Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A note on degenerate poly-Genocchi numbers and polynomials

Hye Kyung Kim[1] and Lee-Chae Jang[2*]

*Correspondence:
lcjang@konkuk.ac.kr
[2]Graduate School of Education, Konkuk University, Seoul, 143-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract

Recently, some mathematicians have been studying a lot of degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects. Our research is also interested in this field. In this paper, we introduce a new type of the degenerate poly-Genocchi polynomials and numbers, based on Kim and Kim's (*J. Math. Anal. Appl.* 487(2):124017, 2020) modified polyexponential function. The paper is divided into two parts. In Sect. 2, we consider a new type of the degenerate poly-Genocchi polynomials and numbers constructed from the modified polyexponential function. We also show several combinatorial identities related to the degenerate poly-Genocchi polynomials and numbers. Some of them include the degenerate and other special polynomials and numbers such as the Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind, degenerate Euler polynomials, degenerate Bernoulli polynomials and Bernoulli numbers of order α, etc. In Sect. 3, we also introduce the degenerate unipoly Genocchi polynomials attached to an arithmetic function by using the degenerate polylogarithm function. We give some new explicit expressions and identities related to degenerate unipoly Genocchi polynomials and special numbers and polynomials.

MSC: 11B73; 11B83; 05A19

Keywords: Degenerate poly-Genocchi polynomials and numbers; Modified degenerate polyexponential functions; Higher-order Bernoulli polynomials and numbers; Unipoly functions; Unipoly Genocchi polynomials

1 Introduction

The study of degenerate versions of some special polynomials and numbers, namely degenerate Bernoulli and Euler polynomials and numbers, was initiated by Carlitz [2]. Since then, many mathematicians have been studying degenerate versions of special polynomials and numbers such as Bernoulli, Euler, and Genocchi polynomials and numbers [1, 3–14]. Recently, Kim et al. studied polynomials and numbers mentioned above in terms of Jindalrae and Gaenari numbers and polynomials, discrete harmonic numbers and polynomials [15, 16]. In particular, Genocchi numbers have been extensively studied in many different contexts in such branches of mathematics as, for instance, elementary number theory, complex analytic number theory, differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory (p-adic L-functions), and quantum physics (quantum groups). The works of Genocchi numbers and their combinatorial relations have received much attention [13, 17–20]. In the paper, we focus on a new type of degenerate poly-Genocchi polynomial and numbers.

As is well known, the Bernoulli polynomials of order $\alpha \in \mathbb{R}$ are defined by means of the following generating function:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \bigl(\text{see } [3, 4, 6, 21]\bigr). \tag{1}$$

We note that, for $\alpha = 1$, $B_n(x) = B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials.

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α.

The Euler polynomials are defined by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \bigl(\text{see } [2, 5]\bigr). \tag{2}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

The Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \quad \bigl(\text{see } [13, 14, 17]\bigr). \tag{3}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers.

The degenerate poly-Bernoulli polynomials are defined by using the polyexponential functions (see [11]) and they are reduced to the degenerate Bernoulli polynomials if $k = 1$. The poly-exponential functions were first studied by Hardy [22] and reconsidered by Kim and Kim [1, 9, 10] in the view of an inverse to the polylogarithm functions which were studied by Jaonquière [23], Lewis [24], and Zagier [25]. In 1997, Kaneko [21] introduced poly-Bernoulli numbers which are defined by the polylogarithm function.

Recently, Kim and Kim [1] introduced the modified polyexponential function as follows:

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k} \quad (k \in \mathbb{Z}). \tag{4}$$

By using these functions, they also defined type 2 poly-Bernoulli and type 2 unipoly-Bernoulli polynomials and obtained several interesting properties of them (see [9]).

Kim et al. [10] introduced poly-Genocchi polynomials arising from the modified poly-exponential function as follows:

$$\frac{2\,\mathrm{Ei}_k(\log(1+t))}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}. \tag{5}$$

When $x = 0$, $G_n^{(k)} = G_n^{(k)}(0)$ are called the poly-Genocchi numbers. Note that $G_n(x) = G_n^{(1)}(x)$ $(n \geq 0)$ are the Genocchi polynomials.

The degenerate exponential functions are defined as follows:

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}} \quad \bigl(\text{see } [1, 8\text{–}12]\bigr). \tag{6}$$

Here, we note that

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\quad\bigl(\text{see [10, 12]}\bigr),\tag{7}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ $(n\geq 1)$.

In [2], Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials respectively given by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!},\qquad \frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}.\tag{8}$$

When $x=0$, then $B_{n,\lambda}=B_{n,\lambda}(0)$ and $E_{n,\lambda}=E_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers and the degenerate Euler numbers, respectively.

Kim et al. [10] considered the degenerate poly-Bernoulli polynomials as follows:

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.\tag{9}$$

When $x=0$, $\beta_{n,\lambda}^{(k)}=\beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}^{(1)}(x)=B_n(x)$ $(n\geq 0)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\bigl(\text{see [2–4, 26]}\bigr).\tag{10}$$

In [5], Kim et al. considered the degenerate Genocchi polynomials given by

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!}.\tag{11}$$

When $x=0$, $G_{n,\lambda}=G_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

For $n\geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l\quad\bigl(\text{see [9, 10, 24]}\bigr),\tag{12}$$

where $(x)_0=1$, $(x)_n=x(x-1)\cdots(x-n+1)$ $(n\geq 1)$.

From (12), it is easy to see that

$$\frac{1}{k!}\bigl(\log(1+t)\bigr)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}.\tag{13}$$

In the inverse expression to (12), for $n\geq 0$, the Stirling numbers of the second kind are defined by

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l.\tag{14}$$

From (14), it is easy to see that

$$\frac{1}{k!}\left(e^t-1\right)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}. \tag{15}$$

Recently, Kim and Kim [6] introduced the degenerate Stirling numbers of the second kind as follows:

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l \quad (n\geq 0). \tag{16}$$

As an inversion formula of (16), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n\geq 0)\ \bigl(\text{see [6]}\bigr). \tag{17}$$

From (16) and (17), Kim and Kim observed that

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad \bigl(\text{see [6]}\bigr). \tag{18}$$

The paper is divided into two parts. In Sect. 2, we define a new type of the degenerate poly-Genocchi polynomials and numbers constructed from the modified polyexponential function. We also show several combinatorial identities related to the degenerate poly-Genocchi polynomials and numbers. Some of them include the degenerate and other special polynomials and numbers such as the Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind, degenerate Euler polynomials, degenerate Bernoulli polynomials and Bernoulli numbers of order α, etc. In Sect. 3, we also introduce the degenerate unipoly Genocchi polynomials attached to an arithmetic function, by using the degenerate polylogarithm function. We give some new explicit expressions and identities related to degenerate unipoly Genocchi polynomials and special numbers and polynomials.

2 Degenerate poly-Genocchi numbers and polynomials

In this section, we consider the poly-Genocchi polynomials and the degenerate poly-Genocchi polynomials respectively as follows:

$$\frac{\mathrm{Ei}_k(\log(1+2t))}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n^{(k)}(x)\frac{t^n}{n!}, \tag{19}$$

and

$$\frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{20}$$

When $x=0$, $G_n^{(k)}=G_n^{(k)}(0)$ are called the poly-Genocchi numbers.

It is easy to show that $G_n(x) = G_n^{(1)}(x)$ $(n \geq 0)$ are the Genocchi polynomials because of

$$\mathrm{Ei}_1\big(\log(1+2t)\big) = 2t.$$

When $x = 0$, $G_{n,\lambda}^{(k)} = G_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.
It is easy to show that $G_{n,\lambda}(x) = G_{n,\lambda}^{(1)}(x)$ $(n \geq 0)$ are the degenerate Genocchi polynomials.

Theorem 1 *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\begin{aligned} &G_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} 2^l (1)_{n-l,\lambda} \beta_{l,\frac{\lambda}{2}}^{(k)}\left(\frac{x}{2}\right), \\ &G_{0,\lambda}^{(k)}(x) = 0. \end{aligned} \tag{21}$$

Proof From (6) and (9), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1} e_\lambda^x(t) \\ &= \frac{\mathrm{Ei}_k(\log(1+2t))}{(e_\lambda(t)+1)(e_\lambda(t)-1)} \big(e_\lambda(t)-1\big) e_\lambda^x(t) \\ &= \frac{\mathrm{Ei}_k(\log(1+2t))}{(e_{\frac{\lambda}{2}}(2t)-1)} e_\lambda^x(t) \left(\sum_{m=1}^{\infty} (1)_{m,\lambda} \frac{t^m}{m!} \right) \\ &= \left(\sum_{l=0}^{\infty} \beta_{l,\frac{\lambda}{2}}^{(k)}\left(\frac{x}{2}\right) \frac{2^l t^l}{l!} \right) \left(\sum_{m=1}^{\infty} (1)_{m,\lambda} \frac{t^m}{m!} \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} \binom{n-1}{l} 2^l (1)_{n-l,\lambda} \beta_{l,\frac{\lambda}{2}}^{(k)}\left(\frac{x}{2}\right) \right) \frac{t^n}{n!}. \end{aligned} \tag{22}$$

Therefore, by comparing the coefficients on both sides of (22), we get the desired result. □

When $x = 0$, we have

$$G_{n,\lambda}^{(k)} = \sum_{l=0}^{n-1} \binom{n-1}{l} 2^l (1)_{n-l,\lambda} \beta_{l,\frac{\lambda}{2}}^{(k)}. \tag{23}$$

Theorem 2 *For $n \geq 0$, we get*

$$\begin{aligned} &G_{n,\lambda}^{(k)}(x) = \sum_{j=1}^{n} \sum_{m=1}^{j} \binom{n}{j} \frac{2^{j-1}}{m^{k-1}} S_1(j,m) E_{n-m,\lambda}(x), \\ &G_{0,\lambda}^{(k)} = 0. \end{aligned} \tag{24}$$

Proof By using (4) and (8), we get

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{1}{(e_\lambda(t)+1)} \mathrm{Ei}_k\big(\log(1+2t)\big) e_\lambda^x(t)$$

$$= \frac{1}{2}\left(\sum_{l=0}^{\infty} E_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\frac{(\log(1+2t))^m}{(m-1)!m^k}\right)$$
$$= \frac{1}{2}\left(\sum_{l=0}^{\infty} E_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\frac{1}{m^{k-1}}\sum_{j=m}^{\infty} S_1(j,m)\frac{2^j t^j}{j!}\right)$$
$$= \frac{1}{2}\left(\sum_{l=0}^{\infty} E_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{j=1}^{\infty}\sum_{m=1}^{j} 2^j\frac{1}{m^{k-1}} S_1(j,m)\frac{t^j}{j!}\right)$$
$$= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{m=1}^{j}\binom{n}{j}2^{j-1}\frac{1}{m^{k-1}} S_1(j,m)E_{n-m,\lambda}(x)\right)\frac{t^n}{n!}. \tag{25}$$

Therefore, by comparing the coefficients on both sides of (25), we get the result that we wanted. □

Theorem 3 *For $n \geq 0$, we have*

$$G_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}2^l\frac{1}{(l+1)m^{k-1}} S_1(l+1,m)G_{n-l,\lambda}(x). \tag{26}$$

Proof From (13) and (19), we observe that

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1}e_\lambda^x(t)$$
$$= \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\cdot\frac{1}{2t}\,\mathrm{Ei}_k\bigl(\log(1+2t)\bigr)$$
$$= \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\cdot\frac{1}{2t}\left(\sum_{l=1}^{\infty}\sum_{m=1}^{l}2^l\frac{1}{m^{k-1}}S_1(l,m)\frac{t^l}{l!}\right)$$
$$= \sum_{j=0}^{\infty} G_{j,\lambda}(x)\frac{t^j}{j!}\frac{1}{2t}\left(\sum_{l=0}^{\infty}\sum_{m=1}^{l+1}2^{l+1}\frac{1}{m^{k-1}}S_1(l+1,m)\frac{t^{l+1}}{(l+1)!}\right)$$
$$= \sum_{j=0}^{\infty} G_{j,\lambda}(x)\frac{t^j}{j!}\left(\sum_{l=0}^{\infty}\sum_{m=1}^{l+1}2^{l}\frac{1}{(l+1)m^{k-1}}S_1(l+1,m)\frac{t^{l}}{l!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}2^l\frac{1}{(l+1)m^{k-1}}S_1(l+1,m)G_{n-l,\lambda}(x)\right)\frac{t^n}{n!}. \tag{27}$$

Therefore, by comparing the coefficients on both sides of (27), we get the desired result. □

For the next theorem, we need the following well-known identity:

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^{x-1} = \sum_{\alpha=0}^{\infty} B_\alpha^{(\alpha-r+1)}(x)\frac{t^\alpha}{\alpha!}, \tag{28}$$

where $B_\alpha^{(\alpha)}(x)$ is the Bernoulli polynomials of order α in (1).

Theorem 4 *For $n \geq 0$, $k \in \mathbb{Z}$, we get*

$$G_{n,\lambda}^{(k)} = \sum_{m=0}^{n} \binom{n}{m} 2^m \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\ldots,m_{k-1}} \times \frac{B_{m_1}^{(m_1)}}{m_1+1} \frac{B_{m_2}^{(m_2)}}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1} G_{n-l,\lambda} \frac{x^n}{n!}, \tag{29}$$

where $B_m^{(m)}$ is the Bernoulli numbers of order m at $x = o$.

Proof First, we note that

$$\begin{aligned} \frac{d}{dx}\mathrm{Ei}_k\bigl(\log(1+2x)\bigr) &= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(\log(1+2x))^n}{(n-1)!n^k} \\ &= \frac{2}{1+2x}\sum_{n=1}^{\infty}\frac{n(\log(1+2x))^{n-1}}{(n-1)!n^k} \\ &= \frac{2}{(1+2x)\log(1+2x)}\sum_{n=1}^{\infty}\frac{(\log(1+2x))^n}{(n-1)!n^{k-1}} \\ &= \frac{2}{(1+2x)\log(1+2x)}\mathrm{Ei}_{k-1}\bigl(\log(1+2x)\bigr)\,dt. \end{aligned} \tag{30}$$

From (28) and (30), we obtain the following equation:

$$\begin{aligned} &\mathrm{Ei}_k\bigl(\log(1+2x)\bigr) \\ &= \int_0^x \frac{2}{(1+2t)\log(1+2t)} \\ &\quad \times \underbrace{\int_0^t \frac{2}{(1+2t)\log(1+2t)}\cdots\int_0^t \frac{2}{(1+2t)\log(1+2t)}\int_0^{2t}}_{(k-2)\text{-times}} \frac{2\,\mathrm{Ei}_1(\log(1+2t))}{(1+2t)\log(1+2t)}\,dt\cdots dt \\ &= \int_0^x \frac{2}{(1+2t)\log(1+2t)} \\ &\quad \times \underbrace{\int_0^t \frac{2}{(1+2t)\log(1+2t)}\cdots\int_0^t \frac{2}{(1+2t)\log(1+2t)}\int_0^{2t}}_{(k-2)\text{-times}} \frac{4t}{(1+2t)\log(1+2t)}\,dt\cdots dt \\ &= 2x\sum_{m=0}^{\infty}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\ &\quad \times \frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!}. \end{aligned} \tag{31}$$

From equation (31), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}\frac{t^n}{n!} &= \frac{1}{e_\lambda(t)+1}\mathrm{Ei}_k\bigl(\log(1+2t)\bigr) \\ &= \frac{2t}{e_\lambda(t)+1}\sum_{m=0}^{\infty}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \end{aligned}$$

$$\begin{aligned}
&\times \frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}\frac{t^m}{m!} \\
&= \left(\sum_{l=0}^{\infty} G_{l,\lambda}\frac{t^l}{l!}\right)\sum_{m=0}^{\infty}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\
&\times \frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\
&\times \frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}G_{n-l,\lambda}\frac{t^n}{n!}. \qquad (32)
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (32), we get the desired result. □

Corollary 5 *For $k = 2$, we have*

$$G_{n,\lambda}^{(2)} = \sum_{l=0}^{n}\binom{n}{l}2^l\frac{B_{l+1}^{(l)}}{l+1}G_{n-l,\lambda}. \qquad (33)$$

Theorem 6 *For $n \geq 0$, $k \in \mathbb{Z}$, we get*

$$G_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}G_{l,\lambda}^{(k)}. \qquad (34)$$

Proof From (7) and (19), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1}e_\lambda^x(t) \\
&= \sum_{l=0}^{\infty}G_{l,\lambda}^{(k)}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}G_{l,\lambda}^{(k)}\right)\frac{t^n}{n!}. \qquad (35)
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (35), we obtain the desired result. □

Theorem 7 *For $n \geq 1$, $k \in \mathbb{Z}$, we have*

$$G_{n-1,\lambda}^{(k)}(1) + G_{n-1,\lambda}^{(k)} = \sum_{m=1}^{n}\frac{2^n S_1(n,m)}{m^{k-1}}. \qquad (36)$$

Moreover, when $k = 1$,

$$G_{n-1,\lambda}(1) + G_{n-1,\lambda} = \sum_{m=1}^{n}2^n S_1(n,m). \qquad (37)$$

Proof By using (6) and Theorem 6,

$$\begin{aligned}
\mathrm{Ei}_k\big(\log(1+2t)\big) &= \big(e_\lambda(t)+1\big)\frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1}\\
&= \left(\sum_{l=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}G_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)+\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}G_{m,\lambda}^{(k)}+G_{n,\lambda}^{(k)}\right)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}\big(G_{n-1,\lambda}^{(k)}(1)+G_{n-1,\lambda}^{(k)}\big)\frac{t^n}{n!}.
\end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned}
\mathrm{Ei}_k\big(\log(1+2t)\big) &= \sum_{m=1}^{\infty}\frac{(\log(1+2t))^m}{(m-1)!m^k}\\
&= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{1}{m^{k-1}}S_1(n,m)\frac{2^nt^n}{n!}=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{2^n}{m^{k-1}}S_1(n,m)\frac{t^n}{n!}.
\end{aligned} \tag{39}$$

Now, by comparing the coefficients of (38) and (39), we get what we wanted. □

Theorem 8 *For $n \geq 1$, $k = 1$, we have*

$$\sum_{m=1}^{n}2^nS_1(n,m)=2\delta_{n,1}, \tag{40}$$

where $\delta_{n,k}$ is the Kronecker delta.

Proof From (39), we obtain

$$\mathrm{Ei}_1\big(\log(1+2t)\big)=2t=\sum_{n=1}^{\infty}\sum_{m=1}^{n}2^nS_1(n,m)\frac{t^n}{n!}. \tag{41}$$

Hence, by comparing the coefficients of (41), we get the desired result. □

Corollary 9 *For $n \geq 1$, $k = 1$, we have*

$$G_{n-1,\lambda}(1)+G_{n-1,\lambda}=\begin{cases}2, & \text{if } n=1,\\ 0, & \text{otherwise.}\end{cases} \tag{42}$$

Theorem 10 *For $n \geq 0$, $k \in \mathbb{Z}$, we get*

$$G_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_mS_{2,\lambda}(m,l)G_{n-l,\lambda}^{(k)}. \tag{43}$$

Proof By using (21), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1} e_\lambda^x(t) \\
&= \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1} (e_\lambda - 1 + 1)^x \\
&= \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1} \left(\sum_{m=0}^{\infty} \binom{x}{m} \bigl(e_\lambda(t) - 1\bigr)^m \right) \\
&= \frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1} \left(\sum_{m=0}^{\infty} (x)_m \frac{(e_\lambda(t)-1)^m}{m!} \right) \\
&= \left(\sum_{i=0}^{\infty} G_{i,\lambda}^{(k)} \frac{x^i}{i!} \right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(m,l) \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_{2,\lambda}(m,l) G_{n-l,\lambda}^{(k)} \right) \frac{t^n}{n!}.
\end{aligned} \tag{44}$$

Now, by comparing the coefficients of (44), we get what we wanted. □

3 The unipoly Genocchi polynomials and numbers

Let p be any arithmetic function which is real- or complex-valued function defined on the set of positive integers $\mathbb{N}$. Then Kim and Kim [27] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)x^n}{n^k} \quad (k \in \mathbb{Z}). \tag{45}$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x) \tag{46}$$

is the ordinary polylogarithm function, and for $k \geq 2$,

$$\frac{d}{dx} u_k(x|p) = \frac{1}{x} u_{k-1}(x|p) \tag{47}$$

and

$$u_k(x|p) = \int_0^x \frac{1}{t} \underbrace{\int_0^t \frac{1}{t} \cdots \int_0^t \frac{1}{t}}_{(k-2)\text{-times}} u_1(t|p)\, dt\, dt \cdots dt. \tag{48}$$

In [28], Dolgy and Jang introduced the unipoly Genocchi polynomials as follows:

$$\frac{2}{e^t+1} u_k\bigl(\log(1+t)|p\bigr) e^{xt} = \sum_{n=0}^{\infty} G_{n,p}^{(k)}(x) \frac{t^n}{n!}. \tag{49}$$

In this section, we define the degenerate unipoly Genocchi polynomial by

$$\frac{u_k(\log(1+2t)|p)}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}. \tag{50}$$

When $x=0$, $G_{n,\lambda,p}^{(k)}=G_{n,\lambda,p}^{(k)}(0)$ is the degenerate unipoly Genocchi number.

When $p=1$, $G_{n,\lambda,1}^{(k)}(x)=G_{n,\lambda}^{(k)}(x)$ is the degenerate poly-Genocchi polynomial of (19).

Theorem 11 *Let $p(n)=\frac{1}{\Gamma(n)}$ for $n\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{Z}$, then we have*

$$G_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x)=G_{n,\lambda}^{(k)}(x). \tag{51}$$

Proof Let $p(n)=\frac{1}{\Gamma(n)}=\frac{1}{(n-1)!}$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x)\frac{t^n}{n!}&=\frac{u_k(\log(1+2t)|\frac{1}{\Gamma})}{e_\lambda(t)+1}e_\lambda^x(t)\\&=\frac{\mathrm{Ei}_k(\log(1+2t))}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{52}$$

Thus, we have what we wanted. □

Theorem 12 *For $n\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{Z}$, we have*

$$G_{n,\lambda,p}^{(k)}=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{2^l}{l+1}S_1(l+1,m+1)G_{n-l,\lambda}. \tag{53}$$

Proof From (11) and (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,\lambda,p}^{(k)}\frac{t^n}{n!}&=\frac{1}{e_\lambda(t)+1}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}\big(\log(1+2t)\big)^m\\&=\frac{1}{e_\lambda(t)+1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m+1}^{\infty}S_1(l,m+1)\frac{2^lt^l}{l!}\\&=\frac{2t}{e_\lambda(t)+1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m}^{\infty}S_1(l+1,m+1)\frac{2^lt^l}{(l+1)!}\\&=\left(\sum_{j=0}^{\infty}G_{j,\lambda}\frac{t^j}{j!}\right)\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}\frac{p(m+1)(m+1)!}{(m+1)^k}S_1(l+1,m+1)\frac{2^l}{(l+1)}\right)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{2^l}{l+1}S_1(l+1,m+1)G_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{54}$$

Therefore, by comparing the coefficients on both sides of (54), we obtain the result of this theorem. □

Theorem 13 *For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, we have*

$$G_{n,\lambda,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l+1} \binom{n}{l+1} \frac{p(m+1)(m+1)!}{(m+1)^k} 2^{l-1} S_1(l, m+1) E_{n-l,\lambda}. \tag{55}$$

Proof From (8) and (13), we have

$$\begin{aligned}
&\frac{1}{e_\lambda(t)+1} \sum_{m=1}^{\infty} \frac{p(m)(\log(1+2t))^m}{m^k} \cdot \frac{m!}{m!} \\
&= \frac{1}{e_\lambda(t)+1} \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{(\log(1+2t))^{m+1}}{(m+1)!} \\
&= \frac{1}{2} \left(\sum_{j=0}^{\infty} E_{j,\lambda} \frac{t^j}{j!} \right) \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{2^l t^l}{l!} \\
&= \frac{1}{2} \left(\sum_{j=0}^{\infty} E_{j,\lambda} \frac{t^j}{j!} \right) \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l+1} \frac{p(m+1)(m+1)!}{(m+1)^k} 2^l S_1(l, m+1) \right) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l+1} \binom{n}{l+1} \frac{p(m+1)(m+1)!}{(m+1)^k} 2^{l-1} S_1(l, m+1) E_{n-l,\lambda} \right) \frac{t^n}{n!}.
\end{aligned} \tag{56}$$

Thus, by comparing the coefficients on both sides of (56), we obtain the desired theorem. □

Theorem 14 *For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, we have*

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{\alpha=0}^{n} \sum_{l=0}^{\alpha} \sum_{m=1}^{l+1} \binom{n}{\alpha} \binom{\alpha}{l} (1)_{\alpha-1,\lambda} \frac{p(m)m!}{m^k} 2^{n-\alpha+l} S_1(l+1, m) B_{n-\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right). \tag{57}$$

Proof From (7), (8), and (13), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} \\
&= \frac{u_k(\log(1+2t)|p)}{e_\lambda(t)+1} e_\lambda^x(t) \\
&= \frac{1}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=1}^{\infty} \frac{p(m)(\log(1+2t))^m}{m^k} \frac{m!}{m!} \\
&= \frac{1}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=1}^{\infty} \frac{p(m)m!}{m^k} \sum_{l=m}^{\infty} S_1(l, m) \frac{2^l t^l}{l!} \\
&= \frac{e_\lambda^x(t)}{e_\lambda(t)+1} \frac{e_\lambda(t)-1}{e_\lambda(t)-1} \sum_{l=0}^{\infty} \sum_{m=1}^{l+1} \frac{p(m)m!}{m^k} S_1(l+1, m) \frac{2^{l+1} t^{l+1}}{l!} \\
&= \frac{2t e_\lambda^x(t)}{e_{\frac{\lambda}{2}}(2t)-1} (e_\lambda(t)-1) \sum_{l=0}^{\infty} \sum_{m=1}^{l+1} \frac{p(m)m!}{m^k} S_1(l+1, m) \frac{2^l t^l}{l!} \\
&= \left(\sum_{i=0}^{\infty} B_{i,\frac{\lambda}{2}}\left(\frac{x}{2}\right) \frac{2^i t^i}{i!} \right) \left(\sum_{j=1}^{\infty} (1)_{j,\lambda} \frac{t^j}{j!} \right) \left(\sum_{l=0}^{\infty} \sum_{m=1}^{l+1} \frac{p(m)m!}{m^k} 2^l S_1(l+1, m) \cdot \frac{t^l}{l!} \right)
\end{aligned}$$

$$= \left(\sum_{i=0}^{\infty} B_{i,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\frac{2^i t^i}{i!}\right)\left(\sum_{\alpha=1}^{\infty}\sum_{l=0}^{\alpha}\sum_{m=1}^{l+1}\binom{\alpha}{l}(1)_{\alpha-l,\lambda}\frac{p(m)m!}{m^k}2^l S_1(l+1,m)\frac{t^\alpha}{\alpha!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{\alpha=0}^{n}\sum_{l=0}^{\alpha}\sum_{m=1}^{l+1}\binom{n}{\alpha}\binom{\alpha}{l}(1)_{\alpha-1,\lambda}\frac{p(m)m!}{m^k}2^{n-\alpha+l}S_1(l+1,m)B_{n-\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\right)\frac{t^n}{n!}. \quad (58)$$

Thus, by comparing the coefficients on both sides of (58), we obtain the desired theorem. □

4 Conclusion

In this paper, we introduced the degenerate poly-Genocchi polynomials by using the modified degenerate polyexponential function. We expressed those polynomials and numbers in relation to: the degenerate poly-Bernoulli polynomials in Theorem 1; the degenerate Euler polynomials and the Stirling numbers of the first kind in Theorem 2; the degenerate Genocchi numbers and the Stirling numbers of the first kind in Theorem 3; the Stirling numbers of the first kind in Theorems 7, 8; the degenerate poly-Genocchi numbers and Bernoulli numbers of order n in Theorem 4; and the degenerate Stirling numbers of the second kind in Theorem 10. Furthermore, we defined the degenerate unipoly Genocchi polynomials and obtained some of their properties. Not to mention, we also obtained the identity for degenerate unipoly Genocchi polynomials and numbers for: the degenerate Genocchi numbers and the Stirling numbers of the first kind in Theorem 12; the degenerate Euler numbers and the Stirling numbers of the first kind in Theorem 13; the degenerate Bernoulli polynomials and the Stirling numbers of the first kind in Theorems 14.

It is important that the study of the degenerate version is widely applied not only to numerical theory and combinatorial theory, but also to symmetric identity, differential equations and probability theory. In particular, many symmetric identities have been studied for degenerate versions of many special polynomials [1, 5–12]. Genocchi numbers have been also extensively studied in many different branches of mathematics. The works of Genocchi numbers and their combinatorial relations have received much attention [13, 17–20]. With this in mind, as a future project, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, economics, and engineering as well as mathematics.

Acknowledgements
The authors would like to thank the referees for the detailed and valuable comments that helped improve the original manuscript in its present form. Also, the authors thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding
This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, the Ministry of Education (NRF-2018R1D1A1B07049584).

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
All authors reveal that there is no ethical problem in the production of this paper.

Competing interests
The authors declare no conflict of interest.

Consent for publication
All authors want to publish this paper in this journal.

Authors' contributions

HKK conceived of the framework and structed the whole paper. L-CJ and HKK checked the results of the paper and completed the revision of the article. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics Education, Daegu Catholic University, Gyeongsan, 38430, Republic of Korea. [2]Graduate School of Education, Konkuk University, Seoul, 143-701, Republic of Korea.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 May 2020 Accepted: 21 July 2020 Published online: 03 August 2020

References

1. Kim, T., Kim, D.S.: Degenerate polyexponential functions and degenerate Bell polynomials. J. Math. Anal. Appl. **487**(2), Article ID 124017 (2020)
2. Carlitz, L.: Degenerate Stirling, Bernoulli and Eulerian numbers. Util. Math. **15**, 51–88 (1979)
3. Kim, D.S., Kim, T., Kwon, J., Lee, H.: A note on λ-Bernoulli numbers of the second kind. Adv. Stud. Contemp. Math. (Kyungshang) **30**(2), 187–195 (2020)
4. Kim, D.S., Kim, T.: Higher-order Bernoulli and poly-Bernoulli mixed type polynomials. Georgian Math. J. **22**(2), 265–272 (2015)
5. Kim, T., Kim, D.S., Dolgy, D.V., Kwon, J.: Some identities on degenerate Genocchi and Euler numbers. Informatica **31**(4), 42–51 (2020)
6. Kim, D.S., Kim, T.: A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. **27**(2), 227–235 (2020)
7. Kim, T., Kim, D.S.: Some identities of extended degenerate *r*-central Bell polynomials arising from umbral calculus. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**(1), Article ID 1 (2020)
8. Kim, T., Kim, D.S.: A note on central Bell numbers and polynomials. Russ. J. Math. Phys. **27**(1), 76–81 (2020)
9. Kim, T., Kim, D.S., Kwon, J.K., Lee, H.S.: Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials. Adv. Differ. Equ. **2020**, Article ID 168 (2020)
10. Kim, T., Kim, D.S., Kim, H.Y., Jang, L.C.: Degenerate poly-Bernoulli numbers and polynomials. Informatica **31**(3), 2–8 (2020)
11. Kim, T., Kim, D.S., Kwon, J.K., Kim, H.Y.: A note on degenerate Genocchi and poly-Genocchi numbers and polynomials. J. Inequal. Appl. **2020**, Article ID 110 (2020)
12. Kim, T., Kim, D.S., Lee, H., Kwon, J.: Degenerate binomial coefficients and degenerate hypergeometric functions. Adv. Differ. Equ. **2020**, Article ID 115 (2020)
13. Kim, T.: Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **20**(1), 23–28 (2020)
14. Kurt, B.: Some identities for the generalized poly-Genocchi polynomials with the parameters a, b and c. J. Math. Anal. **8**(1), 156–163 (2017)
15. Kim, T., Kim, D.S., Jang, L.C., Lee, H.: Jindalrae and Gaenari numbers and polynomials in connection with Jindalrae–Stirling numbers. Adv. Differ. Equ. **2020**, Article ID 245 (2020)
16. Kim, T., Kim, D.S.: Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials. Symmetry **12**(6), Article ID 905 (2020)
17. Dere, R., Simsek, Y.: Genocchi polynomials associated with the Umbral algebra. Appl. Math. Comput. **218**(3), 756–761 (2011)
18. Araci, S.: Novel identities for q-Genocchi numbers and polynomials. J. Funct. Spaces Appl. **2012**, Article ID 214961 (2012)
19. Araci, S.: Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. Appl. Math. Comput. **233**, 599–607 (2014)
20. Araci, S., Acikgoz, M., Sen, E.: On the von Staudt–Clausen's theorem associated with q-Genocchi numbers. Appl. Math. Comput. **247**, 780–785 (2014)
21. Kaneko, M.: Poly-Bernoulli numbers. J. Théor. Nr. Bordx. **9**(1), 221–228 (1997)
22. Hardy, G.H.: On a class a functions. Proc. Lond. Math. Soc. **3**, 441–460 (1905)
23. Jonquière, A.: Note sur la serie $\sum_{n=1}^{\infty} \frac{x^n}{n^s}$. Bull. Soc. Math. Fr. **17**, 142–152 (1889)
24. Lewin, L.: Polylogarithms and Associated Functions. North-Holland, New York (1981). With a foreword by A.J. Van der Poorten
25. Zagier, D.: The Bloch–Wigner–Ramakrishnan polylogarithm function. Math. Ann. **286**(1–3), 613–624 (1990)
26. Bayad, A., Hamahata, Y.: Polylogarithms and poly-Bernoulli polynomials. Kyushu J. Math. **65**(1), 15–24 (2012)
27. Kim, D.S., Kim, T.: A note on polyexponential and unipoly functions. Russ. J. Math. Phys. **26**(1), 40–49 (2019)
28. Dolgy, D.V., Jang, L.-C.: A note on the polyexponential Genocchi polynomials and numbers. Symmetry **12**, 1007 (2020). https://doi.org/10.3390/sym12061007