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Stability and bifurcation analysis of two-species competitive model with Michaelis–Menten type harvesting in the first species

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Abstract

In this paper, a two-species competitive model with Michaelis–Menten type harvesting in the first species is studied. We have made a detailed mathematical analysis of the model to describe some important results that may be produced by the interaction of biological resources. The permanence, stability, and bifurcation (saddle-node bifurcation and transcritical bifurcation) of the model are investigated. The results show that with the change of parameters, two species could eventually coexist, become extinct or one species will be driven to extinction and the other species will coexist. Moreover, by constructing the Lyapunov function, sufficient conditions to ensure the global asymptotic stability of the positive equilibrium are given. Our study shows that compared with linear harvesting, nonlinear harvesting can exhibit more complex dynamic behavior. Numerical simulations are presented to illustrate the theoretical results.

Keywords: Competitive; Michaelis–Menten type harvesting; Stability; Bifurcation

1 Introduction

During the past decade, a competitive system has been extensively investigated by many scholars [1–24], many excellent results concerned with the permanence, extinction, and global attractivity of the competition system have been obtained.

Traditional two-species Lotka–Volterra competitive system is as follows:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(b_1 - a_{11}x_1 - a_{12}x_2), \\ \frac{dx_2}{dt} &= x_2(b_2 - a_{21}x_1 - a_{22}x_2),\end{aligned}\tag{1.1}$$

where x_1 and x_2 denote the population density of the first and second species at time t , respectively. b_i , a_{ij} , $i, j = 1, 2$, are positive constants. System (1.1) has been investigated in mathematical biology books [25]. Depending on the relationship of the coefficients of system (1.1), it has the following dynamic behaviors.

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(1) If

$$\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}},$$

holds, system (1.1) has a unique positive equilibrium, which is globally attractive.

(2) If

$$\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}}, \quad \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}},$$

holds, then the second species will be driven to extinction, while the first species will approach $\frac{b_1}{a_{11}}$.

On the other hand, the study of resource-management, including fisheries, forestry, and wildlife management, has great importance as human activity is the main cause of the extinction of endangered species. It is necessary to harvest the population, but harvesting should be regulated so that both the ecological sustainability and conservation of the species can be implemented in a long run. Many scholars are interested in establishing appropriate biological models to further understand the scientific management of renewable resources. For example, based on model (1.1), Sharma and Samanta [26] further considered the harvesting and obtained the following model:

$$\begin{aligned} \frac{dx(t;p)}{dt} &= x(t)[(r_{1l})^{(1-p)}(r_{1u})^p - (b_{11l})^{(1-p)}(b_{11u})^p x(t) \\ &\quad - (b_{12l})^{(1-p)}(b_{12u})^p y(t)] - q_1 E x(t), \\ \frac{dy(t;p)}{dt} &= y(t)[(r_{2l})^{(1-p)}(r_{2u})^p - (b_{22l})^{(1-p)}(b_{22u})^p y(t) \\ &\quad - (b_{21l})^{(1-p)}(b_{21u})^p x(t)] - q_2 E y(t), \end{aligned} \tag{1.2}$$

where $p \in [0, 1]$. Due to the difficulty in the estimation of the model parameters, the authors argued that taking account of the imprecise of biological parameter values makes some situations more realistic. They developed a method to handle imprecise parameters. In addition, they discussed the existence and stability of the system equilibria, as well as the bionomic equilibrium and optimal harvesting policy. The equilibrium solution of the control problem was given and the harvest strategy was optimized dynamically.

Ecosystem with harvesting has been extensively investigated by many scholars [27–37]. May et al. [31] proposed two types of harvesting: (1) constant harvest and (2) linear harvest. For the first case, it is impossible to harvest a certain number of species every year. Although this type of harvest may be relatively easy to study, it is not a reality. For the second case, the harvesting term takes the form $h(x) = qEx$, obviously, when x or E tends to infinity, $h(x)$ tends to infinity. Clearly this contradicts the facts, because in reality there is limited harvesting capacity or number of species, so the amount of species that can be harvested is limited. To overcome the drawback of the two kinds of harvesting, Clark [32] proposed the Michaelis–Menten type harvesting $h(x) = \frac{qEx}{mE+nx}$ when x or E tends to infinity, $h(x)$ tends to $\frac{qE}{n}$ or $\frac{qx}{m}$. In this type of harvesting, if the number of species tends to infinity,

the final harvest depends on the harvesting capacity, and if the harvesting capacity tends to infinity, the final harvest depends on the number of species, which is in accordance with the actual situation. Since such kind of harvesting is more suitable, it brings many scholars to do works in this direction (see [33–37] and the references cited therein). For example, Yu, Chen, and Lai [35] introduced Michaelis–Menten type harvesting into a May type cooperative system and discussed the extinction of the first species and the global attraction of the unique positive equilibrium. Chen [36] studied the Lotka–Volterra commensal symbiosis model with Michaelis–Menten type harvesting; the modified model takes the following:

$$\begin{aligned} \frac{dx}{dt} &= \gamma_1 x \left(1 - \frac{x}{k_1} + \alpha \frac{y}{k_1} \right) - \frac{qEx}{m_1 E + m_2 x}, \\ \frac{dy}{dt} &= \gamma_2 y \left(1 - \frac{y}{k_2} \right), \end{aligned} \tag{1.3}$$

where x and y denote the population density of the two species at time t , respectively. q denotes the fishing coefficient of the first species and E denotes the fishing effort. $r_1, r_2, K_1, K_2, \alpha, m_1, m_2$ are all positive constants. The results show that the system has a globally asymptotically stable positive equilibrium. In addition, the two species can coexist stably when α and K_2 are large enough. On the contrary, the first species will be driven to extinction.

After that, Liu et al. [37] considered the two-species amensalism model with Michaelis–Menten type harvesting and a cover for the first species; the modified model takes the following:

$$\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1(1 - k)xy - \frac{qE(1 - k)x}{m_1 E + m_2(1 - k)x}, \\ \frac{dy}{dt} &= a_2 y - b_2 y^2. \end{aligned} \tag{1.4}$$

Here, x and y denote the population density of the two species at time t , and k is the refuge ($0 < k < 1$). When the parameters satisfy certain conditions, saddle-node bifurcation and transcritical bifurcation will occur in the system, and the maximum threshold of species without extinction risk under continuous fishing is obtained. Scholars’ research shows that compared with the linear harvesting model, the Michaelis–Menten harvesting type model can not only reflect the harvesting process more realistically, but also show richer dynamic behavior.

Nevertheless, to the best of our knowledge, no scholars have proposed and studied the dynamic behaviors of the two-species competitive model with Michaelis–Menten type harvesting. So in this paper, based on model (1.1), we propose the following system:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x \left(1 - \frac{x}{k_1} \right) - \alpha_1 xy - \frac{q_1 Ex}{m_1 E + h_1 x}, \\ \frac{dy}{dt} &= r_2 y \left(1 - \frac{y}{k_2} \right) - \alpha_2 xy, \end{aligned} \tag{1.5}$$

where $r_1, r_2, k_1, k_2, \alpha_1, \alpha_2, q_1, m_1, h_1$, and E are all positive. For simplicity, we make the following nondimensional scheme:

$$\bar{t} = r_1 t, \quad \bar{x} = \frac{1}{k_1} x, \quad \bar{y} = \frac{1}{k_2} y.$$

Dropping the bars, we have the following system:

$$\begin{aligned} \frac{dx}{dt} &= x \left(1 - x - a_1 y - \frac{b_1}{c_1 + x} \right), \\ \frac{dy}{dt} &= \rho y (1 - y - a_2 x), \end{aligned} \tag{1.6}$$

where $a_1 = \frac{\alpha_1 k_2}{r_1}, b_1 = \frac{q_1 E}{k_1 r_1 h_1}, c_1 = \frac{m_1 E}{h_1 k_1}, \rho = \frac{r_2}{r_1}, a_2 = \frac{k_1 \alpha_2}{r_2}$, and the initial conditions

$$x(0) = x_0 > 0, \quad y(0) = y_0 > 0. \tag{1.7}$$

Biologically, we consider system (1.6) is defined on the set

$$R_0^+ \times R_0^+ = \{(x, y) \in R^2 \mid x \geq 0, y \geq 0\}.$$

The organization of this paper is as follows. The basic properties of the model are discussed in the next section. We analyze the existence of the equilibria of the system in Sect. 3. The local stability of the equilibria of the system are investigated in Sect. 4. We consider the global stability of the positive equilibrium in Sect. 5. The possible bifurcation of the system is studied in Sect. 6. Numerical simulation is presented to show the feasibility of theoretical results in Sect. 7, and a brief discussion of our results is given in the last section.

2 Basic properties of the model

Lemma 2.1 ([38]) *When $a, b > 0$ and $\frac{dx}{dt} \leq (\geq) x(t)(a - bx(t))$ with $x(0) > 0$, then*

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{a}{b} \left(\liminf_{t \rightarrow +\infty} x(t) \geq \frac{a}{b} \right).$$

Definition 2.1 System (1.6) is said to be permanent if there exist two constants m and M ($0 < m < M$) such that each positive solution $(x(t, x_0, y_0), y(t, x_0, y_0))$ of system (1.6) under the initial condition $(x_0, y_0) \in \text{Int}(R_+^2)$ satisfies

$$\begin{aligned} \min \left\{ \liminf_{t \rightarrow +\infty} x(t, x_0, y_0), \liminf_{t \rightarrow +\infty} y(t, x_0, y_0) \right\} &\geq m, \\ \max \left\{ \limsup_{t \rightarrow +\infty} x(t, x_0, y_0), \limsup_{t \rightarrow +\infty} y(t, x_0, y_0) \right\} &\leq M. \end{aligned}$$

Theorem 2.1 *All solutions $(x(t), y(t))$ of system (1.6) are positive under initial condition (1.7), i.e., $x(t) > 0, y(t) > 0$ for all $t \geq 0$.*

Proof Since

$$x(t) = x(0) \exp \left\{ \int_0^t \left(1 - x(s) - a_1 y(s) - \frac{b_1}{c_1 + x(s)} \right) ds \right\} > 0$$

and

$$y(t) = y(0) \exp \left\{ \int_0^t \rho(1 - y(s) - a_2x(s)) ds \right\} > 0.$$

So $x(t) > 0, y(t) > 0$ for all $t \geq 0$ with initial condition (1.7).

This completes the proof. □

Theorem 2.2 *All solutions $(x(t), y(t))$ of system (1.6) that satisfy initial condition (1.7) are bounded, for all $t \geq 0$.*

Proof Based on the positivity of variable x, y , from system (1.6), we have

$$\frac{dx}{dt} = x \left(1 - x - a_1y - \frac{b_1}{c_1 + x} \right) \leq x(1 - x). \tag{2.1}$$

From Lemma 2.1, we can obtain

$$\limsup_{t \rightarrow +\infty} x(t) \leq 1. \tag{2.2}$$

Meanwhile, from system (1.6), we have

$$\frac{dy}{dt} = \rho y(1 - y - a_2x) \leq \rho y(1 - y). \tag{2.3}$$

Again from the same Lemma 2.1, we can get

$$\limsup_{t \rightarrow +\infty} y(t) \leq 1. \tag{2.4}$$

This completes the proof. □

Theorem 2.3 *Assume that $a_1 + \frac{b_1}{c_1} < 1, 0 < a_2 < 1$, system (1.6) is permanent with initial condition (1.7).*

Proof From (2.2) and (2.4), for $\varepsilon > 0$ small enough, there is $T > 0$ such that, for $t > T$, we have

$$x(t) \leq 1 + \varepsilon, \quad y(t) \leq 1 + \varepsilon.$$

Then from the first equation of system (1.6), one could get

$$\frac{dx}{dt} = x \left(1 - x - a_1y - \frac{b_1}{c_1 + x} \right) \geq x \left(1 - a_1(1 + \varepsilon) - \frac{b_1}{c_1} - x \right).$$

From Lemma 2.1, when $1 - a_1(1 + \varepsilon) - \frac{b_1}{c_1} > 0$, we can get

$$\liminf_{t \rightarrow +\infty} x(t) \geq 1 - a_1(1 + \varepsilon) - \frac{b_1}{c_1}.$$

Let $\varepsilon \rightarrow 0$ in this inequality, we can obtain

$$\liminf_{t \rightarrow +\infty} x(t) \geq 1 - a_1 - \frac{b_1}{c_1} = \omega_1.$$

In the same way, we can obtain the following results for $y(t)$:

$$\liminf_{t \rightarrow +\infty} y(t) \geq \omega_2,$$

where $\omega_2 = 1 - a_2 > 0$, i.e., $0 < a_2 < 1$. So we choose $m = \min(\omega_1, \omega_2)$, $M = 1$.

This completes the proof. □

3 The existence of equilibria

The equilibria of system (1.6) are determined by the following equations:

$$\begin{aligned} x \left(1 - x - a_1 y - \frac{b_1}{c_1 + x} \right) &= 0, \\ \rho y (1 - y - a_2 x) &= 0. \end{aligned} \tag{3.1}$$

Obviously, system (1.6) always has two boundary equilibria $E_0(0, 0)$ and $E_1(0, 1)$ for all parameters. For other possible boundary equilibria and positive equilibria, we consider the following cases:

(i) When $x \neq 0, y = 0$, the other boundary equilibria of system (1.6) satisfy the following equation:

$$x^2 - (1 - c_1)x + b_1 - c_1 = 0. \tag{3.2}$$

Let Δ_1 denote the discriminant with express Δ_1 in terms of b_1 , i.e.,

$$\Delta_1(b_1) = (1 + c_1)^2 - 4b_1.$$

If the equilibria for system (1.6) exist, then $\Delta_1(b_1) \geq 0$, i.e.,

$$(H_1) \quad b_1 \leq \frac{1+c_1}{4} := b_1^*,$$

and

$$x_{21} = \frac{1 - c_1 - \sqrt{\Delta_1(b_1)}}{2}, \quad x_{22} = \frac{1 - c_1 + \sqrt{\Delta_1(b_1)}}{2}.$$

If $(1 - c_1)^2 = \Delta_1(b_1)$, one could calculate that

$$(H_2) \quad b_1 = c_1.$$

Therefore, we can calculate that

$$(H_3) \quad \text{if } b_1 > c_1, \text{ then } (1 - c_1)^2 > \Delta_1(b_1);$$

$$(H_4) \quad \text{if } b_1 < c_1, \text{ then } (1 - c_1)^2 < \Delta_1(b_1).$$

Besides, we have $c_1 \leq b_1^*$ and $c_1 = b_1^*$ if and only if $c_1 = 1$.

From conditions $(H_1), (H_2), (H_3),$ and (H_4) , we can get that:

(1) If $0 < b_1 < c_1, x_{21} < 0, x_{22} > 0$.

(2) If $b_1 = c_1$, when $0 < c_1 < 1, x_{21} = 0, x_{22} > 0$; when $c_1 = 1, x_{21} = x_{22} = 0$; when $c_1 > 1, x_{21} < 0, x_{22} = 0$.

(3) If $c_1 < b_1 < b_1^*$, when $0 < c_1 < 1, x_{21} > 0, x_{22} > 0$; when $c_1 > 1, x_{21} < 0, x_{22} < 0$.

(4) If $b_1 = b_1^*$, when $0 < c_1 < 1, x_{23} > 0$.

(5) If $b_1 > b_1^*$, system (1.6) has no other boundary equilibria.

(ii) When $x \neq 0, y \neq 0$, the possible positive equilibria of system (1.6) are determined by the following equation:

$$1 - x - a_1y - \frac{b_1}{c_1 + x} = 0, \tag{3.3}$$

$$1 - y - a_2x = 0,$$

i.e.,

$$Ax^2 - Bx + C = 0, \tag{3.4}$$

where $A = a_1a_2 - 1, B = a_1 + c_1 - a_1a_2c_1 - 1, C = c_1 - a_1c_1 - b_1$.

Let the discriminant of (3.4) be denoted by Δ_2 , i.e.,

$$\Delta_2 = B^2 - 4AC = (-c_1A - a_1 + 1)^2 + 4b_1A.$$

It is obvious that $\Delta_2 > 0$ if $A > 0$.

When $\Delta_2 \geq 0$, the existence of positive equilibria of system (1.6), and

$$x_{31} = \frac{B - \sqrt{\Delta_2}}{2A}, x_{32} = \frac{B + \sqrt{\Delta_2}}{2A}. \tag{3.5}$$

(6) If $\Delta_2 > 0$, let us discuss the following cases:

Case 1: When (a) $A > 0, B > 0, C < 0$ or (b) $A > 0, B < 0, C < 0$, then $x^* = x_{32} > 0, x_{31} < 0$ and system (1.6) has only one positive equilibrium $E^*(x^*, y^*) = (x^*, 1 - a_2x^*)$ if $\frac{1}{a_2} > x^*$.

Case 2: When (c) $A < 0, B > 0, C > 0$ or (d) $A < 0, B < 0, C > 0$, then $x^* = x_{31} > 0, x_{32} < 0$ and system (1.6) has only one positive equilibrium $E^*(x^*, y^*) = (x^*, 1 - a_2x^*)$ if $\frac{1}{a_2} > x^*$.

Case 3: When $A < 0, B < 0, C < 0$, then $x_{31} > x_{32} > 0$ and system (1.6) has two positive equilibria:

$$E_{31}(x_{31}, y_{31}) = (x_{31}, 1 - a_2x_{31})$$

and

$$E_{32}(x_{32}, y_{32}) = (x_{32}, 1 - a_2x_{32}).$$

Both E_{31} and E_{32} will exist if $\frac{1}{a_2} > \max\{x_{31}, x_{32}\}$.

Case 4: When $A > 0, B < 0, C = 0$, then $x_{32} = 0 > x_{31}$ and system (1.6) has no positive equilibria.

Case 5: When $A < 0, B > 0, C = 0$, then $x_{31} = 0 > x_{32}$ and system (1.6) has no positive equilibria.

Case 6: When $A < 0, B < 0, C = 0$, then $x_{31} > 0 = x_{32}$ and system (1.6) has only one positive equilibrium $E_{31}(x_{31}, y_{31}) = (x_{31}, 1 - a_2x_{31})$ if $\frac{1}{a_2} > x_{31}$.

(7) If $\Delta_2 = 0, B < 0$, then $x_{31} = x_{32} = x_{33} > 0$ and system (1.6) has only one positive equilibrium $E_{33}(x_{33}, y_{33}) = (\frac{B}{2A}, 1 - a_2 \frac{B}{2A})$ if $\frac{1}{a_2} > \frac{B}{2A}$.

(8) If $\Delta_2 < 0$, then system (1.6) has no positive equilibria.

From what has been discussed above, we can get the following theorem.

Theorem 3.1 *System (1.6) always has two boundary equilibria $E_0(0, 0)$ and $E_1(0, 1)$ for all parameters. The other possible boundary equilibria and positive equilibria are as follows:*

- (i) *For other possible boundary equilibria:*
 - (1) *If $0 < b_1 < c_1$, system (1.6) has only one other boundary equilibrium $E_{22}(x_{22}, 0)$.*
 - (2) *If $b_1 = c_1$ and $0 < c_1 < 1$, system (1.6) has only one other boundary equilibrium $E_{22}(x_{22}, 0)$.*
 - (3) *If $c_1 < b_1 < b_1^*$ and $0 < c_1 < 1$, system (1.6) has two other boundary equilibria $E_{21}(x_{21}, 0)$ and $E_{22}(x_{22}, 0)$.*
 - (4) *If $b_1 = b_1^*$ and $0 < c_1 < 1$, system (1.6) has only one other boundary equilibrium $E_{23}(x_{23}, 0)$.*
 - (5) *If $b_1 > b_1^*$, system (1.6) has no other boundary equilibria.*
- (ii) *For positive boundary equilibria:*
 - (6) *If $\Delta_2 \geq 0$, we have the following.*
 - (a) *When $A > 0, B > 0, C < 0$ or $A > 0, B < 0, C < 0$ or $A < 0, B < 0, C \geq 0$ or $A < 0, B > 0, C > 0$, then system (1.6) has only one positive equilibrium $E^*(x^*, y^*)$ if $\frac{1}{a_2} > x^*$.*
 - (b) *When $A < 0, B < 0, C < 0$, then system (1.6) has two positive equilibria $E_{31}(x_{31}, y_{31})$ and $E_{32}(x_{32}, y_{32})$ if $\frac{1}{a_2} > \max\{x_{31}, x_{32}\}$.*
 - (7) *If $\Delta_2 = 0$, then system (1.6) has only one positive equilibrium $E_{33}(x_{33}, y_{33})$.*
 - (8) *If $\Delta_2 < 0$, then system (1.6) has no positive equilibria.*

4 Stability of equilibria

Theorem 4.1 *For the boundary equilibria E_0 and E_1 of system (1.6), which always exist, we have:*

- (1) E_0 is always unstable.
- (2) *For E_1 , we have the following results:*
 - (a) *If $b_1 < c_1(1 - a_1)$, E_1 is a saddle;*
 - (b) *If $b_1 > c_1(1 - a_1)$, E_1 is a stable node;*
 - (c) *If $b_1 < c_1(1 - a_1)$, E_1 is a saddle node with $a_1 a_2 - 1 + \frac{1 - a_1}{c_1} \neq 0$; E_1 is a saddle with $a_1 a_2 - 1 + \frac{1 - a_1}{c_1} = 0$.*

Proof Firstly, we calculate the Jacobi matrix of system (1.6):

$$J(x, y) = \begin{pmatrix} 1 - 2x - a_1 y - \frac{b_1 c_1}{(c_1 + x)^2} & -a_1 x \\ -\rho a_2 y & \rho(-a_2 x - 2y + 1) \end{pmatrix}. \tag{4.1}$$

(1) The Jacobian matrix of system (1.6) at E_0 is

$$J(E_0) = \begin{pmatrix} 1 - \frac{b_1}{c_1} & 0 \\ 0 & \rho \end{pmatrix}. \tag{4.2}$$

Obviously, the eigenvalues of $J(E_0)$ are $\lambda_1 = 1 - \frac{b_1}{c_1}$ and $\lambda_2 = \rho > 0$, so E_0 is always unstable.

(2) The Jacobian matrix of system (1.6) at E_1 is given by

$$J(E_1) = \begin{pmatrix} 1 - a_1 - \frac{b_1}{c_1} & 0 \\ -\rho a_2 & -\rho \end{pmatrix}. \tag{4.3}$$

It is obvious that the eigenvalues of $J(E_1)$ are $\lambda_1 = 1 - a_1 - \frac{b_1}{c_1}$ and $\lambda_2 = -\rho < 0$, so the stability of $J(E_1)$ depends on λ_1 . E_1 is a saddle if $b_1 < c_1(1 - a_1)$ and a stable node if $b_1 > c_1(1 - a_1)$. When $b_1 = c_1(1 - a_1)$, we cannot directly come to the conclusion.

Let us first move E_1 to the origin by transforming $(X, Y) = (x, y - 1)$ and make Taylor’s expansion around the origin, under which system (1.6) is as follows:

$$\begin{aligned} \frac{dX}{dt} &= \left(\frac{b_1}{c_1^2} - 1\right)x^2 - a_1xy - \frac{b_1}{c_1^3}x^3 + \frac{b_1}{c_1^4}x^4, \\ \frac{dY}{dt} &= -\rho a_2x - \rho y - \rho y^2 - \rho a_2xy. \end{aligned} \tag{4.4}$$

Next, making the following transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ -a_2^2 & -a_2 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \tag{4.5}$$

and letting $\tau = \rho t$, system (4.4) becomes

$$\begin{aligned} \frac{dX_1}{d\tau} &= a_{01}X_1Y_1 + a_{02}X_1^2 + a_{03}X_1^3 + P_1(X_1, Y_1), \\ \frac{dY_1}{d\tau} &= Y_1 + b_{01}X_1Y_1 + b_{02}X_1^2 + b_{03}Y_1^2 + b_{04}X_1^3 + Q_1(X_1, Y_1) := P(X_1, Y_1), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} a_{01} &= -\frac{a_1a_2}{\rho}, & a_{02} &= \frac{a_2(c_1(a_1a_2 - 1) + 1 - a_1)}{\rho c_1}, & a_{03} &= -\frac{b_1a_2^2}{\rho c_1^3}, \\ b_{01} &= \frac{a_2^2(a_1 - \rho)}{\rho}, & b_{02} &= -\frac{a_2^2(c_1(a_1a_2 - 1) + 1 - a_1)}{\rho c_1}, & b_{03} &= a_1, & b_{04} &= \frac{b_1a_2^3}{\rho c_1^3}. \end{aligned}$$

And $P_1(X_1, Y_1)$ and $Q_1(X_1, Y_1)$ are power series in (X_1, Y_1) with terms $X_1^i Y_1^j$ satisfying $i + j \geq 4$.

From the implicit function theorem, there exists a function $Y_1 = \varphi(X_1)$ that satisfies $\varphi(0) = 0$ and $P(X_1, \varphi(X_1)) = 0$. By using the second equation of system (4.6), we can obtain

$$Y_1 = \varphi(X_1) = -b_{02}X_1^2 + o(X_1^3). \tag{4.7}$$

Substituting (4.7) into the first equation of system (4.6), we have

$$\frac{dX_1}{d\tau} = a_{02}X_1^2 - \left(a_{01}b_{02} + \frac{a_2^2(1 - a_1)}{\rho c_1^2}\right)X_1^3 + \dots$$

By Theorem 7.1 in Chap. 2 in [39], when $a_{02} \neq 0$, i.e., $m = 2$, E_1 is a saddle node; when $a_{02} = 0$, then we have $m = 3$, $a_m = \frac{a_2^2(a_1-1)}{\rho c_1^2} < 0$ and E_1 is a saddle.

This completes the proof. □

Theorem 4.2 *Assume that other boundary equilibria E_{21}, E_{22} , and E_{23} of system (1.6) exist, we have:*

- (1) E_{21} is always unstable.
- (2) For E_{22} , we have the following results:
 - (a) If $\frac{1}{a_2} > x_{22}$, E_{22} is a saddle;
 - (b) If $\frac{1}{a_2} < x_{22}$, E_{22} is a stable node;
 - (c) If $\frac{1}{a_2} = x_{22}$, E_{22} is a saddle node.
- (3) If $a_2(c_1 - 1) \neq -2$, E_{23} is a saddle node; on the contrary, E_{23} consists of a hyperbolic sector and an elliptic sector.

Proof When other boundary equilibrium points exist, we have

$$1 - x_{2i} - a_1 y_{2i} - \frac{b_1}{c_1 + x_{2i}} = 0,$$

here, $i = 1, 2, 3$.

So (4.1) can be written as

$$J(E_{2i}) = \begin{pmatrix} \frac{x_{2i}}{(c_1+x_{2i})^2}(b_1 - (c_1 + x_{2i})^2) & -a_1 x_{2i} \\ 0 & \rho(1 - a_2 x_{2i}) \end{pmatrix}. \tag{4.8}$$

(1) It is easy to get from (4.8) that the eigenvalues of $J(E_{21})$ are $\lambda_1 = \frac{x_{21}}{(c_1+x_{21})^2}(b_1 - (c_1 + x_{21})^2)$ and $\lambda_2 = \rho(1 - a_2 x_{21})$, while

$$\begin{aligned} \lambda_1 &= \frac{x_{21}}{(c_1 + x_{21})^2} (b_1 - (c_1 + x_{21})^2) \\ &= \frac{x_{21} \sqrt{\Delta_1(b_1)} (\sqrt{\Delta_1(b_1) + 4b_1} - \sqrt{\Delta_1(b_1)})}{2(c_1 + x_{21})^2} > 0, \end{aligned}$$

so E_{21} is unstable.

(2) From (4.8), one could obtain $\lambda_1 = \frac{x_{22}}{(c_1+x_{22})^2}(b_1 - (c_1 + x_{22})^2)$ and $\lambda_2 = \rho(1 - a_2 x_{22})$ are the eigenvalues of $J(E_{22})$; nevertheless,

$$\begin{aligned} \lambda_1 &= \frac{x_{22}}{(c_1 + x_{22})^2} (b_1 - (c_1 + x_{22})^2) \\ &= \frac{-x_{22} \sqrt{\Delta_1(b_1)} (\sqrt{\Delta_1(b_1) + 4b_1} + \sqrt{\Delta_1(b_1)})}{2(c_1 + x_{22})^2} < 0, \end{aligned}$$

so the stability of $J(E_{22})$ depends on λ_2 . If $\lambda_2 > 0$, i.e., $\frac{1}{a_2} > x_{22}$, then E_{22} is a saddle; if $\lambda_2 < 0$, i.e., $\frac{1}{a_2} < x_{22}$, then E_{22} is a stable node; if $\lambda_2 = 0$, i.e., $\frac{1}{a_2} = x_{22}$, it is difficult for us to draw the conclusion.

Now, let us use Theorem 7.1 in Chap. 2 in [39] to determine its stability. First of all, we make transformation $(X, Y) = (x - x_{22}, y)$ to move E_{22} to the original, and then expand in

power series up to the third order around the origin, which makes the system as follows:

$$\begin{aligned} \frac{dX}{dt} &= c_{01}X - Y - a_1XY + c_{02}X^2 + c_{03}X^3, \\ \frac{dY}{dt} &= -\rho a_2XY - \rho Y^2, \end{aligned} \tag{4.9}$$

where $c_{01} = x_{22}(\frac{b_1}{(c_1+x_{22})^2} - 1)$, $c_{02} = \frac{b_1c_1}{(c_1+x_{22})^3} - 1$, $c_{03} = -\frac{b_1c_1}{(c_1+x_{22})^4}$.

Next, we make the following transformation:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -c_{01} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \tag{4.10}$$

and let $\tau = c_{01}t$, system (4.9) becomes

$$\begin{aligned} \frac{dX_2}{d\tau} &= X_2 + d_{01}X_2Y_2 + d_{02}X_2^2 + d_{03}Y_2^2 + P_2(X_2, Y_2) := P(X_2, Y_2), \\ \frac{dY_2}{d\tau} &= e_{01}X_2Y_2 + e_{02}Y_2^2 + Q_2(X_2, Y_2), \end{aligned} \tag{4.11}$$

where $d_{01} = \frac{a_1c_{01}-\rho a_2-2c_{02}}{c_{01}}$, $d_{02} = \frac{c_{02}}{c_{01}}$, $d_{03} = \rho - a_1 + \frac{c_{02}+\rho a_2}{c_{01}}$, $e_{01} = -\frac{\rho a_2}{c_{01}}$, $e_{02} = \rho(\frac{a_2}{c_{01}} + 1)$, and $P_2(X_2, Y_2)$ and $Q_2(X_2, Y_2)$ are power series in (X_2, Y_2) with terms $X_2^i Y_2^j$ satisfying $i + j \geq 3$.

According to the implicit function theorem, there is a function $X_2 = \varphi(Y_2)$ such that $\varphi(0) = 0$ and $P(\varphi(Y_2), Y_2) = 0$. From the first equation of system (4.11), we have

$$X_2 = \varphi(Y_2) = -d_{03}Y_2^2 + o(Y_2^3). \tag{4.12}$$

Substituting (4.12) into the second equation of system (4.11), we can get

$$\frac{dY_2}{d\tau} = e_{02}Y_2^2 + \dots$$

By Theorem 7.1 in Chap. 2 in [39], because $e_{02} > 0$ and E_{22} is the saddle node.

(3) From (4.8) and $x_{23} = \frac{1-c_1}{2}$, one could get the eigenvalues of $J(E_{23})$ are $\lambda_1 = 0$ and $\lambda_2 = \rho(1 - a_2x_{23})$. In order to get the stability of E_{23} , firstly, we make transformation $(X, Y) = (x - x_{23}, y_{23})$ to move E_{23} to the origin, and then expand in power series up to the third order around the origin, which makes the system as follows:

$$\begin{aligned} \frac{dX}{dt} &= a_{01}Y + a_{02}XY + a_{03}X^2 + a_{04}X^3, \\ \frac{dY}{dt} &= b_{01}Y + b_{02}XY + b_{03}Y^2, \end{aligned} \tag{4.13}$$

where $a_{01} = \frac{a_1(c_1-1)}{2}$, $a_{02} = -a_1$, $a_{03} = \frac{c_1-1}{c_1+1}$, $a_{04} = \frac{-4c_1}{(c_1+1)^2}$, $b_{01} = \frac{a_2(c_1-1)+2}{2}$, $b_{02} = -a_2\rho$, $b_{03} = -\rho$. Secondly, we discuss it in two cases.

Case 1: If $\lambda_2 \neq 0$, the stability of E_{23} can be proved by the same method in Theorem 4.2(2), so E_{23} is a saddle node if $x_{23} \neq \frac{1}{a_2}$.

Case 2: If $\lambda_2 = 0$, let $\tau = -\frac{a_1}{a_2}t$, then system (4.13) becomes

$$\begin{aligned} \frac{dX}{d\tau} &= Y + a_2XY + c_{01}X^2 + c_{02}X^3 + P_3(X, Y) := P(X, Y), \\ \frac{dY}{d\tau} &= d_{01}XY + d_{02}Y^2 + Q_3(X, Y). \end{aligned} \tag{4.14}$$

Here, $c_{01} = \frac{2}{a_1(c_1+1)}$, $c_{02} = \frac{4a_2c_1}{a_1(c_1+1)^2}$, $d_{01} = \frac{\rho a_2^2}{a_1}$, $d_{02} = \frac{\rho a_2}{a_1}$, and $P_3(X, Y)$ and $Q_3(X, Y)$ are power series in (X, Y) with terms $X^i Y^j$ satisfying $i + j \geq 3$.

According to the implicit function theorem, there is a function $Y = \varphi(X)$ such that $\varphi(0) = 0$ and $P(X, \varphi(X)) = 0$. From the first equation of system (4.14), we have

$$Y = \varphi(X) = \frac{a_2(c_1 - 1)}{a_1(c_1 + 1)}X^2 + o(X^3). \tag{4.15}$$

Substituting (4.15) into the second equation of system (4.14), we can get

$$\frac{dY}{d\tau} = \frac{a_2^3 \rho (c_1 - 1)}{a_1^2 (c_1 + 1)}X^3 + \dots$$

and

$$v(X) = \frac{a_2}{a_1} \left(\frac{2(1 - c_1)}{1 + c_1} + \rho a_2 \right) X + \dots$$

By Theorem 7.3 in Chap. 2 in [39], because $k = 2m + 1 = 3$, $m = 1$, $a_k = \frac{a_2^3 \rho (c_1 - 1)}{a_1^2 (c_1 + 1)} < 0$, $N = 1$, $B_N = \frac{a_2}{a_1} \left(\frac{2(1 - c_1)}{1 + c_1} + \rho a_2 \right) > 0$, $\lambda = B_N^2 + 4(m + 1)a_{2m+1} = \left(\frac{a_2}{a_1} \left(\frac{2(1 - c_1)}{1 + c_1} - \rho a_2 \right) \right)^2 > 0$, then E_{23} consists of a hyperbolic sector and an elliptic sector.

This completes the proof. □

Theorem 4.3 *When the positive equilibria exist, we have:*

- (1) E_{31} is a stable node.
- (2) E_{32} is a saddle.
- (3) E_{33} is a saddle node.

Proof Notice that when the positive equilibria exist, (4.1) can be simplified as follows:

$$J(E_{3i}) = \begin{pmatrix} x_{3i} \left(\frac{b_1}{(c_1 + x_{3i})^2} - 1 \right) & -a_1 x_{3i} \\ -\rho a_2 y_{3i} & -\rho y_{3i} \end{pmatrix}, \tag{4.16}$$

where $i = 1, 2, 3$.

Thus,

$$\begin{aligned} \text{tr} J(E_{3i}) &= x_{3i} \left(\frac{b_1}{(c_1 + x_{3i})^2} - 1 \right) - \rho y_{3i}, \\ \det J(E_{3i}) &= -\rho x_{3i} y_{3i} \left(\frac{b_1}{(c_1 + x_{3i})^2} - 1 \right) - \rho a_1 a_2 x_{3i} y_{3i} \\ &= -\rho x_{3i} y_{3i} \left(\frac{b_1}{(c_1 + x_{3i})^2} - 1 + a_1 a_2 \right). \end{aligned}$$

Using

$$1 - x_{3i} - a_1 y_{3i} - \frac{b_1}{c_1 + x_{3i}} = 0$$

and

$$1 - y_{3i} - a_2 x_{3i} = 0,$$

we have

$$\det J(E_{3i}) = \frac{-\rho x_{3i} y_{3i}}{c_1 + x_{3i}} (2A x_{3i} - B).$$

(1) According to (3.5), we get that

$$\det J(E_{32}) = \frac{-\rho x_{32} y_{32} \sqrt{\Delta_2}}{c_1 + x_{32}} < 0,$$

so E_{32} is a saddle.

(2) Similarly, we have

$$\det J(E_{31}) = \frac{\rho x_{31} y_{31} \sqrt{\Delta_2}}{c_1 + x_{31}} > 0.$$

Therefore, we consider the sign of $\text{tr} J(E_{31})$. Let $N(x) = \frac{b_1}{(c_1+x)^2} - 1, x \in (0, +\infty)$, then we have

$$\frac{dN(x)}{dx} = \frac{-2b_1}{(c_1 + x)^3} < 0.$$

Note that $N(x)$ is monotonically decreasing in the interval $(0, +\infty)$. It is clear that

$$0 < x_{32} < x_{33} < x_{31}$$

and

$$N(x_{33}) = \frac{b_1}{(c_1 + x_{33})^2} - 1 = -a_1 a_2 < 0.$$

So we can obtain $N(x_{31}) < 0$, and it is easy to calculate that

$$\text{tr} J(E_{31}) = x_{31} N(x_{31}) - \rho y_{31} < 0.$$

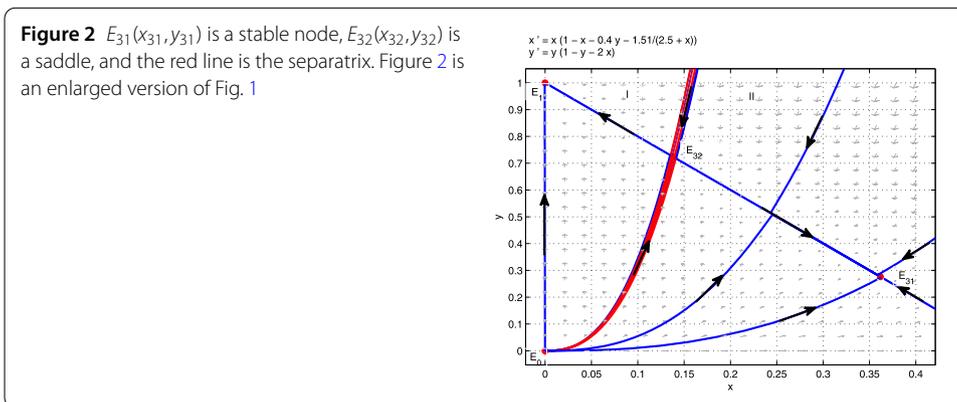
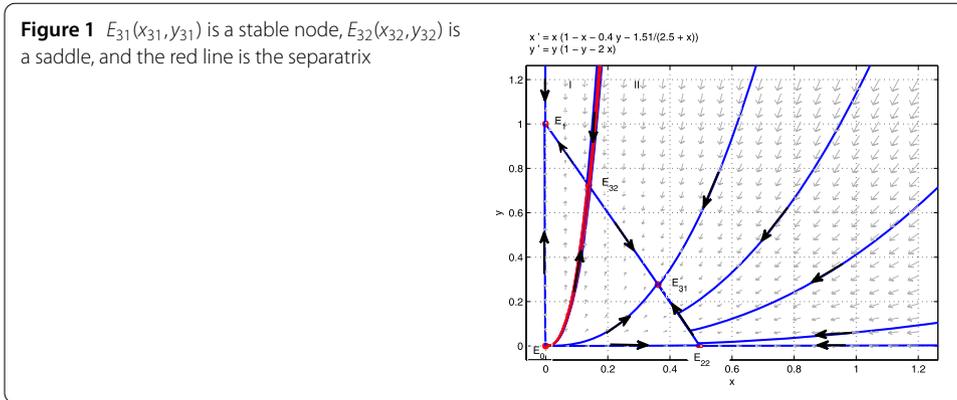
The above analysis shows that E_{31} is a stable node.

(3) From $x_{33} = \frac{B}{2A}$, we can get

$$\det J(E_{33}) = \frac{-\rho x_{33} y_{33}}{c_1 + x_{33}} \left(2A * \frac{B}{2A} - B \right) = 0,$$

$$\text{tr} J(E_{33}) = x_{33} \left(\frac{b_1}{(c_1 + x_{33})^2} - 1 \right) - \rho y_{33}$$

$$= -a_1 a_2 x_{33} - \rho y_{33} < 0,$$



so the eigenvalues of $J(E_{33})$ are $\lambda_1 < 0$ and $\lambda_2 = 0$, we use the same method as Theorem 4.2(2) and easily get E_{33} is a saddle node.

In order to verify the above results, let $a_1 = 0.4, a_2 = 2, b_1 = 1.51, c_1 = 2.5, \rho = 1$. By simple computation, we have $A < 0, B < 0, C < 0$, and $\Delta_2 > 0$, $E_{31}(0.3618, 0.27639)$ is a stable node, $E_{32}(0.138197, 0.72361)$ is a saddle. From Fig. 1, it is easy to get the red line that divides the first quadrant into two parts, recorded as I (left one) and II (right one). Assume that the initial conditions are located in region I, all the solutions tend to $E_1(0, 1)$ which is a stable manifold and $E_{32}(0.138197, 0.72361)$ is an unstable manifold. From the biological point of view, when the initial values are located in region I, the first species will be driven to extinction. On the contrary, assume that the initial conditions are located in region II, all the solutions tend to $E_{31}(0.3618, 0.27639)$ which is a stable manifold, $E_{32}(0.138197, 0.72361)$ and $E_{22}(0.49599, 0)$ are unstable manifolds. From the biological point of view, when the initial values are located in region II, two species can always coexist. This is the bistable phenomenon which is shown in Figs. 1 and 2.

This completes the proof. □

Theorem 4.4 *For the unique positive equilibrium that exists for system (1.6), our results are as follows:*

- (1) *When $A > 0, B > 0, C < 0$ or $A > 0, B < 0, C < 0$ or $A < 0, B < 0, C \geq 0$, E^* is a saddle;*
- (2) *When $A < 0, B > 0, C > 0$, E^* is a stable node.*

Proof Through the discussion in Theorem 4.3, we can get

$$\begin{aligned} \det J(E^*) &= \frac{-\rho x^* y^*}{c_1 + x^*} (2Ax^* - B), \\ \operatorname{tr} J(E^*) &= x^* \left(\frac{b_1}{(c_1 + x^*)^2} - 1 \right) - \rho y^* \\ &= \frac{x^*}{c_1 + x^*} ((A - 1)x^* - (B + a_1 a_2 c_1)) - \rho y^*. \end{aligned}$$

In order to analyze the stability of E^* , let us consider the functions

$$f(x) = 2Ax - B$$

and

$$g(x) = (A - 1)x - (B + a_1 a_2 c_1).$$

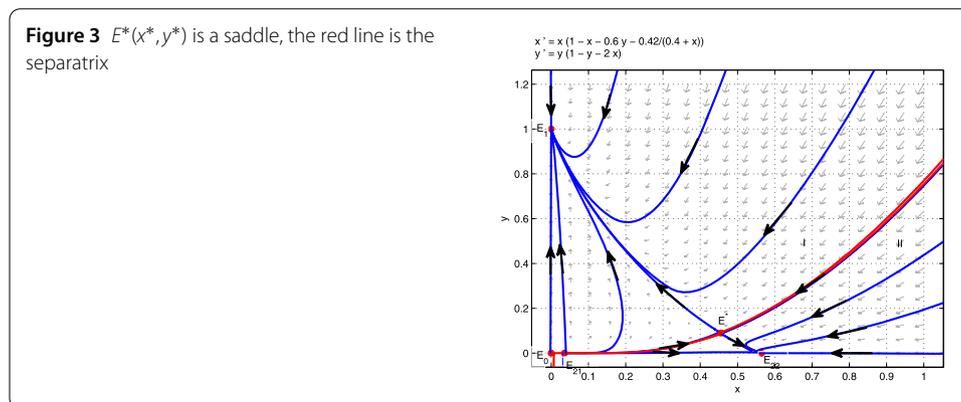
Next let us discuss them in different cases:

Case 1: When $A > 0, B > 0, C < 0$ or $A > 0, B < 0, C < 0$, note that $x^* = x_{32} > \frac{B}{2A}$, thus it is easy for us to get $f(x^*) > 0$, i.e., $\det J(E^*) < 0$, so E^* is a saddle.

Case 2: When $A < 0, B < 0, C \geq 0$, we have $x^* = x_{31} < \frac{B}{2A}$, which gives $f(x^*) > 0$, i.e., $\det J(E^*) < 0$, so E^* is a saddle.

Case 3: When $A < 0, B > 0, C > 0$, we have $x^* = x_{31} > 0$, which implies that $f(x^*) < 0, g(x^*) < 0$, i.e., $\det J(E^*) > 0, \operatorname{tr} J(E^*) < 0$, so E^* is a stable node.

In order to verify the above results, let $a_1 = 0.6, a_2 = 2, b_1 = 0.42, c_1 = 0.4, \rho = 1$. By simple computation, we have $A > 0, B < 0, C < 0$. $E^*(x^*, y^*)$ and $E_{21}(0.03542, 0)$ are saddle, $E_1(0, 1)$ and $E_{22}(0.56458, 0)$ are stable nodes. From Fig. 3, it is easy to get the red line that divides the first quadrant into two parts, recorded as I (left one) and II (right one). Assume that the initial conditions are located in region I, all the solutions tend to $E_1(0, 1)$ which is a stable manifold, $E^*(x^*, y^*)$ and $E_{21}(0.03542, 0)$ are unstable manifolds. From the biological point of view, when the initial values are located in region I, the first species will be driven to extinction. On the contrary, assume that the initial conditions are located in region II, all the solutions tend to $E_{22}(0.56458, 0)$ which is a stable manifold, $E^*(x^*, y^*)$ and $E_{21}(0.03542, 0)$ are unstable manifolds. From the biological point of view, when the initial



values are located in region II, the second species will be driven to extinction. This is the bistable phenomenon which is shown in Fig. 3.

This completes the proof. □

5 Global stability of equilibrium

Theorem 5.1 *When E^* is locally stable, which is globally asymptotically stable.*

Proof We will adapt the idea of Yu [40] to prove Theorem 5.1. More precisely, we will consider a Lyapunov function

$$V(x, y) = (c_1 + x^*) \left(x - x^* - x^* \ln \frac{x}{x^*} \right) + \rho a_2 \left(y - y^* - y^* \ln \frac{y}{y^*} \right).$$

It is easy to see that the function $V(x, y)$ is zero at the equilibrium $E^*(x^*, y^*)$, which is positive everywhere in the first quadrant except at E^* . Then the time derivative of $V(x, y)$ along the trajectories of (1.6) is

$$\begin{aligned} D^+ V(t) &= (c_1 + x^*) \frac{x - x^*}{x} \left(x \left(1 - x - a_1 y - \frac{b_1}{c_1 + x} \right) \right) \\ &\quad + \rho a_2 \frac{y - y^*}{y} (\rho y (1 - y - a_2 x)) \\ &= (c_1 + x^*) (x - x^*) \left(x^* + a_1 y^* + \frac{b_1}{c_1 + x^*} - x - a_1 y - \frac{b_1}{c_1 + x} \right) \\ &\quad + \rho^2 a_2 (y - y^*) (y^* + a_2 x^* - y - a_2 x) \\ &= (x - x^*)^2 \left(\frac{b_1}{c_1 + x} - (c_1 + x^*) \right) - \rho^2 a_2 (y - y^*)^2 \\ &\quad - (a_1 (c_1 + x^*) + \rho a_2^2) (x - x^*) (y - y^*) \\ &< (x - x^*)^2 \left(\frac{b_1}{c_1} - c_1 \right) - \rho^2 a_2 (y - y^*)^2. \end{aligned}$$

Note that $B > 0, C > 0$ when E^* exists, and we have $\frac{b_1}{c_1} < 1 - a_1 < c_1(1 - a_1 a_2)$, i.e., $\frac{b_1}{c_1} - c_1 < 0$, so one could obtain $D^+ V(t) < 0$ strictly for all $x, y > 0$ except the equilibrium $E^*(x^*, y^*)$, where $D^+ V(t) = 0$. Hence, $V(x, y)$ satisfies Lyapunov’s asymptotic stability theorem, then the equilibrium $E^*(x^*, y^*)$ of system (1.6) is globally asymptotically stable. From the biological point of view, two species can always coexist (see Fig. 4).

This completes the proof. □

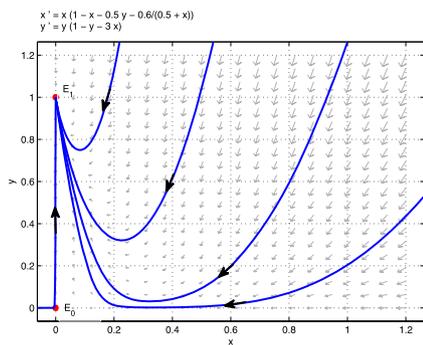
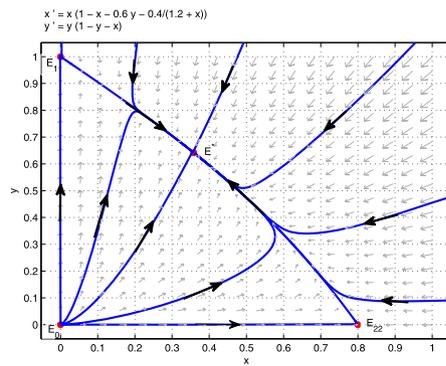
6 Bifurcation analysis

In this section, we are interested in studying the various possible bifurcations of system (1.6). From Theorem 3.1, we know that system (1.6) may undergo saddle-node bifurcation at E_{23} and E_{23} , respectively, and transcritical bifurcation around the equilibria E_0 and E_1 , which is a very interesting phenomenon.

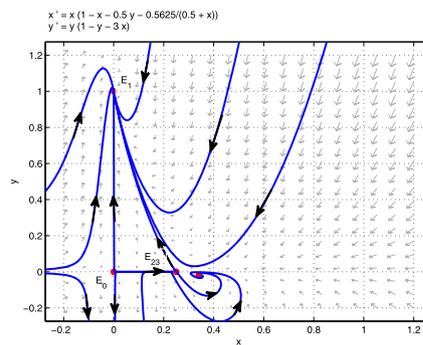
6.1 Saddle-node bifurcation

From Theorem 3.1, it is easy to find that when $c_1 < b_1 < b_1^*$ and $0 < c_1 < 1$ the system has two different boundary equilibria, which may coincide if $b_1 = b_1^*$ and $0 < c_1 < 1$ or which

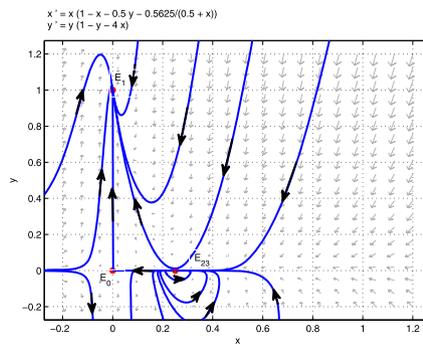
Figure 4 $E^*(x^*, y^*)$ is globally asymptotically stable



(a)



(b)



(c)

Figure 5 Dynamics behaviors of system (1.6)

may disappear if $b_1 > b_1^*$. The emergence or appearance of the equilibria is due to the saddle-node bifurcation at E_{23} (see Fig. 5 and Fig. 6).

Theorem 6.1 *System (1.6) undergoes a saddle-node bifurcation around E_{23} when $b_1 = b_{1SN} = \frac{(1+c_1)^2}{4}$ and $0 < c_1 < 1$, where b_1 is the bifurcation parameter.*

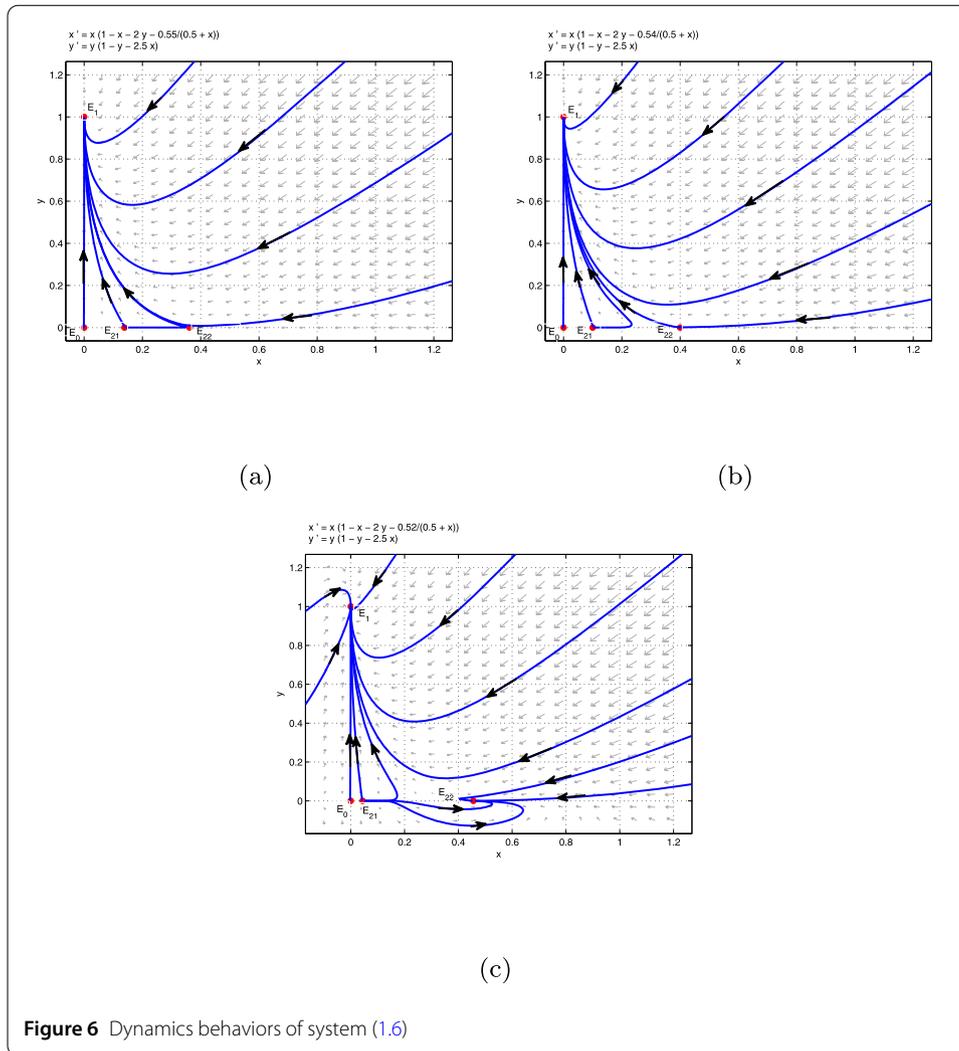


Figure 6 Dynamics behaviors of system (1.6)

Proof By the proof of Theorem 4.2, we have an eigenvalue of $J(E_{23})$ that is zero, named λ_1 . Let V_1 and W_1 represent the eigenvectors of λ_1 for the matrices $J(E_{23})$ and $J(E_{23})^T$, respectively. After simple calculation, we have

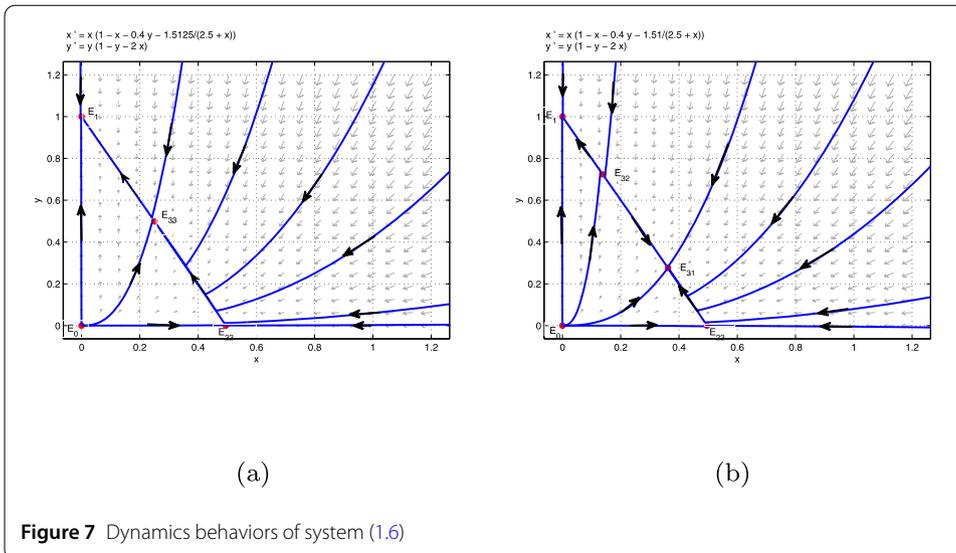
$$V_1 = \begin{pmatrix} V_{11} \\ V_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad W_1 = \begin{pmatrix} W_{11} \\ W_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{a_1(1-c_1)}{\rho(a_2(c_1-1)+2)} \end{pmatrix}.$$

Moreover,

$$F_{b_1}(E_{23}; b_{1SN}) = \begin{pmatrix} \frac{-x}{c_1+x} \\ 0 \end{pmatrix}_{x=x_{23}} = \begin{pmatrix} \frac{c_1-1}{c_1+1} \\ 0 \end{pmatrix},$$

$$D^2F(E_{23}; b_{1SN})(V_1, V_1) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_{11}^2 + 2 \frac{\partial^2 F_1}{\partial x \partial y} V_{11} V_{12} + \frac{\partial^2 F_1}{\partial y^2} V_{12}^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_{11}^2 + 2 \frac{\partial^2 F_2}{\partial x \partial y} V_{11} V_{12} + \frac{\partial^2 F_2}{\partial y^2} V_{12}^2 \end{pmatrix}_{(E_{23}; b_{1SN})}$$

$$= \begin{pmatrix} \frac{2b_1 c_1}{(c_1+x_{23})^3} - 2 \\ 0 \end{pmatrix}.$$



Clearly, we can get that V_1 and W_1 satisfy

$$W_1^T F_{b_1}(E_{23}; b_{1SN}) = \frac{c_1 - 1}{c_1 + 1} \neq 0,$$

$$W_1^T [D^2 F(E_{23}; b_{1SN})(V_1, V_1)] = \frac{2(c_1 - 1)}{c_1 + 1} \neq 0.$$

This implies that when $b_1 = b_{1SN}$, the saddle-node bifurcation occurs at E_{23} .

This completes the proof. □

Similarly, the conditions for the existence of positive equilibria of system (1.6) are given in Theorem 3.1, and we could find that when $\Delta_2 > 0, A < 0, B < 0, C < 0$, the system has two different positive equilibria, which may coincide if $\Delta_2 = 0$ and $A < 0, B < 0$ or which may disappear if $\Delta_2 < 0$. The emergence or appearance of the equilibria is due to the saddle-node bifurcation at E_{33} (see Fig. 5 (a) and Fig. 7).

Theorem 6.2 *System (1.6) undergoes a saddle-node bifurcation around E_{33} when $b_1 = \tilde{b}_{1SN}$ and $A < 0, B < 0$, where b_1 is the bifurcation parameter.*

Proof By the proof of Theorem 4.3, we know that an eigenvalue of $J(E_{23})$ is zero, named λ_1 . Let V_2 and W_2 represent the eigenvectors of λ_1 for the matrices $J(E_{33})$ and $J(E_{33})^T$, respectively. After simple calculation, we have

$$V_2 = \begin{pmatrix} V_{21} \\ V_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -a_2 \end{pmatrix}; \quad W_2 = \begin{pmatrix} W_{21} \\ W_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-a_1 x_{33}}{\rho y_{33}} \end{pmatrix}.$$

Moreover,

$$F_{b_1}(E_{33}; b_{1SN}) = \begin{pmatrix} \frac{-x}{c_1 + x} \\ 0 \end{pmatrix}_{x=x_{33}} = \begin{pmatrix} \frac{-x_{33}}{c_1 + x_{33}} \\ 0 \end{pmatrix},$$

$$\begin{aligned}
 D^2F(E_{33}; b_{1SN})(V_2, V_2) &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_{21}^2 + 2 \frac{\partial^2 F_1}{\partial x \partial y} V_{21} V_{22} + \frac{\partial^2 F_1}{\partial y^2} V_{22}^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_{21}^2 + 2 \frac{\partial^2 F_2}{\partial x \partial y} V_{21} V_{22} + \frac{\partial^2 F_2}{\partial y^2} V_{22}^2 \end{pmatrix}_{(E_{33}; b_{1SN})} \\
 &= \begin{pmatrix} 2 \left(\frac{b_1 c_1}{(c_1 + x_{33})^3} + a_1 a_2 - 1 \right) \\ 0 \end{pmatrix}.
 \end{aligned}$$

So, we can obtain V_2 and W_2 satisfy

$$\begin{aligned}
 W_2^T F_{b_1}(E_{33}; b_{1SN}) &= \frac{-x_{33}}{c_1 + x_{33}} \neq 0, \\
 W_2^T [D^2F(E_{33}; b_{1SN})(V_2, V_2)] &= \frac{AB}{2(a_1 - 1) - B} \neq 0,
 \end{aligned}$$

which means that when $b_1 = b_{1SN}$, the saddle-node bifurcation occurs at E_{33} .

This completes the proof. □

6.2 Transcritical bifurcation

Through the discussion of Theorem 3.1, we find an interesting phenomenon: when $b_1 = c_1$, E_{21} will coincide with E_0 if $0 < c_1 < 1$; E_{22} will coincide with E_0 if $c_1 > 1$. Therefore, the emergence of this phenomenon is owing to the transcritical bifurcation at E_0 (see Fig. 8). Then we obtain the following.

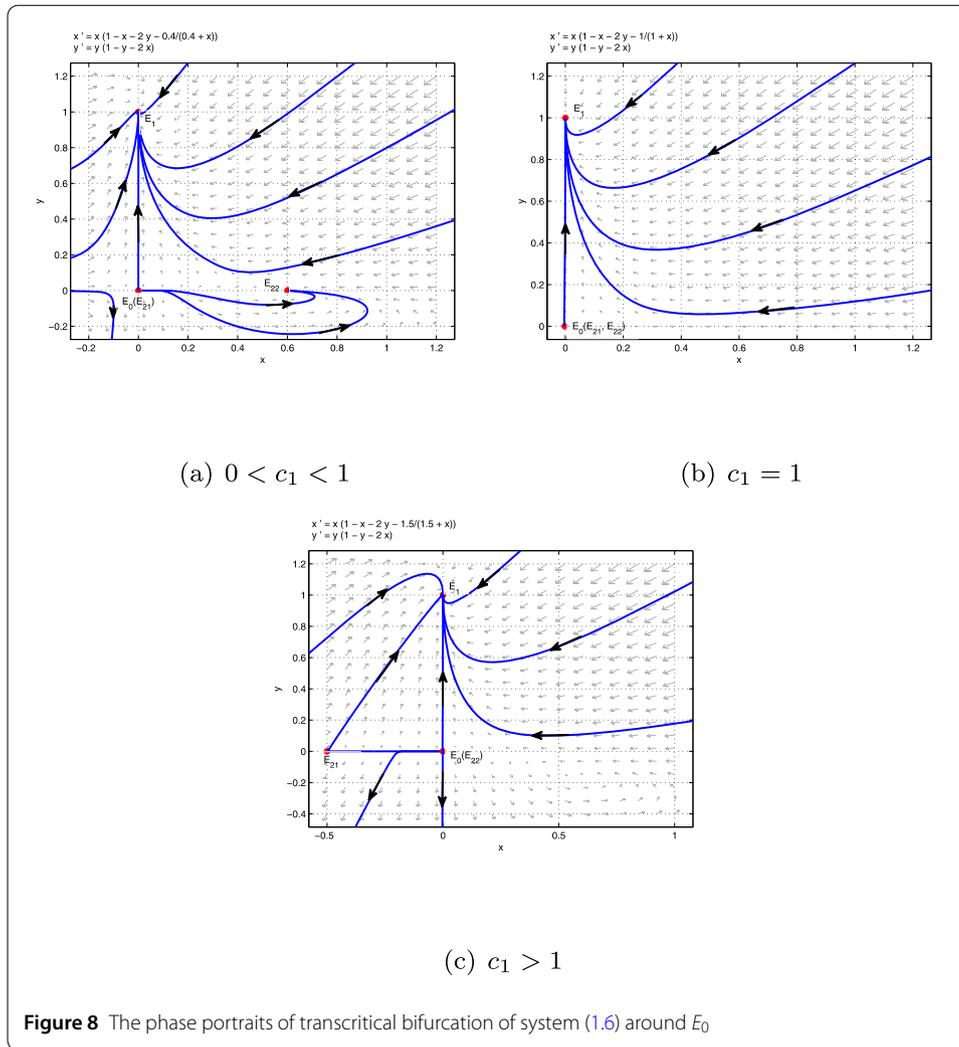
Theorem 6.3 *System (1.6) undergoes a transcritical bifurcation around E_0 with b_1 as a bifurcation parameter, when $b_1 = b_{1SN} = c_1$.*

Proof Here, we use Sotomayor’s theorem to verify the transversality conditions for transcritical bifurcation. The Jacobian matrix of system (1.6) evaluated at the point E_0 is given by (3.2). Obviously, the eigenvalue $\lambda_1 = 0$ of $J(E_0)$ if $b_1 = c_1$. Let V_3 and W_3 be the eigenvectors of $J(E_0)$ and $J(E_0)^T$ corresponding to $\lambda_1 = 0$, respectively. Then we can obtain

$$V_3 = \begin{pmatrix} V_{31} \\ V_{32} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} W_{31} \\ W_{32} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}
 F_{b_1}(E_0; b_{1TC}) &= \begin{pmatrix} \frac{-x}{c_1 + x} \\ 0 \end{pmatrix}_{x=x_0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
 DF_{b_1}(E_0; b_{1TC})V_3 &= \begin{pmatrix} \frac{-c_1}{(c_1 + x)^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(E_0; b_{1TC})} = \begin{pmatrix} -\frac{1}{c_1} \\ 0 \end{pmatrix}, \\
 D^2F(E_0; b_{1TC})(V_3, V_3) &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_{31}^2 + 2 \frac{\partial^2 F_1}{\partial x \partial y} V_{31} V_{32} + \frac{\partial^2 F_1}{\partial y^2} V_{32}^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_{31}^2 + 2 \frac{\partial^2 F_2}{\partial x \partial y} V_{31} V_{32} + \frac{\partial^2 F_2}{\partial y^2} V_{32}^2 \end{pmatrix}_{(E_0; b_{1TC})} \\
 &= \begin{pmatrix} 2 \left(\frac{1}{c_1} - 1 \right) \\ 0 \end{pmatrix}.
 \end{aligned}$$



Thus, we have

$$W_3^T F_{b_1}(E_0; b_{1TC}) = 0,$$

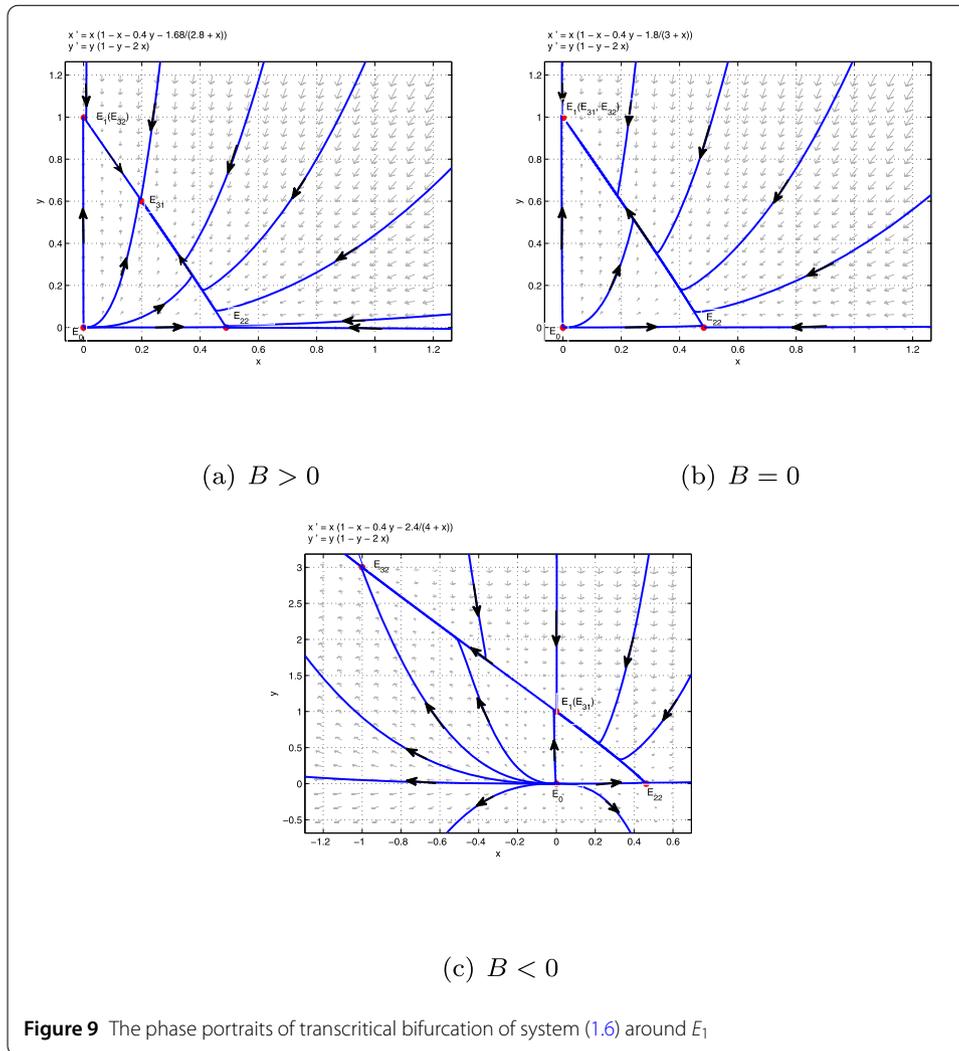
$$W_3^T [DF_{b_1}(E_0; b_{1TC}) V_3] = \frac{1}{c_1} \neq 0,$$

$$W_3^T [D^2 F(E_0; b_{1TC})(V_3, V_3)] = 2 \left(1 - \frac{1}{c_1} \right) \neq 0.$$

So from Sotomayor's theorem system (1.6) undergoes a transcritical bifurcation at E_0 . This completes the proof. □

In the same way, for the positive equilibria of system (1.6), when $b_1 = c_1(1 - a_1)$ and $A < 0$, E_{32} will coincide with E_1 if $B > 0$; E_{31} will coincide with E_1 if $B < 0$. Hence, the appearance of this phenomenon is owing to the transcritical bifurcation at E_1 (see Fig. 9). Then we can get the following.

Theorem 6.4 *System (1.6) undergoes a transcritical bifurcation around E_1 with b_1 as a bifurcation parameter, when $b_1 = b_{1SN} = c_1(1 - a_1)$ and $A < 0$.*



Proof Here, we use Sotomayor’s theorem to verify the transversality conditions for transcritical bifurcation. The Jacobian matrix of system (1.6) evaluated at the point E_1 is given by (3.2). Clearly, the eigenvalue $\lambda_1 = 0$ of $J(E_1)$ if $b_1 = c_1(1 - a_1)$. Let V_4 and W_4 be the eigenvectors of $J(E_1)$ and $J(E_1)^T$ corresponding to $\lambda_1 = 0$, respectively. Then we can get

$$V_4 = \begin{pmatrix} V_{41} \\ V_{42} \end{pmatrix} = \begin{pmatrix} 1 \\ -a_2 \end{pmatrix}, \quad W_4 = \begin{pmatrix} W_{41} \\ W_{42} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Furthermore,

$$F_{b_1}(E_1; b_{1TC}) = \begin{pmatrix} \frac{-x}{c_1+x} \\ 0 \end{pmatrix}_{x=x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_{b_1}(E_1; b_{1TC})V_4 = \begin{pmatrix} \frac{-c_1}{(c_1+x)^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -a_2 \end{pmatrix}_{(E_1; b_{1TC})} = \begin{pmatrix} -\frac{1}{c_1} \\ 0 \end{pmatrix},$$

$$\begin{aligned}
 D^2F(E_1; b_{1TC})(V_4, V_4) &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_{41}^2 + 2 \frac{\partial^2 F_1}{\partial x \partial y} V_{41} V_{42} + \frac{\partial^2 F_1}{\partial y^2} V_{42}^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_{41}^2 + 2 \frac{\partial^2 F_2}{\partial x \partial y} V_{41} V_{42} + \frac{\partial^2 F_2}{\partial y^2} V_{42}^2 \end{pmatrix}_{(E_1; b_{1TC})} \\
 &= \begin{pmatrix} -\frac{2B}{c_1} \\ 0 \end{pmatrix}.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 W_4^T F_{b_1}(E_1; b_{1TC}) &= 0, \\
 W_4^T [DF_{b_1}(E_1; b_{1TC})V_4] &= -\frac{1}{c_1} \neq 0, \\
 W_4^T [D^2F(E_1; b_{1TC})(V_4, V_4)] &= -\frac{2B}{c_1} \neq 0.
 \end{aligned}$$

So from Sotomayor’s theorem system (1.6) undergoes a transcritical bifurcation at E_1 . This completes the proof. □

7 Numeric simulations

Now, let us give the following examples to illustrate the main results.

Example 7.1 Let $a_1 = 0.6, a_2 = 1, b_1 = 0.4, c_1 = 1.2, \rho = 1$, then we have E^* is globally asymptotically stable (Fig. 4).

Example 7.2 Let $a_1 = 0.5, c_1 = 0.5, \rho = 1$, by simple computation, we have:

- (1) For $a_2 = 1, b_1 = 0.6$, we get $b_1^* = 0.375 < b_1, \Delta_2 = -0.6375 < 0$, and the system has no other boundary equilibria and positive equilibria.
- (2) For $a_2 = 3$, we get E_{23} is a saddle.
- (3) For $a_2 = 4$, we get E_{23} consists of a hyperbolic sector and an elliptic sector.

Figure 5 shows the above results.

Example 7.3 Let $a_1 = 2, a_2 = 2.5, c_1 = 0.5, \rho = 1$, by simple computation, we have $\frac{1}{a_2} = 0.4$,

- (1) For $b_1 = 0.55$, we get $x_{22} = 0.3618 < \frac{1}{a_2}$ and E_{22} is a saddle, E_{21} is unstable.
- (2) For $b_1 = 0.54$, we get $x_{22} = 0.4 = \frac{1}{a_2}$ and E_{22} is a saddle node, E_{21} is unstable.
- (3) For $b_1 = 0.52$, we get $x_{22} = 0.4562 > \frac{1}{a_2}$ and E_{22} is a stable node, E_{21} is unstable.

Figure 6 shows the above results.

Example 7.4 Let $a_1 = 0.4, a_2 = 2, c_1 = 2.5, \rho = 1$, we have:

- (1) For $b_1 = 1.5125$, we get $\Delta_2 = 0$ and the unique positive equilibrium E_{33} is a saddle node.
- (2) For $b_1 = 1.51$, we get $\Delta_2 > 0$ and E_{31} is a stable node, E_{32} is a saddle.

Figure 7 shows the above results.

Example 7.5 Let $a_1 = 2, a_2 = 2, \rho = 1$, by simple computation, we have:

- (1) For $b_1 = 0.4, c_1 = 0.4$, we get E_{21} coincides with E_0 .
- (2) For $b_1 = 1, c_1 = 1$, we get both E_{21} and E_{22} coincide with E_0 .
- (3) For $b_1 = 1.5, c_1 = 1.5$, we get E_{22} coincides with E_0 .

Figure 8 shows that system (1.6) undergoes transcritical bifurcation at E_0 .

Example 7.6 Let $a_1 = 0.4$, $a_2 = 2$, $\rho = 1$, by simple computation, we have:

- (1) For $c_1 = 2.8$, we get $b_1 = 1.68$ and E_{32} coincide with E_1 .
- (2) For $c_1 = 3$, we get $b_1 = 1.8$ and both E_{31} and E_{32} coincide with E_1 .
- (3) For $c_1 = 4$, we get $b_1 = 2.4$ and E_{31} coincides with E_1 .

Figure 9 shows that system (1.6) undergoes transcritical bifurcation at E_1 .

8 Conclusion

In this paper, we have considered a two-species competitive system with Michaelis–Menten type harvesting. The model shows rich dynamic behaviors. We have studied the permanence condition of the system, and by analyzing the stability of the system equilibria, we obtained that the system cannot collapse for any parameter value as the origin is always unstable. In addition, from the global asymptotic stability of the positive equilibrium, it can be seen that two species can coexist stably under appropriate conditions. We also get that there are two different cases of bistability in the system: on the one hand, a boundary equilibrium and a positive equilibrium are globally asymptotically stable; on the other hand, the two boundary equilibria are globally asymptotically stable.

Qualitative analysis indicates that Michaelis–Menten type harvesting plays an important role in the dynamic behaviors and bifurcations of the system. Firstly, the parameters b_1 and c_1 of the Michaelis–Menten type harvesting term will affect the number and stability of the system equilibria, compared with system (1.1), the boundary equilibria and positive equilibria of system (1.6) are both increased. Secondly, from the Michaelis–Menten type harvesting term $h(x) = \frac{b_1 x}{c_1 + x}$, we know that system (1.6) is supplementary to system (1.1). Because system (1.6) becomes unharvested situation if $b_1 = 0$, system (1.6) becomes constant harvested situation if $c_1 = 0$. Thirdly, we give a strict proof of the bifurcation of system (1.6) by Sotomayor's theorem, which has important ecological significance. Through saddle-node bifurcation and transcritical bifurcation, one could obtain the maximum threshold without extinction risk of species in continuous harvest. This provides important reference for decision makers to make reasonable strategies to ensure the sustainable development of ecosystem and maximize economic benefits.

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Ethics approval and consent to participate

Not applicable.

Competing interests

The authors declare that there is no conflict of interests.

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