{z} = B_k z + \begin{pmatrix} B_{k1}z_1\varepsilon \\ \bar{B}_{k1}z_2\varepsilon \end{pmatrix} + \begin{pmatrix} B_{k2}z_1^2z_2 \\ \bar{B}_{k2}z_1z_2^2 \end{pmatrix} + O(|z|\varepsilon^2 + |z|^4),$$

where

$$B_{k2} = B_{k21} + \frac{3}{2}(C_{k21} + D_{k21}) = \begin{cases} \frac{1}{2\pi}b_{021} + \frac{1}{4\pi}c_{021} + \frac{1}{2\sqrt{\pi}}E_{(0,0)}, & k = 0, \\ \frac{3}{4\pi}b_{k21} + \frac{1}{2\sqrt{\pi}}E_{(k,0)} + \frac{1}{2\sqrt{2\pi}}E_{(k,2k)}, & k \neq 0, \end{cases}$$

which can be written down in real coordinates w by letting $z_1 = w_1 - iw_2$, $z_2 = w_1 + iw_2$, and transforming to polar coordinates $w_1 = \rho \cos \xi$, $w_2 = \rho \sin \xi$. Then, this normal form becomes

$$\begin{cases} \dot{\rho} = v_{k1}\rho\varepsilon + v_{k2}\rho^3 + O(\varepsilon^2\rho + |(\rho, \varepsilon)|^4), \\ \dot{\xi} = -\omega_k + O(|(\rho, \varepsilon)|), \end{cases}$$

with $v_{k1} = \text{Re}(B_{k1})$, $v_{k2} = \text{Re}(B_{k2})$.

It's know that the direction of the bifurcation is determined by the sign of $v_{k1}v_{k2}$ (supercritical if $v_{k1}v_{k2} < 0$, subcritical $v_{k1}v_{k2} > 0$), and the stability of the nontrivial periodic orbits can be determined by the sign of v_{k2} (stable if $v_{k2} < 0$, unstable if $v_{k2} > 0$).

4 Normal form of the delay-induced Hopf bifurcation for a diffusive model

In this subsection, we shall study the directions, stability and period of bifurcating periodic solutions by moderately and scrupulously applying the normal form theory and the center manifold theory of partial functional differential equations which are presented in [15–18] in detail. For fixed $j \in N_0$, $0 \leq k \leq N_1$, we denote $\tau^* = \tau_k^j$, and then introduce a new parameter $\varepsilon \in \mathbb{R}$ by setting $\varepsilon = \tau - \tau^*$ such that $\varepsilon = 0$ becomes the Hopf bifurcation value obviously.

Setting $\tilde{u}(\cdot, t) = u(\cdot, \tau t) - u^*$, $\tilde{v}(\cdot, t) = v(\cdot, \tau t) - v^*$, $\tilde{U}(t) = (\tilde{u}(\cdot, \tau t), \tilde{v}(\cdot, \tau t))$ and $\mathcal{C} = \mathcal{C}([-1, 0], X)$, and then dropping the tildes for the simplification of notation, system (1.2) can be written as follows:

$$\frac{dU(t)}{dt} = \tau D_0 \Delta U(t) + L(\tau)(U_t) + \tilde{F}(U_t, \tau), \quad \varphi = (\varphi_1, \varphi_2)^T,$$

where

$$D_0 \Delta U = \begin{pmatrix} d_1 \Delta u \\ d_2 \Delta v \end{pmatrix}, \quad L(\tau)\varphi = \begin{pmatrix} a_{11}\varphi_1(0) & a_{12}\varphi_2(0) \\ a_{21}\varphi_1(-1) & a_{22}\varphi_2(0) \end{pmatrix},$$

$$\tilde{F}(\varphi, \varepsilon) = \tau \begin{pmatrix} f^{(1)}(\tau) \\ f^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(1)} \varphi_1^i(0) \varphi_2^j(0) \varphi_1^l(-1) \\ \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} \varphi_1^i(0) \varphi_2^j(0) \varphi_1^l(-1) \end{pmatrix},$$

with $f_{ijl}^{(n)} = \frac{\partial^{i+j+l} \tilde{f}^{(n)}(u^*, v^*, w^*)}{\partial u^i \partial v^j \partial w^l}$, $n = 1, 2$, and

$$\tilde{f}^{(1)}(u, v, w) = \frac{(u + u^*)(1 - u - u^*)}{1 + c(u + u^*)} - \sqrt{u + u^*}(v + v^*),$$

$$\tilde{f}^{(2)}(u, v, w) = \gamma(v + v^*)(-\beta + \sqrt{w + u^*}).$$

By direct computation, we can obtain $f_{020} = f_{210} = f_{120} = f_{030} = 0$.

Setting $\tau = \tau^* + \varepsilon$, $\Lambda_0 = \{-i\tau^* \omega^*, i\tau^* \omega^*\}$ gives

$$\frac{dU(t)}{dt} = \tau^* D_0 \Delta U(t) + L(\tau^*)(U_t),$$

$$\frac{dU(t)}{dt} = \tau D_0 \Delta U(t) + L(\tau)(U_t) + \tilde{F}(U_t, \varepsilon), \quad \varphi = (\varphi_1, \varphi_2)^T,$$

$$\tilde{F}(U_t, \varepsilon) = \varepsilon D_0 \Delta \varphi(0) + L(\varepsilon)(\varphi) + f(\varphi, \tau^* + \varepsilon), \quad \text{for } \varphi \in \mathcal{C}.$$

The eigenvalues of $\tau^* D \Delta$ on X are $\mu_k^i = -d_i \tau^* k^2$, $i = 1, 2, k \in N_0$.

$$\beta_k^1 = \begin{pmatrix} \gamma_k(x) \\ 0 \end{pmatrix}, \quad \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k(x) \end{pmatrix}, \quad \gamma_k(x) = \frac{\cos kx}{\|\cos kx\|_{2,2}}, \quad k \in N_0.$$

$\mathbf{B}_k = \text{span}\{\langle v(\cdot), \beta_k^i \rangle v \in \mathcal{C}, i = 1, 2\}$, $z_i(\theta) \in \mathcal{C} = \mathcal{C}([-1, 0], \mathbb{R}^2)$,

$$z_i^T(\theta) \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \in \mathbf{B}_k.$$

The linear PDE restricted on $\mathbf{B}_k$ is equivalent to the FDE on $\mathcal{C} = C([-1, 0], \mathbb{R}^2)$,

$$\dot{z}(t) = \begin{pmatrix} \mu_k^1 & 0 \\ 0 & \mu_k^2 \end{pmatrix} z(t) + L(\tau^*)(z_t).$$

When $\tau = \tau^*$, define $\eta(\theta) \in BV([-1, 0], \mathbb{R})$, such that

$$\mu_k \varphi(0) + L(\tau^*)\varphi = \int_{-1}^0 d\eta(\theta)\varphi(\theta),$$

and the adjoint bilinear form on $\mathcal{C}^* \times \mathcal{C}$, $\mathcal{C}^* = C([0, 1], \mathbb{R}^{2*})$ as follows:

$$\begin{aligned} \langle \psi(s), \phi(\theta) \rangle &= \psi(0)\phi(0) - \int_{-1}^0 \int_0^\theta \psi(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \\ \Phi_k &= (pe^{i\omega^* \tau^* \theta}, \bar{p}e^{-i\omega^* \tau^* \theta}), \quad \Psi_k = \text{col}(q^T e^{-i\omega^* \tau^* s}, \bar{q}^T e^{i\omega^* \tau^* s}), \end{aligned}$$

where $\langle \Phi_k, \Psi_k \rangle = I_2$, and

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega^* + d_1 k^2 - a_{11}}{a_{12}} \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = D \begin{pmatrix} 1 \\ \frac{i\omega^* + d_1 k^2 - a_{11}}{a_{21}} e^{i\omega^* \tau^*} \end{pmatrix},$$

with $D = (1 + \tau^*(i\omega^* + d_1 k^2 - a_{11}) + \frac{(i\omega^* + d_1 k^2 - a_{11})e^{i\omega^* \tau^*}}{a_{12}a_{21}})^{-1}$,

$$\dot{z} = Bz + \begin{pmatrix} A_{k1}z_1\varepsilon \\ \frac{A_{k1}z_1\varepsilon}{A_{k1}z_2\varepsilon} \end{pmatrix} + \begin{pmatrix} A_{k2}z_1^2z_2 \\ \frac{A_{k2}z_1^2z_2}{A_{k2}z_1z_2^2} \end{pmatrix} + O(|z|\varepsilon^2 + |z|^4),$$

where

$$A_{k1} = -k^2(d_1q_1p_1 + d_2q_2p_2) + i\omega^* q^T p,$$

and

$$\begin{aligned} A_{k2} &= \frac{i}{2\omega^* \tau^*} \left(B_{k20}B_{k11} - 2|B_{k11}|^2 - \frac{1}{3}|B_{k02}|^2 \right) + \frac{1}{2}(B_{k21} + D_{k21}), \\ B_{k20} &= \begin{cases} \frac{\tau^*}{\sqrt{\pi}}(c_1q_1 + c_2q_2), & k = 0, \\ 0, & k \neq 0, \end{cases} \quad B_{k11} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}}(c_3q_1 + c_4q_2), & k = 0, \\ 0, & k \neq 0. \end{cases} \\ B_{k02} &= \begin{cases} \frac{\tau^*}{\sqrt{\pi}}(\bar{c}_1q_1 + \bar{c}_2q_2), & k = 0, \\ 0, & k \neq 0, \end{cases} \quad B_{k21} = \begin{cases} \frac{\tau^*}{\pi}c_5, & k = 0, \\ \frac{3\tau^*}{2\pi}c_5, & k \neq 0. \end{cases} \\ c_1 &= f_{200}^{(1)}p_1^2 + 2f_{110}^{(1)}p_1p_2, \quad c_2 = f_{002}^{(2)}p_1^2e^{-2i\omega^* \tau^*} + 2f_{110}^{(2)}p_1p_2e^{-i\omega^* \tau^*}, \\ c_3 &= f_{200}^{(1)}|p_1|^2 + 2f_{110}^{(1)}\text{Re}\{p_1\bar{p}_2\}, \quad c_4 = f_{002}^{(2)}|p_1|^2 + 2f_{110}^{(2)}\text{Re}\{p_1\bar{p}_2e^{-i\omega^* \tau^*}\}, \\ c_5 &= q_1(f_{300}^{(1)}p_1|p_1|^2 + f_{210}^{(1)}(p_1^2\bar{p}_2 + 2|p_1|^2p_2)) + q_2(f_{003}^{(2)}p_1|p_1|^2e^{-i\omega^* \tau^*} \\ &\quad + f_{012}^{(2)}(p_1^2\bar{p}_2e^{-2i\omega^* \tau^*} + 2|p_1|^2p_2)), \end{aligned}$$

and

$$D_{k21} = \begin{cases} E_0, & k = 0, \\ E_0 + \frac{\sqrt{2}}{2} E_{2k}, & k \neq 0, \end{cases}$$

$$E_j = \frac{2\tau^*}{\sqrt{\pi}} \begin{pmatrix} F_1 h_{j11}^{(1)}(0) + \overline{F_1} h_{j20}^{(1)}(0) + F_2 h_{j11}^{(2)}(0) + \overline{F_2} h_{j20}^{(2)}(0) \\ F_3 h_{j11}^{(1)}(-1) + \overline{F_3} h_{j20}^{(1)}(-1) + F_4 h_{j11}^{(2)}(0) + \overline{F_4} h_{j20}^{(2)}(0) \end{pmatrix},$$

where

$$F_1 = f_{200}^{(1)} p_1 + f_{110}^{(1)} p_2, \quad F_2 = f_{110}^{(1)} p_1, \quad F_3 = f_{011}^{(2)} p_2 + f_{002}^{(2)} p_1, \quad F_4 = f_{011}^{(2)} p_1 e^{-i\omega^* \tau^*},$$

where $h_{k20}(\theta)$ and $h_{k11}(\theta)$ are all determined by the following equations:

$$\begin{cases} \dot{h}_{k20}(\theta) - 2i\omega^* \tau^* h_{k20}(\theta) = \Phi_k(\theta) \begin{pmatrix} B_{k20} \\ \bar{B}_{k20} \end{pmatrix}, \\ \dot{h}_{k20}(0) - L(\tau^*)(h_{k20}) = \tau^* c_{kj} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \\ \dot{h}_{k11}(\theta) = 2\Phi_k(\theta) \begin{pmatrix} B_{k11} \\ \bar{B}_{k11} \end{pmatrix}, \\ \dot{h}_{k11}(0) - L(\tau^*)(h_{k11}) = 2\tau^* c_{kj} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{cases}$$

with

$$c_{kj} = \begin{cases} \frac{1}{\sqrt{\pi}}, & j = k = 0, \\ \frac{1}{\sqrt{\pi}}, & j = 0, k \neq 0, \\ \frac{1}{\sqrt{2\pi}}, & j = 2k \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Through the change of variables $z_1 = \omega_1 - i\omega_2$, $z_2 = \omega_1 + i\omega_2$ and $\omega_1 = \rho_1 \cos \phi$, $\omega_2 = \rho_1 \sin \phi$, the normal form becomes the following polar coordinate system:

$$\begin{cases} \dot{\rho} = \kappa_{k1} \alpha \rho + \kappa_{k2} \rho^3 + O(\alpha^2 \rho + |(\rho, \alpha)|^4), \\ \dot{\phi} = -\omega^* \tau^* + O(|(\rho, \alpha)|), \end{cases}$$

with the parameters $\kappa_{k1} = \operatorname{Re} A_{k1}$, $\kappa_{k2} = \operatorname{Re} A_{k2}$. Thus, from [15], we know that the sign of $\kappa_{k1} \kappa_{k2}$ determines the direction of the bifurcation and the sign of the parameter κ_{k2} can determine the stability of the nontrivial periodic orbits, and so we have following results.

Theorem 4.1

- (a) When $\kappa_{k1} \kappa_{k2} < 0$, the Hopf bifurcation that the system undergoes at the critical value $\tau = \tau^*$ is a supercritical bifurcation. Moreover, if $\kappa_{k2} < 0$, then the bifurcating periodic solution is always stable; if $\kappa_{k2} > 0$, then the bifurcating periodic solution is unstable.

- (b) When $\kappa_{k1}\kappa_{k2} > 0$, the Hopf bifurcation that the system undergoes at the critical value $\tau = \tau^*$ is a subcritical bifurcation. Moreover, if $\kappa_{k2} < 0$, then the bifurcating periodic solution is stable; if $\kappa_{k2} > 0$, then the bifurcating periodic solution is an unstable one.

In the next section, we perform some accurate numerical simulations, together with dynamical analysis for Hopf bifurcation, of systems (1.1) and (1.2).

5 Numerical simulations

In this section, by using mathematical software Matlab, we present some numerical simulations to support and extend our analytical results.

6 Hopf bifurcation

For system (1.1), choosing the parameters $d_1 = 0.05$, $d_2 = 0.1$, $\gamma = 1$, $\beta = 0.5048$, and doing simple calculation, we can obtain that the Hopf bifurcation value is $c_0 = 0.7366$. We also can get the values $\kappa_{k1} = -0.0567$, $\kappa_{k2} = 0.0186$. This implies us that if the parameter c goes across the critical value 0.7366, then a family of unstable spatially homogeneous periodic solutions will bifurcate from the positive equilibrium E^* , which are clearly depicted by Figs. 1 and 2.

7 Delay-induced Hopf bifurcation

For system (1.2), we chose to set the values $d_1 = 0.05, d_2 = 0.1, \gamma = 1, c = 0.75, \beta = 0.6$. Then, a long series of exact calculations shows that $(u^*, v^*) = (0.4094, 0.3517)$, and also $\tau_* = 0.9363, \kappa_{01} = 0.0625, \kappa_{02} = -0.1026$. Hence, $(u^*, v^*) = (0.4094, 0.3517)$ is showed to be locally stable when the parameter $\tau \in [0, \tau_*)$. When the parameter τ crosses the critical value τ_* , $(u^*, v^*) = (0.4094, 0.3517)$ will lose its stability and at the same time Hopf bifurcation will occur, so a family of stable spatially homogeneous periodic solutions are shown to be bifurcating from $(u^*, v^*) = (0.4094, 0.3517)$, which are respectively depicted by Figs. 3 and 4.

8 Conclusions

In this paper, we mainly have dealt with a predator–prey model along with a spatial diffusion, Smith growth rate and herd behavior, subject to Neumann boundary condition. We have discussed the stability of positive constant equilibria, and the existence and stability of Hopf bifurcation near the positive constant equilibria. Meanwhile, regarding time delay as an influential bifurcation parameter, and by using the normal form and the center manifold theorem of the functional and partial differential equations, we have concluded the existence, stability and direction of periodic solutions of functional differential equations and some partial differential equations, respectively. In fact, we have obtained a critical value of time delay, and the interesting conclusions have shown us that the critical value can really affect the stability of the positive constant equilibrium. At last, exact numerical

simulations have been carried out in order to depict our theoretical analysis. For system (1.1), we have chosen to set $d_1 = 0.05$, $d_2 = 0.1$, $\gamma = 1$, $\beta = 0.5048$, and, together with a direct computation, we have obtained the critical value $c_0 = 0.7366$, and have also gotten the values $\kappa_{k1} = -0.0567$, $\kappa_{k2} = 0.0186$. We can conclude from the result that the positive equilibrium $(u^*, v^*) = (0.4094, 0.3517)$ is asymptotically stable for the parameter $c = 0.8 > 0.7366$, and there exist unstable spatially homogenous periodic solutions that are bifurcating from the positive equilibrium (u^*, v^*) when the parameter $c = 0.75 > 0.7366$, which is shown in Figs. 1 and 2. For system (1.2), choosing to set the values $d_1 = 0.05$, $d_2 = 0.1$, $\gamma = 1$, $c = 0.75$, $\beta = 0.6$, together with a very direct computation, we have also obtained the critical value of delay for the parameter $k = 0$, that is, $\tau_* = 0.9363$, $\kappa_{k1} = 0.0625$, $\kappa_{k2} = -0.1026$. We have concluded that the positive equilibrium $(u^*, v^*) = (0.4094, 0.3517)$ is asymptotically stable for the parameter $\tau = 0.9063 < 0.9363$ and unstable for $\tau = 0.9463 > 0.9363$. So, system (1.2) undergoes Hopf bifurcation near the positive equilibrium (u^*, v^*) at the time when the delay τ increasingly exceeds the critical value τ_* . By computing $\kappa_{01} = 0.0625$, $\kappa_{02} = -0.1026$, and combining with the results in Sect. 3, we know that the direction of Hopf bifurcation is supercritical, and the bifurcating periodic solution is stable, and there also exist stable spatially homogenous periodic solutions, which is shown in Figs. 3 and 4.

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Authors' contributions

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