Chaharpashlou et al. *Advances in Difference Equations* (2020) 2020:326
https://doi.org/10.1186/s13662-020-02780-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

C^*-Algebra valued fuzzy normed spaces with application of Hyers–Ulam stability of a random integral equation

R. Chaharpashlou[1], Donal O'Regan[2], Choonkil Park[3*] and Reza Saadati[1]

*Correspondence:
baak@hanyang.ac.kr
[3]Research Institute for Natural Sciences, Hanyang University, 04763, Seoul, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we consider C^*-algebra valued fuzzy normed spaces. We study the random integral equation $(\frac{1}{2c})\int_{x-cd}^{x+cd} u(\gamma,\tau,d_0)\,d\tau = u(\gamma,x,d)$ which is related to the stochastic wave equation. In addition, using a C^*-algebra valued fuzzy controller function, we consider its C^*-algebra valued fuzzy Hyers–Ulam stability.

MSC: 54E50; 39B52; 39B62; 46L05; 47H10; 39B82

Keywords: Nonlinear random integral equation; Stochastic wave equation; Iterative method; Random operator; Fuzzy normed space; C^*-algebra valued fuzzy set

1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for additive groups in Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6–11]).

Let A be a C^*-algebra and x be a self-adjoint element in A. Then if x is of the form yy^* for some $y \in A$, then x is called a positive element. Denote by A^+ the cone of positive elements of A. We will denote $z \preceq w$ when $w - z \in A^+$ (see [12]).

Using random normed spaces introduced by Šerstnev [13] and studied by Muštari [14] and Radu [15], Cheng and Mordeson [16] defined fuzzy normed spaces.

In this paper, we generalize a recent paper of Saadati [17] using C^*-algebra valued fuzzy sets and applying t-norms on C^*-algebras (see [18, 19]).

2 C^*-Algebra valued fuzzy normed spaces

In this section, we discuss C^*-algebra. For more details, we refer the reader to [20–22].

Definition 1 Let $\mathcal{A}$ be an order commutative C^*-algebra and $\mathcal{A}^+$ be the positive section of $\mathcal{A}$. Let $U \neq \emptyset$. A C^*-algebra valued fuzzy set $\mathcal{C}$ on U is a function $\mathcal{C} : U \longrightarrow \mathcal{A}^+$. For each u in U, $\mathcal{C}(u)$ represents the degree (in $\mathcal{A}^+$) to which u satisfies $\mathcal{A}^+$.

We put $\mathbf{0} = \inf \mathcal{A}^+$ and $\mathbf{1} = \sup \mathcal{A}^+$. Now, we define the triangular norm (t-norm) on $\mathcal{A}^+$.

Definition 2 A function $\mathcal{T} : \mathcal{A}^+ \times \mathcal{A}^+ \to \mathcal{A}^+$ which satisfies

(i) $(\forall u \in \mathcal{A}^+)$ $(\mathcal{T}(u, \mathbf{1}) = u)$; (boundary condition)
(ii) $(\forall (u, v) \in \mathcal{A}^+ \times \mathcal{A}^+)$ $(\mathcal{T}(u, v) = \mathcal{T}(v, u))$; (commutativity)
(iii) $(\forall (u, v, w) \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+)$ $(\mathcal{T}(u, \mathcal{T}(v, w)) = \mathcal{T}(\mathcal{T}(u, v), w))$; (associativity)
(iv) $(\forall (u, u', v, v') \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+)$ $(u \preceq u'$ and $v \preceq v' \Rightarrow \mathcal{T}(u, v) \preceq \mathcal{T}(u', v'))$, (monotonicity)

is called a t-norm.

If, for every $u, v \in \mathcal{A}^+$ and sequences $\{u_n\}$ and $\{v_n\}$ converging to u and v, we have

$$\lim_n \mathcal{T}(u_n, v_n) = \mathcal{T}(u, v),$$

then we say $\mathcal{T}$ on $\mathcal{A}^+$ is continuous (in short, a ct-norm).

Definition 3 Assume that $\mathcal{F} : \mathcal{A}^+ \to \mathcal{A}^+$ satisfies $\mathcal{F}(\mathbf{0}) = \mathbf{1}$ and $\mathcal{F}(\mathbf{1}) = \mathbf{0}$ and is decreasing. Then $\mathcal{F}$ is called a negation on $\mathcal{A}^+$.

Example 4 Let

$$\operatorname{diag} M_n\big([0,1]\big) = \left\{ \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix} = \operatorname{diag}[u_1, \dots, u_n], u_1, \dots, u_n \in [0,1] \right\}.$$

We say $\operatorname{diag}[u_1, \dots, u_n] \preceq \operatorname{diag}[b_1, \dots, b_n]$ if and only if $a_i \leq b_i$ for all $i = 1, \dots, n$ and also $\mathbf{1} = \operatorname{diag}[1, \dots, 1]$ and $\mathbf{0} = \operatorname{diag}[0, \dots, 0]$. Now, we see that if $\mathcal{A} = \operatorname{diag} M_n([0,1])$, then $\operatorname{diag} M_n([0,1]) = \mathcal{A}^+$. Let $\mathcal{T}_P : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ be

$$\mathcal{T}_P\big(\operatorname{diag}[u_1, \dots, u_n], \operatorname{diag}[v_1, \dots, v_n]\big) = \operatorname{diag}[u_1 \cdot v_1, \dots, u_n \cdot v_n].$$

Then $\mathcal{T}_P$ is a t-norm (product t-norm). Note that this t-norm is continuous.

Example 5 Let $\operatorname{diag} M_n([0,1]) = \mathcal{A}^+$ and $\mathcal{T}_M : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ be

$$\mathcal{T}_M\big(\operatorname{diag}[u_1, \dots, u_n], \operatorname{diag}[v_1, \dots, v_n]\big) = \operatorname{diag}\big[\min(u_1, v_1), \dots, \min(u_n, v_n)\big].$$

Then $\mathcal{T}_M$ is a t-norm (minimum t-norm). Note that this t-norm is continuous.

Definition 6 The triple $(S, \eta, \mathcal{T})$ is called a *C^*-algebra valued fuzzy normed space* (in short, C^*AVFN-space) if $S \neq \emptyset$, $\mathcal{T}$ is a ct-norm on $\mathcal{A}^+$ and η is a C^*-algebra valued fuzzy set on $S^2 \times]0, +\infty[$ such that, for each $t, s, p \in T$ and τ, ς in $]0, +\infty[$, we have

(a) $\eta(s,\tau) \succ \mathbf{0}$;
(b) $\eta(s,\tau) = \mathbf{1}$ for all $\tau > 0$ if and only if $s = 0$;
(c) $\eta(as,\tau) = \eta(s, \frac{\tau}{|a|})$ for all $s \in S$ and $a \in \mathbb{R}$ with $a \neq 0$;
(d) $\eta(t+s,\tau+\varsigma) \succeq \mathcal{T}(\eta(t,\tau),\eta(s,\varsigma))$ for all $t,s \in S$ and $\tau,\varsigma \geq 0$;
(e) $\eta(s,\cdot) : (0,\infty) \to \mathcal{A}^+ \setminus \{\mathbf{0}\}$ is left continuous;
(f) $\lim_{t\longrightarrow\infty} \eta(s,\tau) = \mathbf{1}$.

Also, η is a *C^*-algebra valued fuzzy norm.*

Let $(S,\eta,\mathcal{T})$ be a C^*AVFN-space. For $\tau > 0$, define the *open ball* $B(t,\varrho,\tau)$ as

$$B(s,\varrho,\tau) = \bigl\{t \in S : \eta(t-s,\tau) \succeq \mathcal{F}(\varrho)\bigr\},$$

in which $s \in S$ is the center and $\varrho \in \mathcal{A}^+ \setminus \{\mathbf{0},\mathbf{1}\}$ is the radius. We say that $A \subseteq S$ is *open* if, for each $s \in A$, there exist $\tau > 0$ and $\varrho \in \mathcal{A}^+ \setminus \{\mathbf{0},\mathbf{1}\}$ such that $B(s,\varrho,\tau) \subseteq A$. We denote the family of all open subsets of S by τ_η, and so τ_η is the *C^*-fuzzy topology induced by the C^*-algebra valued fuzzy norm η.*

Example 7 Consider the linear normed space $(S,\|\cdot\|)$. Let $\mathcal{T} = \mathcal{T}_M$ and define the fuzzy set η on $S^2 \times (0,\infty)$ as follows:

$$\eta(s,\tau) = \operatorname{diag}\left[\frac{\tau}{\tau + \|s\|}, \exp\left(-\frac{\|s\|}{\tau}\right)\right]$$

for all $\tau \in \mathbf{R}^+$. Then $(S,\eta,\mathcal{T}_M)$ is a C^*AVFN-space.

Lemma 8 ([23]) *Let $(S,\eta,\mathcal{T})$ be a C^*AVFN-space. Then $\eta(s,\tau)$ is nondecreasing with respect to τ for all $s \in S$.*

Definition 9 Let $\{s_n\}_{n\in\mathbf{N}}$ be a sequence in a C^*AVFN-space $(S,\eta,\mathcal{T})$. If, for all $\varepsilon \in \mathcal{A}^+ \setminus \{\mathbf{0}\}$ and $\tau > 0$, there exists $n_0 \in \mathbf{N}$ such that, for all $m \geq n \geq n_0$,

$$\eta(s_m - s_n,\tau) \succeq \mathcal{F}(\varepsilon),$$

then $\{s_n\}_{n\in\mathbf{N}}$ is said to be Cauchy.

Also $\{s_n\}_{n\in\mathbf{N}}$ is said to be convergent to $s \in S$ ($s_n \xrightarrow{\eta} s$) if $\eta(s_n - s,\tau) = \eta(s - s_n,\tau) \to \mathbf{1}$ as $n \to +\infty$ for every $\tau > 0$. If every Cauchy sequence is convergent in a C^*AVFN-space, then the space is said to be *complete*. A complete C^*AVFN-space is called a C^*-algebra valued fuzzy Banach space (in short, a C^*AVFB-space).

3 Random operators in C^*AVFB-spaces

Let (Γ,Σ,ξ) be a probability measure space. Assume that $(T,\mathcal{B}_T)$ and $(S,\mathcal{B}_S)$ are Borel measurable spaces, in which T and S are C^*AVFB-spaces. A mapping $F : \Gamma \times T \to S$ is said to be a random operator if $\{\gamma : F(\gamma,t) \in B\} \in \Sigma$ for all t in T and $B \in \mathcal{B}_S$. Also, F is a random operator if $F(\gamma,t) = s(\gamma)$ is an S-valued random variable for every t in T. A random operator $F : \Gamma \times T \to S$ is called *linear* if $F(\gamma,at_1 + bt_2) = aF(\gamma,t_1) + bF(\gamma,t_2)$ for almost every γ for each t_1, t_2 in T and scalars a, b and *bounded* if there exists a nonnegative real-valued random variable $M(\gamma)$ such that

$$\eta\big(F(\gamma,t_1)-F(\gamma,t_2),M(\gamma)\tau\big)\succeq\eta(t_1-t_2,\tau),$$

almost every γ for each t_1, t_2 in T and $\tau>0$.

Recently, some authors discussed the approximation of functional equations in several spaces by using a direct technique and a fixed point technique; for fuzzy Menger normed algebras, see [24]; for fuzzy metric spaces, see [25, 26]; for FN spaces, see [27]; for non-Archimedean random Lie C^*-algebras, see [28]; for non-Archimedean random normed spaces, see [29]; for random multi-normed space, see [30]; and we also refer the reader to [31–34].

Note that a $[0,\infty]$-valued metric is called a generalized metric.

Theorem 10 ([35, 36]) *Consider a complete generalized metric space (T,δ) and a strictly contractive function $\Lambda:T\to T$ with Lipschitz constant $L<1$. For every given element $t\in T$, either*

$$\delta\big(\Lambda^n t,\Lambda^{n+1}t\big)=\infty$$

for each $n\in\mathbb{N}$ or there is $n_0\in\mathbb{N}$ such that

(1) $\delta(\Lambda^n t,\Lambda^{n+1}t)<\infty$, $\forall n\geq n_0$;
(2) *the fixed point s^* of Λ is the convergent point of sequence $\{\Lambda^n t\}$;*
(3) *in the set $V=\{s\in T\mid\delta(\Lambda^{n_0}t,s)<\infty\}$, s^* is the unique fixed point of Λ;*
(4) $(1-L)\delta(s,s^*)\leq\delta(s,\Lambda s)$ *for every $s\in V$.*

4 Random integral equation related to the stochastic wave equation

Let (Γ,Σ,ξ) be a probability space and $(S,\eta,\mathcal{T}_M)$ be a C^*AVFB-space. Assume that the real numbers $c>0$ and d_0 are fixed, and suppose that $\gamma\in\Gamma$. Consider the stochastic wave equation

$$u_{dd}(\gamma,x,d)=c^2u_{xx}(\gamma,x,d). \tag{4.1}$$

Since

$$\begin{aligned}
u_d(\gamma,x,d)&=\frac{1}{2c}\frac{\partial}{\partial d}\int_{x-cd}^{x+cd}H(\gamma,\tau,d_0)\,d\tau\\
&=\frac{1}{2}H(\gamma,x+cd,d_0)+\frac{1}{2}H(\gamma,x-cd,d_0),\\
u_{dd}(\gamma,x,d)&=\frac{c}{2}H_d(\gamma,x+cd,d_0)-\frac{c}{2}H_d(\gamma,x-cd,d_0),\\
u_x(\gamma,x,d)&=\frac{1}{2c}H(\gamma,x+cd,d_0)-\frac{1}{2c}H(\gamma,x-cd,d_0),\\
u_{xx}(\gamma,x,d)&=\frac{1}{2c}H_x(\gamma,x+cd,d_0)-\frac{1}{2c}H_x(\gamma,x-cd,d_0),
\end{aligned} \tag{4.2}$$

we have that

$$u(\gamma,x,d):=\frac{1}{2c}\int_{x-cd}^{x+cd}H(\gamma,\tau,d_0)\,d\tau \tag{4.3}$$

is a solution of (4.1) for any random differentiable S-valued function H on $\Gamma \times \mathbb{R}$.

On the other hand, Jung [37] showed that if the S-valued functions F and G on $\Gamma \times \mathbb{R}^2$ are twice differentiable, then the S-valued solution u on $\Gamma \times \mathbb{R}^2$ of (4.1) has a representation of the form

$$u(\gamma, x, d) = F(\gamma, x + cd) + G(\gamma, x - cd), \tag{4.4}$$

in which

$$\begin{aligned} &\frac{1}{2c}\int_{x-cd}^{x+cd} F(\gamma, \tau)\, d\tau = F(\gamma, x + cd), \\ &\frac{1}{2c}\int_{x-cd}^{x+cd} G(\gamma, \tau)\, d\tau = G(\gamma, x - cd). \end{aligned} \tag{4.5}$$

Consider the random integral equation

$$\frac{1}{2c}\int_{x-cd}^{x+cd} u(\gamma, \tau, d_0)\, d\tau = u(\gamma, x, d), \tag{4.6}$$

which is controlled by the continuous fuzzy set $\varphi(x, d, t)$ as

$$\eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd} u(\gamma, \tau, d_0)\, d\tau - u(\gamma, x, d), t\right) \succeq \varphi(x, d, t). \tag{4.7}$$

We say that the random integral equation (4.6) has fuzzy Hyers–Ulam stability if there are $u_0(\gamma, x, d)$ and $\lambda > 0$ such that

$$\begin{aligned} &\frac{1}{2c}\int_{x-cd}^{x+cd} u_0(\gamma, \tau, d_0)\, d\tau = u_0(\gamma, x, d), \\ &\eta\big(u(\gamma, x, d) - u_0(\gamma, x, d), t\big) \succeq \varphi\left(x, d, \frac{t}{\lambda}\right). \end{aligned} \tag{4.8}$$

5 C^*-Algebra-valued fuzzy Hyers–Ulam stability

Let $c > 0$, $d_0 > 0$, and $a + cd_0 < b - cd_0$. Let (Γ, Σ, ξ) be a probability measure space, $(S, \eta, \mathcal{T}_M)$ be a C^*AVFB-space, $\alpha := [a, b]$, $\beta := (0, d_0]$, and $\alpha_0 := [a + cd_0, b - cd_0]$. Let $M > 0$ and $0 < L < 1$. Consider a continuous C^*-algebra-valued fuzzy set $\varphi : \alpha \times \beta \times (0, \infty) \to J$ which is increasing in the second and third components and satisfies

$$\inf_{\tau \in [x-cd, x+cd]} \varphi\left(\tau, d, \frac{t}{d}\right) \succeq \varphi\left(x, d, \frac{t}{L}\right) \tag{5.1}$$

for all $x \in \alpha_0$, $d \in \beta$, and $t > 0$.

The set T consists of all random operators $F : \Gamma \times \alpha \times \beta \to S$ which satisfy the following:

(a) $F(\gamma, x, d)$ is continuous for each $x \in \alpha_0$, $d \in \beta$, and $\gamma \in \Gamma$;
(b) $F(\gamma, x, d) = 0_S$ for all $x \in \alpha \setminus \alpha_0$, $d \in \beta$, and $\gamma \in \Gamma$;
(c) $\eta(F(\gamma, x, d), t) \succeq \varphi(x, d, \frac{t}{M})$ for all $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$.

Theorem 11 *Suppose that a random operator $u \in T$ satisfies the random integral inequality*

$$\eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd} u(\gamma,\tau,d_0)\,d\tau - u(\gamma,x,d),t\right) \succeq \varphi(x,d,t) \tag{5.2}$$

for all $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$. Then there is a unique random operator $u_0 \in T$ which satisfies

$$\frac{1}{2c}\int_{x-cd}^{x+cd} u_0(\gamma,\tau,d_0)\,d\tau = u_0(\gamma,x,d), \tag{5.3}$$

$$\eta\big(u(\gamma,x,d) - u_0(\gamma,x,d)\big) \succeq \varphi\big(x,d,(1-L)t\big) \tag{5.4}$$

for all $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$.

Proof We consider the $[0,\infty]$-valued metric δ on T defined by

$$\begin{aligned} &\delta(F,G) \\ &\quad := \inf\left\{\lambda \in [0,\infty] \;\middle|\; \eta\big(F(\gamma,x,d) - G(\gamma,x,d),t\big) \succeq \varphi\left(x,d,\frac{t}{\lambda}\right)\right. \\ &\qquad \left.\forall x \in \alpha_0, d \in \beta, \gamma \in \Gamma, t > 0\right\}. \end{aligned} \tag{5.5}$$

In [38], Miheț and Radu proved that (B,δ) is complete (see also [39]).

Consider the operator $\Lambda : T \to T$ given by

$$(\Lambda H)(\gamma,x,d) := \begin{cases} \frac{1}{2c}\int_{x-cd}^{x+cd} H(\gamma,\tau,d_0)\,d\tau, & (x \in \alpha_0, d \in \beta, \gamma \in \Gamma), \\ 0, & (\text{otherwise}). \end{cases} \tag{5.6}$$

It is easy to show that ΛH is continuous on $\Gamma \times \alpha_0 \times \beta$. Let $x - cd = \xi_1 < \xi_2 < \cdots < \xi_k = x + cd$, $\triangle s_i = \xi_i - \xi_{i-1}$, $i = 1,2,\ldots,k$. Using (5.1), (c), and (5.6), we obtain

$$\begin{aligned} \eta(\Lambda H)(\gamma,x,d),t) &= \eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd} H(\gamma,\tau,d_0)\,d\tau,t\right) \\ &= \eta\left(\frac{1}{2c}\lim_{\|\Delta s\|\to 0}\sum_{i=1}^{k} H(\gamma,\xi_i,d_0)\Delta s_i,t\right) \\ &= \eta\left(\lim_{\|\Delta s\|\to 0}\sum_{i=1}^{k} H(\gamma,\xi_i,d_0)\Delta s_i,2ct\right) \\ &= \lim_{\|\Delta s\|\to 0}\eta\left(\sum_{i=1}^{k} H(\gamma,\xi_i,d_0)\Delta s_i,2ct\right) \\ &\succeq \lim_{\|\Delta s\|\to 0}\mathcal{T}_M\eta\left(H(\gamma,\xi_i,d_0)\Delta s_i,\frac{2ct}{k}\right) \\ &\succeq \inf_{\tau\in[x-cd,x+cd]}\eta\left(H(\gamma,\xi_i,d_0),\frac{2ct}{|\Delta s_i|k}\right) \end{aligned}$$

$$
\begin{aligned}
&\succeq \inf_{\tau\in[x-cd,x+cd]} \eta\left(H(\gamma,\tau,d_0), \frac{2ctk}{2cdk}\right) \\
&\succeq \inf_{\tau\in[x-cd,x+cd]} \varphi\left(\tau,d_0,\frac{t}{dM}\right) \\
&\succeq \varphi\left(x,d,\frac{t}{LM}\right) \\
&\succ \varphi\left(x,d,\frac{t}{M}\right)
\end{aligned}
\tag{5.7}
$$

for any given $x\in\alpha_0$, $d\in\beta$, $t>0$, and $\gamma\in\Gamma$, and then $\Lambda H\in T$. Let $F,G\in T$ and $\lambda_{FG}\in[0,\infty]$ such that $\delta(F,G)\leq\lambda_{FG}$. Then we have

$$
\eta\big(F(\gamma,x,d)-G(\gamma,x,d),t\big)\succeq\varphi\left(x,d,\frac{t}{\lambda_{FG}}\right)
\tag{5.8}
$$

for all $x\in\alpha_0$, $d\in\beta$, $t>0$, and $\gamma\in\Gamma$, i.e., Λ is strictly contractive on T. From (5.1), (5.6), and (5.8), we get

$$
\begin{aligned}
\eta\big((\Lambda F)(\gamma,x,d)-(\Lambda G)(\gamma,x,d),t\big) &= \eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd}\big(F(\gamma,\tau,d_0)-G(\gamma,\tau,d_0)\big)\,d\tau,t\right) \\
&= \eta\left(\frac{1}{2c}\lim_{\|\Delta s\|\to 0}\sum_{i=1}^{k}\big(F(\gamma,\xi_i,d_0)-G(\gamma,\xi_i,d_0)\big)\Delta s_i,t\right) \\
&= \eta\left(\lim_{\|\Delta s\|\to 0}\sum_{i=1}^{k}\big(F(\gamma,\xi_i,d_0)-G(\gamma,\xi_i,d_0)\big)\Delta s_i,2ct\right) \\
&= \lim_{\|\Delta s\|\to 0}\eta\left(\sum_{i=1}^{k}\big(F(\gamma,\xi_i,d_0)-G(\gamma,\xi_i,d_0)\big)\Delta s_i,2ct\right) \\
&\succeq \lim_{\|\Delta s\|\to 0}\mathcal{T}_M\eta\left(\big(F(\gamma,\xi_i,d_0)-G(\gamma,\xi_i,d_0)\big),\frac{2ct}{|\Delta s_i|k}\right) \\
&\succeq \inf_{\tau\in[x-cd,x+cd]}\eta\left(\big(F(\gamma,\tau,d_0)-G(\gamma,\tau,d_0)\big),\frac{2ctk}{2cdk}\right) \\
&\succeq \inf_{\tau\in[x-cd,x+cd]}\varphi\left(\tau,d_0,\frac{t}{d\lambda_{FG}}\right) \\
&\succeq \varphi\left(x,d_0,\frac{t}{L\lambda_{FG}}\right) \\
&\succ \varphi\left(x,d,\frac{t}{L\lambda_{FG}}\right)
\end{aligned}
\tag{5.9}
$$

for any given $x\in\alpha_0$, $d\in\beta$, $t>0$, and $\gamma\in\Gamma$, which implies that $\delta(\Lambda F,\Lambda G)\leq L\lambda_{FG}$, and so $\delta(\Lambda F,\Lambda G)\leq Ld(F,G)$. Suppose $H_0\in T$. Using (5.2) and (5.5), we get

$$
\begin{aligned}
\eta\big((\Lambda H_0)(\gamma,x,d)-H_0(\gamma,x,d),t\big) &\succeq \eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd}H_0(\gamma,\tau,d_0)\,d\tau-H_0(\gamma,x,d),t\right) \\
&\succeq \varphi(x,d,t)
\end{aligned}
\tag{5.10}
$$

for any $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$. Thus (5.5) implies that

$$\delta(\Lambda H_0, H_0) \leq 1 < \infty. \tag{5.11}$$

Now,

(1) Theorem 10 (2) implies that there is $u_0 \in T$ such that $\Lambda^n H_0 \to u_0$ in (T, δ) and $\Lambda u_0 = u_0$.
(2) Theorem 10 (3) implies that u_0 is the unique element of T which satisfies $(\Lambda u_0)(\gamma, x, d) = u_0(\gamma, x, d)$ for any $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$.
(3) Theorem 10 (3), together with (5.5) and (5.2), implies that

$$\delta(u, u_0) \leq \frac{1}{1-L}\delta(\Lambda u, u) \leq \frac{1}{1-L}, \tag{5.12}$$

since (5.2) means that $\delta(\Lambda u, u) \leq 1$. In view of (5.5), we can conclude that (5.4) holds for all $x \in \alpha_0$ and $d \in \beta$.

□

6 Conclusion

In this paper, we modified and generalized fuzzy normed spaces and introduced the concept of a C^*AVFN-space. As an application, we studied the Hyers–Ulam stability of a random integral equation related to the stochastic wave equation in C^*AVFB-spaces.

Acknowledgements
Not applicable.

Funding
This work was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science, and Technology (NRF-2017R1D1A1B04032937).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Author details
[1]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran. [2]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, University Road, Galway, Ireland. [3]Research Institute for Natural Sciences, Hanyang University, 04763, Seoul, Korea.

Received: 9 April 2020 Accepted: 17 June 2020 Published online: 02 July 2020

References

1. Ulam, S.M.: Problems in Modern Mathematics. Wiley, New York (1960)
2. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222–224 (1941)
3. Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64–66 (1950)
4. Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
5. Găvruta, P.: A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)
6. Jung, S., Popa, D., Rassias, M.T.: On the stability of the linear functional equation in a single variable on complete metric spaces. J. Glob. Optim. **59**, 13–16 (2014)

7. Lee, Y., Jung, S., Rassias, M.T.: Uniqueness theorems on functional inequalities concerning cubic–quadratic-additive equation. J. Math. Inequal. **12**, 43–61 (2018)
8. Li, T., Zada, A.: Connections between Hyers–Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces. Adv. Differ. Equ. **2016**, 153 (2016)
9. Li, T., Zada, A., Faisal, S.: Hyers–Ulam stability of nth order linear differential equations. J. Nonlinear Sci. Appl. **9**, 2070–2075 (2016)
10. Zada, A., Yar, M., Li, T.: Existence and stability analysis of nonlinear sequential coupled system of Caputo fractional differential equations with integral boundary conditions. Ann. Univ. Paedagog. Crac. Stud. Math. **17**, 103–125 (2018)
11. Zada, A., Ali, S., Li, T.: Analysis of a new class of impulsive implicit sequential fractional differential equations. Int. J. Nonlinear Sci. Numer. Simul. (in press). https://doi.org/10.1515/ijnsns-2019-0030
12. Dixmier, J.: C^*-Algebras. North-Holland, New York (1977)
13. Šerstnev, A.N.: On the notion of a random normed space. Dokl. Akad. Nauk USSR **149**, 280–283 (1963)
14. Muštari, D.H.: Almost sure convergence in linear spaces of random variables. Teor. Veroâtn. Primen. **15**, 351–357 (1970)
15. Radu, V.: Linear operators in random normed spaces. Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) **17**(65), 217–220 (1975)
16. Cheng, S.C., Mordeson, J.N.: Fuzzy linear operators and fuzzy normed linear spaces. Bull. Calcutta Math. Soc. **86**(5), 429–436 (1994)
17. Saadati, R.: Nonlinear contraction and fuzzy compact operator in fuzzy Banach algebras. Fixed Point Theory **20**, 289–297 (2019)
18. Beg, I., Ahmed, M., Nafadi, H.: Fixed points of $\mathcal{L}$-fuzzy mappings in ordered b-metric spaces. J. Funct. Spaces **2018**, Article ID 5650242 (2018)
19. Huang, H., Wu, C.: Characterizations of compact sets in fuzzy set spaces with L_p metric. Fuzzy Sets Syst. **330**, 16–40 (2018)
20. Glück, J.: A note on lattice ordered C^*-algebra and Perron–Frobenius theory. Math. Nachr. **291**(11–12), 1727–1732 (2020)
21. Green, M.D.: The lattice structure of C^*-algebras and their duals. Math. Proc. Camb. Philos. Soc. **81**(2), 245–248 (1977)
22. Hussain, S.: Fixed point and common fixed point theorems on ordered cone b-metric space over Banach algebra. J. Nonlinear Sci. Appl. **13**, 22–33 (2020)
23. Saadati, R., Vaezpour, S.M.: Some results on fuzzy Banach spaces. J. Appl. Math. Comput. **17**(1–2), 475–484 (2005)
24. Mirmostafaee, A.K.: Perturbation of generalized derivations in fuzzy Menger normed algebras. Fuzzy Sets Syst. **195**, 109–117 (2012)
25. Naeem, R., Anwar, M.: Jensen type functionals and exponential convexity. J. Math. Comput. Sci. **17**, 429–436 (2017)
26. Park, C., Shin, D., Saadati, R., Lee, R.: A fixed point approach to the fuzzy stability of an AQCQ-functional equation. Filomat **30**(7), 1833–1851 (2016)
27. Naeem, R., Anwar, M.: Weighted Jensen's functionals and exponential convexity. J. Math. Comput. Sci. **19**, 171–180 (2019)
28. Shoaib, A., Azam, A., Arshad, M., Ameer, E.: Fixed point results for multivalued mappings on a sequence in a closed ball with applications. J. Math. Comput. Sci. **17**, 308–316 (2017)
29. Ciepliski, K.: On a functional equation connected with bi-linear mappings and its Hyers–Ulam stability. J. Nonlinear Sci. Appl. **10**(11), 5914–5921 (2017)
30. Agarwal, R.P., Saadati, R., Salamati, A.: Approximation of the multiplicatives on random multi-normed space. J. Inequal. Appl. **2017**, 204 (2017)
31. EL-Fassi, I.: Solution and approximation of radical quintic functional equation related to quintic mapping in quasi-β-Banach spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**, 675–687 (2019)
32. Jang, S., Saadati, R.: Approximation of an additive (ϱ_1, ϱ_2)-random operator inequality. J. Funct. Spaces **2020**, Article ID 7540303 (2020)
33. Saadati, R., Park, C.: Approximation of derivations and the superstability in random Banach $*$-algebras. Adv. Differ. Equ. **2018**, 418 (2018)
34. Wang, Z., Saadati, R.: Approximation of additive functional equations in NA Lie C^*-algebras. Demonstr. Math. **51**, 37–44 (2018)
35. Cădariu, L., Radu, V.: Fixed point methods for the generalized stability of functional equations in a single variable. Fixed Point Theory Appl. **2008**, Article ID 749392 (2008)
36. Diaz, J.B., Margolis, B.: A fixed point theorem of the alternative, for contractions on a generalized complete metric space. Bull. Am. Math. Soc. **74**, 305–309 (1968)
37. Jung, S.: A fixed point approach to the stability of an integral equation related to the wave equation. Abstr. Appl. Anal. **2013**, Article ID 612576 (2013)
38. Miheţ, D., Radu, V.: On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl. **343**, 567–572 (2008)
39. Miheţ, D., Saadati, R.: On the stability of the additive Cauchy functional equation in random normed spaces. Appl. Math. Lett. **24**, 2005–2009 (2011)