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# Stability, bifurcation, and chaos control of a novel discrete-time model involving Allee effect and cannibalism

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## Abstract

This paper is related to some dynamical aspects of a class of predator–prey interactions incorporating cannibalism and Allee effects for non-overlapping generations. Cannibalism has been frequently observed in natural populations, and it has an ability to alter the functional response concerning prey–predator interactions. On the other hand, from dynamical point of view cannibalism is considered as a procedure of stabilization or destabilization within predator–prey models. Taking into account the cannibalism in prey population and with addition of Allee effects, a new discrete-time system is proposed and studied in this paper. Moreover, existence of fixed points and their local dynamics are carried out. It is verified that the proposed model undergoes transcritical bifurcation about its trivial fixed point and period-doubling bifurcation around its boundary fixed point. Furthermore, it is also proved that the proposed system undergoes both period-doubling and Neimark–Sacker bifurcations (NSB) around its interior fixed point. Our study demonstrates that outbreaks of periodic nature may appear due to implementation of cannibalism in prey population, and these periodic oscillations are limited to prey density only without leaving an influence on predation. To restrain this periodic disturbance in prey population density, and other fluctuating and bifurcating behaviors of the model, various chaos control methods are applied. At the end, numerical simulations are presented to illustrate the effectiveness of our theoretical findings.

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## 1 Introduction

The most interesting and fascinating topic of current research in mathematical biology is the inclusion of Allee effect as well as cannibalism in prey and predator population. The occurrence of Allee effect is the most essential phenomenon in the biological world, and it has been treated as the crucial and extremely significant factor in ecology and population dynamics [1]. Initially, in 1930, the famous ecologist Allee illustrated the Allee effect at low

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population densities, which has been acknowledged as a prominent factor of positive density dependence in low-density population [2, 3]. The existence of Allee effect represents that it is mandatory for a population to sustain at least a minimum size of population itself in the natural world. There exist numerous populations in the universe, in which Allee effect has been extensively investigated, including insects [4], birds and mammals [5], plants [6], and marine invertebrates [7]. Attention to the interaction among small size populations such as the Allee effect has developed quickly in recent decades [8–12], including predator–prey interaction models. Recently, the development in this era has proved that the inclusion of Allee effect factor in a predator–prey model affects the dynamics of the system and may be the cause of destabilization, but it depends upon where Allee effects are attached [13]. For more interesting dynamical results related to Allee effects, we refer to [14–18] and the references therein.

On the other hand, cannibalism is also an important and intriguing topic in the case of predator–prey interaction, and it plays a key role in the dynamics of such interaction. Among humans, the motivations for cannibalism factor can vary as in human populations it has been documented all around the world. It has been practiced as a social norm in various indigenous South American, African, and New Guinean tribes [19]. It has also been practiced in Northern India among a sect of ascetics or witch doctors “Aghoris” in the hope of achieving immortality. The emphasizing behavior of cannibalism has been observed in a substantial diversity of animals organized as noncarnivorous insects, flour beetles, spider, fish, and locusts [13, 20–23]. Generally, the cannibals and their sufferers are in various maturation stages of life such as adult and teenage, immature and mature, and diverse sorts of categories. This occurrence throws back a predator–prey interaction within the identical species, and the equivalent mathematical models are dissimilar structurally from the predator–prey models only for different species [24, 25]. Polis [25] has tremendous contribution related to cannibalism and has mentioned around thirteen hundred different species involving this factor. In the case of predator–prey interaction, the inclusion of cannibalism is considered as a mechanics of natural selection which is in fact a familiar phenomenon [26]. According to many ecologists and biologists, the behavior of population dynamics has been extremely affected in response to the impact of cannibalism, and these incorporate the lifeboat instrument, where the affected role of cannibalism precedes to perseverance in population destruction [27]. It can also be helpful to get stability in cycling populations [28]. In numerous species, cannibalism factor appears when minimal resources are available corresponding to high level population densities of the species [29]. In the beginning, cannibalism was stimulated as an impact of obstruction in the predator population only, and accordingly it is occasionally competition mediated [28, 30, 31]. Despite ecological support in experimental work as well as in field work, this phenomenon is often observable in prey population [32–34]. However, the experimental work immensely encourages researchers to formulate some innovative ideas in the present scenario. These experimental findings greatly inspire to develop new ideas in current research. A comprehensive study of the present and previous mathematical surveys related to cannibalism indicates that the appearance of cannibalism in population models has many applications. These models comprise ODE, PDE, and discrete models representing two and three species population models, where cannibalism is involved in both prey and predator populations, a ratio-dependent type functional response, and some recent emerging developed models incorporating diseased predators associated with cannibalism [27, 28, 35–39]. In

[40] the authors studied a well-known Lotka–Volterra model involving cannibalism in a predator population and discussed the stability analysis of the proposed model. Additionally, these investigations reveal how stability has been affected by cannibalism. Zhang *et al.* [41] proposed a new method based on non-dimensionalization and applied it on a stage-structure model involving the predator cannibalism factor. The authors also studied the dynamics of the predator–prey model of stage structure including global stability analysis, subcritical and supercritical Hopf bifurcation along with biological meaning of parameters involved in the system.

For the purpose of investigating a class of population models related to non-overlapping generation, Danca *et al.* [42] discussed the following system:

$$\begin{cases} x_{n+1} := rx_n(1 - x_n) - \alpha x_n y_n, \\ y_{n+1} := dx_n y_n, \end{cases} \tag{1}$$

where  $x_n$  and  $y_n$  represent prey and predator population respectively with  $n$ th generation. Furthermore,

- $r$  represents intrinsic growth rate of prey;
- $\alpha$  indicates per capita searching efficiency;
- $d$  is the conversion rate of predator.

Taking into account the rate of natural death  $c$  for predator, system (1) takes the form [43]

$$\begin{cases} x_{n+1} := rx_n(1 - x_n) - \alpha x_n y_n, \\ y_{n+1} := dx_n y_n - cy_n. \end{cases} \tag{2}$$

Recently, Shabbir *et al.* [44] discussed the dynamical complexity of model (2) along with prey cannibalism. Moreover, Seval Işık [45] further modified (2) with addition of Allee effect in prey equation and stated it as follows:

$$\begin{cases} x_{n+1} := rx_n(1 - x_n) - \alpha x_n y_n \left(\frac{x_n}{x_n + m}\right), \\ y_{n+1} := dx_n y_n - cy_n, \end{cases} \tag{3}$$

where constant term  $m$  imposed on the prey equation is known as Allee constant.

At present, some remarkable continued work related to the modification of system (3) including asymptotic stability, bifurcation analysis, and chaos control study has been carried out. Liu [46] examined the existence of periodic solutions for a discrete semi-ratio-dependent predator–prey system. Moreover, the permanence and existence of unique uniformly asymptotic stability of positive almost periodic solutions in a discrete predator–prey system with time delays were determined in [47]. Din [48] considered a Leslie–Gower predator–prey model and studied bifurcation along with feedback control methodologies to control chaos and bifurcation. For further similar fascinating results related to discrete-time predator–prey models, we refer to [49–54] and the references therein.

In this article, our aim is to discuss the dynamics of our proposed model developed from the inclusion of cannibalism in prey population of discrete-time predator–prey model (3)

and is expressed by

$$\begin{cases} x_{n+1} := rx_n(1 - x_n) + bx_n - \frac{\alpha x_n^2 y_n}{x_n + m} - \frac{\beta x_n^2}{x_n + \gamma}, \\ y_{n+1} := dx_n y_n - cy_n. \end{cases} \tag{4}$$

Clearly, the addition of the term  $\beta \times x \times \frac{x}{x+\gamma}$  is known as a generic cannibalism factor. The cannibalism rate is denoted by  $\beta$ , whereas prey cannibalism has Holling-II type functional response. The term  $bx$  represents the birth rate of prey, and the condition  $\beta > b$  is imposed because it takes depredation of prey. Note that the  $x(t)$  population is depredating on its own species.

The rest of this manuscript can be summarized as follows: Sect. 2 deals with the existence of biologically possible equilibria and the conditions of asymptotic stability. Section 3 is associated with the study of bifurcation analysis for system (4). OGY and hybrid control methods are utilized in Sect. 4. Finally, extensive numerical simulations are imposed in Sect. 5 to justify our analytical results.

### 2 Stability analysis of steady-states

This section is dedicated for the exploration of local stability analysis of system (4). To investigate the solution of system (4), we consider the following algebraic system:

$$\begin{aligned} x &= rx(1 - x) + bx - \frac{\alpha x^2 y}{x + m} - \frac{\beta x^2}{x + \gamma}, \\ y &= dxy - cy. \end{aligned}$$

Simple computations yield the following equilibria for system (4):

$$E^0 = (0, 0), \quad E^1 = (k, 0), \quad E^* = (x^*, y^*),$$

where

$$\begin{aligned} k &:= \frac{r(1 - \beta) - (1 + \alpha) + c + \sqrt{4r(c + r - 1)\beta + (r(1 - \beta) - (1 + \alpha) + c)^2}}{2r}, & x^* &:= \frac{1 + c}{d}, \\ y^* &:= \frac{(1 + c + dm)(d^2(b + r - 1)\gamma + (1 + c)d(b + r - \beta - r\gamma - 1) - (1 + c)^2 r)}{(1 + c)d\alpha(1 + c + d\gamma)}. \end{aligned}$$

Moreover, trivial equilibrium  $E^0$  always exists, the boundary equilibrium  $E^1$  exists only for  $k > 0$ , that is,  $k$  is the solution of  $rx^2 + (1 + \beta + r\gamma - b - r)x + \gamma - r\gamma - b\gamma = 0$  with  $b + r > 1$ , and unique positive equilibrium  $E^*$  exists only for  $\gamma d^2 + (1 + c)(1 + \beta + \gamma r)d + (1 + c)^2 r < (b + r)(1 + c + \gamma d)d$ . The variational matrix at any arbitrary point  $(x, y)$  of system (4) is given by

$$V(x, y) = \begin{bmatrix} b + r - 2rx + x\left(\frac{x\beta}{(x+\gamma)^2} - \frac{(2m+x)y\alpha}{(m+x)^2} - \frac{2\beta}{x+\gamma}\right) & \frac{-\alpha x^2}{x+m} \\ dy & dx - c \end{bmatrix}.$$

Assume that  $R^*$  is any arbitrary solution of system (4) with Jacobian

$$V(R^*) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

and the characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^2 - I\lambda + \wp,$$

where  $I = \alpha_{11} + \alpha_{22}$  and  $\wp = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$ , then Lemma 1 gives insight about the exploration of local stability analysis of system (4).

**Lemma 1** ([55]) *Let  $\mathcal{P}(\lambda) = \lambda^2 - I\lambda + \wp$  and  $\mathcal{P}(1) > 0$ . Moreover,  $\lambda_1, \lambda_2$  are roots of equation  $\mathcal{P}(\lambda) = 0$ , then:*

- (i)  $|\lambda_1| < 1$  and  $|\lambda_2| < 1$  if and only if  $\mathcal{P}(-1) > 0$  and  $\wp < 1$ ;
- (ii)  $|\lambda_1| > 1$  and  $|\lambda_2| > 1$  if and only if  $\mathcal{P}(-1) > 0$  and  $\wp > 1$ ;
- (iii)  $|\lambda_1| < 1$  and  $|\lambda_2| > 1$  or  $(|\lambda_1| > 1$  and  $|\lambda_2| < 1)$  if and only if  $\mathcal{P}(-1) < 0$ ;
- (iv)  $\lambda_1 = -1$  and  $|\lambda_2| \neq 1$  if and only if  $\mathcal{P}(-1) = 0$  and  $I \neq 0, 2$ ;
- (v)  $\lambda_1, \lambda_2$  are complex conjugates with  $|\lambda_1| = 1 = |\lambda_2|$  if and only if  $I^2 - 4\wp < 0$  and  $\wp = 1$ .

Since  $\lambda_1$  and  $\lambda_2$  are eigenvalues of system (4), we have elaborated the following topological findings interconnected to the stability of  $R^*$ .  $R^*$  is known as sink if  $|\lambda_1| < 1$  and  $|\lambda_2| < 1$ , as sink is the point of suction which is stable. The equilibrium point  $R^*$  is recognized as source if  $|\lambda_1| > 1$  and  $|\lambda_2| > 1$ , as it always remains unstable. The equilibrium  $R^*$  is saddle if  $|\lambda_1| > 1$  and  $|\lambda_2| < 1$  or vice versa ( $|\lambda_1| < 1$  and  $|\lambda_2| > 1$ ), whereas it is non-hyperbolic if conditions (iv) and (v) from Lemma 1 are fulfilled.

It can be easily observed that the trivial equilibrium  $E^0 = (0, 0)$  of system (4) has eigenvalues  $b + r$  and  $-c$ , then the following assumptions hold:

- $(0, 0)$  is a sink if and only if  $b + r \in (0, 1)$  and  $c \in (0, 1)$ ;
- $(0, 0)$  is a source if and only if  $b + r > 1$  and  $c > 1$ ;
- $(0, 0)$  represents a saddle point if and only if  $b + r > 1$  and  $c < 1$  and vice versa;
- $(0, 0)$  is non-hyperbolic for  $b + r = 1$  or  $c = 1$ .

Also, for  $r = 0.1$ , the topological classification of trivial equilibrium in  $bc$ -plane is plotted in Fig. 1(a).

Furthermore, at boundary equilibrium point  $E^1 = (k, 0)$ ,  $\mathcal{V}(k, 0)$  is computed as follows:

$$\mathcal{V}(k, 0) := \begin{bmatrix} b + r - 2kr - \frac{k\beta(k+2\gamma)}{(k+\gamma)^2} & -\frac{k^2\alpha}{k+m} \\ 0 & dk - c \end{bmatrix}.$$

Furthermore, the following topological results for boundary equilibrium are satisfied.

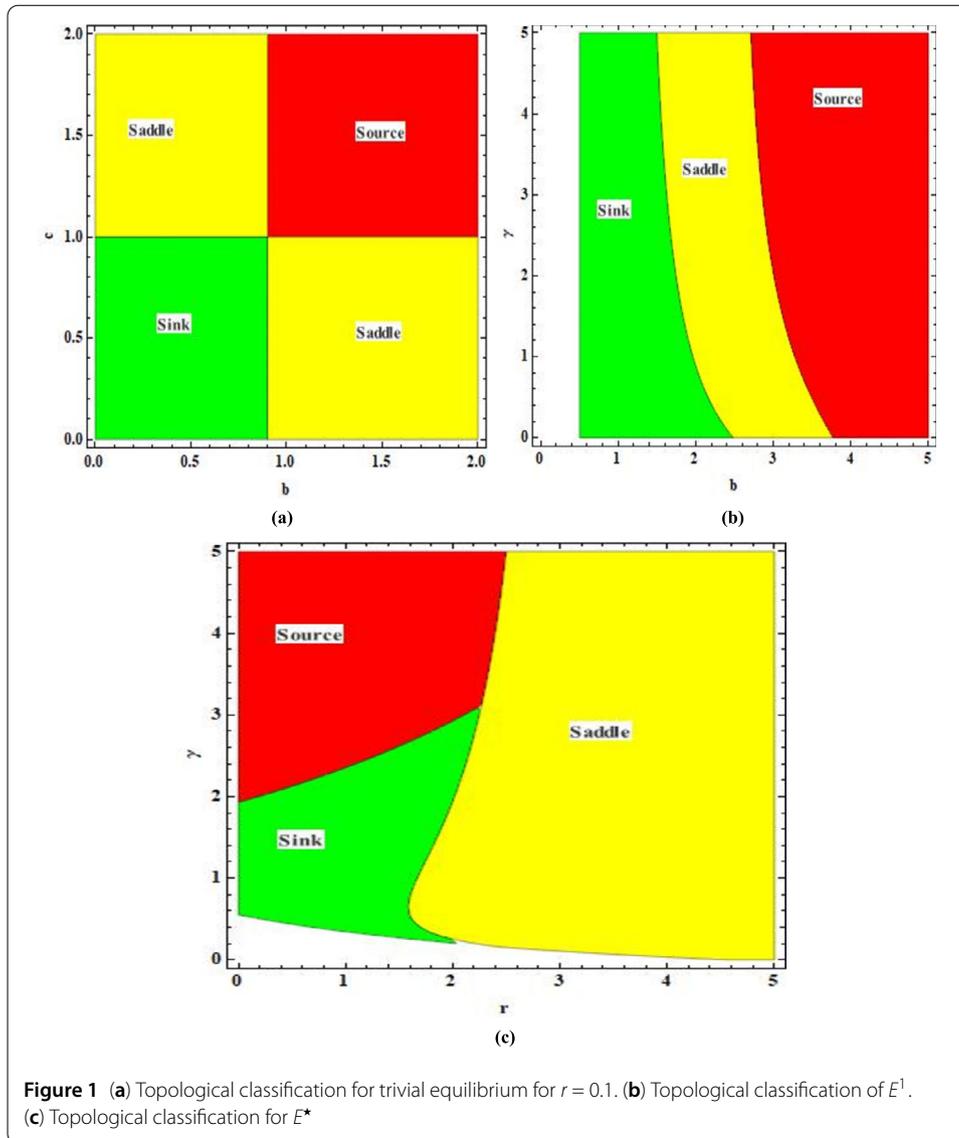
**Theorem 1** *Suppose that  $b + r > 1$  and  $\Psi = \frac{k\beta(k+2\gamma)}{(k+\gamma)^2}$ , then the following results hold:*

- $E^1$  is a sink  $\Leftrightarrow 2kr + \Psi < b + r + 1 < 2(1 + kr) + \Psi$  &  $0 < dk < 1 + c$ .
- $E^1$  is a source  $\Leftrightarrow 1 + b + r > 2(1 + kr) + \Psi$  &  $dk > 1 + c$ .
- $E^1$  is a saddle point  $\Leftrightarrow 2kr + \Psi < b + r + 1 < 2(1 + kr) + \Psi$  &  $dk > 1 + c$ .
- $E^1$  is non-hyperbolic  $\Leftrightarrow b + r = 1 + 2kr + \Psi$  or  $dk = 1 + c$ .

Taking  $(\beta, r, c, d) = (1.27, 0.5, 0.82, 1.27)$ , the topological classification for boundary equilibrium is depicted in Fig. 1(b) in  $b\gamma$ -plane.

Moreover,  $\mathcal{V}(x, y)$  about interior equilibrium  $E^*$  is expressed by

$$\mathcal{V}(E^*) := \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix},$$



where

$$\begin{aligned}
 u_{11} &:= -\frac{(1+c)^4 r + (1+c)d^3(2m(-2+b+r) - \gamma)\gamma + d^4 m(-2+b+r)\gamma^2}{d(1+c+dm)(1+c+d\gamma)^2} \\
 &\quad - \frac{(1+c)^3 d(-1+2r\gamma) + (1+c)^2 d^2(m(-2+b+r-\beta) + \gamma(-2+\beta+r\gamma))}{d(1+c+dm)(1+c+d\gamma)^2}, \\
 u_{21} &:= \frac{(1+c+dm)(-(1+c)^2 r + d^2(-1+b+r)\gamma + (1+c)d(-1+b+r-\beta-r\gamma))}{(1+c)\alpha(1+c+d\gamma)}, \\
 u_{12} &:= -\frac{(1+c)^2 \alpha}{d(1+c+dm)}; \quad u_{22} := 1.
 \end{aligned}$$

The characteristic equation of Jacobian  $V(E^*)$  is given by

$$\mathcal{F}(\lambda) := \lambda^2 - (u_{11} + u_{22})\lambda + u_{11}u_{22} - u_{12}u_{21}. \tag{5}$$

By performing simple algebraic calculations and letting  $b + r > 1$  and  $d(b + r) > (1 + c)r + d(1 + \beta + r\gamma)$ , we get

$$\begin{aligned} \mathcal{F}(1) &:= \frac{(1 + c)(d^2(b + r - 1)\gamma + d(1 + c)(b + r - \beta - r\gamma - 1) - (1 + c)^2r)}{d(1 + c + d\gamma)}, \\ \mathcal{F}(-1) &:= \frac{d^2m(5 + (c - 1)(b + r) - c) - (1 + c)^2(3 + c)r}{d(1 + c + dm)} + \frac{2(1 + c)^2\beta}{(1 + c + d\gamma)^2} \\ &\quad - \frac{(1 + c)^2(3 + c + dm)\beta}{(1 + c + dm)(1 + c + d\gamma)} + \frac{(1 + c)(3 + (1 + c)(b - (m - 1)r) - c)}{1 + c + dm}. \end{aligned} \tag{6}$$

From (6), we see that  $\mathcal{F}(1) > 0$ . Therefore, the following topological classification can be made by applying Lemma 1.

**Theorem 2** *Assume that  $(1 + c)(d(1 + \beta + r\gamma) + (1 + c)r) + d^2\gamma < d(b + r)(1 + c + d\gamma)$  and put  $1 + c + dm = \eta$ ,  $1 + c + d\gamma = \xi$ , and  $1 + c = \Omega$  such that the equilibrium point  $E^* = (x^*, y^*)$  of map (4) exists, then the following findings remain accurate:*

(a)  $E^*$  is asymptotically stable  $\Leftrightarrow$

$$\left. \begin{aligned} &d\Omega^2(1 + mr + \xi\beta(2 + \eta)) + d^2m(1 + 2(b + r)) + \xi^2\Omega^2r(\Omega + 2) \\ &< 2d\Omega^2\beta\eta + \xi^2\{d^2m(\Omega(1 + b + r) + 5) + d\Omega(\Omega(b + r) + 4)\}; \end{aligned} \right\}$$

and

$$\left. \begin{aligned} &d[\beta\eta\Omega^2 + \xi^2\{\Omega(1 + b\Omega) + (2 + (b + r)\Omega)md\}] \\ &< r(d^2m + \Omega^2(1 + d + \Omega)) + d\xi\{\xi(\eta + dm(b + \Omega)) + \Omega^2(1 + \xi + \eta\beta)\}. \end{aligned} \right\}$$

(b)  $E^*$  is an unstable equilibrium point  $\Leftrightarrow$

$$\left. \begin{aligned} &d\Omega^2(1 + mr + \xi\beta(2 + \eta)) + d^2m(1 + 2(b + r)) + \xi^2\Omega^2r(\Omega + 2) \\ &< 2d\Omega^2\beta\eta + \xi^2\{d^2m(\Omega(1 + b + r) + 5) + d\Omega(\Omega(b + r) + 4)\} \end{aligned} \right\}$$

and

$$\left. \begin{aligned} &d\xi\{\xi(\eta + dm(b + \Omega)) + \Omega^2(1 + \xi + \eta\beta)\} + r(d^2m + \Omega^2(1 + d + \Omega)) \\ &< d[\beta\eta\Omega^2 + \xi^2\{\Omega(1 + b\Omega) + (2 + (b + r)\Omega)md\}]. \end{aligned} \right\}$$

(c)  $E^*$  is a saddle point if and only if

$$\left. \begin{aligned} &2d\Omega^2\beta\eta + \xi^2\{d^2m(\Omega(1 + b + r) + 5) + d\Omega(\Omega(b + r) + 4)\} \\ &< d\Omega^2(1 + mr + \xi\beta(2 + \eta)) + d^2m(1 + 2(b + r)) + \xi^2\Omega^2r(\Omega + 2). \end{aligned} \right\}$$

(d)  $E^*$  is non-hyperbolic  $\Leftrightarrow u_{11} + u_{22} \neq 0, 2$  and

$$r := - \frac{d\eta(-5 + c + \frac{2\Omega}{\eta} + b(1 - c - \frac{2\Omega}{\eta}) - \frac{2\Omega^2\beta}{\xi^2} + \frac{\Omega^2(2+\eta)\beta}{\eta\xi})}{(1 + c)^2(3 + c - d) + d(1 + c(2 + c - d) + d)m}$$

or

$$|u_{11} + u_{22}| \leq 2, \quad \text{and}$$

$$\beta := -\frac{(\Omega^2(2+c)r - cd^2m(b+r-1) - \Omega^2d(b+r-mr-1))\xi^2}{\Omega^2d(1+(1+\eta)(c+d\gamma))}.$$

Furthermore, taking  $(\beta, b, c, d, m) = (4.47, 3.07, 0.5, 3.15, 4.4)$  topological classification for positive equilibrium is depicted in Fig. 1(c).

### 3 Bifurcation analysis

This section is devoted to the study of bifurcation in which three different types of bifurcations are investigated. We explore transcritical bifurcation, periodic-doubling bifurcation, and Neimark–Sacker bifurcation of system (4) at  $E^0, E^1$ , and  $E^*$ .

#### 3.1 Transcritical bifurcation at $E^0$

In this section, our claim is that fixed point  $E^0$  undergoes transcritical bifurcation. Hence we assume that

$$r \equiv r^0 := 1 - b$$

and consider the set

$$B_T := \{(r^0, b, c, d, m, \alpha, \beta, \gamma) \in \mathbb{R}_+^8 : r^0 := 1 - b\}.$$

As  $(r^0, b, c, d, m, \alpha, \beta, \gamma) \in B_T$ , then (4) is alternatively described by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} (r^0 + \hat{r})x(1-x) + bx - \frac{\alpha x^2 y}{x+m} - \frac{\beta x^2}{x+\gamma} \\ dxy - cy \end{pmatrix},$$

where parameter  $\hat{r}$  represents a very small perturbation in  $r^0$ . Therefore, an application of the Taylor series expansion about  $(x, y, \hat{r}) = (0, 0, 0)$  yields

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} b + \hat{r} & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_0(x, y, \hat{r}) \\ g_0(x, y) \end{pmatrix}, \tag{7}$$

where

$$\begin{cases} f_0(x, y, \hat{r}) := -(\hat{r} + \frac{\beta}{\gamma})x^2 - \frac{\alpha}{m}x^2y + \frac{\beta}{\gamma^2}x^3 + x\hat{r} + O((|x| + |y| + |\hat{r}|)^4), \\ g_0(x, y, \hat{r}) := dxy. \end{cases}$$

The linear portion of map (7) is in a canonical form as  $r^0 := 1 - b$ . Consequently, the implementation of center manifold  $W^c(0, 0, 0)$  for map (7) is approximated by

$$W^c(0, 0, 0) := \{u, v, \hat{r} \in \mathbb{R}^3 : v = h_1u^2 + h_2u\hat{r} + h_3\hat{r}^2 + O((|u| + |\hat{r}|)^3)\},$$

where  $h_1 = h_2 = h_3 = 0$ .

Additionally, we introduce the map restricted to the center manifold  $\mathcal{W}^c(0, 0, 0)$ :

$$F : u \rightarrow u + \hat{r} + k_1 u^2 + k_2 u \hat{r} + k_3 \hat{r}^2 + O((|u| + |\hat{r}|)^4),$$

where  $k_1 = a - 1 - \frac{\beta}{\gamma}$ ,  $k_2 = 1$ ,  $k_3 = 0$ .

Now, here we establish  $\mathcal{L}_1 \neq 0$  and  $\mathcal{L}_2 \neq 0$  as follows:

$$\begin{cases} \mathcal{L}_1 := (\frac{\partial^2 F}{\partial u^2})_{(0,0)} = 2(b - 1 - \frac{\beta}{\gamma}) \neq 0, \\ \mathcal{L}_2 := (\frac{\partial^2 F}{\partial u \partial \hat{r}})_{(0,0)} = 1 \neq 0. \end{cases}$$

Thus, we can state the following theorem related to transcritical bifurcation.

**Theorem 3** *Suppose that  $r = 1 - b$  and  $b - 1 - \frac{\beta}{\gamma} \neq 0$ , then (4) undergoes transcritical bifurcation at its trivial equilibrium  $E^0$ .*

### 3.2 Period-doubling bifurcation at $E^1$

In the present section, we study period-doubling bifurcation about  $E^1 = (k, 0)$ . Therefore the Jacobian matrix of corresponding system (4) with respect to  $E^1$  is given by

$$\mathbb{J}(E^1) := \begin{pmatrix} b + r - 2kr - \frac{k\beta(k+2\gamma)}{(k+\gamma)^2} & -\frac{k^2\alpha}{k+m} \\ 0 & dk - c \end{pmatrix}.$$

Using Lemma 1 for the case of non-hyperbolic steady-state, when one eigenvalue  $\lambda_1 = -1$  implies that

$$\beta := \frac{(1 + b + r - 2kr)(k + \gamma)^2}{k(k + 2\gamma)},$$

we consider the set

$$B_{P_{\mathbb{B}}} := \left\{ (r, b, c, d, m, \alpha, \beta, \gamma) \in \mathcal{R}_+^8 : \beta := \frac{(1 + b + r - 2kr)(k + \gamma)^2}{k(k + 2\gamma)} \right\}.$$

Moreover, suppose that  $(r, b, c, d, m, \alpha, \beta, \gamma) \in B_{P_{\mathbb{B}}}$ . Then the boundary fixed point  $E^1$  of system (4) sustains period-doubling bifurcation whenever  $\beta$  is chosen as a bifurcation parameter and it varies in a small neighborhood of  $\beta_1 := \frac{(1+b+r-2kr)(k+\gamma)^2}{k(k+2\gamma)}$ . Therefore, in terms of parameters  $(\beta_1, r, b, c, d, m, \alpha, \gamma)$ , system (4) can be demonstrated as follows:

$$\begin{pmatrix} H \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} rH(1 - H) + bH - \frac{\alpha H^2 Z}{H+m} - \frac{\beta_1 H^2}{H+\gamma} \\ dHZ - cZ \end{pmatrix}. \tag{8}$$

After a small perturbation  $\tilde{\beta}$  in  $\beta_1$ , map (8) can be rearranged as

$$\begin{pmatrix} H \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} rH(1 - H) + bH - \frac{\alpha H^2 Z}{H+m} - \frac{(\beta_1 + \tilde{\beta}) H^2}{H+\gamma} \\ dHZ - cZ \end{pmatrix}, \tag{9}$$

where  $|\tilde{\beta}| \ll 1$ . Suppose that  $x = H - k$  and  $y = Z$ , then map (9) is reshaped as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x, y, \tilde{\beta}) \\ g_1(x, y) \end{pmatrix}, \tag{10}$$

where

$$\begin{cases} f_1(x, y, \tilde{\beta}) := c_{13}x^2 + c_{14}xy + c_{15}x^3 + c_{16}x^2y + c_{17}x\tilde{\beta} + c_{18}x^2\tilde{\beta} + O((|x| + |y| + |\tilde{\beta}|)^4), \\ g_1(x, y) := c_{23}xy; \\ c_{11} := b + r - 2kr - \frac{k\tilde{\beta}(k+2\gamma)}{(k+\gamma)^2}; \quad c_{12} := -\frac{k^2\alpha}{k+m}; \quad c_{21} := 0; \quad c_{22} := dk - c; \\ c_{13} := -r - \frac{\tilde{\beta}}{\gamma} + \frac{4k\tilde{\beta}}{\gamma^2} - \frac{4k^2\tilde{\beta}}{\gamma^3}; \quad a_{14} := \frac{\alpha k(k+2m)}{(k+m)^2}; \quad a_{15} := \frac{2\tilde{\beta}(\gamma^2 - 4k\gamma + 4k^2)}{\gamma^4}; \\ a_{16} := \frac{-\alpha m^2}{(k+m)^3}; \quad a_{17} := -\frac{2k(\gamma-k)}{\gamma^2}; \quad a_{18} := -\frac{\gamma^2 + 4k^2 - 4k\gamma}{\gamma^3}; \quad c_{23} := d. \end{cases}$$

Additionally, we establish the following translation map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow T \begin{pmatrix} u \\ v \end{pmatrix}, \tag{11}$$

where  $T := \begin{pmatrix} c_{12} & c_{12} \\ -c_{11}-1 & \lambda_2 - c_{11} \end{pmatrix}$  is a nonsingular matrix. Map (11) under translation (10) can be prepared as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_2(u, v, \tilde{\beta}) \\ g_2(u, v, \tilde{\beta}) \end{pmatrix}, \tag{12}$$

where

$$\begin{cases} f_2(u, v, \tilde{\beta}) := \left(\frac{(\lambda_2 - c_{11})c_{15}}{(\lambda_2 + 1)c_{12}}\right)x^3 + \left(\frac{(\lambda_2 - c_{11})c_{16}}{(\lambda_2 + 1)c_{12}}\right)x^2y + \left(\frac{(\lambda_2 - c_{11})(\tilde{\beta}c_{18} + c_{13})}{c_{12}(\lambda_2 + 1)}\right)x^2 \\ \quad + \left(\frac{(\lambda_2 - c_{11})c_{14}}{c_{12}(\lambda_2 + 1)} - \frac{c_{23}}{\lambda_2 + 1}\right)xy + \left(\frac{(\lambda_2 - c_{11})\tilde{\beta}c_{17}}{c_{12}(\lambda_2 + 1)}\right)x + O((|u| + |v| + |\tilde{\beta}|)^4); \\ g_2(u, v, \tilde{\beta}) := \left(\frac{(1 + c_{11})c_{15}}{(1 + \lambda_2)c_{12}}\right)x^3 + \left(\frac{(1 + c_{11})c_{16}}{(1 + \lambda_2)c_{12}}\right)x^2y + \left(\frac{(1 + c_{11})(\tilde{\beta}c_{18} + c_{13})}{(\lambda_2 + 1)c_{12}}\right)x^2 \\ \quad + \left(\frac{(1 + c_{11})c_{14}}{(1 + \lambda_2)c_{12}} + \frac{c_{23}}{\lambda_2 + 1}\right)xy + \left(\frac{(1 + c_{11})\tilde{\beta}c_{17}}{(1 + \lambda_2)c_{12}}\right)x + O((|u| + |v| + |\tilde{\beta}|)^4) \end{cases}$$

and  $x := c_{12}(u + v)$ ;  $y := (\lambda_2 - c_{11})v - (1 + c_{11})u$ .

In addition, we execute the center manifold  $w^c(0, 0, 0)$  for map (12) computed at  $(0, 0)$  and within a small neighborhood of  $\tilde{\beta} = 0$ , then we have

$$w^c(0, 0, 0) := \{u, v, \tilde{\beta} \in R^3 : v = h_1u^2 + h_2u\tilde{\beta} + h_3\tilde{\beta}^2 + O((|u| + |\tilde{\beta}|)^3)\}, \quad \text{where}$$

$$h_1 := \frac{(1 + c_{11})(c_{11}c_{14} - c_{12}c_{13} + c_{12}c_{23} + c_{14})}{\lambda_2^2 - 1}, \quad h_2 := -\frac{(1 + c_{11})c_{17}}{\lambda_2^2 - 1}, \quad h_3 := 0.$$

Consequently, the restricted map to the center manifold  $w^c(0, 0, 0)$  is given by

$$F : u \rightarrow -u + s_1u^2 + s_2u\tilde{\beta} + s_3u^2\tilde{\beta} + s_4u\tilde{\beta}^2 + s_5u^3 + O((|u| + |\tilde{\beta}|)^4), \quad \text{where}$$

$$\begin{cases} s_1 := \left(\frac{c_{12}c_{23}}{\lambda_2 + 1} - \frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1}\right)(1 + c_{11}) + \left(\frac{\lambda_2 - c_{11}}{\lambda_2 + 1}\right)c_{12}c_{13}; \quad s_2 := \left(\frac{\lambda_2 - c_{11}}{\lambda_2 + 1}\right)c_{17}; \\ s_3 := \left(\frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1} - \frac{c_{12}c_{23}}{\lambda_2 + 1}\right)(\lambda_2 - c_{11})h_2 + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1}2c_{12}c_{13}h_2 + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1}c_{12}c_{18} \\ \quad + \left(\frac{c_{12}c_{23}}{\lambda_2 + 1} - \frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1}\right)(1 + c_{11})h_2 + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1}c_{17}h_1; \end{cases}$$

and

$$\begin{cases} s_4 := \left(\frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1} - \frac{c_{12}c_{23}}{\lambda_2 + 1}\right)(\lambda_2 - c_{11})h_3 + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} 2c_{12}c_{13}h_3 + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} c_{16}h_2 \\ \quad + \left(\frac{c_{12}c_{23}}{\lambda_2 + 1} - \frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1}\right)(1 + c_{11})h_3; \\ s_5 := \left(\frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1} - \frac{c_{12}c_{23}}{\lambda_2 + 1}\right)(\lambda_2 - c_{11})h_1 + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} c_{12}^2 c_{15} + \frac{\lambda_2 - c_{11}}{\lambda_2 + 1} 2c_{12}c_{13}h_1 \\ \quad + \left(\frac{c_{12}c_{23}}{\lambda_2 + 1} - \frac{(\lambda_2 - c_{11})c_{14}}{\lambda_2 + 1}\right)(1 + c_{11})h_1 + \frac{(c_{11} - \lambda_2)(1 + c_{11})c_{12}c_{16}}{\lambda_2 + 1}. \end{cases}$$

Now, here we establish once more that  $L_1$  and  $L_2$  both are nonzero:

$$\begin{cases} L_1 := \left(\frac{\partial^2 f}{\partial u \partial \beta} + \frac{1}{2} \frac{\partial F}{\partial \beta} \frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = \frac{(\lambda_2 - c_{11})c_{17}}{\lambda_2 + 1} \neq 0; \\ L_2 := \left(\frac{1}{6} \frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2} \frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = s_5 + s_1^2 \neq 0. \end{cases}$$

The aforementioned analysis yields the following conclusion.

**Theorem 4** *If  $L_1 \& L_2 \neq 0$ , then system (4) undergoes period-doubling bifurcation at  $E^1$  whenever  $\beta$  varies in a small neighborhood of  $\beta_1 := \frac{(1+b+r-2kr)(k+\gamma)^2}{k(k+2\gamma)}$ . Moreover, in the case of  $L_2 > 0$ , there exist period-two orbits which bifurcate from equilibrium  $E^1$ , which are stable orbits, whereas for  $L_2 < 0$  unstable orbits are generated.*

### 3.3 Period-doubling bifurcation at $E^*$

In this section, we discuss the existence of period-doubling bifurcation in (4) at equilibrium  $E^*$ . In addition, the characteristic equation of the Jacobian matrix about  $E^* = (x^*, y^*)$  is expressed by

$$\begin{aligned} \mathcal{F}(\lambda) &:= \lambda^2 - p(x^*, y^*)\lambda + q(x^*, y^*), \quad \text{where} \\ \begin{cases} p(x^*, y^*) &:= \frac{(1+c)^2\beta}{(1+c+d\gamma)^2} - \frac{(1+c)^2r+d^2m(b+r-3)-2(1+c)d}{d(1+c+dm)} \\ &\quad - \frac{(1+c)^2\beta}{(1+c+dm)(1+c+d\gamma)}; \\ q(x^*, y^*) &:= \frac{(1+c)(b+(-1+b)c)+(1+(-1+b)c)dm}{(1+c+dm)} + \frac{(1+c)^2\beta}{(1+c+d\gamma)^2} \\ &\quad + \frac{((1+c)^2(2+c-d)+d((1+c)^2-cd)m)r}{d(1+c+dm)} - \frac{(1+c)^2(2+c+dm)\beta}{(1+c+dm)(1+c+d\gamma)}. \end{cases} \end{aligned} \tag{13}$$

Suppose that

$$p^2(x^*, y^*) > 4q(x^*, y^*), \tag{14}$$

and  $\mathcal{F}(-1) = 0$ , then it follows that

$$r := -\frac{d(1+c+dm)\left(-5+c+\frac{2(1+c)}{1+c+dm}+b\left(1-c-\frac{2(1+c)}{1+c+dm}\right)-\frac{2(1+c)^2\beta}{(1+c+d\gamma)^2}+\frac{(1+c)^2(3+c+dm)\beta}{(1+c+dm)(1+c+d\gamma)}\right)}{(1+c)^2(3+c-d)+d(1+c(2+c-d)+d)m}. \tag{15}$$

From Eq. (13), assume that  $\mathcal{F}(\lambda) = 0$ . Then one root is  $\lambda_1 = -1$  and the condition  $|\lambda_2| \neq 1$  implies that

$$q(x^*, y^*) \neq \pm 1. \tag{16}$$

Again we consider the following set:

$$B_{P_U} := \{ (r, b, c, d, m, \alpha, \beta, \gamma) \in \mathcal{R}_+^8 : (14), (15), \text{ and } (16) \text{ are fulfilled} \}.$$

The fixed point  $E^*$  of map (4) undergoes period-doubling bifurcation whenever the parameter  $r$  varies in a small neighborhood of the set  $B_{P_U}$ .

Let

$$r_1 := - \frac{d(1+c+dm)(-5+c+\frac{2(1+c)}{1+c+dm}+b(1-c-\frac{2(1+c)}{1+c+dm})-\frac{2(1+c)^2\beta}{(1+c+d\gamma)^2}+\frac{(1+c)^2(3+c+dm)\beta}{(1+c+dm)(1+c+d\gamma)})}{(1+c)^2(3+c-d)+d(1+c(2+c-d)+d)m},$$

and take the arbitrary parameters  $(r_1, b, c, d, m, \alpha, \beta, \gamma) \in B_{P_U}$ , then map (4) can be expressed by

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} r_1P(1-P) + bP - \frac{\alpha P^2 Q}{P+m} - \frac{\beta P^2}{P+\gamma} \\ dPQ - cQ \end{pmatrix}. \tag{17}$$

Assume a small perturbation  $\tilde{r}$  bifurcation parameter. Then map (17) can be examined by

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} (r_1 + \tilde{r})P(1-P) + bP - \frac{\alpha P^2 Q}{P+m} - \frac{\beta P^2}{P+\gamma} \\ dPQ - cQ \end{pmatrix}, \tag{18}$$

where  $|\tilde{r}| \ll 1$  is a small perturbation parameter.

Putting  $x = P - x^*$  and  $y = Q - y^*$ , then system (18) is transformed into the following form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_3(x, y, \tilde{r}) \\ g_3(x, y, \tilde{r}) \end{pmatrix}, \tag{19}$$

where

$$\begin{cases} f_3(x, y, \tilde{r}) := a_{13}x^2 + a_{14}xy + a_{15}x^3 + a_{16}x^2y + a_{17}x\tilde{r} + a_{18}x^2\tilde{r} + O((|x| + |y| + |\tilde{r}|)^4), \\ g_3(x, y, \tilde{r}) := a_{23}xy + a_{24}x\tilde{r} + O((|x| + |y| + |\tilde{r}|)^4), \end{cases}$$

$$\begin{cases} a_{11} := - \frac{(1+c)^4\tilde{r}+(1+c)d^3(2m(-2+b+\tilde{r})-\gamma)+d^4m(-2+b+\tilde{r})\gamma^2}{d(1+c+dm)(1+c+d\gamma)^2} \\ \quad - \frac{(1+c)^3d(-1+2\tilde{r}\gamma)+(1+c)^2d^2(m(-2+b+\tilde{r}-\beta)+\gamma(-2+\beta+\tilde{r}\gamma))}{d(1+c+dm)(1+c+d\gamma)^2}; \\ a_{21} := \frac{(1+c+dm)(-(1+c)^2\tilde{r}+d^2(-1+b+\tilde{r})\gamma+(1+c)d(-1+b+\tilde{r}-\beta-\tilde{r}\gamma))}{(1+c)\alpha(1+c+d\gamma)}, \\ a_{13} := \frac{2(d^2\gamma(b+\tilde{r}-1)+d(1+c)(b+\tilde{r}-\beta-\tilde{r}\gamma-1)-(1+c)^2\tilde{r})}{(1+c+dm)(1+c+d\gamma)} + \frac{2\beta d(1+c)}{(1+c+d\gamma)^2} \\ \quad - \frac{(1+c)(d^2\gamma(b+\tilde{r}-1)+d(1+c)(b+\tilde{r}-\beta-\tilde{r}\gamma-1)-(1+c)^2\tilde{r})}{(1+c+dm)^2(1+c+d\gamma)} - \frac{\beta d}{1+c+d\gamma} \\ \quad - \frac{d^2\gamma(b+\tilde{r}-1)+d(1+c)(b+\tilde{r}-\beta-\tilde{r}\gamma-1)-(1+c)^2\tilde{r}}{(1+c)(1+c+d\gamma)} - \frac{\beta d(1+c)^2}{(1+c+d\gamma)^3} - \tilde{r}; \\ a_{15} := \frac{d(1+c)(d^2\gamma(b+\tilde{r}-1)+d(1+c)(b+\tilde{r}-\beta-\tilde{r}\gamma-1)-(1+c)^2\tilde{r})}{(1+c+dm)^2(1+c+d\gamma)} \\ \quad + \frac{(d^2\gamma(b+\tilde{r}-1)+d(1+c)(b+\tilde{r}-\beta-\tilde{r}\gamma-1)-(1+c)^2\tilde{r})d}{(1+c+dm)(1+c+d\gamma)(1+c)} + \frac{\beta d^2(1+c)^2}{(1+c+d\gamma)^4} \\ \quad - \frac{2d(d^2\gamma(b+\tilde{r}-1)+d(1+c)(b+\tilde{r}-\beta-\tilde{r}\gamma-1)-(1+c)^2\tilde{r})}{(1+c+dm)^2(1+c+d\gamma)} + \frac{\beta d^2}{(1+c+d\gamma)^2} \\ \quad - \frac{2\beta d^2(1+c)}{(1+c+d\gamma)^3}; \quad a_{14} := - \frac{\alpha(1+c)(1+c+2dm)}{(1+c+dm)^2}; \quad a_{16} := - \frac{\alpha m^2 d^3}{(1+c+dm)^3}; \\ a_{18} := - \frac{1+3c+2dm+4cdm+3c^2+c^3+2c^2dm+d^3m^2}{(1+c+dm)^2(1+c)}; \quad a_{22} := 1; \quad a_{23} := d; \\ a_{24} := - \frac{(1+c+dm)(d^2\gamma+d(1+c)(1-\gamma)-(1+c)^2)}{\alpha(1+c+d\gamma)(1+c)}; \quad a_{12} := - \frac{(1+c)^2\alpha}{d(1+c+dm)}; \\ a_{17} := - \frac{1+2c+c^2+md^2}{d(1+c+dm)}. \end{cases}$$

Now, we develop the following map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow T \begin{pmatrix} u \\ v \end{pmatrix}, \tag{20}$$

where  $T := \begin{pmatrix} a_{12} & a_{12} \\ -a_{11}-1 & \lambda_2-a_{11} \end{pmatrix}$  is an invertible matrix. Translation (20) under (19) can be written as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_4(u, v, \tilde{r}) \\ g_4(u, v, \tilde{r}) \end{pmatrix}, \tag{21}$$

where

$$\begin{cases} f_4(u, v, \tilde{r}) := \left(\frac{\lambda_2-a_{11}}{\lambda_2+1}a_{15}\right)x^3 + \left(\frac{\lambda_2-a_{11}}{a_{12}(\lambda_2+1)}(\tilde{r}a_{18}+a_{13})\right)x^2 + \left(\frac{\lambda_2-a_{11}}{a_{12}(\lambda_2+1)}a_{16}\right)x^2y \\ \quad + \left(\frac{\lambda_2-a_{11}}{a_{12}(\lambda_2+1)}a_{14} - \frac{a_{24}}{\lambda_2+1}\right)xy + \left(\frac{\lambda_2-a_{11}}{a_{12}(\lambda_2+1)}\tilde{r}a_{17} - \frac{\tilde{r}a_{25}}{\lambda_2+1}\right)x + O((|u| + |v| + |\tilde{r}|)^4); \\ g_4(u, v, \tilde{r}) := \left(\frac{1+a_{11}}{a_{12}(\lambda_2+1)}a_{15}\right)x^3 + \left(\frac{1+a_{11}}{a_{12}(\lambda_2+1)}(\tilde{r}a_{18}+a_{13})\right)x^2 + \left(\frac{1+a_{11}}{a_{12}(\lambda_2+1)}a_{16}\right)x^2y \\ \quad + \left(\frac{1+a_{11}}{a_{12}(\lambda_2+1)}a_{14} + \frac{a_{24}}{\lambda_2+1}\right)xy + \left(\frac{1+a_{11}}{a_{12}(\lambda_2+1)}\tilde{r}a_{17} + \frac{\tilde{r}a_{25}}{\lambda_2+1}\right)x + O((|u| + |v| + |\tilde{r}|)^4) \end{cases}$$

and  $x := a_{12}(u + v)$ ;  $y := -(1 + a_{11})u + (\lambda_2 - a_{11})v$ .

Now, considering the center manifold  $\mathbb{W}^c(0, 0, 0)$  of (21) in a small neighborhood of  $\tilde{r} = 0$ , then  $\mathbb{W}^c(0, 0, 0)$  can be embellished by

$$\mathbb{W}^c(0, 0, 0) := \{u, v, \tilde{r} \in \mathbb{R}^3 : v = h_1u^2 + h_2u\tilde{r} + h_3\tilde{r}^2 + O((|u| + |\tilde{r}|)^3)\},$$

in which

$$h_1 := \frac{(1 + a_{11})(a_{11}a_{14} - a_{12}a_{13} + a_{12}a_{24} + a_{14})}{\lambda_2^2 - 1};$$

$$h_2 := -\frac{a_{11}a_{17} + a_{12}a_{25} + a_{17}}{\lambda_2^2 - 1}; \quad h_3 = 0.$$

Thus, the map restricted to  $\mathbb{W}^c(0, 0, 0)$  is prescribed as follows:

$$F : u \rightarrow -u + s_1u^2 + s_2u\tilde{r} + s_3u^2\tilde{r} + s_4u\tilde{r}^2 + s_5u^3 + O((|u| + |\tilde{r}|)^4),$$

where

$$\begin{cases} s_1 := \left(\frac{a_{12}a_{24}}{\lambda_2+1} - \frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1}\right)(1 + a_{11}) + \frac{(\lambda_2-a_{11})a_{12}a_{13}}{\lambda_2+1}; & s_2 := \frac{\lambda_2a_{17}-a_{11}a_{17}-a_{12}a_{25}}{\lambda_2+1}; \\ s_3 := \left(\frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1} - \frac{a_{12}a_{24}}{\lambda_2+1}\right)(\lambda_2 - a_{11})h_2 + \frac{2(\lambda_2-a_{11})a_{12}a_{13}h_2}{\lambda_2+1} + \frac{(\lambda_2-a_{11})a_{12}a_{18}}{\lambda_2+1} \\ \quad + \left(\frac{a_{12}a_{24}}{\lambda_2+1} - \frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1}\right)(1 + a_{11})h_2 + \left(\frac{\lambda_2-a_{11}}{\lambda_2+1}a_{17} - \frac{a_{12}a_{25}}{\lambda_2+1}\right)h_1; \\ s_4 := \left(\frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1} - \frac{a_{12}a_{24}}{\lambda_2+1}\right)(\lambda_2 - a_{11})h_3 + \left(\frac{a_{12}a_{24}}{\lambda_2+1} - \frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1}\right)(1 + a_{11})h_3 \\ \quad + \left(\frac{\lambda_2-a_{11}}{\lambda_2+1}a_{17} - \frac{a_{12}a_{25}}{\lambda_2+1}\right)h_2 + \frac{2(\lambda_2-a_{11})a_{12}a_{13}h_3}{\lambda_2+1}; \\ s_5 := \left(\frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1} - \frac{a_{12}a_{24}}{\lambda_2+1}\right)(\lambda_2 - a_{11})h_1 + \frac{(\lambda_2-a_{11})a_{12}^2a_{15}}{\lambda_2+1} + \frac{2(\lambda_2-a_{11})a_{12}a_{13}h_1}{\lambda_2+1} \\ \quad + \left(\frac{a_{12}a_{24}}{\lambda_2+1} - \frac{(\lambda_2-a_{11})a_{14}}{\lambda_2+1}\right)(1 + a_{11})h_1 + \frac{(\lambda_2-a_{11})a_{12}a_{16}(-1-a_{11})}{\lambda_2+1}. \end{cases}$$

Now, here we define two nonzero real numbers as follows:

$$\begin{cases} \mathbb{L}_1 := \left(\frac{\partial^2 f}{\partial u \partial r} + \frac{1}{2} \frac{\partial F}{\partial r} \frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = \frac{\lambda_2 a_{17} - a_{12} a_{25} - a_{11} a_{17}}{\lambda_2 + 1} \neq 0, \\ \mathbb{L}_2 := \left(\frac{1}{6} \frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2} \frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = s_5 + s_1^2 \neq 0. \end{cases}$$

Due to the aforementioned investigations, we state the following theorem related to period-doubling bifurcation.

**Theorem 5** *System (4) undergoes period-doubling bifurcation at  $E^*$  when the parameter  $r$  changes its values around a neighboring point of  $r_1$  whenever  $\mathbb{L}_1 \neq 0$  and  $\mathbb{L}_2 \neq 0$ . Furthermore, the period-two orbits that bifurcate from interior equilibrium  $E^*$  are stable whenever  $\mathbb{L}_2 > 0$  and unstable when  $\mathbb{L}_2 < 0$ .*

### 3.4 Neimark–Sacker bifurcation at $E^*$

This section consists of the existence criteria for Neimark–Sacker bifurcation around an interior fixed point by considering the rate of cannibalism  $\beta$  as a bifurcation parameter. For detailed analysis, we refer to the work done by the authors [56–59]. On the other hand, when Neimark–Sacker bifurcation exists, then as a result, dynamically closed curves appear and attracting steady-states are unstable as varied parameters move towards  $\beta$ . In return, we can discover some isolated orbits along with trajectories and with periodic behavior that thickly overlay these immutable closed curves [60]. In the case of non-hyperbolic fixed points, we have studied the conditions associated with system (4) and a pair of complex eigenvalues having unit modulus. For this, consider (13) and assume that  $\mathcal{F}(\lambda) = 0$  has two roots which are complex conjugate and fulfill the following conditions:

$$\beta := -\frac{((1+c)^2(2+c)r - cd^2m(-1+b+r) - (1+c)^2d(-1+b+r-mr))(1+c+d\gamma)^2}{(1+c)^2d(1+(2+c+dm)(c+d\gamma))} \tag{22}$$

and

$$\left| \frac{(1+c)^2r + d^2m(b+r-3) - 2(1+c)d}{d(1+c+dm)} - \frac{(1+c)^2\beta}{(1+c+dm)(1+c+d\gamma)} + \frac{(1+c)^2\beta}{(1+c+d\gamma)^2} \right| < 2. \tag{23}$$

Further, suppose that

$$B_{\mathbb{N}} := \{(\alpha, \beta, \gamma, b, c, d, m, r) : (22) \text{ and } (23) \text{ holds}\}.$$

Then equilibrium  $E^*$  of (4) undergoes NSB for different parametric values belonging to the small neighborhood of the set  $B_{\mathbb{N}}$ .

Let us replace  $\beta_1 = \beta$  in (22) and  $(r, b, c, d, \alpha, \beta_1, \gamma, m) \in B_{\mathbb{N}}$ . Then the following modification in system (4) can be ensured:

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} rP(1-P) + bP - \frac{\alpha P^2 Q}{P+m} - \frac{\beta_1 P^2}{P+\gamma} \\ dPQ - cQ \end{pmatrix}. \tag{24}$$

Consider the perturbation of map (24) by selecting  $\hat{\beta}$  as a minimal perturbation parameter, then we have the following map:

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} rP(1 - P) + bP - \frac{\alpha P^2 Q}{P+m} - \frac{(\beta_1 + \hat{\beta})P^2}{P+\gamma} \\ dPQ - cQ \end{pmatrix},$$

where  $|\hat{\beta}| \ll 1$ .

Now, we introduce the transformation and put  $x = P - x^*$  and  $y = Q - y^*$ , where  $E^* = (x^*, y^*)$ , then (24) can be rearranged as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_5(x, y) \\ g_5(x, y) \end{pmatrix}, \tag{25}$$

where

$$\begin{cases} f_5(x, y) := m_{13}x^2 + m_{14}xy + m_{15}x^3 + m_{16}x^2y + O((|x| + |y| + |\hat{\beta}|)^4), \\ g_5(x, y) := m_{23}xy + O((|x| + |y| + |\hat{\beta}|)^4), \end{cases}$$

and  $m_{11}, m_{12}, m_{21}, m_{22}, m_{13}, m_{14}, m_{15}, m_{16}$ , and  $m_{23}$  are given in (19) by replacing  $m_{ij} = a_{ij}$ ,  $i = 1, 2; j = 1, 2, 3, 4, 5, 6$  and  $r_1$  by  $r$ ,  $\beta$  by  $(\beta_1 + \hat{\beta})$ . The characteristic equation corresponding to the system of (25) evaluated at  $(0, 0)$  is expressed by

$$\lambda^2 - p(\hat{\beta})\lambda + q(\hat{\beta}) = 0, \tag{26}$$

where

$$\begin{cases} p(\hat{\beta}) := -\frac{(1+c)^2(\beta_1 + \hat{\beta})}{(1+c+dm)(1+c+d\gamma)} - \frac{(1+c)^2r+d^2m(b+r-3)-2(1+c)d}{d(1+c+dm)} \\ \quad + \frac{(1+c)^2(\beta_1 + \hat{\beta})}{(1+c+d\gamma)^2}; \\ q(\hat{\beta}) := \frac{((1+c)^2(2+c-d)+d((1+c)^2-cd)m)r}{d(1+c+dm)} + \frac{(1+c)^2(\beta_1 + \hat{\beta})}{(1+c+d\gamma)^2} \\ \quad + \frac{(1+c)(b+(-1+b)c)+(1+(-1+b)c)dm}{(1+c+dm)} - \frac{(1+c)^2(2+c+dm)(\beta_1 + \hat{\beta})}{(1+c+dm)(1+c+d\gamma)}. \end{cases}$$

Since  $(r, b, c, d, \alpha, \beta_1, \gamma, m) \in \mathbb{B}_{\mathbb{R}}$ , the solutions of (26) are  $\lambda_1$  and  $\lambda_2$  along with  $|\lambda_1| = |\lambda_2| = 1$ ; consequently one has

$$\lambda_1, \lambda_2 := \frac{p(\hat{\beta})}{2} \pm \frac{i}{2} \sqrt{4q(\hat{\beta}) - p^2(\hat{\beta})}$$

and

$$\begin{aligned} |\lambda_1| &= |\lambda_2| = \sqrt{q(\hat{\beta})}, \\ \left(\frac{d|\lambda_1|}{d\hat{\beta}}\right)_{(\hat{\beta}=0)} &:= \left(\frac{d|\lambda_2|}{d\hat{\beta}}\right)_{(\hat{\beta}=0)} \\ &:= -\frac{\Omega^2(1 + (1 + \eta)(\Omega + d\gamma - 1))}{2\eta\xi^2 \sqrt{\frac{d(\Omega(b+(b-1)(\Omega-1)+(b-1)(\Omega-1))dm)-(\Omega^2(1+\Omega-d)+d(\Omega^2-(-1)d)m)r}{d\eta} + \frac{\Omega^2\beta_1}{\xi^2} - \frac{\Omega^2\beta_1(1+\eta)}{\xi\eta}}}, \end{aligned}$$

where  $1 + c + dm = \eta$ ,  $1 + c + d\gamma = \xi$ , and  $1 + c = \Omega$ .

Further, we assume that

$$\rho(0) := \frac{\Omega^2 \beta_1}{\xi^2} - \frac{\Omega^2 r - 2\Omega d + d^2 m(b+r-3)}{d\eta} - \frac{\Omega^2 \beta_1}{\xi \eta} \neq 0, -1.$$

Moreover,  $(\beta_1, r, b, c, d, \alpha, \gamma, m) \in B_{\mathbb{R}}$  implies that  $-2 < \rho(0) < 2$ . Thus  $\rho(0) \neq \pm 2, 0, -1$  gives  $\lambda_1^n, \lambda_2^n \neq 1$  for all  $n = 1, 2, 3, 4$  at  $\hat{\beta} = 0$ . Consequently, the roots of (26) do not occur in the intersection of the unit circle with the coordinate axes when  $\hat{\beta} = 0$  and if the following conditions hold:

$$\left. \begin{aligned} \frac{\Omega^2 \beta_1}{\xi \eta} + \frac{\Omega^2 r - 2\Omega d + d^2 m(b+r-3)}{d\eta} &\neq \frac{\Omega^2 \beta_1}{\xi^2}, \\ \frac{\Omega^2 \beta_1}{\xi \eta} + \frac{\Omega^2 r - 2\Omega d + d^2 m(b+r-3)}{d\eta} &\neq 1 + \frac{\Omega^2 \beta_1}{\xi^2}. \end{aligned} \right\} \tag{27}$$

Now we study the normal form of (26) at  $\hat{\beta} = 0$ . In order to acquire the normal form, we choose  $\mu = \frac{\rho(0)}{2}$ ,  $\zeta = \frac{1}{2}\sqrt{4q(0) - \rho^2(0)}$  is attained only if we elaborate the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} m_{12} & 0 \\ \mu - m_{11} & -\zeta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{28}$$

The desired form of (25) under conversion (28) can be reorganized as follows:

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &\rightarrow \begin{pmatrix} \mu & -\zeta \\ \zeta & \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tilde{f}(u, v) \\ \tilde{g}(u, v) \end{pmatrix}, \quad \text{where} \\ \begin{cases} \tilde{f}(u, v) := \frac{m_{15}}{m_{12}}x^3 + \frac{m_{16}}{m_{12}}x^2y + \frac{m_{13}}{m_{12}}x^2 + \frac{m_{14}}{m_{12}}xy + O((|u| + |v|)^4); \\ \tilde{g}(u, v) := \frac{(\eta - m_{11})m_{15}}{\zeta m_{12}}x^3 + \frac{(\eta - m_{11})m_{16}}{\zeta m_{12}}x^2y + \frac{(\eta - m_{11})m_{13}}{\zeta m_{12}}x^2 \\ \quad + \left(\frac{(\eta - m_{11})m_{14}}{\zeta m_{12}} - \frac{m_{23}}{\zeta}\right)xy + O((|u| + |v|)^4), \end{cases} \end{aligned}$$

$x = m_{12}u$  and  $y = (\eta - m_{11})u - \zeta v$ . Now, we define nonzero  $L \in R$ (set of real numbers) as follows:

$$L := \left( \left[ -\operatorname{Re} \left( \frac{(1 - 2\lambda_1)\lambda_2^2}{1 - \lambda_1} \Upsilon_{20} \Upsilon_{11} \right) - \frac{1}{2} (|\Upsilon_{11}|^2 + |\Upsilon_{02}|^2 + \operatorname{Re}(\lambda_2 \Upsilon_{21})) \right] \right)_{\hat{\beta}=0},$$

where

$$\begin{aligned} \Upsilon_{20} &:= \frac{1}{8} [\tilde{f}_{uuu} - \tilde{f}_{vv} + 2\tilde{g}_{uv} + i(\tilde{g}_{uu} - \tilde{g}_{vv} - 2\tilde{f}_{uv})], \\ \Upsilon_{11} &:= \frac{1}{4} [\tilde{f}_{uu} + \tilde{f}_{vv} + i(\tilde{g}_{uu} + \tilde{g}_{vv})], \\ \Upsilon_{02} &:= \frac{1}{8} [\tilde{f}_{uu} - \tilde{f}_{vv} - 2\tilde{g}_{uv} + i(\tilde{g}_{uu} - \tilde{g}_{vv} + 2\tilde{f}_{uv})], \end{aligned}$$

and

$$\Upsilon_{21} := \frac{1}{16} [\tilde{f}_{uuu} + \tilde{f}_{uvv} + \tilde{g}_{uuv} + \tilde{g}_{vvv} + i(\tilde{g}_{uuu} + \tilde{g}_{uvv} - \tilde{f}_{uuv} - \tilde{f}_{vvv})].$$

Due to our aforementioned mathematical study, one can state the following theorem cf. [61–65].

**Theorem 6** *Assume that (27) holds and  $L \neq 0$ , then (4) undergoes NSB at  $E^*$ , when  $\beta$  changes its values in a small neighborhood of  $\beta_1$ . Additionally, if  $L < 0$ , then an attracting invariant closed curve bifurcates from  $E^*$  for  $\beta_1 < \beta$ , and for the case of  $L > 0$ , a repelling invariant closed curve bifurcates from  $E^*$  for  $\beta_1 > \beta$ .*

#### 4 Chaos control

The chaos control and theory of bifurcation is one of the most vital and developed areas of the current research. It has significant characteristics in population models especially models associated with biological species. Furthermore, discrete-time population models are more chaotic and complex as compared to their continuous counterparts. Hence, it is obvious to execute chaos control techniques to evade any uncertainty. The current section consists of the following two feedback control techniques:

- (i) OGY feedback control strategy;
- (ii) Hybrid feedback control strategy.

First, we apply the OGY method on system (4).

##### 4.1 OGY control method

In this section, we execute the OGY method which was proposed by Ott *et al.*, for details see also [66, 67]. Now, we implement the OGY method on system (4), then we have the following modified form of system (4):

$$\begin{cases} x_{n+1} := rx_n(1 - x_n) + bx_n - \frac{\alpha x_n^2 y_n}{x+m} - \frac{\beta x_n^2}{x_n+\gamma} = f(x_n, y_n, r), \\ y_{n+1} := dx_n y_n - cy_n = g(x_n, y_n, r), \end{cases} \tag{29}$$

where  $r$  is taken as a control parameter. Here, we restrict  $r$  to a small interval  $r \in (r_0 - \zeta, r_0 + \zeta)$  to achieve the desired control by implementing small perturbations. Also  $r_0$  indicates any value from the chaotic region. Moreover, suppose that  $(x^*, y^*)$  is unstable equilibrium of system (4) in the chaotic region under the influence of period-doubling bifurcation. Then, by applying the following linear map, system (29) can be estimated in the neighborhood of  $(x^*, y^*)$  as follows:

$$\begin{bmatrix} x_{n+1} - x^* \\ y_{n+1} - y^* \end{bmatrix} \approx J(x^*, y^*, r_0) \begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix} + B[r - r_0], \tag{30}$$

where

$$J(x^*, y^*, r_0) := \begin{bmatrix} \frac{\partial f(x^*, y^*, r_0)}{\partial x} & \frac{\partial f(x^*, y^*, r_0)}{\partial y} \\ \frac{\partial g(x^*, y^*, r_0)}{\partial x} & \frac{\partial g(x^*, y^*, r_0)}{\partial y} \end{bmatrix}$$

and

$$B := \begin{bmatrix} \frac{\partial f(x^*, y^*, r_0)}{\partial r} \\ \frac{\partial g(x^*, y^*, r_0)}{\partial r} \end{bmatrix} = \begin{bmatrix} -\frac{(1+c)(1+c-d)}{d^2} \\ 0 \end{bmatrix}.$$

Moreover, controllable system (29) yields that the following matrix

$$C := [B : JB] = \begin{bmatrix} -\frac{(1+c)(1+c-d)}{d^2} & \left(\frac{\partial f(x^*, y^*, r_0)}{\partial x}\right)\left(-\frac{(1+c)(1+c-d)}{d^2}\right) \\ 0 & \left(\frac{\partial g(x^*, y^*, r_0)}{\partial x}\right)\left(-\frac{(1+c)(1+c-d)}{d^2}\right) \end{bmatrix}$$

is of rank 2.

Moreover, taking  $[r - r_0] = -K[(x_n - x^*), (y_n - y^*)]$ , where  $K = [k_1 \ k_2]$ , then system (30) can be written as

$$\begin{bmatrix} (x_{n+1} - x^*) \\ (y_{n+1} - y^*) \end{bmatrix} \approx [J - BK] \begin{bmatrix} (x_n - x^*) \\ (y_n - y^*) \end{bmatrix}.$$

Furthermore, the equivalent controlled system of (4) is stated as follows:

$$\begin{cases} x_{n+1} := x_n(1 - x_n)[r_0 - k_1(x_n - x^*) - k_2(y_n - y^*)] + bx_n - \frac{\alpha x_n^2 y_n}{x+m} - \frac{\beta x_n^2}{x_n + \gamma}, \\ y_{n+1} := dx_n y_n - cy_n. \end{cases} \tag{31}$$

Moreover, the equilibrium point  $(x^*, y^*)$  is asymptotically stable when both eigenvalues of  $'J - BK'$  belong to an open unit disk. The variational matrix  $'J - BK'$  of system (31) can be written as follows:

$$J - BK = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix},$$

and the characteristic equation of  $'J - BK'$  is given by

$$\mathbb{P}(\lambda) := \lambda^2 - (\omega_{11} + \omega_{22})\lambda + \mathcal{D} = 0,$$

where

$$\begin{cases} \omega_{11} := \frac{(1+c)(-(1+c)^2 r + d^2(-1+b+r)\gamma + (1+c)d(-1+b+r-\beta-r\gamma))}{d(1+c+dm)(1+c+d\gamma)} - b \\ \quad + \left(1 - \frac{1+c}{d}\right)r - \frac{(1+c)r}{d} + \frac{2(1+c-d)r}{d} + \frac{(1+c)^2 \beta}{(1+c+d\gamma)^2} + \frac{(1+c)(1+c-d)\rho}{d^2} + 2; \\ \omega_{21} := \frac{(1+c+dm)(-(1+c)^2 r + d^2(-1+b+r)\gamma + (1+c)d(-1+b+r-\beta-r\gamma))}{(1+c)\alpha(1+c+d\gamma)}; \\ \omega_{12} := \frac{(1+c)(-\frac{(1+c)d\alpha}{1+c+dm} + (1+c-d)\sigma)}{d^2}; \quad \omega_{22} = 1; \quad \mathcal{D} = \omega_{11}\omega_{22} - \omega_{12}\omega_{21}, \\ \begin{cases} L_1 : \mathbb{P}(1) = 0 \Rightarrow \mathcal{D} - \omega_{11} = 0 \Rightarrow \omega_{12}\omega_{21} = 0; \\ L_2 : \mathbb{P}(-1) = 0 \Rightarrow 2(1 + \omega_{11}) - \omega_{12}\omega_{21} = 0; \\ L_3 : \mathbb{P}(0) = 1 \Rightarrow \mathcal{D} = \omega_{11}\omega_{22} - \omega_{12}\omega_{21} = 1. \end{cases} \end{cases}$$

### 4.2 Hybrid control method

To control the chaos which develops due to appearance of bifurcation in system (4), we implement a hybrid control strategy [68]. This strategy was primarily developed for controlling the chaos which developed because of period-doubling bifurcation, but in [69] similar methodology is applied for Neimark–Sacker bifurcation. Moreover, assuming that

(4) substantiates bifurcation at  $E^*$ , we get the modified controlled system as follows:

$$\begin{cases} u_{n+1} := \rho [rx_n(1-x_n) + bx_n - \frac{\alpha x_n^2 y_n}{x+m} - \frac{\beta x_n^2}{x_n+\gamma}] + (1-\rho)x_n, \\ v_{n+1} := \rho [dx_n y_n - cy_n] + (1-\rho)y_n, \end{cases} \tag{32}$$

where  $\rho \in (0, 1)$  is a controlled parameter. For some applications of the method defined in (32), we refer to the following references [64, 65, 68–71].

Consider the Jacobian of (32) estimated at  $E^*$  and prescribed as follows:

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix},$$

also the characteristic equation is  $\lambda^2 - (\Gamma_{22} + \Gamma_{11})\lambda + \mathbb{D} = 0$ , where

$$\begin{aligned} \Gamma_{11} &:= 1 - \frac{(r(1+c)^4 + 2(1+c)^3 d r \gamma + 2(1+c)d^3 m(b+r-1)\gamma + d^4 m(b+r-1)\gamma^2)\rho}{d(1+c+dm)(1+c+d\gamma)^2} \\ &\quad - \frac{\rho(1+c)^2 d^2 ((b+r-\beta-1)m + (\beta+r\gamma)\gamma)}{d(1+c+dm)(1+c+d\gamma)^2}; \quad \Gamma_{12} := -\frac{(1+c)^2 \alpha \rho}{d(1+c+dm)}, \\ \Gamma_{21} &:= \frac{(1+c+dm)(-(1+c)^2 r + d^2(-1+b+r)\gamma + (1+c)d(-1+b+r-\beta-r\gamma))\rho}{(1+c)\alpha(1+c+d\gamma)}, \\ \Gamma_{22} &:= 1; \quad \mathbb{D} := \Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21}. \end{aligned}$$

Analogous to the above mathematical computation, we demonstrate the following lemma related to the stability of controlled system.

**Lemma 2** *The interior equilibrium  $E^*$  of (32) is locally asymptotically stable whenever*

$$|\Gamma_{22} + \Gamma_{11}| < 1 + \mathbb{D} < 2.$$

### 5 Numerical simulation and discussion

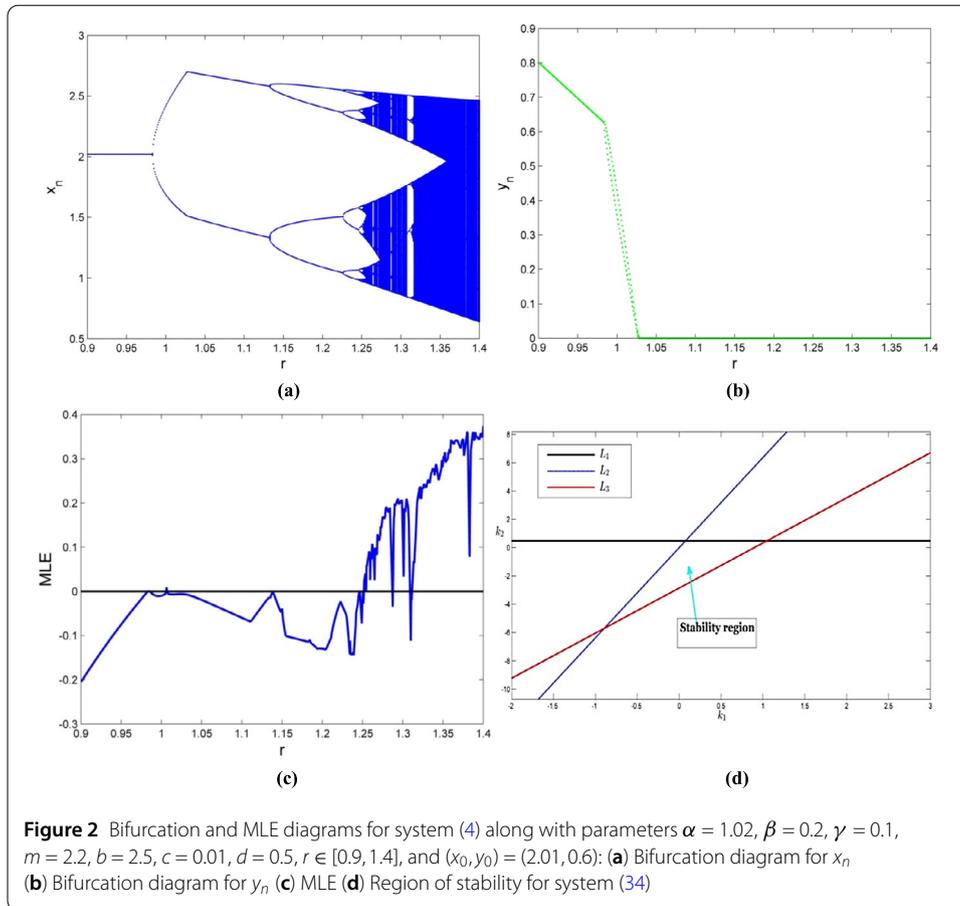
*Example 1* Choosing parameters  $\alpha = 1.02, \beta = 0.2, \gamma = 0.1, m = 2.2, b = 2.5, c = 0.01, d = 0.5, r \in [0.9, 1.4]$  and with initial condition  $(x_0, y_0) = (2.01, 0.6)$ , system (4) undergoes period-doubling bifurcation when  $r \approx 0.9831708806347621$ . Bifurcation diagrams and maximum Lyapunov exponents (MLE) of the corresponding system are depicted in Fig. 2(a), (b), (c). Moreover, system (4) has the unique fixed point  $(x^*, y^*) = (2.02, 0.6279609)$ , and the characteristic equation of the variational matrix calculated at this equilibrium is written as follows:

$$\lambda^2 + 0.1548328303284311\lambda - 0.8451671696715704 = 0. \tag{33}$$

Furthermore, the roots of (33) are  $\lambda_1 = -1$  and  $\lambda_2 = 0.8451671696715698$  with  $|\lambda_2| \neq 1$ . Hence the parameters

$$(r, b, c, d, m, \alpha, \beta, \gamma) = (0.9831708806347621, 2.5, 0.01, 0.5, 2.2, 1.02, 0.2, 0.1) \in B_{\mathbb{U}}.$$

Next, we implement the OGY control strategy to control chaos which appears due to period-doubling bifurcation. For this, taking  $r = 0.9832, \beta = 0.2, b = 2.5, c = 0.01, d = 0.5,$



$\alpha = 1.02, \gamma = 0.1, m = 2.2$ , and  $(x^*, y^*) = (2.02, 0.6279609)$ , the equivalent controlled system is given by

$$\begin{cases} x_{n+1} := x_n(1 - x_n)[0.9832 - k_1(x_n - 2.02) - k_2(y_n - 0.628)] + 2.5x_n \\ \quad - \frac{1.02x_n^2 y_n}{x_n + 2.2} - \frac{0.2x_n^2}{x_n + 0.1}, \\ y_{n+1} := 0.5x_n y_n - 0.01y_n. \end{cases} \tag{34}$$

Then the Jacobian ‘ $J - BK$ ’ of (34) can be evaluated by

$$J - BK := \begin{bmatrix} -1.1548328 + 2.0604k_1 & -0.9862578199 + 2.0604k_2 \\ 0.313980436 & 1 \end{bmatrix}.$$

Moreover,  $\mathbb{L}_1, \mathbb{L}_2$ , and  $\mathbb{L}_3$  represent the marginal stability lines, which are given by

$$\begin{cases} \mathbb{L}_1 : k_2 = 0.47867298578199047; \\ \mathbb{L}_2 : k_2 = 6.369823620824943k_1 + -6.864613518385784 \times 10^{-16}; \\ \mathbb{L}_3 : k_2 = 3.1849118104124714k_1 - 2.8522105954534736. \end{cases}$$

Region bounded by  $\mathbb{L}_1, \mathbb{L}_2$ , and  $\mathbb{L}_3$  represents the stability region and is shown in Fig. 2(d).

Further, assume that  $\alpha = 1.02, \beta = 0.2, \gamma = 0.1, m = 2.2, b = 2.5, c = 0.01, d = 0.5, r \in [0.9, 1.4]$  and with initial conditions  $(x_0, y_0) = (2.01, 0.6)$ , then system (4) undergoes period-

**Table 1** Controllable interval for system (35) with various values of  $r$  in the chaotic region

Values of bifurcation parameter $r$ from the chaotic region	Stability interval $\rho$
0.99	$0 < \rho < 0.9926582978517923$
1.0	$0 < \rho < 0.9821630969476676$
1.1	$0 < \rho < 0.8911247422163916$
1.2	$0 < \rho < 0.8189351789209777$
1.25	$0 < \rho < 0.7879479974702013$
1.27	$0 < \rho < 0.776336265914807$
1.28	$0 < \rho < 0.7706845320467284$

doubling bifurcation. To control bifurcation, we stated the corresponding hybrid control system as follows:

$$\begin{cases} u_{n+1} := \rho[0.9832x_n(1 - x_n) + 2.5x_n - \frac{1.02x_n^2y_n}{x+2.2} - \frac{0.2x_n^2}{x_n+0.1}] + (1 - \rho)x_n \\ v_{n+1} := \rho[0.5x_ny_n - 0.01y_n] + (1 - \rho)y_n. \end{cases} \tag{35}$$

The control interval of stability can be seen in Table 1 whenever  $0.9 \leq r \leq 1.4$ .

Furthermore, for system (35), the bifurcation diagrams are shown in Fig. 4(a) and (b).

*Example 2* Let  $r = 0.2$ ,  $b = 2.5$ ,  $c = 0.01$ ,  $d = 0.5$ ,  $\alpha = 1.02$ ,  $\gamma = 0.1$ ,  $\beta \in [0.2, 0.5]$  and with initial conditions  $(x_0, y_0) = (2.01, 1.7)$ , then in system (4) Neimark–Sacker bifurcation appears when  $\beta \approx 0.4491479133565015$ . On the other hand, the parallel bifurcation diagrams and MLE are dispatched in Fig. 3(a), (b), and (c). Furthermore, system (4) has an interior equilibrium point  $(x^*, y^*) = (2.02, 1.7778691826151725)$  with characteristic equation calculated at  $(2.02, 1.7778691826151725)$  and given by

$$\lambda^2 - 1.123281307938648\lambda + 0.9999999999999999 = 0. \tag{36}$$

Furthermore, the roots of (36) are  $\lambda_1 = 0.561640653969324 - 0.8273812759598261i$  and  $\lambda_2 = 0.561640653969324 + 0.8273812759598261i$  with  $|\lambda_{1,2}| = 1$ . Thus the parameters  $(\beta, r, b, c, d, m, \alpha, \gamma) = (0.4491479133565015, 0.2, 2.5, 0.01, 0.5, 2.2, 1.02, 0.1) \in \mathbb{B}_{\mathbb{R}}$ .

Next, we implement the OGY control strategy to control chaos due to appearance of NSB. For this, taking  $r = 0.2$ ,  $\beta = 0.4491$ ,  $b = 2.5$ ,  $c = 0.01$ ,  $d = 0.5$ ,  $m = 2.2$ ,  $\alpha = 1.02$ ,  $\gamma = 0.1$  and the unique positive equilibrium  $(x^*, y^*) = (2.02, 1.7779)$ , a modified controlled system is given by

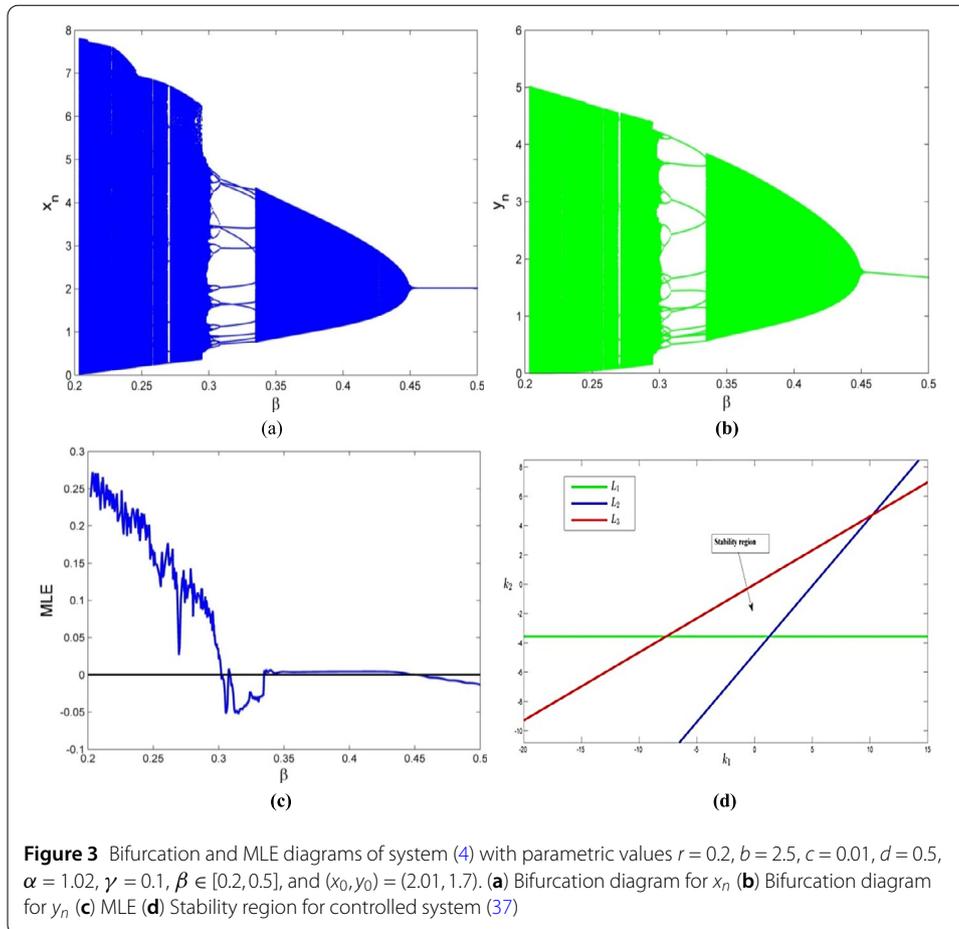
$$\begin{cases} x_{n+1} := 0.2x_n(1 - x_n) - \frac{[0.4491 - k_1(x_n - 2.02) - k_2(y_n - 1.7779)]x_n^2}{x_n + 0.1} + 2.5x_n \\ \quad - \frac{1.02x_n^2y_n}{x+2.2}, \\ y_{n+1} := 0.5x_ny_n - 0.01y_n. \end{cases} \tag{37}$$

Then, the Jacobian matrix ' $J - BK$ ' of updated controlled system (37) reduces to

$$J - BK = \begin{bmatrix} 0.123281308 + 1.92471698k_1 & -0.9862578 + 1.92471698k_2 \\ 0.8889345913075862 & 1 \end{bmatrix}.$$

Moreover, the marginal stability lines  $\mathbb{L}_1$ ,  $\mathbb{L}_2$ , and  $\mathbb{L}_3$  are given as follows:

$$\mathbb{L}_1 : k_2 = 0.5124170616113745,$$



$$\mathbb{L}_2 : k_2 = 2.2498843217002977k_1 + 1.8254688133051193,$$

$$\mathbb{L}_3 : k_2 = 1.1249421608501489k_1 - 2.59557472727822 \times 10^{-16}.$$

In addition, the triangular region of stability bounded by  $\mathbb{L}_1, \mathbb{L}_2,$  and  $\mathbb{L}_3$  is plotted in Fig. 3(d).

Furthermore, we again take  $r = 0.2, b = 2.5, c = 0.01, d = 0.5, \alpha = 1.02, \gamma = 0.1, \beta \in [0.2, 0.5]$  with initial conditions  $(x_0, y_0) = (2.01, 1.7)$ . For these values of parameters, system (4) exhibits Neimark–Sacker bifurcation. Now we perform the hybrid control strategy for the purpose of controlling chaos. For the above numeric values, controlled map (32) takes the form

$$\begin{cases} u_{n+1} := \rho[0.2x_n(1 - x_n) + 2.5x_n - \frac{1.02x_n^2y_n}{x+2.2} - \frac{0.4491x_n^2}{x_n+0.1}] + (1 - \rho)x_n, \\ v_{n+1} := \rho[0.5x_ny_n - 0.01y_n] + (1 - \rho)y_n. \end{cases} \quad (38)$$

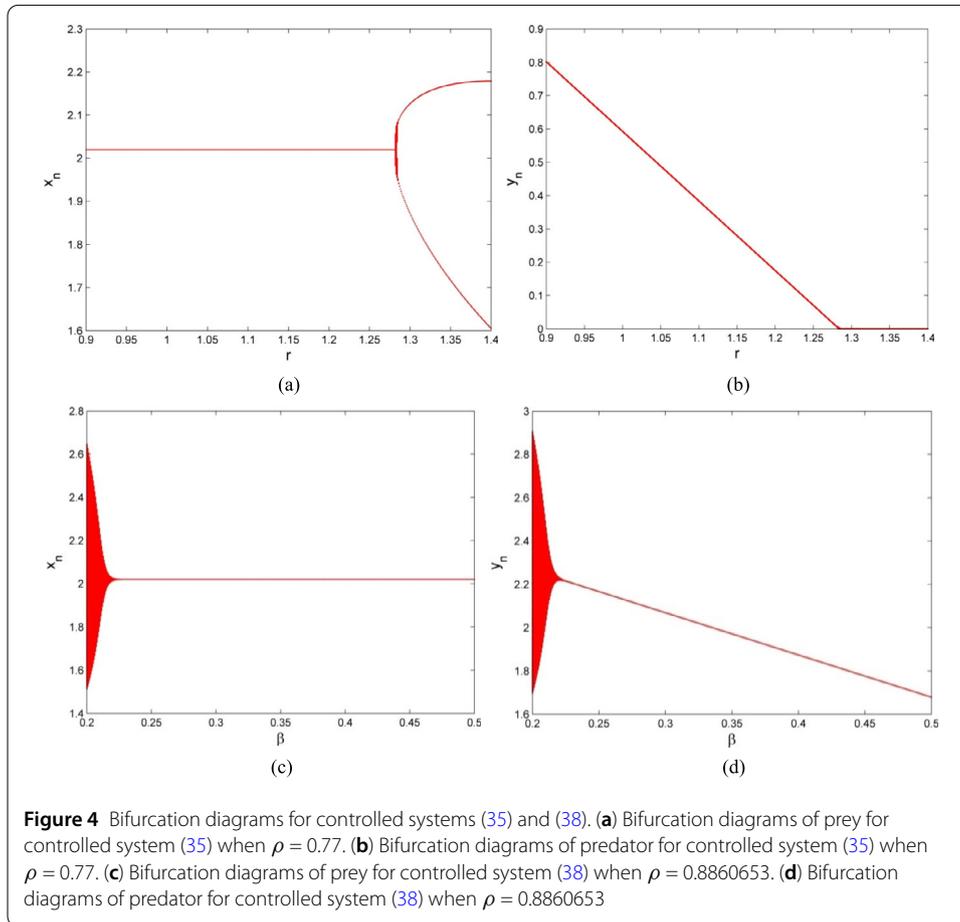
It can be viewed from Table 2 that there exists the control interval of stability when  $0.2 \leq \beta \leq 0.5$ .

Figure 4(c) and (d) illustrates the bifurcation for controlled system (38).

Some phase portraits of system (4) confirm the chaotic behavior of a system where parameter  $\beta$  takes different values from the chaotic region with initial conditions  $(x_0, y_0) = (2.01, 1.7)$ , while parameters  $(r, b, c, d, m, \alpha, \gamma) = (0.2, 2.5, 0.01, 0.5, 2.2, 1.02, 0.1)$  remain the

**Table 2** Controllable interval for system (38) with various values of  $r$  in the chaotic region

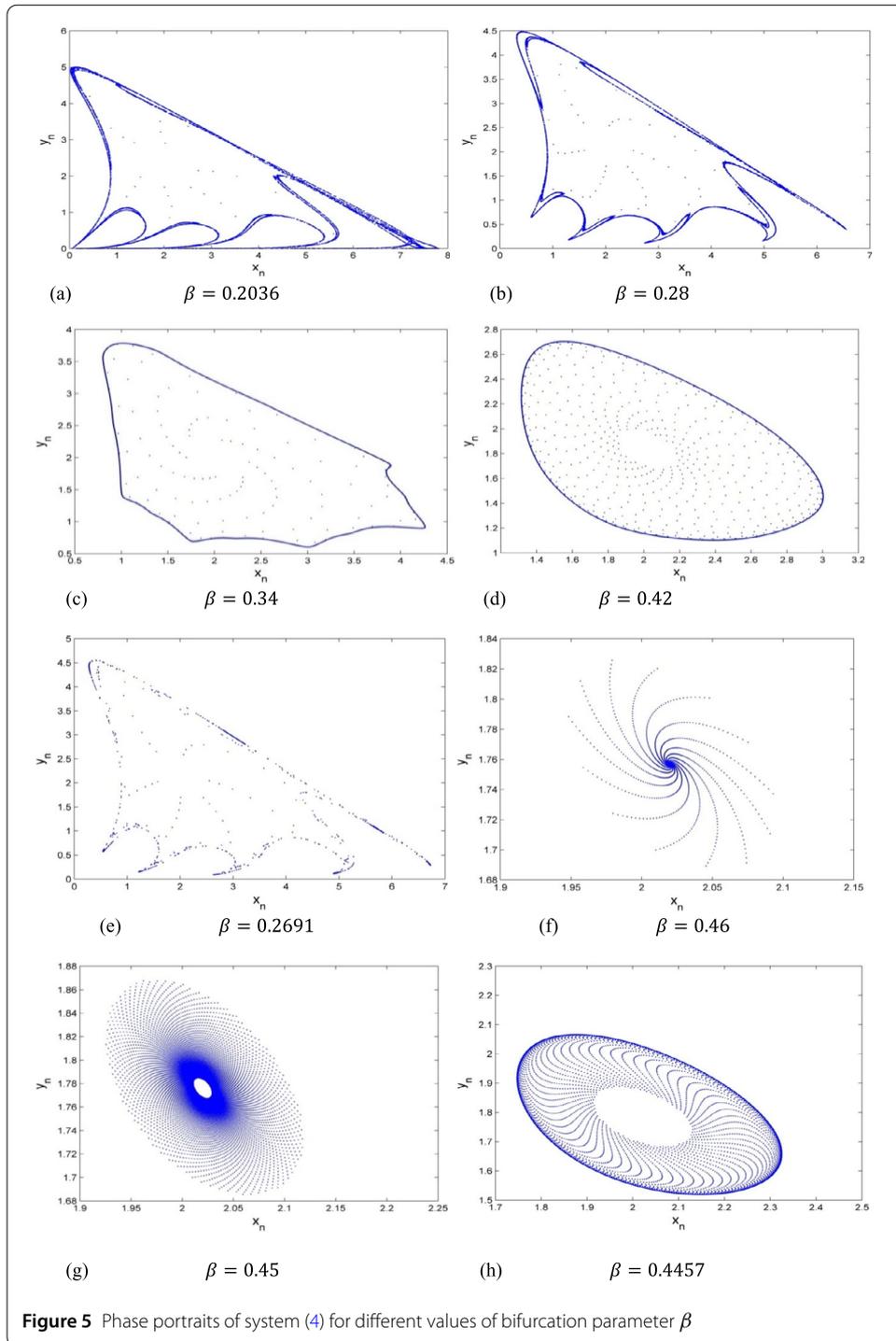
Values of bifurcation parameter $\beta$ from the chaotic region	Stability interval $\rho$
0.20	$0 < \rho < 0.8860653079254625$
0.25	$0 < \rho < 0.9048285188715174$
0.30	$0 < \rho < 0.9253615795556566$
0.35	$0 < \rho < 0.9479272977281787$
0.40	$0 < \rho < 0.9728432237205998$
0.42	$0 < \rho < 0.9835516455804203$



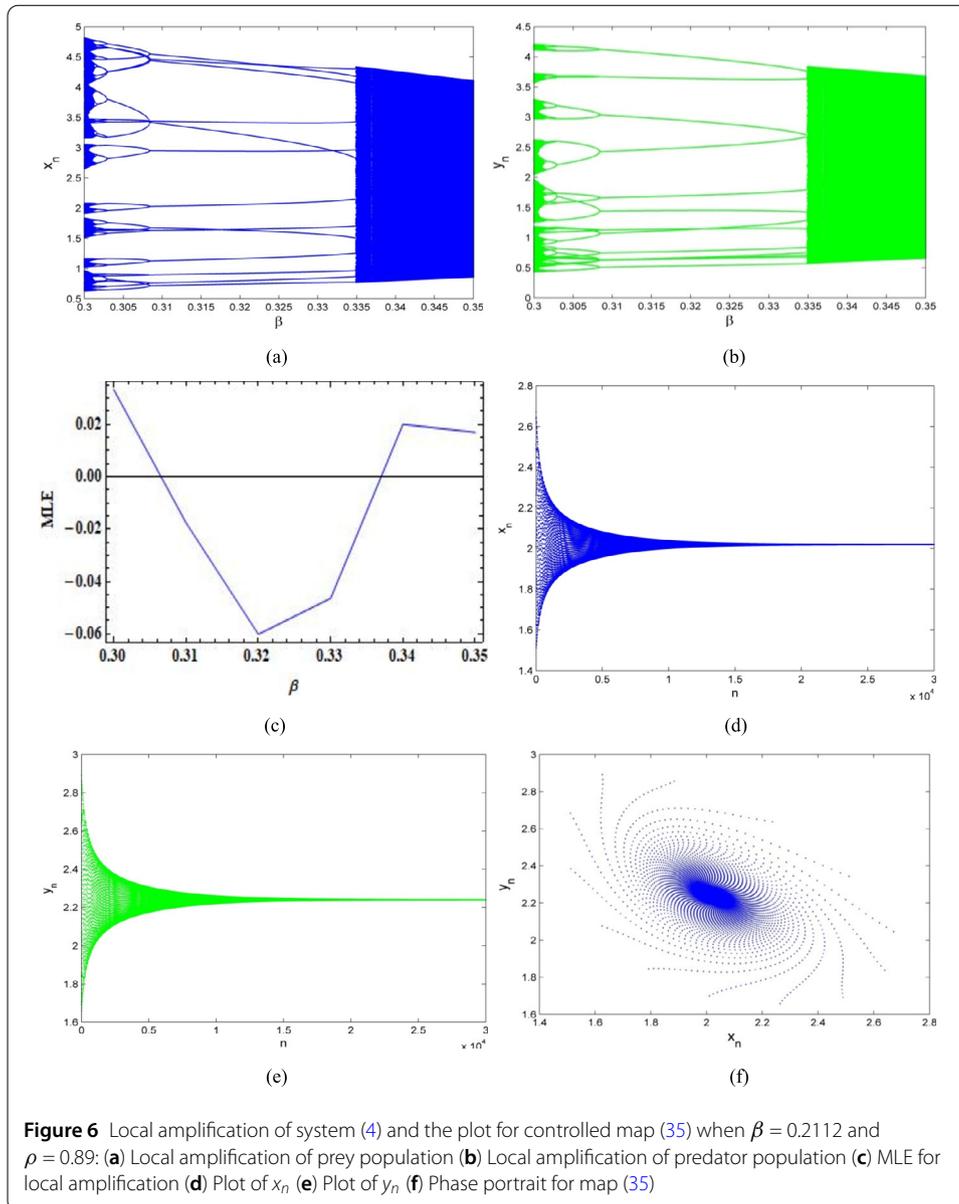
same for each case. Finally, interesting local implication diagrams and the plot of the controlled system are plotted in Fig. 6.

### 6 Concluding remarks

Intra-specific predation or cannibalism is a significant natural process that controls population dynamics. It has immense complex consequences on population dynamics [24, 72]. Considering the Allee effect and cannibalism on prey population, a discrete-time system for predator–prey interaction is proposed and investigated. It is analyzed that model (4) has three steady-states  $E^0, E^1, E^*$ . The linearization technique is utilized to achieve the local stability of the steady-states. Furthermore, if  $\gamma d^2 + (1 + c)(1 + \beta + \gamma r)d + (1 + c)^2 r < (b + r)(1 + c + \gamma d)d$ , then system (4) has a unique positive steady-state  $E^*$ . The topological classification of  $E^0, E^1$ , and  $E^*$  is shown in Fig. 1(a), (b), and (c). It is proved that sys-



tem (4) undergoes transcritical bifurcation at  $E^0$ , period-doubling bifurcation at  $E^1$ , and around  $E^*$  there exist both period-doubling bifurcation and Neimark–Sacker bifurcation by using bifurcation theory. Also, our theoretical results are supported by some figures. In Fig. 2(a), a bifurcation diagram of system (4) is depicted for various parametric values of  $\alpha = 1.02, \beta = 0.2, \gamma = 0.1, m = 2.2, b = 2.5, c = 0.01, d = 0.5, r \in [0.9, 1.4]$  and the initial condition  $(x_0, y_0) = (2.01, 0.6)$ . We observe that  $E^*$  is stable for  $r < 0.9831708806347621$  and



loses its stability at  $r = 0.9831708806347621$ , and the system undergoes period-doubling bifurcation when the growth rate of prey  $r$  exceeds the value  $0.9831708806347621$ . In Figs. 3(a) and (b), the bifurcation diagrams of system (4) are shown. Now, by taking  $\beta$  (cannibalism rate of prey) as a bifurcation parameter with different values of parameters  $r = 0.2, b = 2.5, c = 0.01, d = 0.5, \alpha = 1.02, \gamma = 0.1, \beta \in [0.2, 0.5]$  and the initial condition  $(x_0, y_0) = (2.01, 1.7)$ , the unique steady-state  $E^* = (2.02, 1.7778691826151725)$  of system (4) is stable for  $\beta > 0.4491479133565015$  and loses its stability at  $\beta = 0.4491479133565015$ , also an attracting invariant closed curves appear when  $\beta < 0.4491479133565015$ . In Fig. 5, some phase portraits are plotted for different values of  $\beta$ , which shows the chaotic and complex behavior of the system. Moreover, Figs. 2(d), 3(d), and 4 ensure that our proposed control strategies successfully control the bifurcation. Ultimately, we can say that for predator–prey interaction, the growth rate of prey and the cannibalism rate of prey both have remarkable consequences for the stability of system (4) and for population models.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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