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Ulam–Hyers–Rassias stability for nonlinear Ψ-Hilfer stochastic fractional differential equation with uncertainty

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Abstract

We consider a nonlinear Cauchy problem involving the Ψ-Hilfer stochastic fractional derivative with uncertainty, and we give a stability result. Using fixed point theory, we are able to provide a fuzzy Ulam–Hyers–Rassias stability for the considered nonlinear stochastic fractional differential equations.

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1 Introduction

Fractional analysis is a generalization of classical integer-order differentiation and integration to arbitrary noninteger order. Sousa and Oliveira [1] have recently proposed a fractional differentiation operator, which they called the Ψ-Hilfer operator, unifying several different fractional operators. Stochastic fractional differential equations naturally arise in different fields such as biology, engineering, medicine, physics, and mathematics (for more applications and details, we refer to [2–12]).

We study the nonlinear Ψ-Hilfer stochastic fractional differential equation

$$\begin{cases} {}^H D_{0^+}^{\iota,\kappa;\Psi} \mu(\varrho,\varsigma) \\ \quad = \mathsf{A}\mu(\varrho,\varsigma) + \mathsf{B}\mu(\varrho,\varsigma - \boldsymbol{h}(\varrho,\varsigma)) + \boldsymbol{f}(\varsigma,\mu(\varrho,\varsigma),\mu(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma))), \\ \mu(\varrho,\varsigma) = \Theta(\varrho,\varsigma), \qquad \varsigma \in [-h,0], \end{cases} \tag{1.1}$$

for $\varsigma \in \Xi_5$ and $\varrho \in \Upsilon$, where ${}^H D_{0^+}^{\iota,\kappa;\Psi}(\varrho,\cdot)$ is the Ψ-Hilfer stochastic fractional derivative operator of order $\iota \in (0,1]$ for each $\varrho \in \Upsilon$ with respect to a random operator $\Psi \in \ell(\Upsilon \times \Xi_5, \mathbb{R})$ (see [1, 13]) and type $0 \le \kappa \le 1$, $\mu(\varrho,\varsigma) \in \mathbb{R}^n$, $\boldsymbol{h}(\varrho,\varsigma)$ is a continuous map such that $0 \le \boldsymbol{h}(\varrho,\varsigma) \le h$, $\varsigma \in \Xi_5 = [0,M]$ with $0 < M < +\infty$, $\Theta(\varrho,\varsigma) \in \ell(\Upsilon \times [-h,0], \mathbb{R}^n)$ is a given function, and $\boldsymbol{f} : \Upsilon \times \Xi_5 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathsf{A} \in \mathbb{R}^{n\times n}$, and $\mathsf{B} \in \mathbb{R}^{n\times n}$ are matrices. In this paper, we study the uniqueness of solutions for (1.1) and their Ulam–Hyers–Rassias stability with uncertainty.

2 Preliminaries

Let $\Xi_1 = [\mathbf{a}, \mathbf{b}]$, $\Xi_2 = (\mathbf{a}, \mathbf{b})$, $\Xi_3 = [-h, 0]$, $\Xi_4 = [-h, M]$, $\Xi_5 = [0, M]$, $\Xi_6 = (0, 1]$, $\Xi_7 = [0, \infty]$, and $\Xi_8 = (0, \infty)$.

Definition 2.1 ([14–18]) Suppose that S is a linear space and η is a fuzzy set from $S \times \Xi_8$ to Ξ_6. The ordered pair (S, η) is said to be a *fuzzy normed space* (*FN-space*) whenever

(FN1) $\eta(\xi, \tau) = 1$ for any $\tau \in \Xi_8$ iff $\xi = 0$;
(FN2) $\eta(\mathbf{a}\xi, \tau) = \eta(\xi, \frac{\tau}{|\mathbf{a}|})$ for all $\xi \in S$, $\tau \in \Xi_8$, and $\mathbf{a} \in \mathbb{R}$ with $\mathbf{a} \neq 0$;
(FN3) $\eta(\xi + \zeta, \tau + \theta) \geq \min(\eta(\xi, \tau), \eta(\zeta, \theta))$ for all $\xi, \zeta \in S$ and $\tau, \theta \in \Xi_8$;
(FN4) $\eta(\xi, \cdot) : \Xi_8 \to \Xi_6$ is continuous.

Let (S, η) be an FN-space. A sequence $\{\xi_n\} \subset S$ is fuzzy convergent to $\xi \in S$ in (S, η) if for any $\tau > 0$ and $0 < \epsilon < 1$, there exists a positive integer N_ϵ such that $\eta(\xi_n - \xi, \tau) > 1 - \epsilon$ for $n \geq N_\epsilon$. A sequence $\{\xi_n\} \subset S$ is fuzzy Cauchy in (S, η) if for any $\tau > 0$ and $0 < \epsilon < 1$, there exists a positive integer N_ϵ such that $\eta(\xi_n - \xi_m, \tau) > 1 - \epsilon$ for $n, m \geq N_\epsilon$. An FN-space is Banach if every Cauchy sequence in it is convergent. A Banach FN-space is shortly called an FB-space. Consider the normed space $(S, \|\cdot\|)$. Then

$$\eta(\xi, \tau) = \exp\left(-\frac{\|\xi\|}{\tau}\right)$$

for $\tau \in \Xi_8$ defines a fuzzy norm, and the ordered pair (S, η) is an FN-space.

Consider the probability measure space (Υ, Ξ_8, ξ) and let $(T, \boldsymbol{B}_T)$ and $(S, \boldsymbol{B}_S)$ be Borel measureable spaces, where T and S are FB-spaces. If $\{\varrho : \mathcal{F}(\varrho, \xi) \in B\} \in \Xi_8$ for all $\xi \in T$ and $B \in \boldsymbol{B}_S$, we say that $\mathcal{F} : \Upsilon \times T \to S$ is a random operator. A random operator $\mathcal{F} : \Upsilon \times T \to S$ is said to be *linear* if $\mathcal{F}(\varrho, \mathbf{a}\xi_1 + \mathbf{b}\xi_2) = \mathbf{a}\mathcal{F}(\varrho, \xi_1) + \mathbf{b}\mathcal{F}(\varrho, \xi_2)$ almost everywhere for all $\xi_1, \upsilon_2 \in T$ and scalars $\mathbf{a}, \mathbf{b}$, and *bounded* if there exists a nonnegative real-valued random variable $M(\varrho)$ such that

$$\eta\big(\mathcal{F}(\varrho, \xi_1) - \mathcal{F}(\varrho, \xi_2), M(\varrho)\tau\big) \geq \eta(\xi_1 - \upsilon_2, \tau)$$

almost everywhere for all $\xi_1, \xi_2 \in T$, $\tau \in \Xi_8$, and $\varrho \in \Upsilon$.

The subject of approximation of functional equations in several spaces by direct techniques and fixed point techniques have been studied by some researchers, for instance, fuzzy Menger normed algebras [19], fuzzy metric spaces [20, 21], FN spaces [22], non-Archimedean random Lie C^*-algebras [23], and random multinormed space [24–29]. Some stability results for fractional differential and integral equations have been discussed in [26, 30–38].

Theorem 2.2 (The alternative of fixed point) *Let (T, ρ) be a complete Ξ_7-valued metric space, and let $\Lambda : T \to T$ be a strictly contractive function with Lipschitz constant $\iota < 1$. Then for every given element $\xi \in T$, either*

$$\rho\big(\Lambda^n \xi, \Lambda^{n+1} \xi\big) = \infty$$

for each $n \in \mathbb{N}$, or there is $n_0 \in \mathbb{N}$ such that

(i) $\rho(\Lambda^n \xi, \Lambda^{n+1} \xi) < \infty$ *for all $n \geq n_0$;*

(ii) *the fixed point* ζ^* *of* Λ *is the limit point of the sequence* $\{\Lambda^n\xi\}$;
(iii) *in the set* $V = \{\zeta \in T \mid \rho(\Lambda^{n_0}\xi, \zeta) < \infty\}$, ζ^* *is the unique fixed point of* Λ;
(iv) $(1-\iota)\rho(\zeta, \zeta^*) \le \rho(\zeta, \Lambda\zeta)$ *for every* $\zeta \in V$.

Definition 2.3 (One-parameter Mittag-Leffler function) The Mittag-Leffler function is given by the series

$$\mathcal{E}_\vartheta(\varpi) = \sum_{i=0}^{\infty} \frac{\varpi^i}{\Gamma(\vartheta i + 1)},$$

where $\vartheta \in \mathbb{C}$, $\mathrm{Re}(\vartheta) > 0$, and Γ is the gamma function given by

$$\Gamma(\varpi) = \int_0^\infty e^{-\varsigma}\varsigma^{\varpi-1}\,d\varsigma$$

for $\mathrm{Re}(\varpi) > 0$. In particular, if $\vartheta = 1$, then we have

$$\mathcal{E}_1(\varpi) = \sum_{k=0}^{\infty} \frac{\varpi^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\varpi^k}{k!} = e^{\varpi}.$$

Definition 2.4 ([1, 39]) Consider $\iota > 0$ and the increasing and positive monotone random operator $\Psi(\varrho, \varsigma)$ on $\Upsilon \times (\mathbf{a}, \mathbf{b}]$ with continuous derivative random operator $\Psi'(\varrho, \varsigma)$ on $\Upsilon \times \Xi_2$. Define the LR (left-right) stochastic fractional integrals of a random operator $\boldsymbol{f}$ and random operator Ψ on $\Upsilon \times \Xi_1$ by

$$\mathcal{I}_{\mathbf{a}^+}^{\iota;\Psi}\boldsymbol{f}(\varrho, \varsigma) := \frac{1}{\Gamma(\iota)}\int_{\mathbf{a}}^{\varsigma} \Psi'(\varrho, \upsilon)\big(\Psi(\varrho, \varsigma) - \Psi(\varrho, \upsilon)\big)^{\iota-1}\boldsymbol{f}(\varrho, \upsilon)\,d\upsilon$$

for all $\varsigma \in \Xi_2$ and $\varrho \in \Upsilon$.

Definition 2.5 ([1, 40]) Consider $n \in \mathbb{N}^+$ and let $n-1 < \iota < n$. Let Ξ_1 be the interval such that $-\infty \le \mathbf{a} < \mathbf{b} \le +\infty$, and let $\boldsymbol{f}, \Psi \in \ell^n(\Upsilon \times \Xi_1, \mathbb{R})$ be two random operators, where Ψ is increasing, and $\Psi'(\varrho, \varsigma) \neq 0$ for all $\varsigma \in \Xi_1$ and $\varrho \in \Upsilon$. Define the L-Ψ-Hilfer stochastic fractional derivative operator ${}^H D_{\mathbf{a}^+}^{\iota,\kappa;\Psi}(\varrho, \cdot)$ of order ι and type $0 \le \kappa \le 1$ by

$${}^H D_{\mathbf{a}^+}^{\iota,\kappa;\Psi}\boldsymbol{f}(\varrho, \varsigma) := \mathcal{I}_{\mathbf{a}^+}^{\kappa(n-\iota);\Psi}\left(\frac{1}{\Psi'(\varrho, \varsigma)}\frac{d}{d\varsigma}\right)^n \mathcal{I}_{\mathbf{a}^+}^{(1-\kappa)(n-\iota);\Psi}\boldsymbol{f}(\varrho, \varsigma).$$

We define the R-Ψ-Hilfer stochastic fractional derivative operator as in [1].

Lemma 2.6 ([1]) *If* $\boldsymbol{f} \in \ell^1_{\delta;\Psi}(\Upsilon \times \Xi_1)$, $0 < \iota < 1$, $0 \le \kappa \le 1$, *and* $\delta = \iota + \kappa(1-\iota)$, *then*

$$\mathcal{I}_{\mathbf{a}^+}^{\iota;\Psi}\,{}^H D_{\mathbf{a}^+}^{\iota,\kappa;\Psi}\boldsymbol{f}(\varrho, \varsigma) = \boldsymbol{f}(\varrho, \varsigma) - \frac{(\Psi(\varrho, \varsigma) - \Psi(\varrho, \mathbf{a}))^{\delta-1}}{\Gamma(\delta)}\mathcal{I}_{\mathbf{a}^+}^{(1-\kappa)(1-\iota);\Psi}\boldsymbol{f}(\varrho, \mathbf{a}).$$

Let Ξ_1 $(-\infty < \mathbf{a} < \mathbf{b} < +\infty)$, and let $\eta(\cdot, \tau)_{\ell(\Upsilon\times\Xi_1)}$ denote the fuzzy norm of $\mu = (\mu_1(\varrho, \varsigma), \mu_2(\varrho, \varsigma), \ldots, \mu_n(\varrho, \varsigma))^T \in \mathbb{R}^n$ on $\Upsilon \times \Xi_1$ defined by

$$\eta\big(\mu(\varrho, \varsigma), \tau\big) = \min_{1\le i\le n} \eta_E\big(\mu_i(\varrho, \varsigma), \tau\big),$$

where $\eta(\cdot,\tau)_E$ denotes the Euclidean fuzzy norm of $\mu_i(\varrho,\varsigma)\in\mathbb{R}$ on $\Upsilon\times\Xi_1$. Denote the space of continuous random operators by $\ell(\Upsilon\times\Xi_1)$ and define $\mu\in\mathbb{R}^n$ on $\Upsilon\times\Xi_1$ by

$$\eta\bigl(\mu(\varrho,\varsigma),\tau\bigr)_{\ell(\Upsilon\times\Xi_1)}=\min_{\varsigma\in\Xi_1}\eta\bigl(\mu(\varrho,\varsigma),\tau\bigr).$$

The weighted space $\ell_{1-\delta;\Psi}(\Upsilon\times\Xi_1,\mathbb{R}^n)$ of random operators μ on $\Upsilon\times\Xi_1$ is defined by

$$\ell_{1-\delta;\Psi}(\Upsilon\times\Xi_1)=\bigl\{\mu:\Upsilon\times\Xi_1\to\mathbb{R}^n:\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,0)\bigr)^{1-\delta}\mu(\varrho,\varsigma)\in\ell(\Upsilon\times\Xi_1)\bigr\}$$

for $\delta=\iota+\kappa(1-\iota)$, with the norm

$$\eta\bigl(\mu(\varrho,\varsigma),\tau\bigr)_{\ell_{1-\delta;\Psi}(\Upsilon\times\Xi_1)}=\eta\bigl(\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,0)\bigr)^{1-\delta}\mu(\varrho,\varsigma),\tau\bigr)_{\ell(\Upsilon\times\Xi_1)}.$$

Definition 2.7 ([39]) We say that system (1.1) has the Ulam–Hyers–Rassias stability if for each continuously differentiable random operator $\nu(\varrho,\varsigma)\in\mathbb{R}^n$ satisfying

$$\begin{aligned}&\eta\bigl({}^{H}D_{0^+}^{\iota,\kappa;\Psi}\nu(\varrho,\varsigma)-\mathsf{A}\nu(\varrho,\varsigma)-\mathsf{B}\nu\bigl(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)\bigr)-\boldsymbol{f}\bigl(\varrho,\varsigma,\nu(\varrho,\varsigma),\nu\bigl(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)\bigr)\bigr),\tau\bigr)\\&\quad\geq\varphi(\varsigma,\tau),\end{aligned}\tag{2.1}$$

where $\varphi(\varsigma,\tau)\in\Xi_6$ is a continuous fuzzy set for all $\varsigma\in\Xi_1$, $\tau\in\Xi_8$, and $\varrho\in\Upsilon$, there exist a solution $\mu(\varrho,\varsigma)\in\mathbb{R}^n$ of system (1.1) and a constant $\mathbf{C}>0$ such that

$$\eta\bigl(\mu(\varrho,\varsigma)-\nu(\varrho,\varsigma),\tau\bigr)\geq\varphi\left(\varsigma,\frac{\tau}{\mathbf{C}}\right),$$

where $\mathbf{C}$ is independent of $\mu(\varrho,\varsigma)$ and $\nu(\varrho,\varsigma)$. If $\varphi(\varrho,\varsigma)$ is fixed in the above inequalities, then we get the Ulam–Hyers stability with uncertainty of system (1.1).

Remark 2.8 A random operator $\nu(\varrho,\varsigma)$ is a solution of (2.1) if and only if there is a random operator $\Theta\in\ell(\Upsilon\times\Xi_5,\mathbb{R}^n)$ such that

- $\eta(\Theta(\varrho,\varsigma),\tau)\geq\varphi(\varsigma,\tau)$;
- ${}^{H}D_{0^+}^{\iota,\kappa;\Psi}\nu(\varrho,\varsigma)=\mathsf{A}\nu(\varrho,\varsigma)+\mathsf{B}\nu(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma))+\boldsymbol{f}(\varrho,\varsigma,\nu(\varrho,\varsigma),\nu(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)))+\Theta(\varrho,\varsigma)$.

Lemma 2.9 *Let $\boldsymbol{f}:\Upsilon\times\Xi_5\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be a continuous nonlinear random operator. Then the solution of system* (1.1) *is a continuous random operator $\mu(\varrho,\varsigma):\Upsilon\times\Xi_4\to\mathbb{R}^n$ satisfying*

$$\mu(\varrho,\varsigma)=\begin{cases}\frac{\Theta(\varrho,0)}{\Gamma(\delta)\Gamma(2-\delta)}+\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon))^{\iota-1}[\mathsf{A}\mu(\varrho,\upsilon)\\\quad+\mathsf{B}\mu(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon))]+\boldsymbol{f}(\varrho,\varsigma,\mu(\varrho,\varsigma),\mu(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma))),\\\Theta(\varrho,\varsigma),\qquad\varsigma\in\Xi_3.\end{cases}\tag{2.2}$$

Proof Let

$$\boldsymbol{g}(\varrho,\varsigma)=\mathsf{A}\mu(\varrho,\varsigma)+\mathsf{B}\mu\bigl(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)\bigr)+\boldsymbol{f}\bigl(\varrho,\varsigma,\mu(\varrho,\varsigma),\mu\bigl(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)\bigr)\bigr).$$

From Lemma 2.6 and (1.1), for $\varsigma \geq 0$ and $\varrho \in \Upsilon$, we get

$$\begin{aligned}\mu(\varrho,\varsigma) &= \frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\varrho-1}}{\Gamma(\delta)} \\ &\quad\times \frac{1}{\Gamma(1-\delta)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{-\varrho}\mu(\varrho,0)\,d\upsilon + \mathcal{I}_{0^+}^{\iota;\Psi}g(\varsigma) \\ &= \frac{\Theta(\varrho,0)}{\Gamma(\delta)\Gamma(2-\delta)} \\ &\quad+ \frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\boldsymbol{g}(\varrho,\upsilon)\,d\upsilon, \quad \varsigma \geq 0 \end{aligned} \tag{2.3}$$

when $\varsigma \in \Xi_3$ and $\mu(\varrho,\varsigma) = \Theta(\varrho,\varsigma)$. □

We denote the set of all eigenvalues of A defined as in system (1.1) by $\lambda(\mathsf{A})$ and set $\lambda_{\max}(\mathsf{A}) = \max\{\mathrm{Re}(\lambda) : \lambda \in \lambda(\mathsf{A})\}$ and $\|\mathsf{A}\| = \sqrt{\lambda_{\max}(\mathsf{A}^T\mathsf{A})}$; also, we denote the set of all nonnegative bounded random operators on $\Upsilon \times \Xi_5$ by $\mathcal{B}^+(\Xi_5)$.

(H1) For a nonlinear random operator $\boldsymbol{f} : \Upsilon \times \Xi_5 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, there is a positive map $\boldsymbol{l}(\varsigma) \in \mathcal{B}^+(\Xi_5)$ such that

$$\begin{aligned}&\eta\big(\boldsymbol{f}(\varrho,\varsigma,\mu_1,\nu_1)-\boldsymbol{f}(\varrho,\varsigma,\mu_2,\nu_2),\tau\big)_{\ell(\Upsilon\times\Xi_5)} \\ &\quad\geq \min\left(\eta\left(\mu_1-\mu_2,\frac{\tau}{\boldsymbol{l}(\varsigma)}\right)_{\ell(\Upsilon\times\Xi_5)},\eta\left(\nu_1-\nu_2,\frac{\tau}{\boldsymbol{l}(\varsigma)}\right)_{\ell(\Upsilon\times\Xi_5)}\right);\end{aligned}$$

moreover,

$$\eta\big(\mathsf{A}\mu(\varrho,\varsigma),\tau\big)_{\ell(\Upsilon\times\Xi_5)} \geq \eta\left(\mu(\varrho,\varsigma),\frac{\tau}{\mathbf{a}}\right)_{\ell(\Upsilon\times\Xi_5)}$$

and

$$\eta\big(\mathsf{B}\mu\big(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)\big),\tau\big)_{\ell(\Upsilon\times\Xi_5)} \geq \eta\left(\mu\big(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)\big),\frac{\tau}{\mathbf{b}}\right)_{\ell(\Upsilon\times\Xi_5)}$$

for all $\tau \in \Xi_8$ and $\varrho \in \Upsilon$, where $\|\mathsf{A}\| = \mathbf{a}$, $\|\mathsf{B}\| = \mathbf{b}$, and $\sup_{\upsilon\in[0,\varsigma]}\boldsymbol{l}(\upsilon) = \bar{\boldsymbol{l}}$.

3 Ulam–Hyers–Rassias stability with uncertainty

Using Remark 2.8 and Lemma 2.9, for $\varsigma \in \Xi_5$, we get

$$\begin{aligned}\nu(\varrho,\varsigma) &= \frac{\Theta(\varrho,0)}{\Gamma(\delta)\Gamma(2-\delta)} + \frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma) \\ &\quad-\Psi(\varrho,\upsilon)\big)^{\iota-1}\big(\mathsf{A}\nu(\varrho,\upsilon)+\mathsf{B}\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big) \\ &\quad+\boldsymbol{f}\big(\varrho,\upsilon,\nu(\varrho,\upsilon),\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big)\big)\,d\upsilon.\end{aligned} \tag{3.1}$$

Theorem 3.1 *Assume that* (H1) *holds and*

$$\eta\big(\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,0)\big)^{\iota},\tau\big) \geq \eta\big(\Theta(\varrho,\varsigma),\tau\big).$$

Then (1.1) *has the Ulam–Hyers–Rassias stability with uncertainty on* $\ell(\Upsilon \times \Xi_4)$ *when* $\eta(\Theta(\varrho,\varsigma),\tau)$ *is increasing on* $\Upsilon \times \Xi_5$ *as in Remark* 2.8 .

Proof For all $\varsigma \in \Xi_5$, $\tau \in \Xi_8$, and $\varrho \in \Upsilon$, using (2.2) and (3.1), we have

$$\begin{aligned}
&\eta(\nu-\mu,\tau)_{\ell(\Upsilon\times\Xi_5)} \\
&\quad= \eta\biggl(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\bigr)^{\iota-1}\eta\bigl(\mathsf{A}\bigl(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\bigr) \\
&\qquad+\mathsf{B}\bigl(\nu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)-\mu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr) \\
&\qquad+\bigl(\boldsymbol{f}\bigl(\varrho,\upsilon,\nu(\varrho,\upsilon),\nu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr) \\
&\qquad-\boldsymbol{f}\bigl(\varrho,\upsilon,\mu(\varrho,\upsilon),\mu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr)\bigr),\tau\bigr)\,d\upsilon,\tau\biggr) \\
&\quad\geq \eta\biggl(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\bigr)^{\iota-1}\min\bigl\{\eta\bigl(\mathsf{A}\bigl(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\bigr),\tau\bigr), \\
&\qquad\eta\bigl(\mathsf{B}\bigl(\nu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)-\mu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr),\tau\bigr), \\
&\qquad\eta\bigl(\bigl(\boldsymbol{f}\bigl(\varrho,\upsilon,\nu(\varrho,\upsilon),\nu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr) \\
&\qquad-\boldsymbol{f}\bigl(\varrho,\upsilon,\mu(\varrho,\upsilon),\mu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr)\bigr),\tau\bigr)\bigr\}\,d\upsilon,\tau\biggr) \\
&\quad\geq \eta\biggl(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\bigr)^{\iota-1}\min\biggl\{\eta\biggl(\bigl(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\bigr),\frac{\tau}{\mathbf{a}}\biggr), \\
&\qquad\eta\biggl(\bigl(\nu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)-\mu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr),\frac{\tau}{\mathbf{b}}\biggr), \\
&\qquad\min\biggl\{\eta\biggl(\bigl(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\bigr),\frac{\tau}{\boldsymbol{l}(\varsigma)}\biggr), \\
&\qquad\eta\biggl(\bigl(\nu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)-\mu\bigl(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\bigr)\bigr),\frac{\tau}{\boldsymbol{l}(\varsigma)}\biggr)\biggr\}\biggr\}\,d\upsilon,\tau\biggr) \\
&\quad\geq \eta\biggl(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\bigr)^{\iota-1}\eta\biggl((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\boldsymbol{l}(\varsigma)}\biggr)\,d\upsilon,\tau\biggr) \\
&\quad\geq \eta\biggl(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\bigr)^{\iota-1}\eta\biggl((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}}\biggr)\,d\upsilon,\tau\biggr) \\
&\quad\geq \eta\biggl(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\upsilon)\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\bigr)^{\iota-1}\,d\upsilon,\frac{\tau}{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}\biggr) \\
&\quad\geq \eta\biggl(\frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota}}{\Gamma(\iota+1)},\frac{\tau}{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}\biggr) \\
&\quad= \eta\biggl(\bigl(\Psi(\varrho,\varsigma)-\Psi(\varrho,0)\bigr)^{\iota},\frac{\Gamma(\iota+1)\tau}{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}\biggr).
\end{aligned}$$

Hence, if $\eta((\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota},\tau) \geq \eta(\Theta(\varrho,\varsigma),\tau)$ and $\mathbf{C} = \frac{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}{\Gamma(\iota+1)}$, then we have

$$\eta(\mu-\nu,\tau)_{\ell(\Upsilon\times\Xi_5)} \geq \varphi\biggl(\varsigma,\frac{\tau}{\mathbf{C}}\biggr). \tag{3.2}$$

Now by Definition 2.7, (1.1) has the Ulam–Hyers–Rassias stability with uncertainty on $\Upsilon \times \Xi_5$. □

Now we consider the new condition for constant φ_0:

(H2) $\varphi(\varsigma,\tau) = \varphi_0$

for all $\tau \in \Xi_8$.

Theorem 3.2 *Let* (H1) *and* (H2) *hold. Suppose that*

$$\eta\big((\Psi(\varrho,\varsigma) - \Psi(\varrho,0))^{\iota},\tau\big) \geq \eta\big(\Theta(\varrho,\varsigma),\tau\big).$$

Then (1.1) *has the Ulam–Hyers stability with uncertainty on* $\ell(\Upsilon \times \Xi_4)$.

Proof From (2.2) and (3.1), for all $\varsigma \in \Xi_5$ and $\varrho \in \Upsilon$, we have

$$\begin{aligned}
&\eta(\nu-\mu,\tau)_{\ell(\Upsilon\times\Xi_5)} \\
&\quad= \eta\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\eta\big(\mathsf{A}\big(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\big) \\
&\qquad + \mathsf{B}\big(\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)-\mu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big) \\
&\qquad + \big(\boldsymbol{f}\big(\varrho,\upsilon,\nu(\varrho,\upsilon),\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big) \\
&\qquad - \boldsymbol{f}\big(\varrho,\upsilon,\mu(\varrho,\upsilon),\mu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big)\big),\tau\big)\,d\upsilon,\tau\bigg) \\
&\quad\geq \eta\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\min\big\{\eta\big(\mathsf{A}\big(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\big),\tau\big), \\
&\qquad \eta\big(\mathsf{B}\big(\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)-\mu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big),\tau\big), \\
&\qquad \eta\big(\big(\boldsymbol{f}\big(\varrho,\upsilon,\nu(\varrho,\upsilon),\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big) \\
&\qquad - \boldsymbol{f}\big(\varrho,\upsilon,\mu(\varrho,\upsilon),\mu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big)\big),\tau\big)\big\}\,d\upsilon,\tau\bigg) \\
&\quad\geq \eta\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\min\bigg\{\eta\bigg(\big(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\big),\frac{\tau}{\mathbf{a}}\bigg), \\
&\qquad \eta\bigg(\big(\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)-\mu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big),\frac{\tau}{\mathbf{b}}\bigg), \\
&\qquad \min\bigg\{\eta\bigg(\big(\nu(\varrho,\upsilon)-\mu(\varrho,\upsilon)\big),\frac{\tau}{\boldsymbol{l}(\varsigma)}\bigg), \\
&\qquad \eta\bigg(\big(\nu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)-\mu\big(\varrho,\upsilon-\boldsymbol{h}(\varrho,\upsilon)\big)\big),\frac{\tau}{\boldsymbol{l}(\varsigma)}\bigg)\bigg\}\bigg\}\,d\upsilon,\tau\bigg) \\
&\quad\geq \eta\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\eta\bigg((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\boldsymbol{l}(\varsigma)}\bigg)\,d\upsilon,\tau\bigg) \\
&\quad\geq \eta\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\eta\bigg((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}}\bigg)\,d\upsilon,\tau\bigg) \\
&\quad\geq \eta\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\big(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\big)^{\iota-1}\,d\upsilon,\frac{\tau}{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}\bigg)
\end{aligned}$$

$$\geq \eta\left(\frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota}}{\Gamma(\iota+1)}, \frac{\tau}{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}\right)$$
$$= \eta\left(\left(\Psi(\varrho,\varsigma)-\Psi(\varrho,0)\right)^{\iota}, \frac{\Gamma(\iota+1)\tau}{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}\right).$$

Hence, if $\eta((\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota},\tau) \geq \eta(\Theta(\varrho,\varsigma),\tau)$ and $\mathbf{C} = \frac{\eta((\nu-\mu),\frac{\tau}{\mathbf{a}+\mathbf{b}+\bar{\boldsymbol{l}}})}{\Gamma(\iota+1)}$, then

$$\eta(\mu-\nu,\tau)_{\ell(\Upsilon\times\Xi_5)} \geq \varphi_0. \tag{3.3}$$

Thus (1.1) has the Ulam–Hyers stability with uncertainty on $\Upsilon \times \Xi_5$. □

4 Application

Now we apply our result to the following dynamic fractional-order equation systems with time-varying delay:

$$\begin{cases} {}^{H}D_{0^+}^{\iota,\kappa;\Psi}\mu(\varrho,\varsigma) \\ \quad = \mathsf{A}(\varsigma)\mu(\varrho,\varsigma) + \mathsf{B}(\varsigma)\mu(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma)) + D(\varsigma)\mathcal{G}(\varrho,\varsigma) + \boldsymbol{f}(\varrho,\cdot), & \varsigma \in \Xi_5, \\ \mu(\varrho,\varsigma) = \Theta(\varrho,\varsigma), & \varsigma \in \Xi_3, \end{cases} \tag{4.1}$$

where $\mathcal{G}(\varrho,\varsigma) \in \mathbb{R}^m$, and $\mathsf{A}(\varsigma)$, $\mathsf{B}(\varsigma) \in \mathbb{R}^{n\times n}$, and $D(\varsigma) \in \mathbb{R}^{n\times m}$ are random operator matrices such that $\sup_{\varsigma\in\Xi_5}(\|\mathsf{A}\| + \|\mathsf{B}\|) < \infty$.

Corollary 4.1 *Suppose that* $\|\mathsf{A}\| = \tilde{\mathbf{a}}$, $\|\mathsf{B}\| = \tilde{\mathbf{b}}$, *and there is* $\Theta \in \ell(\Upsilon \times \Xi_5, \mathbb{R}^n)$ *such that*

$$\int_0^{\varsigma} \Psi'(\varrho,\upsilon)\left(\Psi(\varrho,\varsigma)-\Psi(\varrho,\upsilon)\right)^{\iota-1}\eta\left(\Theta(\varrho,\upsilon),\tau\right) > \varphi\left(\varsigma,\frac{\tau}{M}\right).$$

Let (H1) *hold. Then system* (4.1) *has the Ulam–Hyers–Rassias stability with uncertainty if*

$$\eta\left(\left(\Psi(\varrho,\varsigma)-\Psi(\varrho,0)\right)^{\iota},\tau\right) \geq \eta\left(\Theta(\varrho,\varsigma),\tau\right).$$

Also, (H2) *implies that system* (4.1) *has the Ulam–Hyers stability with uncertainty.*

Example 4.2 Suppose that for each $\varrho \in \Upsilon$, $\Psi(\varrho,\varsigma) = \ln(\varsigma+1)$, $\iota = 0.2$, $\kappa \to 1$, $M = 20$, and $\boldsymbol{h}(\varrho,\varsigma) = 2\sin\varsigma$. Let $\Theta(\varrho,\varsigma) = (\varsigma, \sqrt{\varsigma}+1)^T$, and let $\boldsymbol{f}(\varrho,\varsigma,\mu(\varrho,\varsigma),\mu(\varrho,\varsigma-\boldsymbol{h}(\varrho,\varsigma))) = 0.1\sin\mu(\varrho,\varsigma) + 0.1\cos\mu(\varrho,\varsigma-2\sin\varsigma)$ with $\mu(\varrho,\varsigma) = (\mu_1(\varrho,\varsigma),\mu_2(\varrho,\varsigma))^T$. Consider system (1.1) with

$$\mathsf{A} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \tag{4.2}$$

$$\mathsf{B} = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{1}{6} & 0 \end{pmatrix}. \tag{4.3}$$

We have $\|\mathsf{A}\| = \mathbf{a} = \frac{1}{25}$, $\|\mathsf{B}\| = \mathbf{b} = \frac{5}{36}$, and $\bar{\boldsymbol{l}} = 0.1$. By calculation, $(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota} \approx 1.240$, so all the conditions in Theorem 3.1 are satisfied. Then (1.1) has the fuzzy Ulam–Hyers–Rassias stability with uncertainty on $\Upsilon \times [0,20]$. Moreover, the maximum value of

$\eta(\Theta(\varrho,\varsigma),\tau) \approx 0.977$ if $\varphi(\varrho,\varsigma) \leq 0.977$ is a constant continuous fuzzy set, and all the conditions in Theorem 3.2 hold, which implies that (1.1) has the fuzzy Ulam–Hyers stability with uncertainty on $\Upsilon \times [0,20]$.

5 Conclusion

In this paper, we considered a kind of stochastic differential equations involving the Ψ-Hilfer stochastic fractional derivative operator. A fuzzy control function helped us to make stable the stochastic differential equation (1.1). Using the fixed point method, we investigated the Ulam–Hyers–Rassias stability for the nonlinear Ψ-Hilfer stochastic fractional differential equation (1.1) with uncertainty.

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Authors' contributions
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References

1. Sousa, J., Oliveira, E.: On the Ψ-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **60**, 72–91 (2018)
2. Doungmo Goufo, E.F., Kumar, S., Mugisha, S.B.: Similarities in a fifth-order evolution equation with and with no singular kernel. Chaos Solitons Fractals **130**, Article ID 109467 (2020)
3. Ghanbari, B., Kumar, S., Kumar, R.: A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. Chaos Solitons Fractals **133**, Article ID 109619 (2020)
4. Kumar, S., Kumar, R., Cattani, C., Samet, B.: Chaotic behaviour of fractional predator–prey dynamical system. Chaos Solitons Fractals **135**, Article ID 109811 (2020)
5. Kumar, S., Kumar, R., Agarwal, R.P., Samet, B.: A study of fractional Lotka–Volterra population model using Haar wavelet and Adams–Bashforth–Moulton methods. Math. Methods Appl. Sci. **43**, 5564–5578 (2020)
6. Kumar, S., Ghosh, S., Samet, B., Doungmo Goufo, E.F.: An analysis for heat equations arises in diffusion process using new Yang–Abdel–Aty–Cattani fractional operator. Math. Methods Appl. Sci. **43**, 6062–6080 (2020)
7. Kumar, S., Ahmadian, A., Kumar, R., Kumar, D., Singh, J., Baleanu, D., Salimi, M.: An efficient numerical method for fractional SIR epidemic model of infectious disease by using Bernstein wavelets. Mathematics **8**(4), Article ID 558 (2020)
8. Alshabanat, A., Jleli, M., Kumar, S., Samet, B.: Generalization of Caputo–Fabrizio fractional derivative and applications to electrical circuits. Front. Phys. **8**, Article ID 64 (2020)
9. Kumar, S., Ghosh, S., Lotayif, M.S., Samet, B.: A model for describing the velocity of a particle in Brownian motion by Robotnov function based fractional operator. Alex. Eng. J. **59**, 1435–1449 (2020)
10. Malkowsky, E., Rakočević, V.: Advanced Functional Analysis. CRC Press, Boca Raton (2019)
11. Ćirić, L.B.: Some recent results in metrical fixed point theory (2003)
12. Todorčević, V.: Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics. Springer, Cham (2019)
13. Luo, D., Shah, K., Luo, Z.: On the novel Ulam–Hyers stability for a class of nonlinear ψ-Hilfer fractional differential equation with time-varying delays. Mediterr. J. Math. **16**(5), Article ID 112 (2019)
14. Bag, T., Samanta, S.K.: Finite dimensional intuitionistic fuzzy normed linear spaces. Ann. Fuzzy Math. Inform. **8**(2), 245–257 (2014)
15. Hadžić, O., Pap, E.: Fixed Point Theory in Probabilistic Metric Spaces. Mathematics and Its Applications, vol. 536. Kluwer Academic, Dordrecht (2001)

16. Font, J.J., Galindo, J., Macario, S., Sanchis, M.: Mazur–Ulam type theorems for fuzzy normed spaces. J. Nonlinear Sci. Appl. **10**(8), 4499–4506 (2017)
17. Bînzar, T., Pater, F., Nădăban, S.: On fuzzy normed algebras. J. Nonlinear Sci. Appl. **9**(9), 5488–5496 (2016)
18. Saadati, R., Vaezpour, S.M.: Some results on fuzzy Banach spaces. J. Appl. Math. Comput. **17**(1–2), 475–484 (2005)
19. Mirmostafaee, A.K.: Perturbation of generalized derivations in fuzzy Menger normed algebras. Fuzzy Sets Syst. **195**, 109–117 (2012)
20. Naeem, R., Anwar, M.: Jessen type functionals and exponential convexity. J. Math. Comput. Sci. **17**(3), 429–436 (2017)
21. Park, C., Shin, D.Y., Saadati, R., Lee, J.R.: A fixed point approach to the fuzzy stability of an AQCQ-functional equation. Filomat **30**(7), 1833–1851 (2016)
22. Park, C., Anastassiou, G.A., Saadati, R., Yun, S.: Functional inequalities in fuzzy normed spaces. J. Comput. Anal. Appl. **22**(4), 601–612 (2017)
23. Kang, J.I., Saadati, R.: Approximation of homomorphisms and derivations on non-Archimedean random Lie C^*-algebras via fixed point method. J. Inequal. Appl. **2012**, Article ID 251 (2012)
24. Agarwal, R.P., Saadati, R., Salamati, A.: Approximation of the multiplicatives on random multi-normed space. J. Inequal. Appl. **2017**, Article ID 204 (2017)
25. Cho, Y.J., Rassias, T.M., Saadati, R.: Stability of Functional Equations in Random Normed Spaces. Springer Optimization and Its Applications, vol. 86. Springer, Berlin (2013)
26. El-Moneam, M.A., Ibrahim, T.F., Elamody, S.: Stability of a fractional difference equation of high order. J. Nonlinear Sci. Appl. **12**(2), 65–74 (2019)
27. Park, C., Kim, S.O., Alaca, C.: Stability of additive-quadratic ρ-functional equations in Banach spaces: a fixed point approach. J. Nonlinear Sci. Appl. **10**(3), 1252–1262 (2017)
28. Saadati, R., Park, C.: Approximation of derivations and the superstability in random Banach $*$-algebras. Adv. Differ. Equ. **2018**, Article ID 418 (2018)
29. Madadi, M., Saadati, R., De la Sen, M.: Stability of unbounded differential equations in Menger k-normed spaces: a fixed point technique. Mathematics **8**(3), Article ID 400 (2020)
30. Constantinescu, C.D., Ramirez, J.M., Zhu, W.R.: An application of fractional differential equations to risk theory. Finance Stoch. **23**(4), 1001–1024 (2019)
31. El-Sayed, A.M.A., Gaafar, F.M.: Existence and uniqueness of solution for Sturm–Liouville fractional differential equation with multi-point boundary condition via Caputo derivative. Adv. Differ. Equ. **2019**, Article ID 46 (2019)
32. Jiang, J., O'Regan, D., Xu, J., Fu, Z.: Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions. J. Inequal. Appl. **2019**, Article ID 204 (2019)
33. Sene, N.: Stability analysis of the generalized fractional differential equations with and without exogenous inputs. J. Nonlinear Sci. Appl. **12**(9), 562–572 (2019)
34. Sene, N.: Global asymptotic stability of the fractional differential equations. J. Nonlinear Sci. Appl. **13**(3), 171–175 (2020)
35. Agarwal, R.P., Hristova, S., O'Regan, D.: Lyapunov functions to Caputo reaction–diffusion fractional neural networks with time-varying delays. J. Math. Comput. Sci. **18**(3), 328–345 (2018)
36. Ahmed Hamdy, M., El-Borai Mahmoud, M., El-Owaidy, H.M., Ghanem, A.S.: Null controllability of fractional stochastic delay integro-differential equations. J. Math. Comput. Sci. **19**(3), 143–150 (2019)
37. Ali, A., Shah, K., Li, Y. Khan, R.A.: Numerical treatment of coupled system of fractional order partial differential equations. J. Math. Comput. Sci. **19**(2), 74–85 (2019)
38. Ndolane, S.: Exponential form for Lyapunov function and stability analysis of the fractional differential equations. J. Math. Comput. Sci. **18**(4), 388–397 (2018)
39. Vanterler da C. Sousa, J., Capelas de Oliveira, E.: Ulam–Hyers stability of a nonlinear fractional Volterra integro-differential equation. Appl. Math. Lett. **81**, 50–56 (2018)
40. Vanterler da C. Sousa, J., Capelas de Oliveira, E.: On a new operator in fractional calculus and applications. J. Fixed Point Theory Appl. **20**(3), Article ID 96 (2018)