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Optimal consumption and portfolio selection with lower and upper bounds on consumption

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Abstract

We investigate the optimal consumption and investment problem with lower and upper bounds on consumption constraints. We derive closed-form solutions by means of the dynamic programming approach. We also evaluate the effects of the optimal consumption and portfolio on consumption constraints and present some numerical/economic implications. In particular, we see that the upper bound on consumption acts as a bliss level in a quadratic utility model.

Keywords: Consumption constraints; Consumption/Investment problem; CRRA utility; Dynamic programming approach

1 Introduction

After Merton's pioneer research [9, 10] on continuous-time portfolio selection, there have been many studies conducted on the optimal consumption and portfolio selection problem with realistic economic constraints such as borrowing constraints, subsistence consumption constraints, portfolio constraints, etc. In this paper we focus on consumption constraints in particular.

When observing the consumption behavior of the economic agent, it is natural to consider the subsistence consumption requirement. A subsistence consumption constraint means that the agent should consume a positive minimum consumption requirement which the agent can live with. Some authors have previously investigated the portfolio selection problem with a subsistence consumption constraint (Gong and Li [2], Lakner and Nygren [7], Yuan and Hu [15], Shin et al. [13], etc).

When the optimal consumption and investment problem with quadratic utility is considered, a constant bliss level of consumption depending upon the coefficient of the quadratic utility function can be presented. In this level, the agent has zero investment, but consumes a constant bliss level (Koo et al. [6], Shin et al. [12]). This is the reason for our interest in an upper bound on consumption. Since we have already considered the agent who has a constraint which is acting like a bliss level in quadratic utility [12], we investigate the aspects of the optimal consumption and investment policy of agent when considering an upper bound on consumption. In the real financial market, there are certain goods which cannot be infinitely consumed, such as foods, houses, and cars. Thus, the upper bound on consumption has an economic meaning when the agent consumes necessary goods. Jian et al. [4] and Xu and Yi [14] considered the optimal consumption and portfolio selection problem with an upper bound consumption constraint, which is a constant/a linear function of wealth, respectively. Unlike the bliss level on consumption, this constraint implies a free boundary problem. In this paper, we take the portfolio selection problem into account with the constant upper bound on consumption. In this paper we want to combine two previous issues which are a subsistence consumption constraint and a bliss level of consumption in the continuous-time portfolio selection problem, that is, we consider both the upper and lower bounds on consumption simultaneously to formulate these two issues.

Ma et al. [8] also considered the optimal consumption and investment problem with lower and upper bounds on consumption. Mathematically, Ma et al. [8] solved the optimization problem using a dual transformation, while we use the dynamic programming method. Furthermore, we observe the numerical/economic implications of the optimal consumption and portfolio with CRRA utility, and we can observe behaviors of the optimal consumption and portfolio, which are similar to the results with quadratic utility [6, 12].

Bensoussan et al. [1] and Jang et al. [3] solved the optimal consumption and portfolio selection problem in an incomplete market. Bensoussan et al. [1] considered involuntary unemployment risk which was not able to be hedged in a market. In the paper of Jang et al. [3], default risk of the provider of the annuity made a market to be incomplete. Using those approaches, it is possible to extend our results to the labor income problem with voluntary/involuntary retirement jointly.

The rest of this paper is composed as follows. We represent the financial market in Sect. 2. We derive our optimization problem and solve this problem by means of the dynamic programming approach proposed by Karatzas et al. [5] in Sect. 3. Then, Sect. 4 gives some numerical implications, and Sect. 5 brings to a conclusion.

2 The economy

Assume that there are a riskless asset P and a risky asset S in the financial market. A riskless asset P_t at time t follows the ordinary differential equation (ODE)

$$dP_t = rP_t\,dt,$$

where $r > 0$ is an interest rate, and a risky asset S_t at time t evolves according to

$$dS_t = bS_t\,dt + \sigma S_t\,dZ_t,$$

where $b > r$ is a mean rate of return and σ is a volatility of the risky asset. All of the parameters, r, b, and σ, are assumed to be constants. Z_t is the standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by the standard Brownian motion $\{Z_t\}_{t\geq 0}$.

Let π_t be the amounts of money (dollar) invested in the risky asset S_t at time t, and let $c_t \geq 0$ be the consumption rate process at time t. We assume that the portfolio process π_t

at time t is an $\mathcal{F}_t$-measurable process, that the consumption process c_t at time t is an $\mathcal{F}_t$-measurable nonnegative process, and that these processes satisfy the following technical conditions:

$$\int_0^t \pi_s^2\,ds < \infty, \qquad \int_0^t c_s\,ds < \infty \quad \text{for all } t \geq 0 \text{ a.s.}$$

Then, the wealth process of the agent X_t at time t evolves according to

$$dX_t = \big[rX_t + \pi_t(b-r) - c_t\big]\,dt + \sigma\pi_t\,dZ_t, \tag{2.1}$$

with an initial endowment $X_0 = x > 0$.

3 The optimization problem

The goal of the agent is to maximize her/his lifetime discounted expected utility

$$J(x;c,\pi) := \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}\,dt\right]$$

subject to the budget constraint (2.1) and the consumption constraint that restrains the consumption behavior such that

$$\underline{B} \leq c_t \leq \overline{B} \quad \text{for all } t \geq 0, \tag{3.1}$$

where $\underline{B} > 0$ and $\overline{B} > \underline{B}$ are positive constants; that is, the upper bound of consumption as well as the lower bound of consumption. Here, $\beta > 0$ is a subjective discount factor and $\gamma > 0$ ($\gamma \neq 1$) is an agent's coefficient of relative risk aversion. The pair (c,π) of the optimal consumption and portfolio is called *admissible* at initial wealth $x > 0$ if $X_t > 0$ for all $t \geq 0$ and if it satisfies constraint (3.1).

Thus, the value function of our optimization problem can be given by

$$V(x) := \sup_{(c,\pi)\in\mathcal{A}(x)} J(x;c,\pi), \tag{3.2}$$

where $\mathcal{A}(x)$ is the class of all admissible pairs (c,π) at $x > 0$ such that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\frac{c_t^{1-\gamma}}{1-\gamma}\right)^-dt\right] < \infty,$$

where $z^- := \max(-z,0)$.

Assumption 1 (Gong and Li [2]) There exists a positive lower bound of initial wealth $x > 0$ such that

$$x > \frac{\underline{B}}{r}.$$

From the value function (3.2), constraints (2.1) and (3.1), we derive the Bellman equation as follows:

$$\max_{(\underline{B}\leq c\leq\overline{B},\pi)}\left[\big\{rx + \pi(b-r) - c\big\}V'(x) + \frac{1}{2}\sigma^2\pi^2V''(x) - \beta V(x) + \frac{c^{1-\gamma}}{1-\gamma}\right] = 0. \tag{3.3}$$

The first order conditions (FOCs) imply

$$c^* = \begin{cases} \underline{B} & \text{for } \underline{B}/r < x < \underline{z}, \\ (V'(x))^{-\frac{1}{\gamma}} & \text{for } \underline{z} \le x < \overline{z}, \\ \overline{B} & \text{for } x \ge \overline{z}, \end{cases} \tag{3.4}$$

and

$$\pi^* = -\frac{b-r}{\sigma^2}\frac{V'(x)}{V''(x)}, \tag{3.5}$$

where $\underline{z} > 0$ is the wealth level corresponding to the lower consumption level $\underline{B}$, $\overline{z} > 0$ is the wealth level corresponding to the upper consumption level $\overline{B}$, and $\underline{z} < \overline{z}$ ($\underline{z}$ and $\overline{z}$ will be determined later in Theorem 1). Inserting FOCs (3.4) and (3.5) into equation (3.3) yields

$$(rx - \underline{B})V'(x) - \frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)} - \beta V(x) + \frac{\underline{B}^{1-\gamma}}{1-\gamma} = 0 \quad \text{for } \underline{B}/r < x < \underline{z}, \tag{3.6}$$

$$rxV'(x) - \frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)} - \beta V(x) + \frac{\gamma}{1-\gamma}V'(x)^{-\frac{1-\gamma}{\gamma}} = 0 \quad \text{for } \underline{z} \le x < \overline{z}, \tag{3.7}$$

and

$$(rx - \overline{B})V'(x) - \frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)} - \beta V(x) + \frac{\overline{B}^{1-\gamma}}{1-\gamma} = 0 \quad \text{for } x \ge \overline{z}, \tag{3.8}$$

where $\theta := (b-r)/\sigma$ is Sharpe ratio (or market price of risk). Moreover, we define the constant $K > 0$ as follows:

$$K := r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0. \tag{3.9}$$

Remark 1 For later use, we define two quadratic functions as follows:

$$f(m) := rm^2 - \left(\beta + r + \frac{1}{2}\theta^2\right)m + \beta \tag{3.10}$$

and

$$g(n) := \frac{1}{2}\theta^2 n^2 + \left(\beta - r + \frac{1}{2}\theta^2\right)n - r. \tag{3.11}$$

The quadratic equation $f(m) = 0$ has two roots m_1 and m_2 ($m_1 > 1 > m_2 > 0$) and the quadratic equation $g(n) = 0$ has two roots $n_1 > 0$ and $n_2 < -1$. In addition, we have the following relationships between the roots of $f(m) = 0$ and the roots of $g(n) = 0$:

$$n_1 = \frac{1}{m_1 - 1}, \qquad n_2 = \frac{1}{m_2 - 1}, \tag{3.12}$$

and

$$\frac{n_1}{n_1 + 1} = \frac{r - \frac{1}{2}\theta^2 n_1}{\beta}, \qquad \frac{n_2}{n_2 + 1} = \frac{r - \frac{1}{2}\theta^2 n_2}{\beta}. \tag{3.13}$$

Theorem 1 *The value function $V(x)$ of the optimization problem (3.2) is given by*

$$V(x)=\begin{cases} C_2(x-\frac{\underline{B}}{r})^{m_2}+\frac{\underline{B}^{1-\gamma}}{\beta(1-\gamma)} & \textit{for } \underline{B}/r<x<\underline{z},\\ \frac{n_1}{n_1+1}D_1\xi^{-\gamma(n_1+1)}+\frac{n_2}{n_2+1}D_2\xi^{-\gamma(n_2+1)}+\frac{1}{(1-\gamma)K}\xi^{1-\gamma} & \textit{for } \underline{z}\le x\le\overline{z},\\ E_1(\frac{\overline{B}}{r}-x)^{m_1}+\frac{\overline{B}^{1-\gamma}}{\beta(1-\gamma)} & \textit{for } \overline{z}<x<\overline{B}/r,\\ \frac{\overline{B}^{1-\gamma}}{\beta(1-\gamma)} & \textit{for } x\ge\overline{B}/r, \end{cases}$$

where

$$D_1=\frac{\frac{1}{K}+\gamma n_2(\frac{1}{K}-\frac{1}{r})}{\gamma(n_1-n_2)}\underline{B}^{1+\gamma n_1}, \tag{3.14}$$

$$D_2=\frac{\frac{1}{K}+\gamma n_1(\frac{1}{K}-\frac{1}{r})}{\gamma(n_2-n_1)}\overline{B}^{1+\gamma n_2}, \tag{3.15}$$

$$\underline{z}=D_1\underline{B}^{-\gamma n_1}+D_2\underline{B}^{-\gamma n_2}+\frac{\underline{B}}{K},$$

$$\overline{z}=D_1\overline{B}^{-\gamma n_1}+D_2\overline{B}^{-\gamma n_2}+\frac{\overline{B}}{K}, \tag{3.16}$$

$$C_2=\frac{1}{m_2}\left(\underline{z}-\frac{\underline{B}}{r}\right)^{1-m_2}\underline{B}^{-\gamma}>0,$$

and

$$E_1=-\frac{1}{m_1}\left(\frac{\overline{B}}{r}-\overline{z}\right)^{1-m_1}\overline{B}^{-\gamma}<0.$$

For $\underline{z}\le x<\overline{z}$, ξ is determined by the algebraic equation as follows:

$$x=D_1\xi^{-\gamma n_1}+D_2\xi^{-\gamma n_2}+\frac{1}{K}\xi.$$

Proof For $\underline{B}/r<x<\underline{z}$, if we conjecture the form of the solution to equation (3.6) as follows:

$$V(x)=C_1\left(x-\frac{\underline{B}}{r}\right)^{m_1}+C_2\left(x-\frac{\underline{B}}{r}\right)^{m_2}+\frac{\underline{B}^{1-\gamma}}{\beta(1-\gamma)},$$

then it solves equation (3.6) when $C_1=0$ and $C_2>0$. Thus, $V(x)$ is given by

$$V(x)=C_2\left(x-\frac{\underline{B}}{r}\right)^{m_2}+\frac{\underline{B}^{1-\gamma}}{\beta(1-\gamma)}, \tag{3.17}$$

where m_1 and m_2 are the two roots of the quadratic equation $f(m)=0$ in (3.10), and C_2 is a constant which will be determined later.

In the case of $x\ge\overline{z}$, $x-\overline{B}/r$ should be negative in the interval $(\overline{z},\overline{B}/r)$. Hence we suppose that the form of the solution to equation (3.8) is given as follows:

$$V(x)=E_1\left(\frac{\overline{B}}{r}-x\right)^{m_1}+E_2\left(\frac{\overline{B}}{r}-x\right)^{m_2}+\frac{\overline{B}^{1-\gamma}}{\beta(1-\gamma)}.$$

By the property $\lim_{x\to\overline{B}/r} V'(x) < \infty$, $E_2 = 0$. Hence, we obtain $V(x)$ as follows:

$$V(x) = E_1\left(\frac{\overline{B}}{r} - x\right)^{m_1} + \frac{\overline{B}^{1-\gamma}}{\beta(1-\gamma)}.$$

For $x > \overline{B}/r$, similar to the case of $\underline{B}/r < x < \underline{z}$, if we suppose that the form of the solution to equation (3.8) is given as follows:

$$V(x) = F_1\left(x - \frac{\overline{B}}{r}\right)^{m_1} + F_2\left(x - \frac{\overline{B}}{r}\right)^{m_2} + \frac{\overline{B}^{1-\gamma}}{\beta(1-\gamma)},$$

then it solves equation (3.8) when $F_1 = 0$ and $F_2 = 0$ because of $\lim_{x\to\infty} V(x) < \infty$. Thus, $V(x)$ is given by

$$V(x) = \frac{\overline{B}^{1-\gamma}}{\beta(1-\gamma)}.$$

We now find the solution to equation (3.7) for $\underline{z} \le x < \overline{z}$. First, we assume that the optimal consumption is a function of optimal wealth, that is, $c = C(x)$ and $X(\cdot) = C^{-1}(\cdot)$. From FOC (3.4), we have

$$V'(x) = C(x)^{-\gamma}, \qquad V''(x) = -\gamma\frac{C(x)^{-\gamma-1}}{X'(c)}. \tag{3.18}$$

Substituting equations (3.18) into equation (3.7) implies

$$\beta V\bigl(X(c)\bigr) = rc^{-\gamma}X(c) + \frac{1}{2\gamma}\theta^2 c^{1-\gamma}X'(c) + \frac{\gamma}{1-\gamma}c^{1-\gamma}. \tag{3.19}$$

By differentiating equation (3.19) with respect to c, we obtain the following second order ODE:

$$\frac{1}{2\gamma}\theta^2c^2X''(c) - \left(\beta - r - \frac{1-\gamma}{2\gamma}\theta^2\right)cX'(c) - \gamma rX(c) + \gamma c = 0. \tag{3.20}$$

Then, we can obtain the solution to equation (3.20) as follows:

$$X(c) = D_1c^{-\gamma n_1} + D_2c^{-\gamma n_2} + \frac{1}{K}c, \tag{3.21}$$

where n_1 and n_2 are the two roots of the quadratic equation $g(n) = 0$ in (3.11), and D_1 and D_2 are constants which will be determined later. Inserting equation (3.21) into equation (3.19) implies

$$\begin{aligned}
V(x) &= \frac{r - \frac{1}{2}\theta^2 n_1}{\beta}D_1\xi^{-\gamma(n_1+1)} + \frac{r - \frac{1}{2}\theta^2 n_2}{\beta}D_2\xi^{-\gamma(n_2+1)} + \frac{1}{(1-\gamma)K}\xi^{1-\gamma} \\
&= \frac{n_1}{n_1+1}D_1\xi^{-\gamma(n_1+1)} + \frac{n_2}{n_2+1}D_2\xi^{-\gamma(n_2+1)} + \frac{1}{(1-\gamma)K}\xi^{1-\gamma},
\end{aligned}$$

where the second equality is obtained from the relationships (3.13) and ξ is determined from the algebraic equation

$$x = D_1\xi^{-\gamma n_1} + D_2\xi^{-\gamma n_2} + \frac{1}{K}\xi.$$

From (3.21), we obtain the wealth levels $\underline{z}$ and $\overline{z}$:

$$\underline{z} = X(\underline{B}) = D_1\underline{B}^{-\gamma n_1} + D_2\underline{B}^{-\gamma n_2} + \frac{\underline{B}}{K}, \qquad \overline{z} = X(\overline{B}) = D_1\overline{B}^{-\gamma n_1} + D_2\overline{B}^{-\gamma n_2} + \frac{\overline{B}}{K}. \tag{3.22}$$

Based on equations (3.17) and (3.18) as well as the C^1- and C^2-conditions of $V(x)$ at $\underline{z}$, we have the following equations:

$$V'(\underline{z}) = C_2 m_2\left(\underline{z} - \frac{\underline{B}}{r}\right)^{m_2-1} = \underline{B}^{-\gamma} \tag{3.23}$$

and

$$V''(\underline{z}) = C_2 m_2(m_2-1)\left(\underline{z} - \frac{\underline{B}}{r}\right)^{m_2-2} = -\gamma\frac{\underline{B}^{-\gamma-1}}{X'(\underline{B})}. \tag{3.24}$$

Two equations (3.23) and (3.24) imply

$$C_2 = \frac{1}{m_2}\left(\underline{z} - \frac{\underline{B}}{r}\right)^{1-m_2}\underline{B}^{-\gamma} > 0$$

and

$$\underline{B}\cdot X'(\underline{B}) = -\gamma n_1 D_1\underline{B}^{-\gamma n_1} - \gamma n_2 D_2\underline{B}^{-\gamma n_2} + \frac{1}{K}\underline{B} = -\frac{\gamma}{m_2-1}\left(\underline{z} - \frac{\underline{B}}{r}\right). \tag{3.25}$$

Substituting $\underline{z}$ in (3.22) into equation (3.25) implies

$$D_1 = \frac{\frac{1}{K} + \gamma n_2(\frac{1}{K} - \frac{1}{r})}{\gamma(n_1-n_2)}\underline{B}^{1+\gamma n_1}, \tag{3.26}$$

since $n_2 = 1/(m_2-1)$ in (3.12). Similarly, we can also obtain E_1 and D_2 as follows:

$$\begin{aligned} E_1 &= -\frac{1}{m_1}\left(\frac{\overline{B}}{r} - \overline{z}\right)^{1-m_1}\overline{B}^{-\gamma} < 0, \\ D_2 &= \frac{\frac{1}{K} + \gamma n_1(\frac{1}{K} - \frac{1}{r})}{\gamma(n_2-n_1)}\overline{B}^{1+\gamma n_2}. \end{aligned} \tag{3.27}$$

□

According to Remark 3.3 in Shim and Shin [11], we obtain the following lemma.

Lemma 1 $D_1 > 0$ *in* (3.14).

Proof Since $n_1 - n_2 > 0$, it is sufficient to show that

$$\frac{1}{K} + \gamma n_2\left(\frac{1}{K} - \frac{1}{r}\right) > 0. \tag{3.28}$$

For $n_2 < x < n_1$, we define a function

$$h(x) := -\frac{g(x)}{x - n_2} = -\frac{1}{2}\theta^2(x - n_1) > 0,$$

with the quadratic function $g(\cdot)$ in (3.11). Then, $h(x)$ is a linearly decreasing function for $n_2 < x < n_1$. Since $g(-1/\gamma) = -K < 0$ and $n_2 < -1/\gamma < 0 < n_1$, we can see that $h(-1/\gamma) > h(0) > 0$ and

$$-\frac{K}{\frac{1}{\gamma} + n_2} > -\frac{r}{n_2} > 0,$$

that is,

$$0 < \frac{-\frac{1}{\gamma} - n_2}{K} < -\frac{n_2}{r}. \tag{3.29}$$

Thus, inequality (3.29) implies inequality (3.28). □

Lemma 2 $D_2 < 0$ *in* (3.15).

Proof It is similar to the proof of Lemma 1. □

Lemma 3

$$\bar{z} < \frac{\overline{B}}{r}.$$

Proof From (3.16), (3.14), and (3.15), we see that

$$\begin{aligned}
\bar{z} &= \frac{\frac{1}{K} + \gamma n_2(\frac{1}{K} - \frac{1}{r})}{\gamma(n_1 - n_2)} \underline{B}^{1+\gamma n_1}\overline{B}^{-\gamma n_1} + \frac{\frac{1}{K} + \gamma n_1(\frac{1}{K} - \frac{1}{r})}{\gamma(n_2 - n_1)} \underline{B}^{1+\gamma n_2}\overline{B}^{-\gamma n_2} + \frac{\overline{B}}{K} \\
&= \left[\frac{\frac{1}{K} + \gamma n_2(\frac{1}{K} - \frac{1}{r})}{\gamma(n_1 - n_2)} \left(\frac{\underline{B}}{\overline{B}}\right)^{1+\gamma n_1} + \frac{\frac{1}{K} + \gamma n_1(\frac{1}{K} - \frac{1}{r})}{\gamma(n_2 - n_1)} + \frac{1}{K}\right]\overline{B} \\
&< \left[\frac{\frac{1}{K} + \gamma n_2(\frac{1}{K} - \frac{1}{r})}{\gamma(n_1 - n_2)} + \frac{\frac{1}{K} + \gamma n_1(\frac{1}{K} - \frac{1}{r})}{\gamma(n_2 - n_1)} + \frac{1}{K}\right]\overline{B} \\
&= \frac{\overline{B}}{r},
\end{aligned}$$

where the inequality is obtained from (3.28) and $\underline{B}/\overline{B} < 1$. □

Lemma 4 $X'(c) > 0$ *for* $\underline{B} < c < \overline{B}$.

Proof We derive $X''(\cdot)$ from $X(\cdot)$ in (3.21) as follows:

$$X''(c) = c^{-2}\left[\gamma n_1(\gamma n_1 + 1)D_1 c^{-\gamma n_1} + \gamma n_2(\gamma n_2 + 1)D_2 c^{-\gamma n_2}\right] =: c^{-2}F(c). \tag{3.30}$$

Since $D_2 < 0$, $F(c) = 0$ has a unique solution $c^\star > 0$ such that

$$c^\star = \left[-\frac{n_2(\gamma n_2 + 1)D_2}{n_1(\gamma n_1 + 1)D_1}\right]^{-\frac{1}{\gamma(n_1 - n_2)}}.$$

Since $\lim_{c\to 0+} F(c) = +\infty$ and $\lim_{c\to+\infty} F(c) = -\infty$, $X'(c)$ is an increasing function for $\underline{B} < c \le c^\star$ and a decreasing function for $c^\star < c < \overline{B}$. Since $X'(\underline{B}) > 0$, if $X'(\overline{B}) \ge 0$, then $X'(c) > 0$ for $\underline{B} < c < \overline{B}$. Thus it is sufficient to show that $X'(\overline{B}) \ge 0$. So

$$\begin{aligned}
X'(\overline{B}) &= -\gamma n_1 D_1 \overline{B}^{-\gamma n_1 - 1} - \gamma n_2 D_2 \overline{B}^{-\gamma n_2 - 1} + \frac{1}{K} \\
&= -\frac{n_1}{n_1 - n_2}\left[\frac{1}{K} + \gamma n_2\left(\frac{1}{K} - \frac{1}{r}\right)\right]\left(\frac{\underline{B}}{\overline{B}}\right)^{1+\gamma n_1} \\
&\quad - \frac{n_2}{n_1 - n_2}\left[\frac{1}{K} + \gamma n_1\left(\frac{1}{K} - \frac{1}{r}\right)\right] + \frac{1}{K} \\
&> -\frac{n_1}{n_1 - n_2}\left[\frac{1}{K} + \gamma n_2\left(\frac{1}{K} - \frac{1}{r}\right)\right] - \frac{n_2}{n_1 - n_2}\left[\frac{1}{K} + \gamma n_1\left(\frac{1}{K} - \frac{1}{r}\right)\right] + \frac{1}{K} \\
&= 0,
\end{aligned}$$

where the second equality is obtained by inserting D_1 and D_2 in (3.26) and (3.27), respectively, into this equation, and the inequality from inequality $-(\underline{B}/\overline{B})^{1+\gamma n_1} > -1$. □

Lemma 4 shows that the wealth process $X(c)$ is an increasing function of consumption c. Therefore, based on Lemmas 3 and 4, we can see that

$$\frac{\underline{B}}{r} < \underline{z} < \overline{z} < \frac{\overline{B}}{r}.$$

We derive the optimal policies in the next theorem using FOCs (3.4), (3.5), and (3.18).

Theorem 2 *The optimal consumption and portfolio policies* (c_t^*, π_t^*) *are given by*

$$c_t^* = \begin{cases} \underline{B} & \text{for } \underline{B}/r < X_t < \underline{z}, \\ \xi_t & \text{for } \underline{z} \le X_t < \overline{z}, \\ \overline{B} & \text{for } X_t \ge \overline{z}, \end{cases}$$

$$\pi_t^* = \begin{cases} -\frac{\theta}{\sigma(m_2-1)}(x - \frac{\underline{B}}{r}) & \text{for } \underline{B}/r < X_t < \underline{z}, \\ \frac{\theta}{\sigma\gamma}(-\gamma n_1 D_1 \xi_t^{-\gamma n_1} - \gamma n_2 D_2 \xi_t^{-\gamma n_2} + \frac{1}{K}\xi_t) & \text{for } \underline{z} \le X_t < \overline{z}, \\ \frac{\theta}{\sigma(m_1-1)}(\frac{\overline{B}}{r} - x) & \text{for } \overline{z} \le X_t < \overline{B}/r, \\ 0 & \text{for } X_t \ge \overline{B}/r, \end{cases}$$

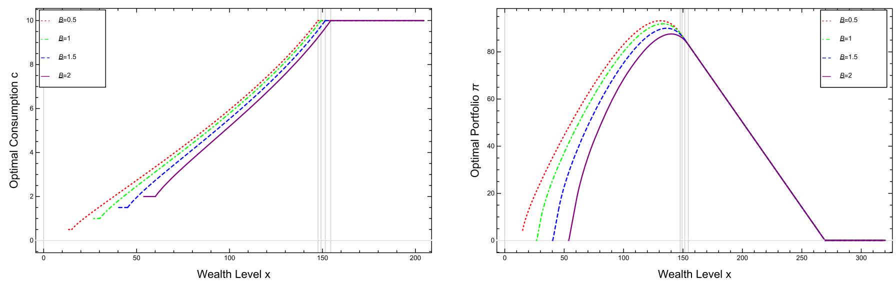
where $\xi_t(\xi_0 = \xi)$ *is determined by the algebraic equation as follows*:

$$X_t = D_1 \xi_t^{-\gamma n_1} + D_2 \xi_t^{-\gamma n_2} + \frac{1}{K}\xi_t.$$

4 Numerical implications

In this section, we take the numerical results about the optimal consumption and portfolio into account. The baseline parameter values from Bensoussan et al. [1] are given as follows:

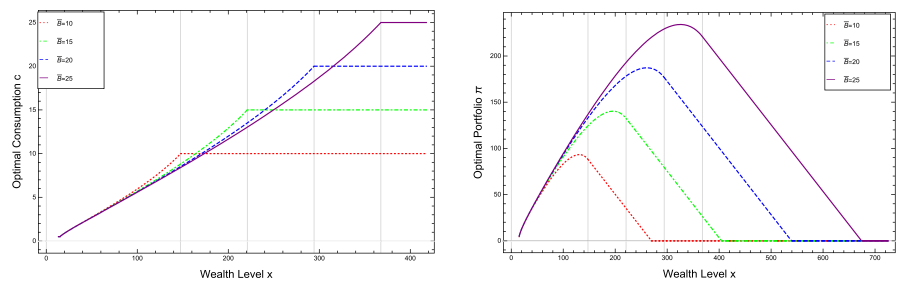
$$\beta = 0.0371, \qquad b = 0.1123, \qquad r = 0.0371, \qquad \sigma = 0.1954, \qquad \gamma = 2.$$



(a) Optimal consumption ($\beta = 0.0371,\ b = 0.1123,\ r = 0.0371,\ \sigma = 0.1954,\ \gamma = 2,\ \overline{B} = 10$)

(b) Optimal portfolio ($\beta = 0.0371,\ b = 0.1123,\ r = 0.0371,\ \sigma = 0.1954,\ \gamma = 2,\ \overline{B} = 10$)

Figure 1 The optimal consumption/portfolio when lower bound on consumption $\underline{B}$ increases



(a) Optimal consumption ($\beta = 0.0371,\ b = 0.1123,\ r = 0.0371,\ \sigma = 0.1954,\ \gamma = 2,\ \underline{B} = 0.5$)

(b) Optimal portfolio ($\beta = 0.0371,\ b = 0.1123,\ r = 0.0371,\ \sigma = 0.1954,\ \gamma = 2,\ \underline{B} = 0.5$)

Figure 2 The optimal consumption/portfolio when upper bound on consumption $\overline{B}$ increases

In Figs. 1(a) and 1(b), we fix $\overline{B} = 10$. As shown in Fig. 1(a), the optimal consumption becomes lower as the subsistence consumption level rises; that is, as the constraint becomes stronger, the agent's consumption shrinks. Similarly, as shown in Fig. 1(b), the optimal portfolio becomes lower as the subsistence consumption level rises; that is, as the constraint becomes stronger, the agent's investment shrinks. In addition, the critical wealth level $\bar{z}$ becomes higher as the subsistence consumption level rises. Since the investment is lower when the subsistence consumption level is higher, a higher critical wealth level is required for the same upper bound on consumption.

In Figs. 2(a) and 2(b), we fix $\underline{B} = 0.5$. As shown in Fig. 2(a), the optimal consumption and critical wealth level $\bar{z}$ become higher as the upper bound on consumption rises. This is because the agent consumes more in order to reach the higher upper bound on consumption and a higher critical wealth level. As shown in Fig. 2(b), the peak of the optimal portfolio becomes higher as the upper bound on consumption rises. This is because the agent invests more in order to reach the critical wealth level $\bar{z}$ quickly when the upper bound on consumption is higher.

From Figs. 1(b) and 2(b), we can observe the decreasing aspect of the optimal portfolio after a certain wealth level. Thus, we can see that the upper bound on consumption provides a bliss level in a quadratic utility model (see [6, 12]). The decreasing features of the optimal portfolio are remarkable effects of an upper bound on consumption.

5 Concluding remarks

Since we have previously taken the optimal consumption and portfolio choice problem into account with quadratic utility [12], we can observe the aspect of the bliss level of consumption. This motivated us to consider an upper bound on consumption with CRRA utility. By investigating the optimal consumption/investment problem under the consumption constraints, we obtain that the upper bound on consumption acts like a bliss level in a quadratic utility model. Furthermore, we consider two boundaries, lower and upper bounds, on consumption and use the dynamic programming method to derive closed-form solutions. We also observe the effects of the optimal consumption and portfolio on consumption constraints and present some numerical implications.

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Competing interests
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Authors' contributions
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References

1. Bensoussan, A., Jang, B.-G., Park, S., Unemployment risks and optimal retirement in an incomplete market. Oper. Res. **64**, 1015–1032 (2016)
2. Gong, N., Li, T.: Role of index bonds in an optimal dynamic asset allocation model with real subsistence consumption. Appl. Math. Comput. **174**, 710–731 (2006)
3. Jang, B.-G., Koo, H.K., Park, S.: Optimal consumption and investment with insurer default risk. Insur. Math. Econ. **88**, 44–56 (2019)
4. Jian, X., Yi, F., Zhang, J.: Investment and consumption problem in finite time with consumption constraint. ESAIM Control Optim. Calc. Var. **23**, 1601–1615 (2017)
5. Karatzas, I., Lehoczky, J.P., Sethi, S.P., Shreve, S.E.: Explicit solution of a general consumption/investment problem. Math. Oper. Res. **11**, 261–294 (1986)
6. Koo, J.L., Ahn, S.R., Koo, B.L., Koo, H.K., Shin, Y.H.: Optimal consumption and portfolio selection with quadratic utility and a subsistence consumption constraint. Stoch. Anal. Appl. **34**, 165–177 (2016)
7. Lakner, P., Nygren, L.M.: Portfolio optimization with downside constraints. Math. Finance **16**, 283–299 (2006)
8. Ma, Q., Yi, F., Guan, C.: A consumption-investment problem with constraints on minimum and maximum consumption rates. J. Comput. Appl. Math. **338**, 185–198 (2018)
9. Merton, R.C.: Lifetime portfolio selection under uncertainty: the continuous-time case. Rev. Econ. Stat. **51**, 247–257 (1969)
10. Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **3**, 373–413 (1971)
11. Shim, G., Shin, Y.H.: An optimal job, consumption/leisure, and investment policy. Oper. Res. Lett. **42**, 145–149 (2014)
12. Shin, Y.H., Koo, J.L., Roh, K.-H.: An optimal consumption and investment problem with quadratic utility and subsistence consumption constraints: a dynamic programming approach. Math. Model. Anal. **23**, 627–638 (2018)
13. Shin, Y.H., Lim, B.H., Choi, U.J.: Optimal consumption and portfolio selection problem with downside consumption constraints. Appl. Math. Comput. **188**, 1801–1811 (2007)
14. Xu, Z.Q., Yi, F.: An optimal consumption-investment model with constraint on consumption. Math. Control Relat. Fields **6**, 517–534 (2016)
15. Yuan, H., Hu, Y.: Optimal consumption and portfolio policies with the consumption habit constraints and the terminal wealth downside constraints. Insur. Math. Econ. **45**, 405–409 (2009)