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Oscillation of solutions of third order nonlinear neutral differential equations

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Abstract

The main objective of this article is to improve and complement some of the oscillation criteria published recently in the literature for third order differential equation of the form

$$(r(t)(z''(t))^{\alpha})' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 > 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$ and α is a ratio of odd positive integers in the two cases $\int_{t_0}^{\infty} r^{\frac{-1}{\alpha}}(s)\,ds < \infty$ and $\int_{t_0}^{\infty} r^{\frac{-1}{\alpha}}(s)\,ds = \infty$. Some illustrative examples are presented.

MSC: 34C10; 34K11

Keywords: Oscillation; Third order differential equation; Nonlinear neutral equation; Nonoscillation

1 Introduction

Consider the nonlinear third order differential equation

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' + q(t)f\bigl(x\bigl(\sigma(t)\bigr)\bigr) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, $z(t) = x(t) + p(t)x(\tau(t))$, and α is a ratio of odd positive integers. We assume that the following conditions hold:

(H$_1$) $r(t), p(t), q(t), \tau(t), \sigma(t) \in C([t_0, \infty))$, $r(t)$, $q(t)$ are positive and $0 \leq p(t) \leq p_0 < \infty$;

(H$_2$) $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, $\sigma(t) > 0$, and $\tau(t) \leq t$;

(H$_3$) $f(u) \in C(\mathbb{R})$ and there exists a positive constant k such that $f(u)/u^{\gamma} \geq k$ for all $u \neq 0$ and γ is a ratio of odd positive integers;

(H$_4$) $\tau'(t) \geq \tau_0 > 0$ and $\tau \circ \sigma = \sigma \circ \tau$.

By a solution of (1.1), we mean a nontrivial function $x(t) \in C([T_x, \infty))$, $T_x \geq t_0$, which has the properties $z(t) \in C^2([T_x, \infty))$, $r(t)(z''(t))^{\alpha} \in C^1([T_x, \infty))$ and satisfies (1.1) on $[T_x, \infty)$. Our attention is restricted to those solutions $x(t)$ of (1.1) satisfying $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is termed nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The oscillatory behavior of solutions of various classes of nonlinear differential and dynamic equations on time scales has received much attention, we refer the reader to [1–17] and the references cited therein.

In 2012, Liu et al. [9] established new oscillation criteria for the second order Emden–Fowler equation

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + q(t)\left|x\left(\sigma(t)\right)\right|^{\gamma-1}x\left(\sigma(t)\right) = 0 \tag{1.2}$$

under the assumptions

$$0 \le p(t) \le 1, \tag{1.3}$$

$$r'(t) \ge 0, \qquad \sigma'(t) > 0, \tag{1.4}$$

and $\alpha \ge \gamma > 0$. In 2016, Wang et al. [16] studied Eq. (1.2) with condition (1.3),

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,\mathrm{d}t = \infty, \tag{1.5}$$

and $\sigma'(t) > 0$ with $\alpha \ge \gamma > 1$ when the condition $r'(t) \ge 0$ is neglected. Meanwhile, Wu et al. [17] established oscillation criteria for (1.2) in the general case when $\alpha > 0$ and $\gamma > 0$ are constants with conditions (1.3) and (1.4). Baculíková et al. [2] considered (1.2) in the more general case when $0 \le p(t) \le p_0 < \infty$ with condition (1.5) and $\sigma'(t) \ge 0$. For the case of third order differential equations, Džurina et al. [18] obtained sufficient conditions for the oscillation of solutions of the differential equation

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}\left(\sigma(t)\right) = 0, \tag{1.6}$$

where

$$0 \le p(t) \le p_0 < 1 \tag{1.7}$$

with condition (1.5). Meanwhile, Baculíková et al. [1] and Su et al. [19] discussed the oscillatory behavior of third order Eq. (1.6) when $r'(t) \ge 0$, (1.7) and (1.5) hold. Also Thandapani et al. [14] studied Eq. (1.6) when (1.7) holds and

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,\mathrm{d}t < \infty. \tag{1.8}$$

Recently, Jiang et al. [7] established new oscillation criteria for Eq. (1.1), where $\gamma = \alpha \ge 1$ and (1.5) hold without requiring (1.4).

More recently, Graef et al. [6] discussed the special case of Eq. (1.1) in which $r = 1$ and $\alpha = \gamma$.

The main goal of this paper is to establish new oscillation criteria motivated by [6, 7], and [17] for Eq. (1.1) under all cases of γ, α (i.e., $\gamma > \alpha$, $\gamma = \alpha$, and $\gamma < \alpha$), $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,\mathrm{d}t < \infty$ and $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,\mathrm{d}t = \infty$ without assumption (1.4). We consider the two cases when (H_4) holds or not.

In the sequel, we give the following notations:

$$Q(t) = \min\{q(t), q(\tau(t))\}, \qquad R(t) = \max\{r(t), r(\tau(t))\},$$
$$\eta'_+(t) = \max\{0, \eta'(t)\}, \qquad p^*(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right),$$
$$p_*(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{m_*(\tau^{-1}(\tau^{-1}(t)))}{m_*(\tau^{-1}(t))}\right), \quad \text{and}$$
$$p_{**}(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{m_{**}(\tau^{-1}(\tau^{-1}(t)))}{m_{**}(\tau^{-1}(t))}\right),$$

where τ^{-1} is the inverse of τ, m_* and m_{**} are functions to be specified later. All functional inequalities considered in this article are assumed to hold eventually, that is, they are satisfied for all t large enough.

2 Some preliminaries

We enlist some known results which will be needed. We first present the following classes of nonoscillatory (let us say positive) solutions of (1.1):

$z(t) \in N_I \Leftrightarrow z'(t) > 0, z''(t) > 0, (r(t)(z''(t))^\alpha)' < 0$,
$z(t) \in N_{II} \Leftrightarrow z'(t) < 0, z''(t) > 0, (r(t)(z''(t))^\alpha)' < 0$, and
$z(t) \in N_{III} \Leftrightarrow z'(t) > 0, z''(t) < 0, (r(t)(z''(t))^\alpha)' < 0$, eventually.

The following lemma comes directly from combining Lemma 1 and Lemma 2 in [13] with Lemma 3 and Lemma 4 in [20].

Lemma 2.1 *Assume that $A \geq 0$ and $B \geq 0$. Then*

$$(A + B)^\lambda \leq A^\lambda + B^\lambda \leq 2^{1-\lambda}(A + B)^\lambda, \quad 0 < \lambda \leq 1, \tag{2.1}$$

and

$$2^{1-\lambda}(A + B)^\lambda \leq A^\lambda + B^\lambda \leq (A + B)^\lambda, \quad \lambda \geq 1. \tag{2.2}$$

Lemma 2.2 *Let $g > 0$. Then*

$$g^r \leq rg + (1 - r) \quad \text{for } 0 < r \leq 1 \tag{2.3}$$

and

$$g^r \geq rg + (1 - r) \quad \text{for } r \geq 1. \tag{2.4}$$

Proof See [21, p. 28]. □

Lemma 2.3 [17]*Assume that $A \geq 0$, $B > 0$, $U \geq 0$, and $\lambda > 0$. Then*

$$AU - BU^{1+\frac{1}{\lambda}} \leq \frac{\lambda^\lambda}{(\lambda + 1)^{\lambda+1}}\frac{A^{\lambda+1}}{B^\lambda}. \tag{2.5}$$

Lemma 2.4 *Assume that x is an eventually positive solution of* (1.1). *If* (1.5) *holds, then $z(t) \in N_I$ or $z(t) \in N_{II}$. While if* (1.8) *holds, then either $z(t) \in N_I$ or $z(t) \in N_{II}$ or $z(t) \in N_{III}$.*

Proof The proof is similar to [22, Theorem 2.1 and Theorem 2.2]. □

Lemma 2.5 ([5, 23]) *Let the function $f(t)$ satisfy $f^{(i)}(t) > 0$, $i = 0, 1, 2, \dots, n$, and $f^{(n+1)}(t) < 0$ eventually, then there exists a constant $k_1 \in (0, 1)$ such that $\frac{f(t)}{f'(t)} \geq \frac{k_1 t}{n}$ eventually.*

3 Oscillation criteria in the case when (H_4) holds

In this section, we establish new oscillation criteria for Eq. (1.1) in the case when (H_4) holds.

Theorem 3.1 *Assume that* (H_1)–(H_4) *hold. If there exists a positive function $\rho(t) \in C^1([t_0, \infty))$ such that*

$$\int_{t_*}^{\infty} \Biggl[K\rho(s)Q(s)\Biggl(\frac{\int_{t_2}^{\lambda_1(s)} \int_{t_1}^{u} \frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\Biggr)^{\gamma} - \biggl(1 + \frac{p_0^{\gamma}}{\tau_0^{\lambda+1}}\biggr) R^g(s) \biggl(\frac{\rho'_+(s)}{\lambda + 1}\biggr)^{\lambda+1} \biggl(\frac{1}{m\rho(s)}\biggr)^{\lambda}\Biggr]\,\mathrm{d}s = \infty, \tag{3.1}$$

where

$$\lambda = \min\{\alpha, \gamma\}, \qquad m = \begin{cases} 1, & \gamma = \alpha, \\ 0 < m \leq 1, & \gamma \neq \alpha, \end{cases} \qquad K = \begin{cases} \frac{k}{2^{\gamma-1}}, & \gamma > 1, \\ k, & \gamma \leq 1, \end{cases} \tag{3.2}$$

$$g = \begin{cases} 1, & \gamma \geq \alpha, \\ \frac{\gamma}{\alpha}, & \gamma < \alpha \end{cases} \quad \text{and} \quad \lambda_1(t) = \begin{cases} t, & \sigma(t) \geq t, \\ \sigma(t), & \sigma(t) < t \end{cases} \tag{3.3}$$

holds for some constant $k > 0$, sufficiently large $t_1 \geq t_0$, and for some $t_* > t_2 > t_1$, then there exists no positive solution $x(t)$ of Eq. (1.1) *satisfying $z(t) \in N_I$.*

Proof Assume that $x(t)$ is a positive solution of Eq. (1.1) satisfying $z(t) \in N_I$ for $t \geq t_1$. Then from (1.1) and (H_3) it follows that

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' = -q(t)f\bigl(x\bigl(\sigma(t)\bigr)\bigr) \leq -kq(t)x^{\gamma}\bigl(\sigma(t)\bigr) < 0. \tag{3.4}$$

Since $(r(\tau(t))(z''(\tau(t)))^{\alpha})' = (r(z'')^{\alpha})'(\tau(t))\tau'(t)$, then in view of ($H_4$) there exists $t_2 \geq t_1$ such that

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' + \frac{p_0^{\gamma}}{\tau_0}\bigl(r\bigl(\tau(t)\bigr)\bigl(z''\bigl(\tau(t)\bigr)\bigr)^{\alpha}\bigr)' \leq -kQ(t)\bigl[x^{\gamma}\bigl(\sigma(t)\bigr) + p_0^{\gamma}x^{\gamma}\bigl(\tau\bigl(\sigma(t)\bigr)\bigr)\bigr] \quad \text{for } t \geq t_2. \tag{3.5}$$

In the following, we consider the two cases $\gamma > 1$ and $\gamma \leq 1$. Firstly, assume that $\gamma > 1$. Using (2.2) with (3.5), we get

$$\begin{aligned}&\big(r(t)\big(z''(t)\big)^{\alpha}\big)' + \frac{p_0^{\gamma}}{\tau_0}\big(r\big(\tau(t)\big)\big(z''\big(\tau(t)\big)\big)^{\alpha}\big)' \\ &\quad \leq -\frac{k}{2^{\gamma-1}}Q(t)\big[x\big(\sigma(t)\big) + p_0 x\big(\tau\big(\sigma(t)\big)\big)\big]^{\gamma} \leq -\frac{k}{2^{\gamma-1}}Q(t)z^{\gamma}\big(\sigma(t)\big).\end{aligned} \tag{3.6}$$

Define the functions $\omega(t)$ and $\nu(t)$ by

$$\omega(t) = \rho(t)\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\gamma}} \tag{3.7}$$

and

$$\nu(t) = \rho(t)\frac{r(\tau(t))(z''(\tau(t)))^{\alpha}}{(z'(\tau(t)))^{\gamma}}, \quad t \geq t_2. \tag{3.8}$$

Then clearly $\omega(t)$ and $\nu(t)$ are positive for $t \geq t_2$ and satisfy

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}} - \gamma\rho(t)r(t)\frac{(z''(t))^{\alpha+1}}{(z'(t))^{\gamma+1}} \tag{3.9}$$

and

$$\nu'(t) = \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{(z'(\tau(t)))^{\gamma}} - \gamma\rho(t)r\big(\tau(t)\big)\tau'(t)\frac{(z''(\tau(t)))^{\alpha+1}}{(z'(\tau(t)))^{\gamma+1}}. \tag{3.10}$$

Now, we consider the two cases $\gamma \geq \alpha$ and $\gamma < \alpha$. We first assume that $\gamma \geq \alpha$. From (3.7), we have

$$z''(t) = \big(z'(t)\big)^{\frac{\gamma}{\alpha}}\left(\frac{\omega(t)}{\rho(t)r(t)}\right)^{\frac{1}{\alpha}}.$$

Substituting into (3.9), we get

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}} - \gamma\rho(t)r(t)\left(\frac{\omega(t)}{\rho(t)r(t)}\right)^{1+\frac{1}{\alpha}}\big(z'(t)\big)^{\frac{\gamma}{\alpha}-1}. \tag{3.11}$$

But since $z'(t)$ is positive and increasing, it follows that there exists a constant $M > 0$ satisfying $z'(t) \geq M$ and

$$\omega'(t) \leq \rho_+'(t)r(t)\left(\frac{\omega(t)}{\rho(t)r(t)}\right) - \gamma M^{\frac{\gamma}{\alpha}-1}\rho(t)r(t)\left(\frac{\omega(t)}{\rho(t)r(t)}\right)^{1+\frac{1}{\alpha}} + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}}.$$

Using inequality (2.5) with $A = \rho_+'(t)r(t)$, $U = \frac{\omega(t)}{\rho(t)r(t)}$, and $B = \gamma M^{\frac{\gamma}{\alpha}-1}\rho(t)r(t)$, it follows that

$$\begin{aligned}\omega'(t) &\leq r(t)\left(\frac{\rho_+'(t)}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\gamma M^{\frac{\gamma}{\alpha}-1}\rho(t)}\right)^{\alpha} + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}} \\ &\leq r(t)\left(\frac{\rho_+'(t)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{M^{\frac{\gamma}{\alpha}-1}\rho(t)}\right)^{\alpha} + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}}.\end{aligned} \tag{3.12}$$

In view of (3.8), we have

$$z''\big(\tau(t)\big) = \big(z'\big(\tau(t)\big)\big)^{\frac{\gamma}{\alpha}}\left(\frac{\nu(t)}{\rho(t)r(\tau(t))}\right)^{\frac{1}{\alpha}}.$$

Substituting into (3.10), we get

$$\begin{aligned}
\nu'(t) &= \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{(z'(\tau(t)))^{\gamma}} \\
&\quad - \gamma\rho(t)r\big(\tau(t)\big)\tau'(t)\big(z'\big(\tau(t)\big)\big)^{\frac{\gamma}{\alpha}-1}\left(\frac{\nu(t)}{\rho(t)r(\tau(t))}\right)^{1+\frac{1}{\alpha}} \\
&\leq \rho'_{+}(t)r\big(\tau(t)\big)\left(\frac{\nu(t)}{\rho(t)r(\tau(t))}\right) + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{(z'(\tau(t)))^{\gamma}} \\
&\quad - \gamma M^{\frac{\gamma}{\alpha}-1}\rho(t)r\big(\tau(t)\big)\tau'(t)\left(\frac{\nu(t)}{\rho(t)r(\tau(t))}\right)^{1+\frac{1}{\alpha}}.
\end{aligned}$$

Again by inequality (2.5), we get

$$\nu'(t) \leq r\big(\tau(t)\big)\left(\frac{\rho'_{+}(t)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{M^{\frac{\gamma}{\alpha}-1}\rho(t)\tau'(t)}\right)^{\alpha} + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{(z'(\tau(t)))^{\gamma}}.$$

But since $z''(t) > 0$ and $\tau(t) \leq t$, we obtain

$$\nu'(t) \leq r\big(\tau(t)\big)\left(\frac{\rho'_{+}(t)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{M^{\frac{\gamma}{\alpha}-1}\rho(t)\tau'(t)}\right)^{\alpha} + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{(z'(t))^{\gamma}}. \tag{3.13}$$

Combining (3.12) and (3.13) and using (3.6), we get

$$\begin{aligned}
\omega'(t) + \frac{p_0^{\gamma}}{\tau_0}\nu'(t) &\leq -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{z(\sigma(t))}{z'(t)}\right)^{\gamma} \\
&\quad + \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\alpha+1}}\right)R(t)\left(\frac{\rho'_{+}(t)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{M^{\frac{\gamma}{\alpha}-1}\rho(t)}\right)^{\alpha}.
\end{aligned} \tag{3.14}$$

Now, assume that $\gamma < \alpha$. Then from (3.7) we have

$$\frac{1}{z'(t)} = \frac{(\frac{\omega(t)}{\rho(t)r(t)})^{\frac{1}{\gamma}}}{(z''(t))^{\frac{\alpha}{\gamma}}}.$$

Substituting into (3.9), we get

$$\begin{aligned}
\omega'(t) &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}} - \gamma\rho(t)r(t)\big(z''(t)\big)^{1-\frac{\alpha}{\gamma}}\left(\frac{\omega(t)}{\rho(t)r(t)}\right)^{1+\frac{1}{\gamma}} \\
&= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\gamma}} \\
&\quad - \gamma\rho(t)\big(r(t)\big)^{1-\frac{1}{\alpha}+\frac{1}{\gamma}}\big(r^{\frac{1}{\alpha}}(t)z''(t)\big)^{1-\frac{\alpha}{\gamma}}\left(\frac{\omega(t)}{\rho(t)r(t)}\right)^{1+\frac{1}{\gamma}}.
\end{aligned}$$

It is clear that $(r^{\frac{1}{\alpha}}(t)z''(t))^{1-\frac{\alpha}{\gamma}}$ is positive and increasing, and so there exists a positive constant m_1 such that

$$\begin{aligned}\omega'(t) &\le \rho'_+(t)r(t)\left(\frac{\omega(t)}{\rho(t)r(t)}\right)+\rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\gamma}\\&\quad-\gamma m_1\rho(t)\bigl(r(t)\bigr)^{1-\frac{1}{\alpha}+\frac{1}{\gamma}}\left(\frac{\omega(t)}{\rho(t)r(t)}\right)^{1+\frac{1}{\gamma}}\end{aligned}$$

for all sufficiently large t. Using inequality (2.5), we conclude that

$$\omega'(t)\le\left(\frac{\rho'_+(t)}{\gamma+1}\right)^{\gamma+1}\left(\frac{r^{\frac{1}{\alpha}}(t)}{m_1\rho(t)}\right)^{\gamma}+\rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\gamma}. \tag{3.15}$$

But since from (3.8) we have

$$\frac{1}{z'(\tau(t))}=\frac{(\frac{\nu(t)}{\rho(t)r(\tau(t))})^{\frac{1}{\gamma}}}{(z''(\tau(t)))^{\frac{\alpha}{\gamma}}},$$

then, by substituting into (3.10), we get

$$\begin{aligned}\nu'(t) &\le \rho'_+(t)r\bigl(\tau(t)\bigr)\left(\frac{\nu(t)}{\rho(t)r(\tau(t))}\right)+\rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\gamma}\\&\quad-\gamma m_1\rho(t)\bigl(r\bigl(\tau(t)\bigr)\bigr)^{1-\frac{1}{\alpha}+\frac{1}{\gamma}}\tau'(t)\left(\frac{\nu(t)}{\rho(t)r(\tau(t))}\right)^{1+\frac{1}{\gamma}}.\end{aligned}$$

This with (2.5) leads to

$$\nu'(t)\le\left(\frac{\rho'_+(t)}{\gamma+1}\right)^{\gamma+1}\left(\frac{r^{\frac{1}{\alpha}}(\tau(t))}{m_1\rho(t)\tau'(t)}\right)^{\gamma}+\rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(t))^\gamma}. \tag{3.16}$$

Combining (3.15) and (3.16), using (3.6), we get

$$\begin{aligned}\omega'(t)+\frac{p_0^\gamma}{\tau_0}\nu'(t) &\le -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{z(\sigma(t))}{z'(t)}\right)^{\gamma}\\&\quad+\left(1+\frac{p_0^\gamma}{\tau_0^{\gamma+1}}\right)R^{\frac{\gamma}{\alpha}}(t)\left(\frac{\rho'_+(t)}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{m_1\rho(t)}\right)^{\gamma}.\end{aligned} \tag{3.17}$$

Combining (3.14) and (3.17), we obtain for any α, γ ratios of odd positive integers that

$$\begin{aligned}\omega'(t)+\frac{p_0^\gamma}{\tau_0}\nu'(t) &\le -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{z(\sigma(t))}{z'(t)}\right)^{\gamma}\\&\quad+\left(1+\frac{p_0^\gamma}{\tau_0^{\lambda+1}}\right)R^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}.\end{aligned} \tag{3.18}$$

Now, we consider the two cases $\sigma(t)<t$ and $\sigma(t)\ge t$. We start by considering the case $\sigma(t)<t$. Since $r(t)(z''(t))^\alpha$ is positive and decreasing, we have

$$z'(t)\ge z'(t)-z'(t_2)=\int_{t_2}^{t}\frac{r^{\frac{1}{\alpha}}(s)z''(s)}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s\ge r^{\frac{1}{\alpha}}(t)z''(t)\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s,$$

i.e.,

$$\left(\frac{z'(t)}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)' \le 0. \tag{3.19}$$

But since $\sigma(t) < t$, then it follows that

$$\frac{z'(\sigma(t))}{z'(t)} \ge \frac{\int_{t_2}^{\sigma(t)}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}. \tag{3.20}$$

Now since by (3.19) we have

$$z(t) \ge z(t) - z(t_3) = \int_{t_3}^{t}\frac{z'(s)\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}{\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\,\mathrm{d}s \ge z'(t)\frac{\int_{t_3}^{t}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s},$$

which means that

$$\frac{z(t)}{z'(t)} \ge \frac{\int_{t_3}^{t}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s} \quad \text{for } t \ge t_3 > t_2, \tag{3.21}$$

then we have

$$\frac{z(\sigma(t))}{z'(\sigma(t))} \ge \frac{\int_{t_3}^{\sigma(t)}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{\sigma(t)}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}. \tag{3.22}$$

This with (3.20) leads to

$$\frac{z(\sigma(t))}{z'(t)} = \frac{z(\sigma(t))}{z'(\sigma(t))}\frac{z'(\sigma(t))}{z'(t)} \ge \frac{\int_{t_3}^{\sigma(t)}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}. \tag{3.23}$$

Substituting into (3.18), we get

$$\begin{aligned}\omega'(t) + \frac{p_0^{\gamma}}{\tau_0}\nu'(t) \le{}& -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{\int_{t_3}^{\sigma(t)}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)^{\gamma} \\ &+ \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\lambda+1}}\right)R^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}.\end{aligned} \tag{3.24}$$

Now, consider the case $\sigma(t) \ge t$. Since $z(t)$ is positive and increasing, it follows from (3.18) that

$$\begin{aligned}\omega'(t) + \frac{p_0^{\gamma}}{\tau_0}\nu'(t) \le{}& -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{z(t)}{z'(t)}\right)^{\gamma} \\ &+ \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\lambda+1}}\right)R^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}.\end{aligned} \tag{3.25}$$

Since $(r(t)(z''(t))^{\alpha})' < 0$, we get (3.19) and consequently we arrive at (3.21). Then, substituting into (3.25), we have

$$\omega'(t) + \frac{p_0^{\gamma}}{\tau_0}\nu'(t) \le -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{\int_{t_3}^{t}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)^{\gamma} + \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\lambda+1}}\right)R^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}. \tag{3.26}$$

Combining (3.24) and (3.26), we get

$$\omega'(t) + \frac{p_0^{\gamma}}{\tau_0}\nu'(t) \le -\frac{k}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{\int_{t_3}^{\lambda_1(t)}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)^{\gamma} + \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\lambda+1}}\right)R^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}.$$

Integrating from t_4 ($>t_3$) to t, we have

$$\omega(t_4) + \frac{p_0^{\gamma}}{\tau_0}\nu(t_4) \ge \int_{t_4}^{t}\left[\frac{k}{2^{\gamma-1}}\rho(s)Q(s)\left(\frac{\int_{t_3}^{\lambda_1(s)}\int_{t_2}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\right)^{\gamma} - \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\lambda+1}}\right)R^{g}(s)\left(\frac{\rho'_+(s)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(s)}\right)^{\lambda}\right]\mathrm{d}s,$$

which contradicts (3.1). Secondly, assume that $\gamma \le 1$. Using (2.1) with (3.5), we obtain

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' + \frac{p_0^{\gamma}}{\tau_0}\bigl(r\bigl(\tau(t)\bigr)\bigl(z''\bigl(\tau(t)\bigr)\bigr)^{\alpha}\bigr)' \le -kQ(t)z^{\gamma}\bigl(\sigma(t)\bigr). \tag{3.27}$$

By completing the proof as the above case of $\gamma > 1$, using (3.27) instead of (3.6), the proof is completed. □

Lemma 3.1 *Assume that conditions* (H$_1$)–(H$_4$) *hold. Let x be an eventually positive solution of Eq.* (1.1) *and the corresponding $z(t)$ satisfies $z(t) \in N_{II}$. If*

$$\int_{t_0}^{\infty} Q(s)\,\mathrm{d}s = \infty \tag{3.28}$$

or

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\left[\frac{1}{r(\tau(s))}\int_{s}^{\infty} Q(u)\,\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s\,\mathrm{d}t = \infty, \tag{3.29}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof Assume that $x(t)$ is a positive solution of Eq. (1.1) satisfying $z(t) \in N_{II}$ for $t \ge t_1$. Going through as in the proof of Theorem 3.1, we arrive at (3.5). In the following, we consider the two cases $\gamma > 1$ and $\gamma \le 1$. Firstly, assume that $\gamma > 1$. Then we have (3.6). Since $z(t)$ is positive and decreasing, we have $\lim_{t\to\infty} z(t) = l \ge 0$ exists. We claim that $l = 0$. If not, then there exists $t_3 \ge t_2$ such that $z(\sigma(t)) > l$ for $t \ge t_3$. Substituting into (3.6), we get

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' + \frac{p_0^{\gamma}}{\tau_0}\bigl(r\bigl(\tau(t)\bigr)\bigl(z''\bigl(\tau(t)\bigr)\bigr)^{\alpha}\bigr)' \le -\frac{kl^{\gamma}}{2^{\gamma-1}}Q(t). \tag{3.30}$$

Integrating (3.30) from t_3 to t and taking into account (3.28), we have

$$\begin{aligned} & r(t)\bigl(z''(t)\bigr)^{\alpha} + \frac{p_0^{\gamma}}{\tau_0} r\bigl(\tau(t)\bigr)\bigl(z''\bigl(\tau(t)\bigr)\bigr)^{\alpha} \\ &\quad \le r(t_3)\bigl(z''(t_3)\bigr)^{\alpha} + \frac{p_0^{\gamma}}{\tau_0} r\bigl(\tau(t_3)\bigr)\bigl(z''\bigl(\tau(t_3)\bigr)\bigr)^{\alpha} - \frac{kl^{\gamma}}{2^{\gamma-1}}\int_{t_3}^{t} Q(s)\,\mathrm{d}s \to -\infty \quad \text{as } t\to\infty, \end{aligned}$$

which is a contradiction. Thus $l = 0$ and consequently $\lim_{t\to\infty} x(t) = 0$. In the following, we obtain the same conclusion in the case when $\int_{t_0}^{\infty} Q(s)\,\mathrm{d}s < \infty$. Integrating (3.30) from t to ∞, we have

$$r(t)\bigl(z''(t)\bigr)^{\alpha} + \frac{p_0^{\gamma}}{\tau_0} r\bigl(\tau(t)\bigr)\bigl(z''\bigl(\tau(t)\bigr)\bigr)^{\alpha} \ge \frac{kl^{\gamma}}{2^{\gamma-1}}\int_{t}^{\infty} Q(s)\,\mathrm{d}s.$$

But since $\tau(t) \le t$, then we can observe that $r(\tau(t))(z''(\tau(t)))^{\alpha} \ge r(t)(z''(t))^{\alpha}$ and consequently we have

$$r\bigl(\tau(t)\bigr)\bigl(z''\bigl(\tau(t)\bigr)\bigr)^{\alpha} \ge \frac{kl^{\gamma}}{2^{\gamma-1}(1+\frac{p_0^{\gamma}}{\tau_0})}\int_{t}^{\infty} Q(s)\,\mathrm{d}s,$$

i.e.,

$$z''\bigl(\tau(t)\bigr) \ge \biggl[\frac{kl^{\gamma}}{2^{\gamma-1}(1+\frac{p_0^{\gamma}}{\tau_0})}\biggr]^{\frac{1}{\alpha}} \biggl[\frac{1}{r(\tau(t))}\int_{t}^{\infty} Q(s)\,\mathrm{d}s\biggr]^{\frac{1}{\alpha}}.$$

Integrating from t to ∞ followed by integrating from t_3 to ∞, we obtain

$$\frac{1}{\tau_0^2} z\bigl(\tau(t_3)\bigr) \ge \biggl[\frac{kl^{\gamma}}{2^{\gamma-1}(1+\frac{p_0^{\gamma}}{\tau_0})}\biggr]^{\frac{1}{\alpha}} \int_{t_3}^{\infty}\int_{t}^{\infty}\biggl[\frac{1}{r(\tau(s))}\int_{s}^{\infty} Q(u)\,\mathrm{d}u\biggr]^{\frac{1}{\alpha}}\,\mathrm{d}s\,\mathrm{d}t,$$

which contradicts (3.29). Thus $\lim_{t\to\infty} x(t) = 0$. Secondly, assume that $\gamma \le 1$. As in the proof of Theorem 3.1, we have (3.27). By completing the proof as in the above case of $\gamma > 1$, using (3.27) instead of (3.6), the proof is completed. □

Theorem 3.2 *Assume that* (H$_1$)–(H$_4$) *hold. If*

$$\int_{t_0}^{\infty}\biggl[\frac{1}{r(t)}\int_{t_0}^{t} Q(s)\bigl(\sigma(s)\bigr)^{\gamma}\biggl(\int_{\sigma(s)}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\biggr)^{\gamma}\,\mathrm{d}s\biggr]^{\frac{1}{\alpha}}\,\mathrm{d}t = \infty, \tag{3.31}$$

then there exists no positive solution $x(t)$ of Eq. (1.1) *satisfying $z(t) \in N_{III}$.*

Proof Let $x(t)$ be an eventually positive solution of Eq. (1.1) satisfying $z(t) \in N_{III}$ for all $t \geq t_1 \geq t_0$. Since $z''(t) < 0$ and $z'(t) > 0$, then by Lemma 2.5, there exist $t_2 \geq t_1$ and a constant k_1 satisfying $0 < k_1 < 1$ such that $z(t) \geq k_1 t z'(t)$ for $t \geq t_2$, i.e.,

$$z(\sigma(t)) \geq k_1 \sigma(t) z'(\sigma(t)), \quad t \geq t_2 \geq t_1. \tag{3.32}$$

Going through as in Theorem 3.1, we arrive at (3.5). In the following, we consider the two cases $\gamma > 1$ and $\gamma \leq 1$. Firstly, assume that $\gamma > 1$. Then we have (3.6), and using (3.32) we get

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + \frac{p_0^{\gamma}}{\tau_0}\left(r(\tau(t))\left(z''(\tau(t))\right)^{\alpha}\right)' \leq -\frac{kk_1^{\gamma}}{2^{\gamma-1}} Q(t)\left(\sigma(t)\right)^{\gamma}\left(z'(\sigma(t))\right)^{\gamma}. \tag{3.33}$$

But since $v(t) = -r^{\frac{1}{\alpha}}(t)z''(t)$ is positive and increasing, then there exists a constant $g_1 > 0$ such that $v(t) \geq g_1$ for $t \geq t_3 \geq t_2$. Hence

$$z'(\sigma(t)) \geq \int_{\sigma(t)}^{\infty} \frac{v(s)}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s \geq g_1 \int_{\sigma(t)}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s. \tag{3.34}$$

Substituting into (3.33) and integrating from t_3 to t, we get

$$\begin{aligned} &-r(t)\left(z''(t)\right)^{\alpha} - \frac{p_0^{\gamma}}{\tau_0} r(\tau(t))\left(z''(\tau(t))\right)^{\alpha} \\ &\quad \geq \frac{kk_1^{\gamma} g_1^{\gamma}}{2^{\gamma-1}} \int_{t_3}^{t} Q(s)\left(\sigma(s)\right)^{\gamma}\left(\int_{\sigma(s)}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right)^{\gamma}\mathrm{d}s. \end{aligned} \tag{3.35}$$

But since $\tau(t) \leq t$, then we can conclude that $r(\tau(t))(z''(\tau(t)))^{\alpha} \geq r(t)(z''(t))^{\alpha}$. Now since from (3.35) we have

$$-r(t)\left(z''(t)\right)^{\alpha} \geq \frac{kk_1^{\gamma} g_1^{\gamma}}{2^{\gamma-1}(1 + \frac{p_0^{\gamma}}{\tau_0})} \int_{t_3}^{t} Q(s)\left(\sigma(s)\right)^{\gamma}\left(\int_{\sigma(s)}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right)^{\gamma}\mathrm{d}s,$$

i.e.,

$$-z''(t) \geq \left(\frac{kk_1^{\gamma} g_1^{\gamma}}{2^{\gamma-1}(1 + \frac{p_0^{\gamma}}{\tau_0})}\right)^{\frac{1}{\alpha}} \left[\frac{1}{r(t)} \int_{t_3}^{t} Q(s)\left(\sigma(s)\right)^{\gamma}\left(\int_{\sigma(s)}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right)^{\gamma}\mathrm{d}s\right]^{\frac{1}{\alpha}}.$$

Then integrating from t_4 $(\geq t_3)$ to t, we get

$$z'(t_4) \geq \left(\frac{kk_1^{\gamma} g_1^{\gamma}}{2^{\gamma-1}(1 + \frac{p_0^{\gamma}}{\tau_0})}\right)^{\frac{1}{\alpha}} \int_{t_4}^{t}\left[\frac{1}{r(s)} \int_{t_3}^{s} Q(u)\left(\sigma(u)\right)^{\gamma}\left(\int_{\sigma(u)}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\right)^{\gamma}\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s,$$

which contradicts (3.31). Secondly, assume that $\gamma \leq 1$. As in the proof of Theorem 3.1, we arrive at (3.27), and then using (3.32) we get

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + \frac{p_0^{\gamma}}{\tau_0}\left(r(\tau(t))\left(z''(\tau(t))\right)^{\alpha}\right)' \leq -kk_1^{\gamma} Q(t)\left(\sigma(t)\right)^{\gamma}\left(z'(\sigma(t))\right)^{\gamma}. \tag{3.36}$$

Going through as in the proof of the case $\gamma > 1$, using (3.36) instead of (3.33), this completes the proof. □

The following results are immediate consequences of Lemma 2.4, Lemma 3.1, Theorem 3.1, and Theorem 3.2.

Theorem 3.3 *Assume that* (1.8) *and all the conditions of Lemma* 3.1, *Theorem* 3.1, *and Theorem* 3.2 *hold. Then every solution* $x(t)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Theorem 3.4 *Assume that* (1.5) *and all the conditions of Lemma* 3.1 *and Theorem* 3.1 *hold. Then every solution* $x(t)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

The following results deal with the special case $\alpha \le 1$ and $\gamma \ge 1$ of Eq. (1.1).

Theorem 3.5 *Assume that conditions* (H_1)–(H_4), $\alpha \le 1$, *and* $\gamma \ge 1$ *hold. If there exists a positive function* $\rho(t) \in C^1([t_0,\infty))$ *such that*

$$\int_{t_*}^{\infty}\left[\frac{k\gamma}{2^{\gamma-1}}\rho(t)Q(t)\left(\frac{\int_{t_2}^{\lambda_1(t)}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)-G_1(t)\right]\mathrm{d}t=\infty \tag{3.37}$$

holds for any positive constants k, M, *sufficiently large* $t_1 \ge t_0$, *and for some* $t_* > t_2 > t_1$, *where* $\lambda_1(t)$ *is defined by* (3.3) *and*

$$\begin{aligned}G_1(t) ={}& \frac{1}{4}\alpha\rho(t)\left[r(t)\left(\frac{\rho'(t)}{\rho(t)}+\frac{1-\alpha}{\alpha M}\right)^2+\frac{p_0^{\gamma}}{\tau_0^2}r\bigl(\tau(t)\bigr)\left(\frac{\rho'(t)}{\rho(t)}+\frac{(1-\alpha)\tau'(t)}{\alpha M}\right)^2\right]\\&+\frac{k(\gamma-1)}{2^{\gamma-1}M}\rho(t)Q(t),\end{aligned}$$

then there exists no positive solution $x(t)$ *of Eq.* (1.1) *satisfying* $z(t) \in N_I$.

Proof Assume that $x(t)$ is an eventually positive solution of Eq. (1.1) satisfying $z(t) \in N_I$. As in the proof of Theorem 3.1, we arrive at (3.6). Now define the function $W(t)$ by

$$W(t)=\rho(t)\frac{r(t)(z''(t))^{\alpha}}{z'(t)},\quad t\ge t_1\ge t_0. \tag{3.38}$$

Then $W(t) > 0$ for $t \ge t_1$ and

$$\begin{aligned}W'(t)&=\frac{\rho'(t)}{\rho(t)}W(t)+\rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{z'(t)}-\rho(t)\frac{r(t)(z''(t))^{\alpha+1}}{(z'(t))^2}\\&=\frac{\rho'(t)}{\rho(t)}W(t)+\rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{z'(t)}-W(t)\frac{z''(t)}{z'(t)}.\end{aligned} \tag{3.39}$$

Since $z'(t)$ and $z''(t)$ are positive, then there exist $t_2 \ge t_1$ and constant $M > 0$ such that $z'(t) \ge M$ for all $t \ge t_2$. Now, from (3.38) and (2.3), we get

$$\frac{z''(t)}{z'(t)}\ge\frac{W(t)}{\alpha\rho(t)r(t)}-\frac{(1-\alpha)}{\alpha M}. \tag{3.40}$$

This with (3.39) yields

$$W'(t) \le \frac{\rho'(t)}{\rho(t)} W(t) + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{z'(t)} - \frac{W^2(t)}{\alpha\rho(t)r(t)} + \frac{(1-\alpha)}{\alpha M} W(t)$$
$$\le \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{z'(t)} + \frac{1}{4}\alpha\rho(t)r(t)\left(\frac{\rho'(t)}{\rho(t)} + \frac{(1-\alpha)}{\alpha M}\right)^2. \tag{3.41}$$

Now define

$$V(t) = \rho(t)\frac{r(\tau(t))(z''(\tau(t)))^{\alpha}}{z'(\tau(t))}. \tag{3.42}$$

As we did for W, we can get

$$V'(t) \le \frac{\rho'(t)}{\rho(t)} V(t) + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{z'(\tau(t))} - \frac{\tau'(t)V^2(t)}{\alpha\rho(t)r(\tau(t))} + \frac{(1-\alpha)\tau'(t)}{\alpha M} V(t).$$

But since z' is increasing and $\tau(t) \le t$, then

$$V'(t) \le \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{z'(t)} + \frac{1}{4}\frac{\alpha\rho(t)r(\tau(t))}{\tau'(t)}\left(\frac{\rho'(t)}{\rho(t)} + \frac{(1-\alpha)\tau'(t)}{\alpha M}\right)^2. \tag{3.43}$$

This with (3.41) leads to

$$W'(t) + \frac{p_0^{\gamma}}{\tau_0} V'(t)$$
$$\le \rho(t)\left[\frac{(r(t)(z''(t))^{\alpha})' + \frac{p_0^{\gamma}}{\tau_0}(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{z'(t)}\right]$$
$$+ \frac{1}{4}\alpha\rho(t)\left[r(t)\left(\frac{\rho'(t)}{\rho(t)} + \frac{(1-\alpha)}{\alpha M}\right)^2 + \frac{p_0^{\gamma} r(\tau(t))}{\tau_0^2}\left(\frac{\rho'(t)}{\rho(t)} + \frac{(1-\alpha)\tau'(t)}{\alpha M}\right)^2\right].$$

Thus, by (3.6) and (2.4), we get

$$W'(t) + \frac{p_0^{\gamma}}{\tau_0} V'(t) \le -\frac{k\gamma}{2^{\gamma-1}}\rho(t)Q(t)\frac{z(\sigma(t))}{z'(t)} + G_1(t). \tag{3.44}$$

Now, we consider the two cases $\sigma(t) < t$ and $\sigma(t) \ge t$.

First assume that $\sigma(t) < t$. As in the proof of Theorem 3.1, we get (3.23). Substituting into (3.44), we have

$$W'(t) + \frac{p_0^{\gamma}}{\tau_0} V'(t) \le -\frac{k\gamma}{2^{\gamma-1}}\rho(t)Q(t)\frac{\int_{t_3}^{\sigma(t)}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s} + G_1(t) \quad \text{for } t \ge t_3 > t_2. \tag{3.45}$$

Secondly, assume that $\sigma(t) \ge t$. Since $z'(t) > 0$, it follows from (3.44) that

$$W'(t) + \frac{p_0^{\gamma}}{\tau_0} V'(t) \le -\frac{k\gamma}{2^{\gamma-1}}\rho(t)Q(t)\frac{z(t)}{z'(t)} + G_1(t). \tag{3.46}$$

As in the proof of Theorem 3.1, we arrive at (3.21). Then, substituting into (3.46), we have

$$W'(t)+\frac{p_0^\gamma}{\tau_0}V'(t)\le -\frac{k\gamma}{2^{\gamma-1}}\rho(t)Q(t)\frac{\int_{t_3}^{t}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}+G_1(t). \tag{3.47}$$

This with (3.45) yields

$$W'(t)+\frac{p_0^\gamma}{\tau_0}V'(t)\le -\frac{k\gamma}{2^{\gamma-1}}\rho(t)Q(t)\frac{\int_{t_3}^{\lambda_1(t)}\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}+G_1(t).$$

Integrating from t_4 ($> t_3$) to t, we get

$$W(t_4)+\frac{p_0^\gamma}{\tau_0}V(t_4)\ge \int_{t_4}^{t}\left[\frac{k\gamma}{2^{\gamma-1}}\rho(s)Q(s)\frac{\int_{t_3}^{\lambda_1(s)}\int_{t_2}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_2}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}-G_1(s)\right]\mathrm{d}s.$$

This contradicts (3.37) and completes the proof. □

Theorem 3.6 *Assume that* (1.8) *and all the conditions of Lemma* 3.1, *Theorem* 3.2, *and Theorem* 3.5 *hold. Then every solution $x(t)$ of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Theorem 3.7 *Assume that* (1.5) *and all the conditions of Lemma* 3.1 *and Theorem* 3.5 *hold. Then every solution $x(t)$ of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

4 Oscillation criteria without condition (H_4)

In this section, we study the oscillation of Eq. (1.1) when either of the two conditions $0\le p(t)\le p_0<1$ or $p(t)\ge 1$, $p(t)\not\equiv 1$ holds for large t. Now, we begin by establishing new oscillation criteria for Eq. (1.1) in the case when $p(t)\ge 1$, $p(t)\not\equiv 1$ for large t with the condition $\tau(t)<t$ and $\tau(t)$ is strictly increasing.

Theorem 4.1 *Assume that* (H_1)–(H_3) *hold*, $p(t)\ge 1$, $p(t)\not\equiv 1$ *for sufficiently large t*, $\tau(t)<t$ *and* $\tau'(t)>0$. *Further assume that there exists a positive function* $m_*(t)\in C^1([t_0,\infty))$ *such that*

$$m_*(t)\int_{t_1}^{t}\frac{\mathrm{d}s}{r^{\frac{1}{\alpha}}(s)}-m_*'(t)\int_{t_2}^{t}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s\le 0 \tag{4.1}$$

and $p_*(t)>0$ *for sufficiently large t. If there exists a positive function* $\rho(t)\in C^1([t_0,\infty))$ *such that*

$$\begin{aligned}\int_{t_*}^{\infty}\Biggl[k\rho(s)q(s)\bigl(p_*(\sigma(s))\bigr)^{\gamma}\left(\frac{\int_{t_2}^{\lambda_2(s)}\int_{t_1}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\right)^{\gamma}\\ -r^{g}(s)\left(\frac{\rho_+'(s)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(s)}\right)^{\lambda}\Biggr]\mathrm{d}s=\infty\end{aligned} \tag{4.2}$$

holds for some constant $k > 0$, *sufficiently large* $t_1 \geq t_0$, *and for some* $t_* > t_2 > t_1$, *where* λ, m, g *are defined by* (3.2), (3.3), *and*

$$\lambda_2(t) = \begin{cases} \tau^{-1}(\sigma(t)), & \sigma(t) < \tau(t), \\ t, & \sigma(t) \geq \tau(t), \end{cases} \tag{4.3}$$

then there exists no positive solution $x(t)$ *of Eq.* (1.1) *satisfying* $z(t) \in N_I$.

Proof Assume that $x(t)$ is an eventually positive solution of Eq. (1.1) satisfying $z(t) \in N_I$ for $t \geq t_1$. From the definition of z (see also (2.2) in [6]), we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\big(z(\tau^{-1}(t)) - x(\tau^{-1}(t))\big) \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}\big(z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))\big) \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{4.4}$$

Define the function $\omega(t)$ as in (3.7). Then $\omega(t) > 0$ for $t \geq t_1$ satisfying (3.9). As in the proof of Theorem 3.1, since $(r(t)(z''(t))^\alpha)' < 0$, we have (3.19) and then

$$\frac{z(t)}{z'(t)} \geq \frac{\int_{t_2}^t \int_{t_1}^s \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s} \quad \text{for } t \geq t_2 > t_1. \tag{4.5}$$

This with (4.1) yields

$$\begin{aligned} \left(\frac{z(t)}{m_*(t)}\right)' &= \frac{1}{m_*^2(t)}\big[z'(t)m_*(t) - z(t)m_*'(t)\big] \\ &\leq \frac{z(t)}{m_*^2(t)}\left[\frac{m_*(t)\int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}{\int_{t_2}^t \int_{t_1}^s \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s} - m_*'(t)\right] \leq 0. \end{aligned}$$

This means that $\frac{z(t)}{m_*(t)}$ is nonincreasing. But since $\tau(t) < t$ and $\tau'(t) > 0$, it follows that $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, and so

$$z(\tau^{-1}(\tau^{-1}(t))) \leq \frac{m_*(\tau^{-1}(\tau^{-1}(t)))z(\tau^{-1}(t))}{m_*(\tau^{-1}(t))}. \tag{4.6}$$

Substituting from (4.6) into (4.4), we get

$$x(t) \geq p_*(t)z(\tau^{-1}(t)). \tag{4.7}$$

This in the view of (1.1) leads to

$$\big(r(t)(z''(t))^\alpha\big)' \leq -kq(t)\big(p_*(\sigma(t))\big)^\gamma z^\gamma\big(\tau^{-1}(\sigma(t))\big). \tag{4.8}$$

In the following, we consider the two cases $\gamma \geq \alpha$ and $\gamma < \alpha$.

First, assume that $\gamma \geq \alpha$. As in the proof of Theorem 3.1, we have (3.12). Then, substituting from (4.8) into (3.12), we obtain

$$\omega'(t) \leq r(t)\left(\frac{\rho'_+(t)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{M^{\frac{\gamma}{\alpha}-1}\rho(t)}\right)^{\alpha} - k\rho(t)q(t)\big(p_*(\sigma(t))\big)^{\gamma}\left(\frac{z(\tau^{-1}(\sigma(t)))}{z'(t)}\right)^{\gamma}. \quad (4.9)$$

Now assume that $\gamma < \alpha$. As in the proof of Theorem 3.1, we have (3.15). Then, substituting from (4.8) into (3.15), we obtain

$$\omega'(t) \leq r^{\frac{\gamma}{\alpha}}(t)\left(\frac{\rho'_+(t)}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{m_1\rho(t)}\right)^{\gamma} - k\rho(t)q(t)\big(p_*(\sigma(t))\big)^{\gamma}\left(\frac{z(\tau^{-1}(\sigma(t)))}{z'(t)}\right)^{\gamma}. \quad (4.10)$$

This with (4.9) yields

$$\omega'(t) \leq r^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda} - k\rho(t)q(t)\big(p_*(\sigma(t))\big)^{\gamma}\left(\frac{z(\tau^{-1}(\sigma(t)))}{z'(t)}\right)^{\gamma}. \quad (4.11)$$

Now, consider the two cases $\sigma(t) < \tau(t)$ and $\sigma(t) \geq \tau(t)$. First assume that $\sigma(t) < \tau(t)$. Since $\tau^{-1}(\sigma(t)) < t$ and $\left(\frac{z'(t)}{\int_{t_1}^{t}\frac{ds}{r^{\frac{1}{\alpha}}(s)}}\right)' \leq 0$, then by (4.5) we have

$$\frac{z(\tau^{-1}(\sigma(t)))}{z'(t)} \geq \frac{\int_{t_2}^{\tau^{-1}(\sigma(t))}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}.$$

Substituting into (4.11), we get

$$\begin{aligned}\omega'(t) \leq{}& r^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}\\ &- k\rho(t)q(t)\big(p_*(\sigma(t))\big)^{\gamma}\left(\frac{\int_{t_2}^{\tau^{-1}(\sigma(t))}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)^{\gamma}. \end{aligned} \quad (4.12)$$

Secondly, assume that $\sigma(t) \geq \tau(t)$. Hence since $z'(t) > 0$ and $\tau^{-1}(\sigma(t)) \geq t$, we have $z(\tau^{-1}(\sigma(t))) \geq z(t)$. Thus it follows from (4.11) and (4.5) that

$$\begin{aligned}\omega'(t) \leq{}& r^{g}(t)\left(\frac{\rho'_+(t)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(t)}\right)^{\lambda}\\ &- k\rho(t)q(t)\big(p_*(\sigma(t))\big)^{\gamma}\left(\frac{\int_{t_2}^{t}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s}\right)^{\gamma}. \end{aligned} \quad (4.13)$$

Combining (4.12) and (4.13) and then integrating from t_3 ($> t_2$) to t, we get

$$\begin{aligned}\omega(t_3) \geq{}& \int_{t_3}^{t}\Bigg[k\rho(s)q(s)\big(p_*(\sigma(s))\big)^{\gamma}\left(\frac{\int_{t_2}^{\lambda_2(s)}\int_{t_1}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\right)^{\gamma}\\ &- r^{g}(s)\left(\frac{\rho'_+(s)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(s)}\right)^{\lambda}\Bigg]\,\mathrm{d}s,\end{aligned}$$

which contradicts (4.2). This completes the proof. □

Theorem 4.2 *Assume that* (H_1)–(H_3) *hold*, $p(t) \geq 1$, $p(t) \not\equiv 1$ *for sufficiently large* t, $\tau(t) < t$, $\tau'(t) > 0$, *and* $p^*(t) > 0$. *If* $x(t)$ *is an eventually positive solution of Eq.* (1.1) *satisfying* $z(t) \in N_{II}$ *with*

$$\int_{t_0}^{\infty} q(s)\bigl(p^*\bigl(\sigma(s)\bigr)\bigr)^{\gamma}\,\mathrm{d}s = \infty \tag{4.14}$$

or

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\left[\frac{1}{r(s)}\int_{s}^{\infty} q(u)\bigl(p^*\bigl(\sigma(u)\bigr)\bigr)^{\gamma}\,\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s\,\mathrm{d}t = \infty, \tag{4.15}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be an eventually positive solution of Eq. (1.1) satisfying $z(t) \in N_{II}$ for $t \geq t_1$. Going through as in the proof of Theorem 4.1, we arrive at (4.4). Since $z(t)$ is decreasing and $\tau(t) < t$, then $z(\tau^{-1}(t)) \geq z(\tau^{-1}(\tau^{-1}(t)))$. Substituting into (4.4), we get

$$x(t) \geq p^*(t) z\bigl(\tau^{-1}(t)\bigr). \tag{4.16}$$

This with (1.1) leads to

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' \leq -kq(t)\bigl(p^*\bigl(\sigma(t)\bigr)\bigr)^{\gamma} z^{\gamma}\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr). \tag{4.17}$$

Since $z(t) > 0$ and $z'(t) < 0$, then $\lim_{t\to\infty} z(t) = l \geq 0$ exists. We claim that $l = 0$. If not, then there exists $t_2 \geq t_1$ such that $\tau^{-1}(\sigma(t)) > t_1$ and $z(\tau^{-1}(\sigma(t))) \geq l$ for $t \geq t_2$. Substituting into (4.17), we get

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' \leq -kl^{\gamma} q(t)\bigl(p^*\bigl(\sigma(t)\bigr)\bigr)^{\gamma}. \tag{4.18}$$

Integrating from t_2 to t and taking into account (4.14), we have

$$r(t)\bigl(z''(t)\bigr)^{\alpha} \leq r(t_2)\bigl(z''(t_2)\bigr)^{\alpha} - kl^{\gamma}\int_{t_2}^{t} q(s)\bigl(p^*\bigl(\sigma(s)\bigr)\bigr)^{\gamma}\,\mathrm{d}s \to -\infty \quad \text{as } t \to \infty,$$

which is a contradiction. Thus $l = 0$ and $\lim_{t\to\infty} x(t) = 0$. In the following, we obtain the same conclusion in the case when $\int_{t_0}^{\infty} q(s)(p^*(\sigma(s)))^{\gamma}\,\mathrm{d}s < \infty$. Integrating (4.18) from t to ∞ and dividing both sides by $r(t)$, we have

$$z''(t) \geq \bigl(kl^{\gamma}\bigr)^{\frac{1}{\alpha}}\left[\frac{1}{r(t)}\int_{t}^{\infty} q(s)\bigl(p^*\bigl(\sigma(s)\bigr)\bigr)^{\gamma}\,\mathrm{d}s\right]^{\frac{1}{\alpha}}, \quad t \geq t_2.$$

Integrating again from t to ∞, we obtain

$$-z'(t) \geq \bigl(kl^{\gamma}\bigr)^{\frac{1}{\alpha}}\int_{t}^{\infty}\left[\frac{1}{r(s)}\int_{s}^{\infty} q(u)\bigl(p^*\bigl(\sigma(u)\bigr)\bigr)^{\gamma}\,\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s, \quad t \geq t_3 \geq t_2.$$

Moreover, by integrating again from t_3 to ∞, we get

$$z(t_3) \geq \bigl(kl^{\gamma}\bigr)^{\frac{1}{\alpha}}\int_{t_3}^{\infty}\int_{t}^{\infty}\left[\frac{1}{r(s)}\int_{s}^{\infty} q(u)\bigl(p^*\bigl(\sigma(u)\bigr)\bigr)^{\gamma}\,\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s\,\mathrm{d}t,$$

which contradicts (4.15). Hence, $l = 0$. So from the fact that $0 < x(t) < z(t)$, it follows that $\lim_{t\to\infty} x(t) = 0$. □

Theorem 4.3 *Assume that* (H_1)–(H_3) *hold*, $p(t) \geq 1$, $p(t) \not\equiv 1$ *for sufficiently large* t, $\tau(t) < t$ *and* $\tau'(t) > 0$. *If for some constant* $k_1 \in (0,1)$ *there exists a function* $m_{**}(t) \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$m_{**}(t) - k_1 t m'_{**}(t) \leq 0, \tag{4.19}$$

$p_{**}(t) > 0$ *for all sufficiently large* t *and*

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t} q(s)\left[\tau^{-1}\bigl(\sigma(s)\bigr)p_{**}\bigl(\sigma(s)\bigr)\int_{\tau^{-1}(\sigma(s))}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right]^{\gamma}\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t = \infty, \tag{4.20}$$

then there exists no positive solution $x(t)$ *of Eq.* (1.1) *satisfying* $z(t) \in N_{III}$.

Proof Let $x(t)$ be an eventually positive solution of Eq. (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, $z(t)$ satisfies $z(t) \in N_{III}$ and $\tau^{-1}(\sigma(t)) > t_0$ for $t \geq t_1 \geq t_0$. From the definition of z, we have (4.4) as in the proof of Theorem 4.1. Since $z''(t) < 0$ and $z'(t) > 0$, then by Lemma 2.5 there exists $t_2 \geq t_1$ such that

$$z(t) \geq k_1 t z'(t), \quad t \geq t_2. \tag{4.21}$$

This with (4.19) yields

$$\begin{aligned}\left(\frac{z(t)}{m_{**}(t)}\right)' &= \frac{1}{m_{**}^2(t)}\bigl[m_{**}(t)z'(t) - z(t)m'_{**}(t)\bigr]\\ &\leq \frac{z(t)}{k_1 t m_{**}^2(t)}\bigl[m_{**}(t) - k_1 t m'_{**}(t)\bigr] \leq 0,\end{aligned}$$

and so $\frac{z(t)}{m_{**}(t)}$ is nonincreasing. Hence $z(\tau^{-1}(\tau^{-1}(t))) \leq \frac{m_{**}(\tau^{-1}(\tau^{-1}(t)))z(\tau^{-1}(t))}{m_{**}(\tau^{-1}(t))}$. Now, from (1.1), (4.4), and (4.21), we have

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' \leq -kk_1^{\gamma} q(t)\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr)^{\gamma}\bigl(p_{**}\bigl(\sigma(t)\bigr)\bigr)^{\gamma}\bigl(z'\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr)\bigr)^{\gamma}. \tag{4.22}$$

But since $-r^{\frac{1}{\alpha}}(t)z''(t)$ is positive and increasing, then we have $-r^{\frac{1}{\alpha}}(t)z''(t) \geq g_1$ for $t \geq t_1$. Hence

$$z'(t) \geq \int_{t}^{\infty}\frac{-r^{\frac{1}{\alpha}}(s)z''(s)}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s \geq g_1\int_{t}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s.$$

Thus

$$z'\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr) \geq g_1\int_{\tau^{-1}(\sigma(t))}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s. \tag{4.23}$$

This with (4.22) leads to

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' \leq -kk_1^{\gamma}g_1^{\gamma} q(t)\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr)^{\gamma}\bigl(p_{**}\bigl(\sigma(t)\bigr)\bigr)^{\gamma}\left(\int_{\tau^{-1}(\sigma(t))}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s\right)^{\gamma}.$$

Integrating from t_2 to t, we get

$$-z''(t) \geq \left(kk_1^\gamma g_1^\gamma\right)^{\frac{1}{\alpha}} \left[\frac{1}{r(t)} \int_{t_2}^{t} q(s)\left[\tau^{-1}\left(\sigma(s)\right)p_{**}\left(\sigma(s)\right) \int_{\tau^{-1}(\sigma(s))}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right]^{\gamma} \mathrm{d}s\right]^{\frac{1}{\alpha}}.$$

Integrating again from t_3 $(\geq t_2)$ to t, we have

$$\frac{z'(t_3)}{(kk_1^\gamma g_1^\gamma)^{\frac{1}{\alpha}}} \geq \int_{t_3}^{t}\left[\frac{1}{r(s)} \int_{t_2}^{s} q(u)\left[\tau^{-1}\left(\sigma(u)\right)p_{**}\left(\sigma(u)\right) \int_{\tau^{-1}(\sigma(u))}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\right]^{\gamma} \mathrm{d}u\right]^{\frac{1}{\alpha}} \mathrm{d}s.$$

This contradicts (4.20) and completes the proof. □

Theorem 4.4 *Assume that* (1.8) *and all the conditions of Theorem* 4.1, *Theorem* 4.2, *and Theorem* 4.3 *hold. Then every solution* $x(t)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Theorem 4.5 *Assume that* (1.5) *and all the conditions of Theorem* 4.1 *and Theorem* 4.2 *hold. Then every solution* $x(t)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Remark 1 The assumptions concerning the existence of the two functions $m_*(t)$ and $m_{**}(t)$ hold, for example, $\mu(t) = \xi(t)$, $\mu(t) = (\xi(t))^\eta$, $\mu(t) = \xi(t)e^{\xi^\epsilon(t)}$, $\mu(t) = (\xi(t))^\eta e^{\epsilon\xi(t)}$ with

$$\xi(t) = \begin{cases} \int_{t_2}^{t}\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\,\mathrm{d}s, & \mu(t) = m_*(t), \\ t^{\frac{1}{k_1}}, & \mu(t) = m_{**}(t), \end{cases}$$

$\eta \geq 1$ and $\epsilon \geq 0$, etc.

Remark 2 From Theorem 4.4 and Theorem 4.5, we can obtain more than one oscillation criterion for Eq. (1.1) in the two theorems with different choices of $m_*(t)$ and $m_{**}(t)$ which are mentioned in Remark 1.

In the following, we discuss the oscillatory behavior of solutions of Eq. (1.1) in the case when $0 \leq p(t) \leq p_0 < 1$.

Theorem 4.6 *Assume that* (H_1)–(H_3) *hold and* $0 \leq p(t) \leq p_0 < 1$. *If there exists a positive function* $\rho(t) \in C^1([t_0,\infty))$ *such that*

$$\begin{aligned} \int_{t_*}^{\infty} \Bigg[& k\rho(s)q(s)\left(1 - p\left(\sigma(s)\right)\right)^\gamma \left(\frac{\int_{t_2}^{\lambda_1(s)}\int_{t_1}^{u} \frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\right)^\gamma \\ & - r^g(s)\left(\frac{\rho_+'(s)}{\lambda+1}\right)^{\lambda+1}\left(\frac{1}{m\rho(s)}\right)^{\lambda}\Bigg]\,\mathrm{d}s = \infty \end{aligned} \tag{4.24}$$

holds for some constant $k > 0$, *for sufficiently large* $t_1 \geq t_0$, *and for some* $t_* > t_2 > t_1$, *where* λ, m, g, $\lambda_1(t)$ *are as defined by* (3.2) *and* (3.3), *then there exists no positive solution* $x(t)$ *of Eq.* (1.1) *satisfying* $z(t) \in N_I$.

Proof Let $x(t)$ be an eventually positive solution of Eq. (1.1) satisfying $z(t) \in N_I$. From the definition of z, we have

$$x(t) = z(t) - p(t)x\bigl(\tau(t)\bigr) \geq \bigl(1 - p(t)\bigr)z(t).$$

This with (1.1) yields

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' \leq -kq(t)\bigl(1 - p\bigl(\sigma(t)\bigr)\bigr)^{\gamma} z^{\gamma}\bigl(\sigma(t)\bigr). \tag{4.25}$$

Defining $\omega(t)$ by (3.7), completing the proof as in the proof of Theorem 4.1 by applying (4.25) instead of (4.8), and considering the two cases $\sigma(t) < t$ and $\sigma(t) \geq t$ instead of the two cases $\sigma(t) < \tau(t)$ and $\sigma(t) \geq \tau(t)$, we get a contradiction to (4.24). □

Theorem 4.7 *Assume that* (H_1)–(H_3) *hold*, $0 \leq p(t) \leq p_0 < 1$, *and* $x(t)$ *is an eventually positive solution of Eq.* (1.1) *satisfying* $z(t) \in N_{II}$. *If*

$$\int_{t_0}^{\infty} q(s)\,\mathrm{d}s = \infty \tag{4.26}$$

or

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\left[\frac{1}{r(s)}\int_{s}^{\infty} q(u)\,\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s\,\mathrm{d}t = \infty, \tag{4.27}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be an eventually positive solution of Eq. (1.1) satisfying $z(t) \in N_{II}$ for $t \geq t_1 \geq t_0$. Since $z(t)$ is positive and decreasing, we have $\lim_{t\to\infty} z(t) = l \geq 0$ exists. We claim that $l = 0$. If not, then for any $\epsilon > 0$ we have $l < z(t) < l + \epsilon$ eventually. Choose $0 < \epsilon < \frac{l(1-p_0)}{p_0}$. It is easy to verify that

$$x(t) = z(t) - p(t)x\bigl(\tau(t)\bigr) \geq z(t) - p(t)z\bigl(\tau(t)\bigr) > l - p_0(l + \epsilon) = k_2(l + \epsilon) > k_2 z(t),$$

where $k_2 = \frac{l-p_0(l+\epsilon)}{(l+\epsilon)} > 0$. Now, it follows from (1.1) that

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' \leq -kk_2^{\gamma} q(t)z^{\gamma}\bigl(\sigma(t)\bigr) \leq -k(k_2 l)^{\gamma} q(t). \tag{4.28}$$

Going through as in the proof of Theorem 4.2 by applying (4.28) instead of (4.18), we can get a contradiction to (4.26) or (4.27). This completes the proof. □

Using a similar technique to the proof of Theorem 4.3 and using (4.25) with (4.21) instead of (4.22), we can get the following result.

Theorem 4.8 *Assume that* (H_1)–(H_3) *hold and* $0 \leq p(t) \leq p_0 < 1$. *If*

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t} q(s)\left[\sigma(s)\bigl(1 - p\bigl(\sigma(s)\bigr)\bigr)\int_{\sigma(s)}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right]^{\gamma}\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t = \infty, \tag{4.29}$$

then there exists no positive solution $x(t)$ *of Eq.* (1.1) *satisfying* $z(t) \in N_{III}$.

Theorem 4.9 *Assume that* (1.8) *and all the conditions of Theorem* 4.6, *Theorem* 4.7, *and Theorem* 4.8 *hold. Then every solution* $x(t)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Theorem 4.10 *Assume that* (1.5) *and all the conditions of Theorem* 4.6 *and Theorem* 4.7 *hold. Then every solution* $x(t)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

5 Examples

Example 1 Consider the third order differential equation

$$\left(\left[\left(x(t) + \frac{25}{4}x\left(\frac{t}{2}\right)\right)''\right]^{\frac{1}{3}}\right)' + \frac{3}{t}x(t) = 0, \quad t \geq 1. \tag{5.1}$$

Here, $r(t) = 1$, $p = \frac{25}{4}$, $\tau(t) = \frac{t}{2}$, $q(t) = \frac{3}{t}$, $\sigma(t) = t$, and $1 = \gamma > \alpha = \frac{1}{3}$. It is clear that $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,\mathrm{d}t = \infty$. Choosing $\rho(t) = \frac{1}{t}$, then we have $\rho'_+(t) = 0$, and

$$\begin{aligned}
&\int_{t_*}^{\infty} \left[k\rho(s)Q(s)\left(\frac{\int_{t_2}^{s}\int_{t_1}^{u} \frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\right)^{\gamma} - \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\alpha+1}}\right)R(s)\left(\frac{\rho'_+(s)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{m\rho(s)}\right)^{\alpha}\right]\mathrm{d}s \\
&= \int_{t_*}^{\infty} \frac{3}{s^2}\left(\frac{\int_{t_2}^{s}\int_{t_1}^{u}\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s}\mathrm{d}u}\right)\mathrm{d}s \geq \int_{t_*}^{\infty}\left(\frac{3}{2s} - \frac{3t_1}{s^2} - \frac{3t_2^2}{2s^3} + \frac{3t_1t_2}{s^3}\right)\mathrm{d}s = \infty.
\end{aligned}$$

Thus, it follows from Theorem 3.4 that every solution $x(t)$ of Eq. (5.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. In fact, $x(t) = \frac{1}{t}$ is a solution of Eq. (5.1).

Example 2 Consider the third order differential equation

$$\left(\left[\left(x(t) + p_0 x\left(t - \frac{1}{2}\right)\right)''\right]^{5}\right)' + \left(t - \frac{1}{2}\right)^{\frac{4}{3}} x^{\frac{1}{3}}\left(t - \frac{1}{2}\right) = 0, \quad t \geq 1, p_0 > 0. \tag{5.2}$$

Here, $r(t) = 1$, $p = p_0$, $\tau(t) = t - \frac{1}{2}$, $q(t) = (t - \frac{1}{2})^{\frac{4}{3}}$, $\sigma(t) = t - \frac{1}{2}$, and $\frac{1}{3} = \gamma < \alpha = 5$. It is clear that $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,\mathrm{d}t = \infty$. Choosing $\rho(t) = 1$, we have $\rho'_+(t) = 0$, and

$$\begin{aligned}
&\int_{t_*}^{\infty} \left[k\rho(s)Q(s)\left(\frac{\int_{t_2}^{\sigma(s)}\int_{t_1}^{u} \frac{1}{r^{\frac{1}{\alpha}}(v)}\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u}\right)^{\gamma}\right. \\
&\qquad \left. - \left(1 + \frac{p_0^{\gamma}}{\tau_0^{\gamma+1}}\right)R^{\frac{\gamma}{\alpha}}(s)\left(\frac{\rho'_+(s)}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{m\rho(s)}\right)^{\gamma}\right]\mathrm{d}s \\
&= \int_{t_*}^{\infty}\left(\frac{1}{2}\right)^{\frac{1}{3}}(s-1)^{\frac{4}{3}}\left(\frac{(s - t_1 - \frac{1}{2})^2 - (t_2 - t_1)^2}{s - t_1}\right)^{\frac{1}{3}}\mathrm{d}s \\
&\geq \int_{t_*}^{\infty}\left(\frac{1}{2}\right)^{\frac{1}{3}}(s-1)\left(\left(s - t_1 - \frac{1}{2}\right)^2 - (t_2 - t_1)^2\right)^{\frac{1}{3}}\mathrm{d}s \\
&> \int_{t_*}^{\infty}\left(\frac{1}{2}\right)^{\frac{1}{3}}\left(s - \left(t_1 + \frac{1}{2}\right)\right)\left(\left(s - t_1 - \frac{1}{2}\right)^2 - (t_2 - t_1)^2\right)^{\frac{1}{3}}\mathrm{d}s = \infty.
\end{aligned}$$

Thus, by Theorem 3.4, it follows that every solution $x(t)$ of Eq. (5.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Example 3 Consider the third order differential equation

$$\left(t\left[\left(x(t) + \frac{1}{3\sqrt{3}}x\left(\frac{t}{3}\right)\right)''\right]^{\frac{1}{3}}\right)' + \lambda t^6 x^3\left(\frac{t}{2}\right) = 0, \quad t > 1, \lambda > 0. \tag{5.3}$$

Here, $r(t) = t$, $p = \frac{1}{3\sqrt{3}}$, $\tau(t) = \frac{t}{3}$, $q(t) = \lambda t^6$, $\sigma(t) = \frac{t}{2}$, and $3 = \gamma > \alpha = \frac{1}{3}$. It is clear that $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \int_1^{\infty} \frac{1}{t^3}\,dt = \frac{1}{2} < \infty$. Choosing $\rho(t) = \frac{1}{t^{10}}$, we have $\rho'_+(t) = 0$, and

$$\begin{aligned}
&\int_{t_*}^{\infty}\left[\frac{k}{2^{\gamma-1}}\rho(s)Q(s)\left(\frac{\int_{t_2}^{\sigma(s)}\int_{t_1}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,dv\,du}{\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,du}\right)^{\gamma}\right.\\
&\qquad\left.-\left(1+\frac{p_0^{\gamma}}{\tau_0^{\alpha+1}}\right)R(s)\left(\frac{\rho'_+(s)}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{m\rho(s)}\right)^{\alpha}\right]ds\\
&\quad=\frac{\lambda t_1^6}{4(3)^6}\int_{t_*}^{\infty}\frac{s^6}{s}\left(\frac{\frac{2}{s^2}+\frac{1}{2t_1^2}-\frac{\frac{1}{t_2}+\frac{t_2}{t_1^2}}{s}}{s^2-t_1^2}\right)^3 ds\\
&\quad>\frac{\lambda t_1^6}{4(3)^6}\int_{t_*}^{\infty}\frac{(s^2-t_1^2)^3}{s}\left(\frac{\frac{2}{s^2}+\frac{1}{2t_1^2}-\frac{\frac{1}{t_2}+\frac{t_2}{t_1^2}}{s}}{s^2-t_1^2}\right)^3 ds=\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}Q(s)\big(\sigma(s)\big)^{\gamma}\left(\int_{\sigma(s)}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,du\right)^{\gamma}ds\right]^{\frac{1}{\alpha}}dt\\
&\quad=\frac{\lambda^3}{(3)^{18}(4)^3}\int_1^{\infty}\frac{(t^4-1)^3}{t^3}\,dt=\frac{\lambda^3}{(3)^{18}(4)^3}\int_1^{\infty}\frac{(t-1)^3(t+1)^3(t^2+1)^3}{t^3}\,dt\\
&\quad>\frac{\lambda^3}{(3)^{18}(4)^3}\int_1^{\infty}(t-1)^3\big(t^2+1\big)^3\,dt>\frac{\lambda^3}{(3)^{18}(4)^3}\int_1^{\infty}(t-1)^3\,dt=\infty.
\end{aligned}$$

Thus, by Theorem 3.3, it follows that every solution $x(t)$ of Eq. (5.3) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. We may note that, for $\lambda = \frac{\sqrt[3]{35}}{\sqrt{2^{17}}}$, we have $x(t) = \frac{1}{t^{\frac{5}{2}}}$ is a solution of Eq. (5.3).

Example 4 Consider the third order neutral delay differential equation

$$\left(t^3\left(x(t) + t^{\frac{5}{3}}\frac{5t+6}{t+1}x\left(\frac{t}{2}\right)\right)''\right)' + t^9 x^3(t-1) = 0, \quad t \ge t_0 = 2. \tag{5.4}$$

Here, $r(t) = t^3$, $p(t) = t^{\frac{5}{3}}\frac{5t+6}{t+1}$, $q(t) = t^9$, $\tau(t) = \frac{t}{2}$, $\sigma(t) = t-1$, $f(u) = u^3$, $\alpha = 1$, and $\gamma = 3$. It is clear that $p(t) = t^{\frac{5}{3}}[5 + \frac{1}{t+1}] \ge (5)(2^{\frac{5}{3}}) \simeq 15.874 > 1$, $\tau \circ \sigma \ne \sigma \circ \tau$, $\sigma(t) \ge \tau(t)$, conditions (H_1)–(H_3), and (1.8) hold, and

$$p\big(\tau^{-1}\big(\tau^{-1}(t)\big)\big) = \frac{20t+6}{4t+1}(4t)^{\frac{5}{3}} = \left[5 + \frac{1}{4t+1}\right](4t)^{\frac{5}{3}} > (5)(4t)^{\frac{5}{3}} > (5)(8)^{\frac{5}{3}} \simeq 160. \tag{5.5}$$

Let $m_*(t) = \int_{t_2}^{t}\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\,du\,ds$ and $m_{**}(t) = t^{\frac{1}{k_1}}$. Thus

$$p_*(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{\int_{t_2}^{\tau^{-1}(\tau^{-1}(t))}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,du\,ds}{\int_{t_2}^{\tau^{-1}(t)}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,du\,ds}\right)$$
$$\geq \frac{1}{p(2t)}\left(1 - \frac{1}{160}\frac{12t^2-13t+3}{6t^2-13t+6}\right) = \frac{1}{p(2t)}\left(1 - \frac{1}{160}\bigl(2+\phi(t)\bigr)\right), \tag{5.6}$$

where $\phi(t) = \frac{13t-9}{6t^2-13t+6}$. Since $\phi'(t) = \frac{-78t^2+108t-39}{(6t^2-13t+6)^2}$, which is negative for $t \geq t_2 = 3 > t_1 = 2$. Thus $\phi(t)$ is positive and decreasing for $t \geq t_2 = 3$. It follows that $\phi(t) \leq \frac{10}{7}$. Thus by (5.6) we have

$$p_*(t) \geq \frac{1}{p(2t)}\left(1 - \frac{1}{160}\frac{24}{7}\right) = \frac{137}{(140)(2t)^{\frac{5}{3}}}\frac{2t+1}{10t+6} > 0 \quad \text{for } t \geq t_2 = 3.$$

By choosing $\rho(t) = \frac{1}{t^8}$, condition (4.2) becomes

$$\int_{t_*}^{\infty}\left[k\rho(s)q(s)\bigl(p_*(\sigma(s))\bigr)^{\gamma}\left(\frac{\int_{t_2}^{\lambda_2(s)}\int_{t_1}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,dv\,du}{\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,du}\right)^{\gamma}\right]ds$$
$$\geq \int_{t_*}^{\infty}\left[\frac{1}{s^8}s^9\left(\frac{137}{(140)(2^{\frac{5}{3}})}\frac{2s-1}{10s-4}\frac{1}{(s-1)^{\frac{5}{3}}}\right)^3\left(\frac{\int_{t_2}^{s}\int_{t_1}^{u}\frac{1}{v^3}\,dv\,du}{\int_{t_1}^{s}\frac{1}{u^3}\,du}\right)^3\right]ds$$
$$\geq \int_{t_*}^{\infty}\left[s\left(\frac{137}{(140)(2^{\frac{5}{3}})}\zeta(s)\frac{1}{s^{\frac{5}{3}}}\right)^3\left(\frac{\int_{t_2}^{s}(\frac{-1}{2u^2}+\frac{1}{2t_1^2})\,du}{\frac{-1}{2s^2}+\frac{1}{2t_1^2}}\right)^3\right]ds, \tag{5.7}$$

where $\zeta(s) = \frac{2s-1}{10s-4}$. Then $\zeta'(s) = \frac{2}{(10s-4)^2} > 0$, i.e., $\zeta(s)$ is positive and increasing and $\zeta(s) \geq \frac{5}{26}$ for $s \geq t_2 = 3$. Now from (5.7) we have

$$\int_{t_*}^{\infty}\left[k\rho(s)q(s)\bigl(p_*(\sigma(s))\bigr)^{\gamma}\left(\frac{\int_{t_2}^{\lambda_2(s)}\int_{t_1}^{u}\frac{1}{r^{\frac{1}{\alpha}}(v)}\,dv\,du}{\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,du}\right)^{\gamma}\right]ds$$
$$\geq \int_{t_*}^{\infty}\left[s\left(\frac{137}{(140)(2^{\frac{5}{3}})}\frac{5}{26}\frac{1}{s^{\frac{5}{3}}}\right)^3\left(\frac{\frac{1}{2s}+\frac{s}{2t_1^2}-\frac{1}{2t_2}-\frac{t_2}{2t_1^2}}{\frac{-1}{2s^2}+\frac{1}{2t_1^2}}\right)^3\right]ds$$
$$\geq \int_{t_*}^{\infty}\left[\bigl(2.082656208\times 10^{-4}\bigr)\frac{1}{s^4}\left(\frac{t_1^2 s + s^3 - \frac{t_1^2}{t_2}s^2 - t_2 s^2}{s^2}\right)^3\right]ds = \infty.$$

But since by (5.5) we have

$$p^*(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right) \geq \frac{159}{(160)(2^{\frac{5}{3}})}\frac{2t+1}{10t+6}\frac{1}{t^{\frac{5}{3}}} > 0,$$

then it follows that condition (4.14) reads

$$\int_{t_0}^{\infty} q(u)\bigl(p^*(\sigma(u))\bigr)^{\gamma}\,du \geq \int_{t_0}^{\infty} u^9\left(\frac{159}{(160)(2^{\frac{5}{3}})}\frac{3}{16}\frac{1}{u^{\frac{5}{3}}}\right)^3 du \simeq \epsilon_1\int_{t_0}^{\infty} u^4\,du = \infty,$$

where $\epsilon_1 = (0.05868968172)^3$. Moreover, since by using (5.5) and letting $k_1 = \frac{1}{2}$ we have

$$p_{**}(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\left[\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right]^{\frac{1}{k_1}}\right)$$
$$\geq \frac{1}{p(2t)}\left(1 - \frac{1}{160}\left(\frac{4t}{2t}\right)^2\right) = \frac{39}{(40)(2^{\frac{5}{3}})}\frac{2t+1}{10t+6}\frac{1}{t^{\frac{5}{3}}} > 0,$$

then condition (4.20) becomes

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t} q(s)\left[\tau^{-1}\big(\sigma(s)\big)p_{**}\big(\sigma(s)\big)\int_{\tau^{-1}(\sigma(s))}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(u)}\,\mathrm{d}u\right]^{\gamma}\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t$$
$$\geq \int_{2}^{\infty}\frac{1}{t^3}\int_{2}^{t} s^9\left[\frac{1}{4(s-1)}\left(\frac{39}{(40)(2^{\frac{5}{3}})}\frac{2s-1}{10s-4}\frac{1}{(s-1)^{\frac{5}{3}}}\right)\right]^3\mathrm{d}s\,\mathrm{d}t$$
$$\geq \int_{2}^{\infty}\frac{1}{t^3}\int_{2}^{t} s^9\left[\frac{1}{4s}\left(\frac{39}{(40)(2^{\frac{5}{3}})}\frac{3}{16}\frac{1}{s^{\frac{5}{3}}}\right)\right]^3\mathrm{d}s\,\mathrm{d}t$$
$$\simeq \epsilon_2\int_{2}^{\infty}\left[\frac{1}{2t} - \frac{2}{t^3}\right]\mathrm{d}t = \infty,\ \text{where } \epsilon_2 = (0.01439558231)^3.$$

Thus, all the conditions of Theorem 4.4 are satisfied, and so every solution $x(t)$ of Eq. (5.4) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

6 General remarks

(1) In this paper, several new oscillation criteria for Eq. (1.1) have been presented which complement and improve the existing results introduced in the cited papers. In fact, our results are applicable in the cases either with $p(t)$ is bounded or unbounded and where the restriction $r'(t) \geq 0$ imposed by the authors in [1, 8, 9, 14, 19], and [17] is dropped in this paper.

(2) It is our belief that the present paper is of significance because it extends most of the cited papers which are concerned with unbounded $p(t)$ and relaxes some of their conditions. For example, Theorem 4.5 includes Theorem 2.6 and Theorem 2.9 of [15], where the author was only concerned with the special case $\alpha = \gamma$ with $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s = \infty$, and with the restriction $\sigma(t)$ is nonincreasing. Moreover, our results in this paper extend those of [5] in the special case $r(t) = 1$, $\alpha = 1$, and $f(u) = u^{\gamma}$, where $\gamma \leq 1$. At the same time it extends those of [4] in the special case $p(t) = 0$, $\alpha = \gamma$, with $\sigma(t)$ being strictly increasing.

(3) Our criteria could be extended to the dynamic equation on time scales. In this case, if we consider $m_*(t) = \int_{t_2}^{t}\int_{t_1}^{s}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u\Delta s$ and $\int_{t_0}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\Delta s = \infty$, then the obtained results will be more general than those of [10], because one may note that the results of [10] are applicable only in the case $\gamma \leq \alpha$, $0 \leq p(t) \leq p_0 < 1$, and $\sigma(t)$ is nondecreasing, while our results are applicable in the case $\gamma > \alpha$ and $p(t) \geq 1$.

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References

1. Baculíková, B., Džurina, J.: Oscillation of third-order neutral differential equations. Math. Comput. Model. **52**, 215–226 (2010)
2. Baculíková, B., Džurina, J.: Oscillation theorems for second-order nonlinear neutral differential equations. Comput. Math. Appl. **62**, 4472–4478 (2011)
3. Bohner, M., Grace, S.R., Sağer, I., Tunç, E.: Oscillation of third-order nonlinear damped delay differential equations. Appl. Math. Comput. **278**, 21–32 (2016)
4. Chatzarakis, G.E., Džurina, J., Jadlovská, I.: Oscillatory and asymptotic properties of third-order quasilinear delay differential equations. J. Inequal. Appl. **2019**, 23 (2019)
5. Chatzarakis, G.E., Grace, S.R., Jadlovská, I., Li, T., Tunç, E.: Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients. Complexity **2019**, Article ID 5691758 (2019)
6. Graef, J.R., Tunç, E., Grace, S.: Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation. Opusc. Math. **37**(6), 839–852 (2017)
7. Jiang, Y., Jiang, C., Li, T.: Oscillatory behavior of third-order nonlinear neutral delay differential equations. Adv. Differ. Equ. **2016**, 171 (2016)
8. Li, T., Rogovchenko, Y.V.: Asymptotic behavior of higher-order quasilinear neutral differential equations. Abstr. Appl. Anal. **2014**, Article ID 395368 (2014)
9. Liu, H., Meng, F., Liu, P.: Oscillation and asymptotic analysis on a new generalized Emden–Fowler equation. Appl. Math. Comput. **219**, 2739–2748 (2012)
10. Grace, S.R., Graef, J.R., Tunç, E.: Oscillatory behavior of a third-order neutral dynamic equation with distributed delays. Electron. J. Qual. Theory Differ. Equ. **2016**, 14 (2016). Proc. 10th Coll. Qualitative Theory of Diff. Equ. (July 1–4, 2015, Szeged, Hungary)
11. Grace, S.R., Graef, J.R., Tunç, E.: On the oscillation of certain third order nonlinear dynamic equations with a nonlinear damping term. Math. Slovaca **67**(2), 501–508 (2017)
12. Grace, S.R., Graef, J.R., Tunç, E.: Oscillatory behaviour of third order nonlinear differential equations with a nonlinear nonpositive neutral term. J. Taibah Univ. Sci. **13**(1), 704–710 (2019)
13. Thandapani, E., Li, T.: On the oscillation of third-order quasi-linear neutral functional differential equations. Arch. Math. **47**, 181–199 (2011)
14. Thandapani, E., Tamilvanan, S., Jambulingam, E.S.: Oscillation of third order half linear neutral delay differential equations. Int. J. Pure Appl. Math. **77**(3), 359–368 (2012)
15. Tunç, E.: Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. Electron. J. Differ. Equ. **2017**, 16 (2017)
16. Wang, R., Li, Q.: Oscillation and asymptotic properties of a class of second-order Emden–Fowler neutral differential equations. SpringerPlus **5**, 1956 (2016)
17. Wu, Y., Yu, Y., Zhang, J., Xiao, J.: Oscillation criteria for second order Emden–Fowler functional differential equations of neutral type. J. Inequal. Appl. **2016**, 328 (2016)
18. Džurina, J., Thandapani, E., Tamilvanan, S.: Oscillation of solutions to third order half-linear neutral differential equations. Electron. J. Differ. Equ. **2012**, 29 (2012)
19. Su, M., Xu, Z.: Oscillation criteria of certain third order neutral differential equations. Differ. Equ. Appl. **4**(2), 221–232 (2012)
20. Džurina, J., Baculíková, B., Jadlovská, I.: Integral oscillation criteria for third-order differential equations with delay argument. Int. J. Pure Appl. Math. **108**(1), 169–183 (2016)
21. Cloud, M.J., Drachman, B.C.: Inequalities with Applications to Engineering. Springer, New York (1998)
22. Grace, S., Agarwal, R.P., Pavani, R., Thandapani, E.: On the oscillation of certain third order nonlinear functional differential equations. Appl. Math. Comput. **202**, 102–112 (2008)
23. Kiguradze, I.T., Chanturia, T.A.: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Kluwer Academic, Dordrecht (1993)