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# Oscillatory and asymptotic behavior of advanced differential equations

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## Abstract

In this paper, a class of fourth-order differential equations with advanced type is studied. Applying the generalized Riccati transformation, integral averaging technique and the theory of comparison, a set of new criteria for oscillation or certain asymptotic behavior of solutions of this equations is given. Our results essentially improve and complement some earlier publications. Some examples are presented to demonstrate the main results.

**MSC:** 34C10; 34K11

**Keywords:** Oscillation; Nonoscillatory solutions; Fourth-order; Advanced differential equations

## 1 Introduction

The present paper deals with the investigation of the oscillatory behavior of fourth-order advanced differential equation

$$(a(t)(y'''(t))^\beta)' + \sum_{i=1}^j q_i(t)g(y(\eta_i(t))) = 0, \quad t \geq t_0, \quad (1)$$

where  $j \geq 1$  and  $\beta$  is a quotient of odd positive integers. Throughout this work, we suppose that  $a \in C^1([t_0, \infty), \mathbb{R})$ ,  $a(t) > 0$ ,  $a'(t) \geq 0$ ,  $q_i, \eta_i \in C([t_0, \infty), \mathbb{R})$ ,  $q_i(t) \geq 0$ ,  $\eta_i(t) \geq t$ ,  $\lim_{t \rightarrow \infty} \eta_i(t) = \infty$ ,  $i = 1, 2, \dots, j$ ,  $g \in C(\mathbb{R}, \mathbb{R})$  such that  $g(x)/x^\beta \geq k > 0$ , for  $x \neq 0$  and under the condition

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \infty. \quad (2)$$

**Definition 1.1** The function  $y \in C^3[t_y, \infty)$ ,  $t_y \geq t_0$  is called a solution of (1), if  $a(t) \times (y'''(t))^\beta \in C^1[t_y, \infty)$ , and  $y(t)$  satisfies (1) on  $[t_y, \infty)$ . Moreover, Eq. (1) is oscillatory if all its solutions oscillate.

**Definition 1.2** Let

$$D = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \quad \text{and} \quad D_0 = \{(t, s) \in \mathbb{R}^2 : t > s \geq t_0\}.$$

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A kernel function  $H_i \in C(D, \mathbb{R})$  is said to belong to the function class  $\mathfrak{S}$ , written  $H \in \mathfrak{S}$ , if, for  $i = 1, 2$ ,

- (i)  $H_i(t, s) = 0$  for  $t \geq t_0, H_i(t, s) > 0, (t, s) \in D_0$ ;
- (ii)  $H_i(t, s)$  has a continuous and nonpositive partial derivative  $\partial H_i / \partial s$  on  $D_0$  and there exist functions  $\tau, \vartheta \in C^1([t_0, \infty), (0, \infty))$  and  $h_i \in C(D_0, \mathbb{R})$  such that

$$\frac{\partial}{\partial s} H_1(t, s) + \frac{\tau'(s)}{\tau(s)} H_1(t, s) = h_1(t, s) H_1^{\beta/(\beta+1)}(t, s) \tag{3}$$

and

$$\frac{\partial}{\partial s} H_2(t, s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(t, s) = h_2(t, s) \sqrt{H_2(t, s)}. \tag{4}$$

In this paper the following methods were used:

- (a) The Riccati transformations technique.
- (b) The method of comparison with second-order differential equations.
- (c) The integral averaging technique.

From them we obtained new criteria for oscillation of Eq. (1).

Advanced differential equations can find application in dynamical systems, mathematics of networks, optimization, as well as, in the mathematical modeling of engineering problems, such as concerning electrical power systems, materials, energy; see [1–4].

During the past few years, there has been constant interest to study the asymptotic properties for oscillation of differential equations in the canonical case, see [5–7], and the non-canonical case, see [8–10]. One active area of research in this decade is the study of the qualitative behavior for oscillation of differential equations, see [11–32].

Our aim in this paper is to complement and improve results in [33–35]. To this end, the following results are presented.

In particular, by using the comparison technique, the equation

$$\left( (y^{(\kappa-1)}(t))^\beta \right)' + q(t) y^\beta(\eta(t)) = 0 \tag{5}$$

has been studied by Agarwal and Grace [33]. They proved that it is oscillatory, if

$$\liminf_{t \rightarrow \infty} \int_t^{\eta(t)} (\eta(s) - s)^{\kappa-2} \left( \int_s^\infty q(t) dt \right)^{1/\beta} ds > \frac{(\kappa - 2)!}{e}. \tag{6}$$

Agarwal et al. in [34] extended the Riccati transformation to obtain new oscillatory criteria for (5) under the condition

$$\limsup_{t \rightarrow \infty} t^{\beta(\kappa-1)} \int_t^\infty q(s) ds > ((\kappa - 1)!)^\beta. \tag{7}$$

Authors in [35] studied the oscillatory behavior of (5), for  $\beta = 1$ . Also, they proved it to be oscillatory, if there exists a function  $\tau \in C^1([t_0, \infty), (0, \infty))$ , by using the Riccati transformation. If

$$\int_{t_0}^\infty \left( \tau(s) q(s) - \frac{(\kappa - 2)! (\tau'(s))^2}{2^{3-2\kappa} s^{\kappa-2} \tau(s)} \right) ds = \infty. \tag{8}$$

To prove this, we apply the previous results to the equation

$$y^{(\kappa)}(t) + by(rt) = 0, \quad t \geq 1, \tag{9}$$

where  $\kappa = 4, b = q_0/t^4$  and  $r = 3$ , and we find:

1. By applying condition (6) in [33], we get

$$q_0 > 13.6.$$

2. By applying condition (7) in [34], we get

$$q_0 > 18.$$

3. By applying condition (8) in [35], we get

$$q_0 > 576.$$

From the above we find that the results in [34] improve the results in [35].

Moreover, the results in [33] improve results [34, 35].

Our aim in the present paper is to employ the Riccati technique, the integral averaging technique and the theory of comparison to establish some new conditions for the oscillation of all solutions of Eq. (1) under the condition (2). Our results essentially improve and complement the results in [33–35]. Some examples are provided to illustrate the main results.

## 2 Some auxiliary lemmas

The proofs of our main results are essentially based on the following lemmas.

**Lemma 2.1** ([36]) *Suppose that  $y \in C^\kappa([t_0, \infty), (0, \infty)), y^{(\kappa)}$  is of a fixed sign on  $[t_0, \infty), y^{(\kappa)}$  not identically zero and there exists a  $t_1 \geq t_0$  such that*

$$y^{(\kappa-1)}(t)y^{(\kappa)}(t) \leq 0,$$

for all  $t \geq t_1$ . If we have  $\lim_{t \rightarrow \infty} y(t) \neq 0$ , then there exists  $t_\theta \geq t_1$  such that

$$y(t) \geq \frac{\theta}{(\kappa - 1)!} t^{\kappa-1} |y^{(\kappa-1)}(t)|,$$

for every  $\theta \in (0, 1)$  and  $t \geq t_\theta$ .

**Lemma 2.2** ([13]) *Let  $\beta$  be a ratio of two odd numbers,  $V > 0$  and  $U$  are constants. Then*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta + 1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta}, \quad V > 0.$$

**Lemma 2.3** ([15]) *Suppose that  $y$  is an eventually positive solution of (1). Then, there exist two possible cases:*

$$(S_1) \quad y(t) > 0, y'(t) > 0, y''(t) > 0, y'''(t) > 0, y^{(4)}(t) < 0,$$

$$(S_2) \quad y(t) > 0, y'(t) > 0, y''(t) < 0, y'''(t) > 0, y^{(4)}(t) < 0,$$

for  $t \geq t_1$ , where  $t_1 \geq t_0$  is sufficiently large.

### 3 Main results

In this section, we shall establish some oscillation criteria for Eq. (1).

*Remark 3.1* ([37]) It is well known that the differential equation

$$[a(t)(y'(t))^\beta]' + q(t)y^\beta(g(t)) = 0, \tag{10}$$

where  $\beta > 0$  is the ratio of odd positive integers,  $a, q \in C([t_0, \infty), \mathbb{R}^+)$ , is nonoscillatory if and only if there exist a number  $t \geq t_0$ , and a function  $\varsigma \in C^1([t, \infty), \mathbb{R})$ , satisfying the inequality

$$\varsigma'(t) + \gamma a^{-1/\beta}(t)(\varsigma(t))^{(1+\beta)/\beta} + q(t) \leq 0.$$

**Theorem 3.1** *Assume that (2) holds. If the differential equations*

$$\left(\frac{2a^{\frac{1}{\beta}}(t)}{(\theta t^2)^\beta}(y'(t))^\beta\right)' + k \sum_{i=1}^j q_i(t)y^\beta(t) = 0 \tag{11}$$

and

$$y''(t) + y(t) \int_t^\infty \left(\frac{1}{a(\varsigma)} \int_\varsigma^\infty \sum_{i=1}^j q_i(s) ds\right)^{1/\beta} d\varsigma = 0 \tag{12}$$

are oscillatory, then every solution of (1) is oscillatory.

*Proof* Assume, for the sake of contradiction, that  $y$  is a positive solution of (1). Then, we can suppose that  $y(t)$  and  $y(\eta_i(t))$  are positive for all  $t \geq t_1$  sufficiently large. From Lemma 2.3, we have two possible cases, (S<sub>1</sub>) and (S<sub>2</sub>).

Let case (S<sub>1</sub>) hold. Using Lemma 2.1, we find

$$y'(t) \geq \frac{\theta}{2} t^2 y'''(t), \tag{13}$$

for every  $\theta \in (0, 1)$  and for all large  $t$ .

Defining

$$\varphi(t) := \tau(t) \left(\frac{a(t)(y'''(t))^\beta}{y^\beta(t)}\right), \tag{14}$$

we see that  $\varphi(t) > 0$  for  $t \geq t_1$ , where  $\tau \in C^1([t_0, \infty), (0, \infty))$  and

$$\begin{aligned} \varphi'(t) &= \tau'(t) \frac{a(t)(y'''(t))^\beta}{y^\beta(t)} + \tau(t) \frac{(a(y''')^\beta)'(t)}{y^\beta(t)} \\ &\quad - \beta \tau(t) \frac{y^{\beta-1}(t)y'(t)a(t)(y'''(t))^\beta}{y^{2\beta}(t)}. \end{aligned}$$

Combining (13) and (14), we obtain

$$\varphi'(t) \leq \frac{\tau'_+(t)}{\tau(t)} \varphi(t) + \tau(t) \frac{(a(t)(y'''(t))^\beta)'}{y^\beta(t)}$$

$$\begin{aligned}
 & -\beta\tau(t)\frac{\theta}{2}t^2\frac{a(t)(y'''(t))^{\beta+1}}{y^{\beta+1}(t)} \\
 & \leq \frac{\tau'(t)}{\tau(t)}\varphi(t) + \tau(t)\frac{(a(t)(y'''(t))^\beta)'}{y^\beta(t)} \\
 & \quad - \frac{\beta\theta t^2}{2(\tau(t)a(t))^{\frac{1}{\beta}}}\varphi^{\frac{\beta+1}{\beta}}(t).
 \end{aligned} \tag{15}$$

From (1) and (15), we get

$$\varphi'(t) \leq \frac{\tau'(t)}{\tau(t)}\varphi(t) - k\tau(t)\frac{\sum_{i=1}^j q_i(t)y^\beta(\eta_i(t))}{y^\beta(t)} - \frac{\beta\theta t^2}{2(\tau(t)a(t))^{\frac{1}{\beta}}}\varphi^{\frac{\beta+1}{\beta}}(t).$$

Note that  $y'(t) > 0$  and  $\eta_i(t) \geq t$ . Thus

$$\varphi'(t) \leq \frac{\tau'(t)}{\tau(t)}\varphi(t) - k\tau(t)\sum_{i=1}^j q_i(t) - \frac{\beta\theta t^2}{2(\tau(t)a(t))^{\frac{1}{\beta}}}\varphi^{\frac{\beta+1}{\beta}}(t). \tag{16}$$

If we set  $\tau(t) = k = 1$  in (16), we obtain

$$\varphi'(t) + \frac{\beta\theta t^2}{2a^{\frac{1}{\beta}}(t)}\varphi^{\frac{\beta+1}{\beta}}(t) + \sum_{i=1}^j q_i(t) \leq 0.$$

Thus, we can see that Eq. (11) is nonoscillatory, which is a contradiction.

Let case (S<sub>2</sub>) hold. Defining

$$\psi(t) := \vartheta(t)\frac{y'(t)}{y(t)},$$

we see that  $\psi(t) > 0$  for  $t \geq t_1$ , where  $\vartheta \in C^1([t_0, \infty), (0, \infty))$ . By differentiating  $\psi(t)$ , we find

$$\psi'(t) = \frac{\vartheta'(t)}{\vartheta(t)}\psi(t) + \vartheta(t)\frac{y''(t)}{y(t)} - \frac{1}{\vartheta(t)}\psi^2(t). \tag{17}$$

Now, by integrating (1) from  $t$  to  $m$  and using  $y'(t) > 0$ , we have

$$a(m)(y'''(m))^\beta - a(t)(y'''(t))^\beta = -\int_t^m \sum_{i=1}^j q_i(s)g(y(\eta_i(s))) \, ds.$$

By virtue of  $y'(t) > 0$  and  $\eta_i(t) \geq t$ , we get

$$a(m)(y'''(m))^\beta - a(t)(y'''(t))^\beta \leq -ky^\beta(t) \int_t^u \sum_{i=1}^j q_i(s) \, ds.$$

Letting  $m \rightarrow \infty$ , we see that

$$a(t)(y'''(t))^\beta \geq ky^\beta(t) \int_t^\infty \sum_{i=1}^j q_i(s) \, ds$$

and so

$$y'''(t) \geq y(t) \left( \frac{k}{a(t)} \int_t^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta}.$$

Integrating again from  $t$  to  $\infty$ , we get

$$y''(t) + y(t) \int_t^\infty \left( \frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \tag{18}$$

Combining (17) and (18), we obtain

$$\psi'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)} \psi(t) - \vartheta(t) \int_t^\infty \left( \frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta - \frac{1}{\vartheta(t)} \psi^2(t). \tag{19}$$

If  $\vartheta(t) = k = 1$  in (19), then we get

$$\psi'(t) + \psi^2(t) + \int_t^\infty \left( \frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0.$$

Hence, we see that Eq. (12) is nonoscillatory, which is a contradiction. The proof of the theorem is complete. □

Based on the above results and Theorem 3.1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with  $\beta = 1$ .

**Theorem 3.2** *Let  $\beta = k = 1$ . Assume that*

$$\int_{t_0}^\infty \frac{\theta t^2}{2a(t)} dt = \infty$$

and

$$\liminf_{t \rightarrow \infty} \left( \int_{t_0}^t \frac{\theta s^2}{2a(s)} ds \right) \int_t^\infty \sum_{i=1}^j q_i(s) ds > \frac{1}{4}, \tag{20}$$

for some constant  $\theta \in (0, 1)$ ,

$$\liminf_{t \rightarrow \infty} t \int_{t_0}^t \int_v^\infty \left( \frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right) d\zeta dv > \frac{1}{4}, \tag{21}$$

then all solutions of (1) are oscillatory.

In this theorem, we employ the integral averaging technique to establish an oscillation criterion for (1).

**Theorem 3.3** *Let (2) hold. If there exist positive functions  $\tau, \vartheta \in C^1([t_0, \infty), \mathbb{R})$  such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{H_1(t, t_1)} \int_{t_1}^t \left( H_1(t, s) k \tau(s) \sum_{i=1}^j q_i(s) - \pi(s) \right) ds = \infty \tag{22}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H_2(t, t_1)} \int_{t_1}^t \left( H_2(t, s) \vartheta(s) \varpi(s) - \frac{\vartheta(s) h_2^2(t, s)}{4} \right) ds = \infty, \tag{23}$$

where

$$\pi(s) = \frac{h_1^{\beta+1}(t, s) H_1^\beta(t, s) 2^\beta \tau(s) a(s)}{(\beta + 1)^{\beta+1} (\theta s^2)^\beta},$$

for all  $\theta \in (0, 1)$ , and

$$\varpi(s) = \int_t^\infty \left( \frac{k}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta,$$

then (1) is oscillatory.

*Proof* Assume, for the sake of contradiction, that  $y$  is a positive solution of (1). Then, we can suppose that  $y(t)$  and  $y(\eta_i(t))$  are positive for all  $t \geq t_1$  sufficiently large. From Lemma 2.3, we have two possible cases, (S<sub>1</sub>) and (S<sub>2</sub>).

Assume that (S<sub>1</sub>) holds. From Theorem 3.1, we find that (16) holds. Multiplying (16) by  $H_1(t, s)$  and integrating the resulting inequality from  $t_1$  to  $t$ , we find that

$$\begin{aligned} \int_{t_1}^t H_1(t, s) k \tau(s) \sum_{i=1}^j q_i(s) ds &\leq \varphi(t_1) H_1(t, t_1) + \int_{t_1}^t \left( \frac{\partial}{\partial s} H_1(t, s) + \frac{\tau'(s)}{\tau(s)} H_1(t, s) \right) \varphi(s) ds \\ &\quad - \int_{t_1}^t \frac{\beta \theta s^2}{2(\tau(s) a(s))^{\frac{1}{\beta}}} H_1(t, s) \varphi^{\frac{\beta+1}{\beta}}(s) ds. \end{aligned}$$

From (3), we get

$$\begin{aligned} \int_{t_1}^t H_1(t, s) k \tau(s) \sum_{i=1}^j q_i(s) ds &\leq \varphi(t_1) H_1(t, t_1) + \int_{t_1}^t h_1(t, s) H_1^{\beta/(\beta+1)}(t, s) \varphi(s) ds \\ &\quad - \int_{t_1}^t \frac{\beta \theta s^2}{2(\tau(s) a(s))^{\frac{1}{\beta}}} H_1(t, s) \varphi^{\frac{\beta+1}{\beta}}(s) ds. \end{aligned} \tag{24}$$

Using Lemma 2.2 with  $V = \beta \theta s^2 / (2(\tau(s) a(s))^{\frac{1}{\beta}}) H_1(t, s)$ ,  $U = h_1(t, s) H_1^{\beta/(\beta+1)}(t, s)$  and  $y = \varphi(s)$ , we get

$$\begin{aligned} h_1(t, s) H_1^{\beta/(\beta+1)}(t, s) \varphi(s) - \frac{\beta \theta s^2}{2(\tau(s) a(s))^{\frac{1}{\beta}}} H_1(t, s) \varphi^{\frac{\beta+1}{\beta}}(s) \\ \leq \frac{h_1^{\beta+1}(t, s) H_1^\beta(t, s) 2^\beta \tau(s) a(s)}{(\beta + 1)^{\beta+1} (\theta s^2)^\beta}, \end{aligned}$$

which with (24) gives

$$\frac{1}{H_1(t, t_1)} \int_{t_1}^t \left( H_1(t, s)k\tau(s) \sum_{i=1}^j q_i(s) - \pi(s) \right) ds \leq \varphi(t_1).$$

This contradicts (22).

Assume that (S<sub>2</sub>) holds. From Theorem 3.1, (19) holds. Multiplying (19) by  $H_2(t, s)$  and integrating the resulting inequality from  $t_1$  to  $t$ , we obtain

$$\begin{aligned} \int_{t_1}^t H_2(t, s)\vartheta(s)\varpi(s) ds &\leq \psi(t_1)H_2(t, t_1) \\ &+ \int_{t_1}^t \left( \frac{\partial}{\partial s}H_2(t, s) + \frac{\vartheta'(s)}{\vartheta(s)}H_2(t, s) \right) \psi(s) ds \\ &- \int_{t_1}^t \frac{1}{\vartheta(s)}H_2(t, s)\psi^2(s) ds. \end{aligned}$$

Thus, from (4), we get

$$\begin{aligned} \int_{t_1}^t H_2(t, s)\vartheta(s)\varpi(s) ds &\leq \psi(t_1)H_2(t, t_1) + \int_{t_1}^t h_2(t, s)\sqrt{H_2(t, s)}\psi(s) ds \\ &- \int_{t_1}^t \frac{1}{\vartheta(s)}H_2(t, s)\psi^2(s) ds \\ &\leq \psi(t_1)H_2(t, t_1) + \int_{t_1}^t \frac{\vartheta(s)h_2^2(t, s)}{4} ds \end{aligned}$$

and so

$$\frac{1}{H_2(t, t_1)} \int_{t_1}^t \left( H_2(t, s)\vartheta(s)\varpi(s) - \frac{\vartheta(s)h_2^2(t, s)}{4} \right) ds \leq \psi(t_1),$$

which contradicts (23). The proof of the theorem is complete. □

### 4 Examples

In this section, we give the following examples.

*Example 4.1* Consider the equation

$$y^{(4)}(t) + \frac{q_0}{t^4}y(2t) = 0, \quad t \geq 1, \tag{25}$$

where  $q_0 > 0$  is a constant. Note that  $\beta = 1, \kappa = 4, a(t) = 1, q(t) = q_0/t^4$  and  $\eta(t) = 2t$ . If we set  $k = 1$ , then conditions (20) and (21) become

$$\begin{aligned} \liminf_{t \rightarrow \infty} \left( \int_{t_0}^t \frac{\theta s^2}{2a(s)} ds \right) \int_t^\infty \sum_{i=1}^j q_i(s) ds &= \liminf_{t \rightarrow \infty} \left( \frac{t^3}{3} \right) \int_t^\infty \frac{q_0}{s^4} ds \\ &= \frac{q_0}{9} > \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} \liminf_{t \rightarrow \infty} t \int_{t_0}^t \int_{\nu}^{\infty} \left( \frac{1}{a(\zeta)} \int_{\zeta}^{\infty} \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta d\nu &= \liminf_{t \rightarrow \infty} t \left( \frac{q_0}{6t} \right), \\ &= \frac{q_0}{6} > \frac{1}{4}, \end{aligned}$$

respectively. From Theorem 3.2, all solutions of (25) are oscillatory, if  $q_0 > 2.25$ .

*Remark 4.1* We compare our result with the known related criteria

Condition	(6)	(7)	(8)
Criterion	$q_0 > 25.5$	$q_0 > 18$	$q_0 > 1728$ .

Therefore, our result improves results [33–35].

*Example 4.2* Consider the differential equation (9), where  $q_0 > 0$  is a constant. Note that  $\beta = 1, \kappa = 4, a(t) = 1, q(t) = q_0/t^4$  and  $\eta(t) = 3t$ . If we set  $k = 1$ , then condition (20) becomes

$$\frac{q_0}{9} > \frac{1}{4}.$$

Therefore, from Theorem 3.2, all solutions of (9) are oscillatory, if  $q_0 > 2.25$ .

*Remark 4.2* Our result improves results [33–35].

**Acknowledgements**

The authors would like to thank the anonymous reviewers for their work and constructive comments that improved the manuscript.

**Funding**

The authors received no direct funding for this work.

**Availability of data and materials**

Not applicable.

**Competing interests**

The authors declares that there are no competing interests.

**Authors' contributions**

The authors declare that they have read and approved the final manuscript.

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Received: 31 March 2020 Accepted: 31 July 2020 Published online: 08 August 2020

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