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Estimates of quantum bounds pertaining to new q-integral identity with applications

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Abstract

In this article, we establish a new generalized q-integral identity involving a q-differentiable function. Using this new auxiliary result, we obtain some new associated quantum bounds essentially using the class of preinvex functions. At the end, we present some applications to the special bivariate means to show the significance of the obtained results. Our approaches and obtained results may lead to further applications in physics.

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1 Introduction and preliminaries

The quantum calculus, which is often regarded as calculus without limits, has emerged as a bridge between mathematics and physics. Although very old, it has experienced a rapid development during the previous century with the research work of Jackson [16]. In recent years many researchers have shown great interest in studying and investigating quantum calculus since it emerged as an interdisciplinary subject. This is, of course, because of the fact that quantum analysis is very helpful in several fields and has huge applications in various areas of natural and applied sciences, such as computer science and particle physics, and additionally acts as a critical tool for researchers working in analytic number theory or in theoretical physics. Many scientists who use quantum calculus are physicists, as quantum calculus has many applications in quantum group theory. For some recent trends in quantum calculus, the interested readers are referred to [12–14, 16, 17, 35]. Recently, Tariboon and Ntouyas [36] introduced the notions of quantum derivative and quantum integral on finite intervals and developed various q-analogues of classical integral inequalities, such as Hölder inequality [49], Hermite–Hadamard inequality [3, 7, 9, 15, 20, 21, 33], Petrović inequality [1], Pólya–Szegö and Čebyšev inequalities [32], Jensen inequality [2, 5, 18].

We now recall some useful concepts and results relating to quantum calculus on finite intervals.

Definition 1.1 ([36]) Let $0 < q < 1$, $J = [\alpha, \beta] \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous function and $\tau \in J$. Then the q-derivative ${}_{\alpha}D_q f(\tau)$ on J of f at τ is defined by

$$ {}_{\alpha}D_q f(\tau) = \frac{f(\tau) - f(q\tau + (1-q)\alpha)}{(1-q)(\tau-\alpha)} \quad (\tau \neq \alpha) $$

and

$$ {}_{\alpha}D_q f(\alpha) = \lim_{\tau \to \alpha} {}_{\alpha}D_q f(\tau). $$

Definition 1.2 ([36]) Let $f : J \to \mathbb{R}$ be a continuous function. Then the q-integral (Riemann-type q-integral) $\int_{\alpha}^{x} f(\tau)\, {}_{\alpha}\mathrm{d}_q\tau$ on J is defined by

$$ \int_{\alpha}^{x} f(\tau)\alpha\, \mathrm{d}_q\tau = (1-q)(x-\alpha)\sum_{n=0}^{\infty} q^n f\bigl(q^n x + \bigl(1-q^n\bigr)\alpha\bigr) $$

if $x \in J$. Moreover, if $c \in (\alpha, x)$, then the definite q-integral $\int_{c}^{x} f(\tau)\, {}_{\alpha}\mathrm{d}_q\tau$ on J is defined by

$$ \int_{c}^{x} f(\tau)\, {}_{\alpha}\mathrm{d}_q\tau = \int_{\alpha}^{x} f(\tau)\, {}_{\alpha}\mathrm{d}_q\tau - \int_{\alpha}^{c} f(\tau)\, {}_{\alpha}\mathrm{d}_q\tau. $$

In recent years several researchers successfully obtained numerous new quantum analogues of different classical inequalities. For instance, Noor et al. [25] and Sudsutad et al. [34] obtained q-analogues of Hermite–Hadamard's inequality using the class of convex functions. Noor et al. [24] obtained q-Hermite–Hadamard's inequality using the class of preinvex functions. Alp et al. [6] gave a corrected q-analogue of Hermite–Hadamard's inequality. Noor et al. [23] obtained quantum analogues of Ostrowski's inequality using the convexity property of functions. Tunç et al. [37] obtained quantum analogues of Simpson's inequality using convex functions and discussed some applications to means. Liu and Zhuang [22] obtained new q-analogues of Hermite–Hadamard's inequality using two times q-differentiable convex functions. Zhang et al. [48] obtained a more generalized and interesting new q-integral identity and obtained several new q-analogues of trapezoid inequalities. Recently, Erden et al. [11] obtained some more new quantum analogues of integral inequalities using convex functions. For more details on recent works regarding q-analogues of integral inequalities, see [4, 38–42].

The aim of this paper is to derive a new generalized quantum integral identity and, applying it as an auxiliary result, we will establish some new estimates of q-bounds using the class of preinvex functions. We will discuss also some new special cases of the obtained results. Finally, we will present some new applications of the main results to special means for different positive real numbers. We hope that the ideas and techniques of this paper will inspire the interested readers working in this fascinating field.

2 Results and discussions

Before we discuss our main results, let us recall the definitions of an invex set and preinvex function.

In what follows, we denote by $\mathcal{X} \subset \mathbb{R}$ a nonempty set, $f : \mathcal{X} \to \mathbb{R}$ is a continuous function, and $\xi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a continuous bifunction.

Definition 2.1 ([10]) A set $\mathcal{X} \subset \mathbb{R}$ is said to be invex with respect to the bifunction $\xi(\cdot,\cdot)$ if $\alpha + \tau\xi(\beta,\alpha) \in \mathcal{X}$ for all $\alpha,\beta \in \mathcal{X}$ and $\tau \in [0,1]$.

Definition 2.2 ([46]) A real-valued function $f : \mathcal{X} \to \mathbb{R}$ is said to be preinvex with respect to the bifunction $\xi(\cdot,\cdot)$ if the inequality

$$f\big(\alpha + \tau\xi(\beta,\alpha)\big) \le (1-\tau)f(\alpha) + \tau f(\beta)$$

holds for all $\alpha,\beta \in \mathcal{X}$ and $\tau \in [0,1]$.

2.1 A key lemma

The following Lemma 2.3 plays a crucial role in deriving our main results.

Lemma 2.3 *Let $q \in (0,1)$, $\xi(\beta,\alpha) > 0$ and $f : \mathcal{B} = [\alpha, \alpha + \xi(\beta,\alpha)] \to \mathbb{R}$ be a q-differentiable function on $\mathcal{B}^\circ$ (the interior of $\mathcal{B}$) such that ${}_\alpha\mathrm{D}_q f$ is q-integrable on $\mathcal{B}$. Then we have the identity*

$$\begin{aligned}
&\frac{1}{8}\left[f(\alpha) + 3f\left(\frac{3\alpha + \xi(\beta,\alpha)}{3}\right) + 3f\left(\frac{3\alpha + 2\xi(\beta,\alpha)}{3}\right) + f\big(\alpha + \xi(\beta,\alpha)\big)\right] \\
&\quad - \frac{1}{\xi(\beta,\alpha)} \int_\alpha^{\alpha+\xi(\beta,\alpha)} f(x)\,{}_\alpha\mathrm{d}_q x \\
&= \xi(\beta,\alpha) \int_0^1 \Phi(\tau)\,{}_\alpha\mathrm{D}_q f\big(\alpha + \tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau,
\end{aligned} \tag{2.1}$$

where

$$\Phi(\tau) = \begin{cases} q\tau - \frac{1}{8}, & \tau \in [0, \frac{1}{3}), \\ q\tau - \frac{1}{2}, & \tau \in [\frac{1}{3}, \frac{2}{3}), \\ q\tau - \frac{7}{8}, & \tau \in [\frac{2}{3}, 1]. \end{cases}$$

Proof Let

$$\mathcal{S}_1 = \int_0^{\frac{1}{3}} \left(q\tau - \frac{1}{8}\right){}_\alpha\mathrm{D}_q f\big(\alpha + \tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau,$$

$$\mathcal{S}_2 = \int_{\frac{1}{3}}^{\frac{2}{3}} \left(q\tau - \frac{1}{2}\right){}_\alpha\mathrm{D}_q f\big(\alpha + \tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau,$$

and

$$\mathcal{S}_3 = \int_{\frac{2}{3}}^{1} \left(q\tau - \frac{7}{8}\right){}_\alpha\mathrm{D}_q f\big(\alpha + \tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau.$$

Then elaborated computations lead to

$$\begin{aligned}
\mathcal{S}_1 &= \int_0^{\frac{1}{3}} q\tau\,{}_\alpha\mathrm{D}_q f\big(\alpha + \tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau - \frac{1}{8}\int_0^{\frac{1}{3}} {}_\alpha\mathrm{D}_q f\big(\alpha + \tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau \\
&= \int_0^{\frac{1}{3}} q\frac{f(\alpha + \tau\xi(\beta,\alpha)) - f(\alpha + q\tau\xi(\beta,\alpha))}{(1-q)\xi(\beta,\alpha)}\,{}_0\mathrm{d}_q\tau
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8}\int_0^{\frac{1}{3}} \frac{f(\alpha+\tau\xi(\beta,\alpha)) - f(\alpha+q\tau\xi(\beta,\alpha))}{\tau(1-q)\xi(\beta,\alpha)}\,{}_0\mathrm{d}_q\tau \\
&= \frac{1}{3}\sum_{n=0}^{\infty} q^{n+1}\frac{f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha)) - f(\alpha+\frac{1}{3}q^{n+1}\xi(\beta,\alpha))}{\xi(\beta,\alpha)} \\
&\quad - \frac{1}{8}\sum_{n=0}^{\infty}\frac{f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha)) - f(\alpha+\frac{1}{3}q^{n+1}\xi(\beta,\alpha))}{\xi(\beta,\alpha)} \\
&= \frac{q\sum_{n=0}^{\infty} q^n f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha)) - \sum_{n=1}^{\infty} q^n f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha))}{3\xi(\beta,\alpha)} \\
&\quad - \frac{1}{8}\frac{\sum_{n=0}^{\infty} f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha)) - \sum_{n=1}^{\infty} f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha))}{\xi(\beta,\alpha)} \\
&= \frac{1}{3}\left[\frac{f(\frac{3\alpha+\xi(\beta,\alpha)}{3})}{\xi(\beta,\alpha)} - (1-q)\sum_{n=0}^{\infty} q^n\frac{f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha))}{\xi(\beta,\alpha)}\right] - \frac{1}{8}\cdot\frac{f(\frac{3\alpha+\xi(\beta,\alpha)}{3}) - f(\alpha)}{\xi(\beta,\alpha)} \\
&= \frac{5}{24}\cdot\frac{f(\frac{3\alpha+\xi(\beta,\alpha)}{3})}{\xi(\beta,\alpha)} + \frac{1}{8\xi(\beta,\alpha)}f(\alpha) - \frac{1}{3}(1-q)\sum_{n=0}^{\infty} q^n\frac{f(\alpha+\frac{1}{3}q^n\xi(\beta,\alpha))}{\xi(\beta,\alpha)} \\
&= \frac{5}{24}\cdot\frac{f(\frac{3\alpha+\xi(\beta,\alpha)}{3})}{\xi(\beta,\alpha)} + \frac{1}{8\xi(\beta,\alpha)}f(\alpha) - \frac{1}{\xi^2(\beta,\alpha)}\int_{\alpha}^{\alpha+\frac{1}{3}\xi(\beta,\alpha)} f(x)_{\alpha}\,\mathrm{d}_q x, \\
\mathcal{S}_2 &= \int_0^{\frac{2}{3}}\left(q\tau - \frac{1}{2}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau - \int_0^{\frac{1}{3}}\left(q\tau - \frac{1}{2}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau \\
&= \frac{1}{6}\cdot\frac{f(\frac{3\alpha+\xi(\beta,\alpha)}{3})}{\xi(\beta,\alpha)} + \frac{1}{6\xi(\beta,\alpha)}f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right) - \frac{1}{\xi^2(\beta,\alpha)}\int_{\alpha+\frac{1}{3}\xi(\beta,\alpha)}^{\alpha+\frac{2}{3}\xi(\beta,\alpha)} f(x)_{\alpha}\,\mathrm{d}_q x,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{S}_3 &= \int_0^{1}\left(q\tau - \frac{7}{8}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau - \int_0^{\frac{2}{3}}\left(q\tau - \frac{7}{8}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau \\
&= \frac{5}{24}\cdot\frac{f(\frac{3\alpha+2\xi(\beta,\alpha)}{3})}{\xi(\beta,\alpha)} + \frac{1}{8\xi(\beta,\alpha)}f\big(\alpha+\xi(\beta,\alpha)\big) - \frac{1}{\xi^2(\beta,\alpha)}\int_{\alpha+\frac{2}{3}\xi(\beta,\alpha)}^{\alpha+\xi(\beta,\alpha)} f(x)_{\alpha}\,\mathrm{d}_q x.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\int_0^1 \Phi(\tau)_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau \\
&\quad = \frac{1}{8\xi(\beta,\alpha)}\left[f(\alpha) + 3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right) + 3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right) + f\big(\alpha+\xi(\beta,\alpha)\big)\right] \\
&\qquad - \frac{1}{\xi^2(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)} f(x)_{\alpha}\,\mathrm{d}_q x.
\end{aligned}
$$

Multiplying both sides of the the above last equality by $\xi(\beta,\alpha)$ leads to the desired result (2.1). □

Corollary 2.4 *Let $q \to 1^-$. Then Lemma 2.3 leads to the conclusion that*

$$\begin{aligned}
&\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\big(\alpha+\xi(\beta,\alpha)\big)\right]\\
&\quad-\frac{1}{\xi(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)}f(x)\,\mathrm{d}x\\
&=\xi(\beta,\alpha)\int_0^1\Psi(\tau)f'\big(\alpha+\tau\xi(\beta,\alpha)\big)\,\mathrm{d}\tau,
\end{aligned}$$

where

$$\Psi(\tau)=\begin{cases}\tau-\frac{1}{8}, & \tau\in[0,\frac{1}{3}),\\ \tau-\frac{1}{2}, & \tau\in[\frac{1}{3},\frac{2}{3}),\\ \tau-\frac{7}{8}, & \tau\in[\frac{2}{3},1].\end{cases}$$

2.2 Estimations of quantum bounds

Theorem 2.5 *Let $q\in(0,1)$, $\xi(\beta,\alpha)>0$ and $f:\mathcal{B}=[\alpha,\alpha+\xi(\beta,\alpha)]\to\mathbb{R}$ be a q-differentiable function on $\mathcal{B}^\circ$ (the interior of $\mathcal{B}$) such that $|{}_\alpha\mathrm{D}_q f|$ is a q-integrable preinvex function. Then one has*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\big(\alpha+\xi(\beta,\alpha)\big)\right]\right.\\
&\quad\left.-\frac{1}{\xi(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)}f(x)\,{}_\alpha\mathrm{d}_q x\right|\\
&\le\xi(\beta,\alpha)\left[\frac{768q^3+432q^2+432q+168}{6912(1+q)(1+q+q^2)}\big|{}_\alpha\mathrm{D}_q f(\alpha)\big|\right.\\
&\quad\left.+\frac{768q^2+768q+432q+264}{6912(1+q)(1+q+q^2)}\big|{}_\alpha\mathrm{D}_q f(\beta)\big|\right].
\end{aligned}\tag{2.2}$$

Proof It follows from Lemma 2.3 and the preinvexity of the function $|{}_\alpha\mathrm{D}_q f|$, together with the properties of the modulus, that

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\big(\alpha+\xi(\beta,\alpha)\big)\right]\right.\\
&\quad\left.-\frac{1}{\xi(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)}f(x)\,{}_\alpha\mathrm{d}_q x\right|\\
&=\xi(\beta,\alpha)\left|\int_0^{\frac{1}{3}}\left(q\tau-\frac{1}{8}\right){}_\alpha\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau\right.\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(q\tau-\frac{1}{2}\right){}_\alpha\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau\\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left(q\tau-\frac{7}{8}\right){}_\alpha\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \xi(\beta,\alpha)\bigg[\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|\left|{}_\alpha D_q f\left(\alpha+\tau\xi(\beta,\alpha)\right)\right|{}_0d_q\tau \\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left|{}_\alpha D_q f\left(\alpha+\tau\xi(\beta,\alpha)\right)\right|{}_0d_q\tau \\
&\quad+\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|\left|{}_\alpha D_q f\left(\alpha+\tau\xi(\beta,\alpha)\right)\right|{}_0d_q\tau\bigg] \\
&\le \xi(\beta,\alpha)\bigg[\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|\left((1-t)\left|{}_\alpha D_q f(\alpha)\right|+t\left|{}_\alpha D_q f(\beta)\right|\right){}_0d_q\tau \\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left((1-t)\left|{}_\alpha D_q f(\alpha)\right|+t\left|{}_\alpha D_q f(\beta)\right|\right){}_0d_q\tau \\
&\quad+\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|\left((1-t)\left|{}_\alpha D_q f(\alpha)\right|+t\left|{}_\alpha D_q f(\beta)\right|\right){}_0d_q\tau\bigg] \\
&= \xi(\beta,\alpha)\bigg(\frac{480q^3+248q^2+248q-3}{6912(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\alpha)\right| \\
&\quad+\frac{160q^2+160q-69}{6912(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\beta)\right|\bigg) \\
&\quad+\xi(\beta,\alpha)\bigg(\frac{6q^3+3}{108(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\alpha)\right| \\
&\quad+\frac{6q^2+6q-3}{108(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\beta)\right|\bigg) \\
&\quad+\xi(\beta,\alpha)\bigg(\frac{-96q^3+184q^2+184q-21}{6912(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\alpha)\right| \\
&\quad+\frac{224q^2+224q+525}{6912(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\beta)\right|\bigg) \\
&= \xi(\beta,\alpha)\bigg[\frac{768q^3+432q^2+432q+168}{6912(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\alpha)\right| \\
&\quad+\frac{768q^2+768q+432q+264}{6912(1+q)(1+q+q^2)}\left|{}_\alpha D_q f(\beta)\right|\bigg],
\end{aligned}
$$

which completes the proof of Theorem 2.5. □

Corollary 2.6 *Let $q \to 1^-$. Then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\left(\alpha+\xi(\beta,\alpha)\right)\right]\right. \\
&\quad\left.-\frac{1}{\xi(\beta,\alpha)}\int_\alpha^{\alpha+\xi(\beta,\alpha)}f(x)\,dx\right| \\
&\le \frac{25\xi(\beta,\alpha)}{576}\left[\left|f'(\alpha)\right|+\left|f'(\beta)\right|\right].
\end{aligned}
$$

Theorem 2.7 *Let $q \in (0,1)$, $p, r > 1$ with $p^{-1}+r^{-1}=1$, $\xi(\beta,\alpha)>0$ and $f:\mathcal{B}=[\alpha,\alpha+\xi(\beta,\alpha)]\to\mathbb{R}$ be a q-differentiable function on $\mathcal{B}^\circ$ (the interior of $\mathcal{B}$) such that $|{}_\alpha D_q f|^r$ is* *a q-integrable preinvex function. Then one has*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\bigl(\alpha+\xi(\beta,\alpha)\bigr)\right]\right.\\
&\quad\left.-\frac{1}{\xi(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)}f(x)\,{}_{\alpha}\mathrm{d}_q x\right|\\
&\le \xi(\beta,\alpha)\left[\left(\frac{[3^{p+1}+(8q-3)^{p+1}](1-q)}{24^{p+1}q(1-q^{p+1})}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\frac{(3q+2)|{}_{\alpha}\mathrm{D}_q f(\alpha)|^r+|{}_{\alpha}\mathrm{D}_q f(\beta)|^r}{9(1+q)}\right)^{\frac{1}{r}}\\
&\quad+\left(\frac{[(3-2q)^{p+1}+(6q-5)^{p+1}](1-q)}{6^{p+1}q(1-q^{p+1})}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{q|{}_{\alpha}\mathrm{D}_q f(\alpha)|^r+|{}_{\alpha}\mathrm{D}_q f(\beta)|^r}{3(1+q)}\right)^{\frac{1}{r}}\\
&\quad+\left(\frac{[(21-16q)^{p+1}+(24q-21)^{p+1}](1-q)}{24^{p+1}q(1-q^{p+1})}\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\frac{(3q-2)|{}_{\alpha}\mathrm{D}_q f(\alpha)|^r+5|{}_{\alpha}\mathrm{D}_q f(\beta)|^r}{9(1+q)}\right)^{\frac{1}{r}}\right].
\end{aligned}\tag{2.3}$$

Proof It follows from Lemma 2.3, Hölder inequality, the preinvexity of the function $|{}_{\alpha}\mathrm{D}_q f|^r$, and the properties of the modulus that

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\bigl(\alpha+\xi(\beta,\alpha)\bigr)\right]\right.\\
&\quad\left.-\frac{1}{\xi(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)}f(x)\,{}_{\alpha}\mathrm{d}_q x\right|\\
&=\xi(\beta,\alpha)\left|\int_0^{\frac{1}{3}}\left(q\tau-\frac{1}{8}\right){}_{\alpha}\mathrm{D}_q f\bigl(\alpha+\tau\xi(\beta,\alpha)\bigr)\,{}_0\mathrm{d}_q\tau\right.\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(q\tau-\frac{1}{2}\right){}_{\alpha}\mathrm{D}_q f\bigl(\alpha+\tau\xi(\beta,\alpha)\bigr)\,{}_0\mathrm{d}_q\tau\\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left(q\tau-\frac{7}{8}\right){}_{\alpha}\mathrm{D}_q f\bigl(\alpha+\tau\xi(\beta,\alpha)\bigr)\,{}_0\mathrm{d}_q\tau\right|\\
&\le\xi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|^p\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|{}_{\alpha}\mathrm{D}_q f\bigl(\alpha+\tau\xi(\beta,\alpha)\bigr)\right|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right.\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|^p\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|{}_{\alpha}\mathrm{D}_q f\bigl(\alpha+\tau\xi(\beta,\alpha)\bigr)\right|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|^p\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|{}_{\alpha}\mathrm{D}_q f\bigl(\alpha+\tau\xi(\beta,\alpha)\bigr)\right|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right]\\
&\le\xi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|^p\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$\times \left(\left|{}_\alpha D_q f(\alpha)\right|^r \int_0^{\frac{1}{3}} (1-\tau)\,{}_0d_q\tau + \left|{}_\alpha D_q f(\beta)\right|^r \int_0^{\frac{1}{3}} \tau\,{}_0d_q\tau \right)^{\frac{1}{r}}$$
$$+ \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| q\tau - \frac{1}{2} \right|^p {}_0d_q\tau \right)^{\frac{1}{p}}$$
$$\times \left(\left|{}_\alpha D_q f(\alpha)\right|^r \int_{\frac{1}{3}}^{\frac{2}{3}} (1-\tau)\,{}_0d_q\tau + \left|{}_\alpha D_q f(\beta)\right|^r \int_{\frac{1}{3}}^{\frac{2}{3}} \tau\,{}_0d_q\tau \right)^{\frac{1}{r}}$$
$$+ \left(\int_{\frac{2}{3}}^{1} \left| q\tau - \frac{7}{8} \right|^p {}_0d_q\tau \right)^{\frac{1}{p}}$$
$$\times \left(\left|{}_\alpha D_q f(\alpha)\right|^r \int_{\frac{2}{3}}^{1} (1-\tau)\,{}_0d_q\tau + \left|{}_\alpha D_q f(\beta)\right|^r \int_{\frac{2}{3}}^{1} \tau\,{}_0d_q\tau \right)^{\frac{1}{r}} \Bigg].$$

Making use of the binomial expansion, we get

$$\begin{aligned}
&\int_0^{\frac{1}{3}} \left| q\tau - \frac{1}{8} \right|^p {}_0d_q\tau \\
&= \left[(-1)^p - 1\right] q^p \int_0^{\frac{1}{8q}} \left(\tau - \frac{1}{8q} \right)^p {}_0d_q\tau + \int_0^{\frac{1}{3}} \left(\tau - \frac{1}{8q} \right)^p {}_0d_q\tau \\
&= \left[(-1)^p - 1\right] q^p \frac{1-q}{1-q^{p+1}} (-1)^p \left(\frac{1}{8q} \right)^{p+1} \\
&\quad + q^p \frac{1-q}{1-q^{p+1}} \left[\left(\frac{1}{3} - \frac{1}{8q} \right)^{p+1} + (-1)^p \left(\frac{1}{8q} \right)^{p+1} \right] \\
&= \frac{[3^{p+1} + (8q-3)^{p+1}](1-q)}{24^{p+1} q (1-q^{p+1})}.
\end{aligned}$$

Similarly,

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \left| q\tau - \frac{1}{2} \right|^p {}_0d_q\tau = \frac{[(3-2q)^{p+1} + (6q-5)^{p+1}](1-q)}{6^{p+1} q (1-q^{p+1})}$$

and

$$\int_{\frac{2}{3}}^{1} \left| q\tau - \frac{7}{8} \right|^p {}_0d_q\tau = \frac{[(21-16q)^{p+1} + (24q-21)^{p+1}](1-q)}{24^{p+1} q (1-q^{p+1})}.$$

Therefore,

$$\begin{aligned}
&\left| \frac{1}{8} \left[f(\alpha) + 3f\left(\frac{3\alpha + \xi(\beta,\alpha)}{3} \right) + 3f\left(\frac{3\alpha + 2\xi(\beta,\alpha)}{3} \right) + f\big(\alpha + \xi(\beta,\alpha)\big) \right] \right. \\
&\quad \left. - \frac{1}{\xi(\beta,\alpha)} \int_\alpha^{\alpha+\xi(\beta,\alpha)} f(x)\,{}_\alpha d_q x \right| \\
&\le \xi(\beta,\alpha) \left[\left(\frac{[3^{p+1} + (8q-3)^{p+1}](1-q)}{24^{p+1} q (1-q^{p+1})} \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\frac{(3q+2)|{}_\alpha D_q f(\alpha)|^r + |{}_\alpha D_q f(\beta)|^r}{9(1+q)} \right)^{\frac{1}{r}}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{[(3-2q)^{p+1} + (6q-5)^{p+1}](1-q)}{6^{p+1}q(1-q^{p+1})}\right)^{\frac{1}{p}} \\
&\times \left(\frac{q|_{\alpha}\mathrm{D}_q f(\alpha)|^r + |_{\alpha}\mathrm{D}_q f(\beta)|^r}{3(1+q)}\right)^{\frac{1}{r}} \\
&+ \left(\frac{[(21-16q)^{p+1} + (24q-21)^{p+1}](1-q)}{24^{p+1}q(1-q^{p+1})}\right)^{\frac{1}{p}} \\
&\times \left(\frac{(3q-2)|_{\alpha}\mathrm{D}_q f(\alpha)|^r + 5|_{\alpha}\mathrm{D}_q f(\beta)|^r}{9(1+q)}\right)^{\frac{1}{r}}\Bigg],
\end{aligned}
$$

and the proof of Theorem 2.7 is completed. □

Corollary 2.8 *Let $q \to 1^-$. Then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha) + 3f\left(\frac{3\alpha + \xi(\beta,\alpha)}{3}\right) + 3f\left(\frac{3\alpha + 2\xi(\beta,\alpha)}{3}\right) + f\big(\alpha + \xi(\beta,\alpha)\big)\right]\right. \\
&\quad \left. - \frac{1}{\xi(\beta,\alpha)} \int_{\alpha}^{\alpha+\xi(\beta,\alpha)} f(x)\,\mathrm{d}x\right| \\
&\leq \xi(\beta,\alpha)\Bigg[\left(\frac{[3^{p+1} + 5^{p+1}]}{24^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\frac{5|f'(\alpha)|^r + |f'(\beta)|^r}{18}\right)^{\frac{1}{r}} \\
&\quad + \left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\frac{|f'(\alpha)|^r + |f'(\beta)|^r}{6}\right)^{\frac{1}{r}} \\
&\quad + \left(\frac{(3^{p+1} + 5)^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\frac{|f'(\alpha)|^r + 5|f'(\beta)|^r}{18}\right)^{\frac{1}{r}}\Bigg].
\end{aligned}
$$

Theorem 2.9 *Let $q \in (0,1)$, $r > 1$, $\xi(\beta,\alpha) > 0$ and $f : \mathcal{B} = [\alpha, \alpha + \xi(\beta,\alpha)] \to \mathbb{R}$ be a q-differentiable function on $\mathcal{B}^\circ$ (the interior of $\mathcal{B}$) such that $|_{\alpha}\mathrm{D}_q f|^r$ is a q-integrable preinvex function. Then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha) + 3f\left(\frac{3\alpha + \xi(\beta,\alpha)}{3}\right) + 3f\left(\frac{3\alpha + 2\xi(\beta,\alpha)}{3}\right) + f\big(\alpha + \xi(\beta,\alpha)\big)\right]\right. \\
&\quad \left. - \frac{1}{\xi(\beta,\alpha)} \int_{\alpha}^{\alpha+\xi(\beta,\alpha)} f(x)\,{}_{\alpha}\mathrm{d}_q x\right| \\
&\leq \xi(\beta,\alpha)\Bigg[\left(\frac{20q-3}{288(1+q)}\right)^{1-\frac{1}{r}} \left(\frac{480q^3 + 248q^2 + 248q - 3}{6912(1+q)(1+q+q^2)}\big|_{\alpha}\mathrm{D}_q f(\alpha)\big|^r\right. \\
&\quad \left. + \frac{160q^2 + 160q - 69}{6912(1+q)(1+q+q^2)}\big|_{\alpha}\mathrm{D}_q f(\beta)\big|^r\right)^{\frac{1}{r}} \\
&\quad + \left(\frac{q}{18(1+q)}\right)^{1-\frac{1}{r}} \left(\frac{6q^3 + 3}{108(1+q)(1+q+q^2)}\big|_{\alpha}\mathrm{D}_q f(\alpha)\big|^r\right. \\
&\quad \left. + \frac{6q^2 + 6q - 3}{108(1+q)(1+q+q^2)}\big|_{\alpha}\mathrm{D}_q f(\beta)\big|^r\right)^{\frac{1}{r}}
\end{aligned}
$$

$$+\left(\frac{21-4q}{288(1+q)}\right)^{1-\frac{1}{r}}\left(\frac{-96q^3+184q^2+184q-21}{6912(1+q)(1+q+q^2)}\left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r\right.$$
$$\left.+\frac{224q^2+224q+525}{6912(1+q)(1+q+q^2)}\left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r\right)^{\frac{1}{r}}\Bigg]. \tag{2.4}$$

Proof It follows from Lemma 2.3, the power mean integral inequality, the preinvexity of the function $|{}_{\alpha}\mathrm{D}_q f|^r$, and the properties of modulus that

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(\alpha)+3f\left(\frac{3\alpha+\xi(\beta,\alpha)}{3}\right)+3f\left(\frac{3\alpha+2\xi(\beta,\alpha)}{3}\right)+f\big(\alpha+\xi(\beta,\alpha)\big)\right]\right.\\
&\qquad\left.-\frac{1}{\xi(\beta,\alpha)}\int_{\alpha}^{\alpha+\xi(\beta,\alpha)}f(x)\,{}_{\alpha}\mathrm{d}_q x\right|\\
&=\xi(\beta,\alpha)\left|\int_0^{\frac{1}{3}}\left(q\tau-\frac{1}{8}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau\right.\\
&\qquad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(q\tau-\frac{1}{2}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau\\
&\qquad\left.+\int_{\frac{2}{3}}^{1}\left(q\tau-\frac{7}{8}\right){}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\,{}_0\mathrm{d}_q\tau\right|\\
&\le\xi(\beta,\alpha)\left[\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|\left|{}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\right|\,{}_0\mathrm{d}_q\tau\right.\\
&\qquad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left|{}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\right|\,{}_0\mathrm{d}_q\tau\\
&\qquad\left.+\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|\left|{}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\right|\,{}_0\mathrm{d}_q\tau\right]\\
&\le\xi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|\,{}_0\mathrm{d}_q\tau\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|\left|{}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\right|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right.\\
&\qquad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\,{}_0\mathrm{d}_q\tau\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left|{}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\right|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\\
&\qquad\left.+\left(\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|\,{}_0\mathrm{d}_q\tau\right)^{1-\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|\left|{}_{\alpha}\mathrm{D}_q f\big(\alpha+\tau\xi(\beta,\alpha)\big)\right|^r\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\right]\\
&\le\xi(\beta,\alpha)\left[\left(\frac{20q-3}{288(1+q)}\right)^{1-\frac{1}{r}}\left(\left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r\int_0^{\frac{1}{3}}(1-\tau)\left|q\tau-\frac{1}{8}\right|\,{}_0\mathrm{d}_q\tau\right.\right.\\
&\qquad\left.+\left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r\int_0^{\frac{1}{3}}\tau\left|q\tau-\frac{1}{8}\right|\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\\
&\qquad+\left(\frac{q}{18(1+q)}\right)^{1-\frac{1}{r}}\left(\left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|q\tau-\frac{1}{2}\right|\,{}_0\mathrm{d}_q\tau\right.\\
&\qquad\left.+\left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\frac{1}{2}\right|\,{}_0\mathrm{d}_q\tau\right)^{\frac{1}{r}}\\
&\qquad+\left(\frac{21-4q}{288(1+q)}\right)^{1-\frac{1}{r}}\left(\left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r\int_{\frac{2}{3}}^{1}(1-\tau)\left|q\tau-\frac{7}{8}\right|\,{}_0\mathrm{d}_q\tau\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r \int_{\frac{2}{3}}^{1} \tau \left|q\tau - \frac{7}{8}\right| {}_0\mathrm{d}_q\tau \Big)^{\frac{1}{r}} \Bigg] \\
&= \xi(\beta,\alpha) \Bigg[\left(\frac{20q-3}{288(1+q)} \right)^{1-\frac{1}{r}} \Bigg(\frac{480q^3 + 248q^2 + 248q - 3}{6912(1+q)(1+q+q^2)} \left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r \\
&+ \frac{160q^2 + 160q - 69}{6912(1+q)(1+q+q^2)} \left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r \Bigg)^{\frac{1}{r}} \\
&+ \left(\frac{q}{18(1+q)} \right)^{1-\frac{1}{r}} \Bigg(\frac{6q^3 + 3}{108(1+q)(1+q+q^2)} \left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r \\
&+ \frac{6q^2 + 6q - 3}{108(1+q)(1+q+q^2)} \left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r \Bigg)^{\frac{1}{r}} \\
&+ \left(\frac{21-4q}{288(1+q)} \right)^{1-\frac{1}{r}} \Bigg(\frac{-96q^3 + 184q^2 + 184q - 21}{6912(1+q)(1+q+q^2)} \left|{}_{\alpha}\mathrm{D}_q f(\alpha)\right|^r \\
&+ \frac{224q^2 + 224q + 525}{6912(1+q)(1+q+q^2)} \left|{}_{\alpha}\mathrm{D}_q f(\beta)\right|^r \Bigg)^{\frac{1}{r}} \Bigg],
\end{aligned}
$$

which completes the proof of Theorem 2.9. □

Corollary 2.10 *Let $q \to 1^-$. Then*

$$
\begin{aligned}
&\left| \frac{1}{8} \left[f(\alpha) + 3f\left(\frac{3\alpha + \xi(\beta,\alpha)}{3} \right) + 3f\left(\frac{3\alpha + 2\xi(\beta,\alpha)}{3} \right) + f\big(\alpha + \xi(\beta,\alpha)\big) \right] \right. \\
&\left. - \frac{1}{\xi(\beta,\alpha)} \int_{\alpha}^{\alpha+\xi(\beta,\alpha)} f(x)\,\mathrm{d}x \right| \\
&\le \xi(\beta,\alpha) \Bigg[\left(\frac{17}{576} \right)^{1-\frac{1}{r}} \left(\frac{973|f'(\alpha)|^r + 251|f'(\beta)|^r}{41472} \right)^{\frac{1}{r}} \\
&+ \left(\frac{1}{36} \right)^{1-\frac{1}{r}} \left(\frac{|f'(\alpha)|^r + |f'(\beta)|^r}{2} \right)^{\frac{1}{r}} \\
&+ \left(\frac{17}{576} \right)^{1-\frac{1}{r}} \left(\frac{251|f'(\alpha)|^r + 973|f'(\beta)|^r}{41472} \right)^{\frac{1}{r}} \Bigg].
\end{aligned}
$$

Remark 2.11 If we assume that $|{}_{\alpha}\mathrm{D}_q f| \le K$ in Theorems 2.5, 2.7, 2.9 and their corollaries, then we can obtain some interesting results. We omit the details here and leave them to the interested readers.

2.3 Applications

In this subsection, we give some applications of our obtained results to special bivariate means.

A bivariate real-valued function $\Upsilon : (0,\infty) \times (0,\infty) \to (0,\infty)$ is said to be a bivariate mean if $\min\{\varepsilon,\zeta\} \le \Upsilon(\varepsilon,\zeta) \le \max\{\varepsilon,\zeta\}$ for all $\varepsilon,\zeta \in (0,\infty)$. It is well known that the bivariate means are closely related to many special functions [8, 19, 26, 30, 31, 44, 45]. Recently, the inequalities between different bivariate means have attracted the attention of many researchers [27–29, 43, 47, 50].

Let $\alpha,\beta>0$ with $\alpha\neq\beta$ and $p\in\mathbb{R}\setminus\{-1,0\}$. Then the arithmetic mean $\mathcal{A}(\alpha,\beta)$ and pth generalized logarithmic mean $\mathcal{L}_p(\alpha,\beta)$ are defined by

$$\mathcal{A}(\alpha,\beta)=\frac{\alpha+\beta}{2}$$

and

$$\mathcal{L}_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}},$$

respectively.

Proposition 2.12 *Let $q\in(0,1)$ and $\beta>\alpha>0$. Then the inequality*

$$\begin{aligned}&\left|\frac{1}{4}\left[\mathcal{A}\big(\alpha^n,\beta^n\big)+\left(\frac{2}{3}\right)^{n-1}\big[\mathcal{A}^n(2\alpha,\beta)+\mathcal{A}^n(\alpha,2\beta)\big]\right]-\frac{(n+1)(1-q)}{1-q^{n+1}}\mathcal{L}_n^n(\alpha,\beta)\right|\\&\quad\leq(\beta-\alpha)\left[\frac{768q^3+432q^2+432q+168}{6912(1+q)(1+q+q^2)}\big(n\alpha^{n-1}\big)\right.\\&\qquad\left.+\frac{768q^2+768q+432q+264}{6912(1+q)(1+q+q^2)}\left(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)}\right)\right]\end{aligned}$$

holds for $n>1$.

Proof Let $f(x)=x^n$ and $\xi(\beta,\alpha)=\beta-\alpha$. Then Proposition 2.12 follows from Theorem 2.5 immediately. □

Proposition 2.13 *Let $\beta>\alpha>0$, $q\in(0,1)$, and $p,r>1$ with $p^{-1}+r^{-1}=1$. Then the inequality*

$$\begin{aligned}&\left|\frac{1}{4}\left[\mathcal{A}\big(\alpha^n,\beta^n\big)+\left(\frac{2}{3}\right)^{n-1}\big[\mathcal{A}^n(2\alpha,\beta)+\mathcal{A}^n(\alpha,2\beta)\big]\right]-\frac{(n+1)(1-q)}{1-q^{n+1}}\mathcal{L}_n^n(\alpha,\beta)\right|\\&\quad\leq(\beta-\alpha)\left[\psi_1^{\frac{1}{p}}\left(\frac{(3q+2)(n\alpha^{n-1})^r+(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)})^r}{9(1+q)}\right)^{\frac{1}{r}}\right.\\&\qquad+\psi_2^{\frac{1}{p}}\left(\frac{q(n\alpha^{n-1})^r+(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)})^r}{3(1+q)}\right)^{\frac{1}{r}}\\&\qquad\left.+\psi_3^{\frac{1}{p}}\left(\frac{(3q-2)(n\alpha^{n-1})^r+5(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)})^r}{9(1+q)}\right)^{\frac{1}{r}}\right]\end{aligned}$$

holds for $n>1$, where

$$\begin{aligned}\psi_1&=\frac{[3^{p+1}+(8q-3)^{p+1}](1-q)}{24^{p+1}q(1-q^{p+1})},\\\psi_2&=\frac{[(3-2q)^{p+1}+(6q-5)^{p+1}](1-q)}{6^{p+1}q(1-q^{p+1})},\end{aligned}$$

and

$$\psi_3 = \frac{[(21-16q)^{p+1} + (24q-21)^{p+1}](1-q)}{24^{p+1}q(1-q^{p+1})}.$$

Proof Let $f(x) = x^n$ and $\xi(\beta,\alpha) = \beta - \alpha$. Then Proposition 2.13 follows easily from Theorem 2.7. □

Proposition 2.14 *Let $\beta > \alpha > 0$, $n > 1$ and $q \in (0,1)$. Then the inequality*

$$\begin{aligned}
&\left|\frac{1}{4}\left[\mathcal{A}(\alpha^n,\beta^n) + \left(\frac{2}{3}\right)^{n-1}\left[\mathcal{A}^n(2\alpha,\beta) + \mathcal{A}^n(\alpha,2\beta)\right]\right] - \frac{(n+1)(1-q)}{1-q^{n+1}}\mathcal{L}_n^n(\alpha,\beta)\right| \\
&\leq (\beta-\alpha)\left[\left(\frac{20q-3}{288(1+q)}\right)^{1-\frac{1}{r}}\left(\frac{480q^3+248q^2+248q-3}{6912(1+q)(1+q+q^2)}\left(n\alpha^{n-1}\right)^r\right.\right. \\
&\quad \left. + \frac{160q^2+160q-69}{6912(1+q)(1+q+q^2)}\left(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)}\right)^r\right)^{\frac{1}{r}} \\
&\quad + \left(\frac{q}{18(1+q)}\right)^{1-\frac{1}{r}}\left(\frac{6q^3+3}{108(1+q)(1+q+q^2)}\left(n\alpha^{n-1}\right)^r\right. \\
&\quad \left. + \frac{6q^2+6q-3}{108(1+q)(1+q+q^2)}\left(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)}\right)^r\right)^{\frac{1}{r}} \\
&\quad + \left(\frac{21-4q}{288(1+q)}\right)^{1-\frac{1}{r}}\left(\frac{-96q^3+184q^2+184q-21}{6912(1+q)(1+q+q^2)}\left(n\alpha^{n-1}\right)^r\right. \\
&\quad \left.\left. + \frac{224q^2+224q+525}{6912(1+q)(1+q+q^2)}\left(\frac{\beta^n-(q\beta+(1-q)\alpha)^n}{(\beta-\alpha)(1-q)}\right)^r\right)^{\frac{1}{r}}\right]
\end{aligned}$$

holds for all $r > 1$.

Proof Let $f(x) = x^n$ and $\xi(\beta,\alpha) = \beta - \alpha$. Then Proposition 2.14 follows from Theorem 2.9. □

3 Conclusions

In this paper, we have derived a new q-integral identity involving a q-differentiable function. Using this new identity as an auxiliary result, we have derived new associated quantum bounds essentially using the class of preinvex functions. We also discussed some special cases of the obtained results which show that the main results obtained in the paper are quite unifying. In order to show the significance of the obtained results, we have also presented applications to special means. Since quantum calculus has extensive applications in many mathematical areas, we hope that our results can be applied in convex analysis, to special functions, in quantum mechanics, related optimization theory, to mathematical inequalities, and may stimulate further research in different areas of pure and applied sciences.

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