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Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms

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Abstract

This work studies the blow-up result of the solution of a coupled nonlocal singular viscoelastic equation with general source and localized frictional damping terms under some suitable conditions. This work is a natural continuation of the previous recent articles by Boulaaras et al. (Appl. Anal., 2020, https://doi.org/10.1080/00036811.2020.1760250; Math. Methods Appl. Sci. 43:6140–6164, 2020; Topol. Methods Nonlinear Anal., 2020, https://doi.org/10.12775/TMNA.2020.014).

MSC: 35L20; 35L35

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1 Introduction

This paper is devoted to a study of the blow-up of the following system of two singular nonlinear viscoelastic equations:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds + \mu(x)u_t = f_1(u,v), & \text{in } Q, \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds + \mu(x)v_t = f_2(u,v), & \text{in } Q, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in (0,L), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in (0,L), \\ u(L,t) = v(L,t) = 0, \qquad \int_0^L xu(x,t)\,dx = \int_0^L xv(x,t)\,dx = 0, \end{cases} \tag{1}$$

where

$$\begin{cases} f_1(u,v) = a_1|u+v|^{2(r+1)}(u+v) + b_1|u|^r.u.|v|^{r+2}, \\ f_2(u,v) = a_1|u+v|^{2(r+1)}(u+v) + b_1|v|^r.v.|u|^{r+2}, \end{cases} \tag{2}$$

and $Q = (0,L) \times (0,T)$, $L < \infty$, $T < \infty$, $\mu \in C^1((0,L))$, $g_1(\cdot)$, $g_2(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and $f_1(\cdot,\cdot)$, $f_2(\cdot,\cdot) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are functions given in (2).

The problems related with localized frictional damping have been extensively studied by many teams [5], where the authors obtained an exponential rate of decay for the solution of the viscoelastic nonlinear wave equation:

$$u_{tt} - \Delta u + f(x,t,u) + \int_0^t g_1(t-s)\Delta u(s)\,ds + a(x)u_t = 0 \quad \text{in } (0,L)\times(0,T),$$

for a damping term $a(x)u_t$ that may be null for some part of the domain.

We used the techniques of [5], and we have proved in [3] the existence of a global solution using the potential well theory for the following viscoelastic system with nonlocal boundary condition and localized frictional damping:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds + a(x)u_t \\ \quad = |v|^{q+1}|u|^{p-1}u, \quad \text{in } (0,L)\times(0,T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds + a(x)v_t \\ \quad = |u|^{p+1}|v|^{q-1}v, \quad \text{in } (0,L)\times(0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in (0,\alpha), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in (0,\alpha), \\ u(\alpha,t) = v(\alpha,t) = 0, \qquad \int_0^\alpha xu(x,t)\,dx = \int_0^\alpha xv(x,t)\,dx = 0. \end{cases} \tag{3}$$

Very recently, in [2] we have studied the following singular one-dimensional nonlinear equations that arise in generalized viscoelasticity with long-term memory:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds = f_1(u,v), \quad \text{in } (0,L)\times(0,T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds = f_2(u,v), \quad \text{in } (0,L)\times(0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in (0,L), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in (0,L), \\ u(L,t) = v(L,t) = 0, \qquad \int_0^L xu(x,t)\,dx = \int_0^L xv(x,t)\,dx = 0. \end{cases} \tag{4}$$

Also in the field of blow-up, in [14], the authors studied the blow-up in finite time of solutions of an initial boundary value problem with nonlocal boundary conditions for a system of nonlinear singular viscoelastic equations.

In view of the articles mentioned above in [2, 3, 5] and a supplement to our recent study in [2], much less effort has been devoted to the blow-up of solutions of two singular nonlinear viscoelastic equations, where nonlocal boundary conditions, general source terms and localized frictional damping are considered.

The structure of the work is as follows: we start by giving the fundamental definitions and theorems on function spaces that we need, then we state the local existence theorem. Finally, we state and prove the main result, which under suitable conditions gives the blow-up in finite time of solutions for system 1.

2 Preliminaries

Let $L_x^p = L_x^p((0,L))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^L x|u|^p\,dx\right)^{\frac{1}{p}}. \tag{5}$$

Let $H = L_x^2((0,L))$ be the Hilbert space of square integral functions having the finite norm

$$\|u\|_H = \left(\int_0^L xu^2\,dx\right)^{\frac{1}{2}}. \tag{6}$$

Let $V = V_x^1((0,L))$ be the Hilbert space equipped with the norm

$$\|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{\frac{1}{2}} \tag{7}$$

and

$$V_0 = \left\{u \in V \text{ such that } u(L) = 0\right\}. \tag{8}$$

Lemma 1 (Poincaré-type inequality) *For any v in V_0 we have*

$$\int_0^L xv^2(x)\,dx \le C_p \int_0^L x\left(v_x(x)\right)^2\,dx \tag{9}$$

and

$$V_0 = \left\{v \in V \textit{ such that } v(L) = 0\right\}.$$

Remark 2 It is clear that $\|u\|_{V_0} = \|u_x\|_H$ defines an equivalent norm on V_0.

Theorem 3 (See [1]) *For any v in V_0 and $2 < p < 4$, we have*

$$\int_0^L x|v|^p\,dx \le C_*\|v_x\|^p_{H=L_x^2(0,L)}, \tag{10}$$

where C_* is a constant depending on L and p only.

We prove the blow-up result under the following suitable assumptions.

(A1) $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable and decreasing functions such that

$$\begin{aligned} &g_1(t) \ge 0, \qquad 1 - \int_0^\infty g_1(s)\,ds = l_1 > 0, \\ &g_2(t) \ge 0, \qquad 1 - \int_0^\infty g_2(s)\,ds = l_2 > 0. \end{aligned} \tag{11}$$

(A2) There exist constants $\xi_1, \xi_2 > 0$ such that

$$\begin{aligned} &g_1'(t) \le -\xi_1 g_1(t), \quad t \ge 0, \\ &g_2'(t) \le -\xi_2 g_2(t), \quad t \ge 0. \end{aligned} \tag{12}$$

(A3) $\mu : [0,L] \to \mathbb{R}_+$ is a C^1 function so that

$$\mu \ge 0, \qquad \mu > 0 \quad \text{in } (L_0, L]. \tag{13}$$

Theorem 4 *Assume* (11), (12), *and* (13) *hold. Let*

$$\begin{cases} -1 < r < \frac{4-n}{n-2}, & n \geq 3; \\ r \geq -1, & n = 1, 2. \end{cases} \tag{14}$$

Then, for any $(u_0, v_0) \in V_0^2$ *and* $(v_1, v_2) \in H^2$, *problem* (1) *has a unique local solution*

$$u \in C\big((0, T^*); V_0\big) \cap C^1\big((0, T^*); H\big),$$

for $T^* > 0$ *small enough.*

Lemma 5 *There exists a function* $F(u, v)$ *such that*

$$\begin{aligned} F(u, v) &= \frac{1}{2(r+2)}\big[uf_1(u, v) + vf_2(u, v)\big] \\ &= \frac{1}{2(r+2)}\big[a_1|u+v|^{2(r+2)} + 2b_1|uv|^{r+2}\big] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u, v), \qquad \frac{\partial F}{\partial v} = f_2(u, v).$$

We take $a_1 = b_1 = 1$ for convenience.

Lemma 6 ([9]) *There exist two positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0}{2(r+2)}\big(|u|^{2(r+2)} + |v|^{2(r+2)}\big) \leq F(u, v) \leq \frac{c_1}{2(r+2)}\big(|u|^{2(r+2)} + |v|^{2(r+2)}\big). \tag{15}$$

We now define the energy functional.

Lemma 7 *Assume* (11), (12), (13), *and* (14) *hold, let* (u, v) *be a solution of* (1), *then* $E(t)$ *is non-increasing, that is,*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_H^2 + \frac{1}{2}\|v_t\|_H^2 + \frac{1}{2}l_1\|u_x\|_H^2 + \frac{1}{2}l_2\|v_x\|_H^2 \\ &\quad + \frac{1}{2}(g_1 o u_x) + \frac{1}{2}(g_2 o v_x) - \int_0^L xF(u, v)\,dx \end{aligned} \tag{16}$$

satisfies

$$\begin{aligned} E'(t) &= -\int_0^L x\mu(x)u_t^2\,dx - \int_0^L x\mu(x)v_t^2\,dx + \frac{1}{2}g_1' \circ u_x + \frac{1}{2}g_2' \circ v_x \\ &\quad - \int_0^t g_1(s)\,ds \int_0^L xu_x^2\,dx - \int_0^t g_2(s)\,ds \int_0^L xv_x^2\,dx \\ &\leq 0, \end{aligned} \tag{17}$$

where

$$\int_0^L xF(u, v)\,dx = \frac{1}{2(r+2)}\big(\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{(r+2)}\big) \tag{18}$$

and

$$(g \circ u_x)(t) = \int_0^L \int_0^t x g(t-s) \left| u_x(x,t) - u_x(x,s) \right|^2 ds\, dx. \tag{19}$$

Proof By multiplying $(1)_1$, $(1)_2$ by xu_t, xv_t, respectively, and integrating over $(0, L)$, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{ \frac{1}{2}\|u_t\|_H^2 + \frac{1}{2}\|v_t\|_H^2 + \frac{1}{2}l_1\|u_x\|_H^2 + \frac{1}{2}l_2\|v_x\|_H^2 + \frac{1}{2}(g_1 \circ u_x) \right. \\
&\qquad \left. + \frac{1}{2}(g_2 \circ u_x) - \int_0^L xF(u,v)\, dx \right\} \\
&\quad = -\int_0^L x\mu(x) u_t^2\, dx - \int_0^L x\mu(x) v_t^2\, dx + \frac{1}{2} g_1' \circ u_x + \frac{1}{2} g_2' \circ v_x \\
&\qquad - \left(\int_0^t g_1(s)\, ds \right) \|u_x\|_H^2 - \left(\int_0^t g_2(s)\, ds \right) \|v_x\|_H^2.
\end{aligned} \tag{20}$$

And by using (11), (12) and (13), we obtain (17). □

3 Blow-up

In this section, we prove the blow-up result of solution of problem (1).

Now we define the functional

$$\begin{aligned}
\mathbb{H}(t) &= -E(t) \\
&= -\frac{1}{2}\|u_t\|_H^2 - \frac{1}{2}\|v_t\|_H^2 - \frac{1}{2}l_1\|u_x\|_H^2 - \frac{1}{2}l_2\|v_x\|_H^2 \\
&\quad - \frac{1}{2}(g_1 o u_x) - \frac{1}{2}(g_2 o v_x) \\
&\quad + \frac{1}{2(r+2)}\left[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{r+2}}^{r+2} \right].
\end{aligned} \tag{21}$$

Theorem 8 *Assume* (11)–(13), *and* (14) *hold. Assume further that* $E(0) < 0$, *then the solution of problem* (1) *blows up in finite time.*

Proof From (17), we have

$$E(t) \leq E(0) \leq 0. \tag{22}$$

Therefore

$$\mathbb{H}'(t) = -E'(t) \geq 0.$$

By (18) and (15), we have

$$\begin{aligned}
0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \frac{1}{2(r+2)}\left[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2} \right] \\
&\leq \frac{c_1}{2(r+2)}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)} \right].
\end{aligned} \tag{23}$$

We set

$$\mathcal{K}(t) = \mathbb{H}^{1-\alpha} + \varepsilon \int_0^L x(uu_t + vv_t)\,dx + \frac{\varepsilon}{2}\int_0^L x\mu(x)\big(u^2+v^2\big)\,dx, \tag{24}$$

where

$$0 < \alpha < \frac{2r+2}{4(r+2)} < 1. \tag{25}$$

By multiplying $(1)_1$, $(1)_2$ by xu, xv and taking the derivative of (24), we get

$$\begin{aligned}\mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\big(\|u_t\|_H^2 + \|v_t\|_H^2\big) - \varepsilon\big(\|u_x\|_H^2 + \|v_x\|_H^2\big)\\ &\quad + \varepsilon\int_0^L u_x \int_0^t g_1(t-s)xu_x(s)\,ds\,dx + \varepsilon\int_0^L v_x\int_0^t g_2(t-s)xv_x(s)\,ds\,dx\\ &\quad + \varepsilon\big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\big],\end{aligned} \tag{26}$$

we have

$$\begin{aligned}&\varepsilon\int_0^t g_1(t-s)\,ds\int_0^L u_x.xu_x(s)\,dx\,ds\\ &\quad = \varepsilon\int_0^t g_1(t-s)\,ds\int_0^L u_x.\big(xu_x(s) - xu_x(t)\big)\,dx\,ds + \varepsilon\left(\int_0^t g_1(s)\,ds\right)\|u_x\|_H^2\\ &\quad \geq \varepsilon\left(\frac{1}{2}\int_0^t g_1(s)\,ds\right)\|u_x\|_H^2 - \frac{\varepsilon}{2}(g_1\circ u_x),\end{aligned} \tag{27}$$

$$\begin{aligned}&\varepsilon\int_0^t g_2(t-s)\,ds\int_0^L v_x.xv_x(s)\,dx\,ds\\ &\quad = \varepsilon\int_0^t g_2(t-s)\,ds\int_0^L v_x.\big(xv_x(s) - xv_x(t)\big)\,dx\,ds + \varepsilon\left(\int_0^t g_2(s)\,ds\right)\|v_x\|_H^2\\ &\quad \geq \varepsilon\left(\frac{1}{2}\int_0^t g_2(s)\,ds\right)\|v_x\|_H^2 - \frac{\varepsilon}{2}(g_2\circ u_x).\end{aligned} \tag{28}$$

We obtain, from (26),

$$\begin{aligned}\mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\big(\|u_t\|_H^2 + \|v_t\|_H^2\big)\\ &\quad - \varepsilon\left(\left(1 - \frac{1}{2}\int_0^t g_1(s)\,ds\right)\|u_x\|_H^2 + \left(1 - \frac{1}{2}\int_0^t g_2(s)\,ds\right)\|v_x\|_H^2\right)\\ &\quad - \frac{\varepsilon}{2}(g_1\circ u_x) - \frac{\varepsilon}{2}(g_2\circ v_x) + \varepsilon\big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\big].\end{aligned} \tag{29}$$

For $0 < a < 1$, from (21)

$$\begin{aligned}\varepsilon\big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\big] &= \varepsilon a\big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\big]\\ &\quad + 2\varepsilon(r+2)(1-a)\mathbb{H}(t)\\ &\quad + \varepsilon(r+2)(1-a)\big(\|u_t\|_H^2 + \|v_t\|_H^2\big)\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon(r+2)(1-a)\left(1-\int_0^t g_1(s)\,ds\right)\|u_x\|_H^2 \\
&+\varepsilon(p+2)(1-a)\left(1-\int_0^t g_2(s)\,ds\right)\|v_x\|_H^2 \\
&+\varepsilon(r+2)(1-a)(g_1\circ u_x) \\
&+\varepsilon(r+2)(1-a)(g_2\circ v_x).
\end{aligned}
\tag{30}
$$

Substituting in (29), we get

$$
\begin{aligned}
\mathcal{K}'(t) \geq{}& (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon\big[(r+2)(1-a)+1\big]\big(\|u_t\|_H^2+\|v_t\|_H^2\big) \\
&+\varepsilon\left[(r+2)(1-a)\left(1-\int_0^t g_1(s)\,ds\right)-\left(1-\frac{1}{2}\int_0^t g_2(s)\,ds\right)\right]\|u_x\|_H^2 \\
&+\varepsilon\left[(r+2)(1-a)\left(1-\int_0^t g_2(s)\,ds\right)-\left(1-\frac{1}{2}\int_0^t g_2(s)\,ds\right)\right]\|v_x\|_H^2 \\
&+\varepsilon\left[(r+2)(1-a)-\frac{1}{2}\right](g_1ou_x+g_2ov_x) \\
&+\varepsilon a\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)}+2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big]+2\varepsilon(r+2)(1-a)\mathbb{H}(t).
\end{aligned}
\tag{31}
$$

In this point, we take $a>0$ small enough so that

$$
\alpha_1=(r+2)(1-a)-1>0
$$

and we assume

$$
\max\left\{\int_0^\infty g_1(s)\,ds,\int_0^\infty g_2(s)\,ds\right\}<\frac{(r+2)(1-a)-1}{((r+2)(1-a)-\frac{1}{2})}=\frac{2\alpha_1}{2\alpha_1+1};
\tag{32}
$$

then we have

$$
\begin{aligned}
\alpha_2&=\left\{(r+2)(1-a)-1)-\int_0^t g_1(s)\,ds\left((r+2)(1-a)-\frac{1}{2}\right)\right\}>0, \\
\alpha_3&=\left\{(r+2)(1-a)-1)-\int_0^t g_2(s)\,ds\left((r+2)(1-a)-\frac{1}{2}\right)\right\}>0,
\end{aligned}
$$

we pick ε small enough such that

$$
\mathbb{H}(0)+\varepsilon\int_0^L x(u_0u_1+v_0v_1)\,dx>0.
$$

Thus, for some $\beta>0$, estimate (31) becomes

$$
\begin{aligned}
\mathcal{K}'(t) \geq{}& \beta\big\{\mathbb{H}(t)+\|u_t\|_H^2+\|v_t\|_H^2+\|u_x\|_H^2+\|v_x\|_H^2 \\
&+(g_1ou_x)+(g_2ov_x)+\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)}+2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big]\big\}.
\end{aligned}
\tag{33}
$$

By (15), for some $\beta_1 > 0$, we obtain

$$\mathcal{K}'(t) \geq \beta_1 \bigl\{ \mathbb{H}(t) + \|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 + (g_1 o u_x) + (g_2 o v_x) + \bigl[\|u\|_{L_x^{2(r+2)}}^{2(p+2)} + \|u\|_{L_x^{2(r+2)}}^{2(r+2)} \bigr] \bigr\} \tag{34}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{35}$$

Next, using Hölder's and Young's inequalities, we have

$$\left| \int_0^L x(uu_t + vv_t)\,dx \right|^{\frac{1}{1-\alpha}} \leq C\bigl[\|u\|_{L_x^{2(r+2)}}^{\frac{\theta}{1-\alpha}} + \|u_t\|_H^{\frac{\mu}{1-\alpha}} + \|v\|_{L_x^{2(r+2)}}^{\frac{\theta}{1-\alpha}} + \|v_t\|_H^{\frac{\mu}{1-\alpha}} \bigr], \tag{36}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

We take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(r+2).$$

Subsequently, for $s = \frac{2}{(1-2\alpha)}$ and by using (21), we obtain

$$\|u\|_{L_x^{2(r+2)}}^{\frac{2}{1-2\alpha}} \leq d\bigl(\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \mathbb{H}(t) \bigr),$$
$$\|v\|_{L_x^{2(r+2)}}^{\frac{2}{1-2\alpha}} \leq d\bigl(\|v\|_{L_x^{2(r+2)}}^{2(r+2)} + \mathbb{H}(t) \bigr), \quad \forall t \geq 0.$$

Therefore,

$$\left| \int_0^L x(uu_t + vv_t)\,dx \right|^{\frac{1}{1-\alpha}} \leq c_3\bigl[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)} + \|u_t\|_H^2 + \|v_t\|_H^2 + \mathbb{H}(t) \bigr].$$

Subsequently,

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(\mathbb{H}^{1-\alpha} + \varepsilon \int_0^L x(uu_t + vv_t)\,dx + \frac{\varepsilon}{2} \int_0 x\mu(x)\bigl(u^2 + v^2\bigr)\,dx \right)^{\frac{1}{1-\alpha}} \\ &\leq c\left\{ \mathbb{H}(t) + \left| \int_0^L x(uu_t + vv_t)\,dx \right|^{\frac{1}{1-\alpha}} + \|u\|_H^{\frac{2}{1-\alpha}} + \|v\|_H^{\frac{2}{1-\alpha}} \right\} \\ &\leq c\bigl[\mathbb{H}(t) + \|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 + (g_1 o u_x) \\ &\quad + (g_2 o v_x) + \|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)} \bigr]. \end{aligned} \tag{37}$$

From (33) and (37), we have

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{38}$$

where $\lambda > 0$, this depends only on β_1 and c.

By integration of (38), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda \frac{\alpha}{(1-\alpha)} t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

4 Conclusion

Mixed non-local problems for hyperbolic and parabolic PDEs have been studied intensively in recent decades. Such equations or systems with constraints modelize many time-dependant physical phenomena. These constraints can be data measured directly on the boundary or giving integral boundary conditions (see for example [1, 4–7, 10–13]). In view of the articles mentioned above in [2, 3, 5] and a supplement to our recent study in [2, 8], we have proved in this work the blow-up of solutions of two singular nonlinear viscoelastic equations, where nonlocal boundary conditions, general source terms and localized frictional damping are considered.

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