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On a unified integral operator for φ-convex functions

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Abstract

Integral operators have a very vital role in diverse fields of science and engineering. In this paper, we use φ-convex functions for unified integral operators to obtain their upper bounds and upper and lower bounds for symmetric φ-convex functions in the form of a Hadamard inequality. Also, for φ-convex functions, we obtain bounds of different known fractional and conformable fractional integrals. The results of this paper are applicable to convex functions.

Keywords: Convex function; φ-convex function; Integral operators; Fractional integrals; Conformable fractional integrals; Bounds

1 Introduction and preliminaries

Some very interesting properties of convex functions make them important in mathematical analysis. It should be noted that in new problems related to convexity, generalized assumptions about convex functions are necessary to obtain applicable results. During the recent era, there have been several attempts to generalize the notion of convex functions. Many important generalizations can be found for convex functions, such as α-convex, m-convex, h-convex, (α, m)-convex, (h, m)-convex, s-convex, (s, m)-convex, GA-convex, GG-convex, and preinvex functions [1, 3, 5, 9, 11, 13, 14, 17, 18, 20, 24].

Definition 1 ([21]) A function $f : I \longrightarrow \mathbb{R}$ is said to be convex if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$, where $I \subseteq \mathbb{R}$ is an interval. If inequality (1.1) is reversed, then f is called a concave function.

Definition 2 ([7]) A function $f : I \longrightarrow \mathbb{R}$ is said to be φ-convex if

$$f\big(tx + (1-t)y\big) \le f(y) + t\varphi\big(f(x), f(y)\big) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$, where $I \subseteq \mathbb{R}$ is a convex set, and $\varphi : f(I) \times f(I) \longrightarrow \mathbb{R}$ is a bifunction.

For $\varphi(x,y) = x - y$, the φ-convex functions reduce to the convex functions. Note that every convex function is φ-convex, but the converse is not true.

Example 1 ([7]) The function

$$f(x) = \begin{cases} -x, & x \geq 0, \\ x, & x < 0, \end{cases} \tag{1.3}$$

where $\varphi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}$ is defined by

$$\varphi(x,y) = \begin{cases} x, & y = 0, \\ -y, & x = 0, \\ -x - y, & x < 0, y < 0, \end{cases} \tag{1.4}$$

is φ-convex but not a convex function.

Integral operators play a very vital role in the study of fractional derivatives and fractional integrals. Next, we give definitions of some integral operators, which will be utilized in the results of this paper.

Definition 3 ([15]) Let $f \in L[x_0, y_0]$, and let g be a positive increasing function on $(x_0, y_0]$ with continuous derivative on (x_0, y_0). The left- and right-sided fractional integral operators of f with respect to g on $[x_0, y_0]$ of order μ, where $\Re(\mu) > 0$, are given by

$$ {}^{\mu}_{g}I_{x_0^+} f(x) = \frac{1}{\Gamma(\mu)} \int_{x_0}^{x} \bigl(g(x) - g(t)\bigr)^{\mu - 1} g'(t) f(t)\, dt, \quad x > x_0, \tag{1.5}$$

$$ {}^{\mu}_{g}I_{y_0^-} f(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{y_0} \bigl(g(t) - g(x)\bigr)^{\mu - 1} g'(t) f(t)\, dt, \quad x < y_0, \tag{1.6}$$

where Γ is the gamma function.

Definition 4 ([16]) Let $f \in L[x_0, y_0]$, and let g be positive increasing function on $(x_0, y_0]$ with continuous derivative on (x_0, y_0). The left- and right-sided k-fractional integral operators of f with respect to g on $[x_0, y_0]$ of order μ, where $\Re(\mu), k > 0$, are given by

$$ {}^{\mu}_{g}I^{k}_{x_0^+} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_{x_0}^{x} \bigl(g(x) - g(t)\bigr)^{\frac{\mu}{k} - 1} g'(t) f(t)\, dt, \quad x > x_0, \tag{1.7}$$

$$ {}^{\mu}_{g}I^{k}_{y_0^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_{x}^{y_0} \bigl(g(t) - g(x)\bigr)^{\frac{\mu}{k} - 1} g'(t) f(t)\, dt, \quad x < y_0, \tag{1.8}$$

where Γ_k is the k-gamma function.

Definition 5 ([2]) Let $f \in L_1[x_0, y_0]$ and $x \in [x_0, y_0]$. Also, let $\eta, \mu, \alpha, \xi, \gamma, \zeta \in C$, $\Re(\mu), \Re(\alpha), \Re(\xi) > 0$, $\Re(\zeta) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then the left- and right-sided generalized fractional integral operators $\epsilon^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,\eta,x_0^+} f$ and $\epsilon^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,\eta,y_0^-} f$ are defined by

$$\bigl(\epsilon^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,\eta,x_0^+} f\bigr)(x;p) = \int_{x_0}^{x} (x-t)^{\alpha - 1} E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi} \bigl(\eta(x-t)^{\mu}; p\bigr) f(t)\, dt, \tag{1.9}$$

$$\left(\epsilon^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,\eta,y_0^-}f\right)(x;p) = \int_x^{y_0} (t-x)^{\alpha-1} E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta(t-x)^{\mu};p\right)f(t)\,dt, \tag{1.10}$$

where $E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}(t;p)$ is the extended generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk,\zeta-\gamma)}{\beta(\gamma,\zeta-\gamma)} \frac{(\zeta)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(\xi)_{n\delta}}. \tag{1.11}$$

Farid [6] (see also [16]) defined unified integral operators and proved that these integral operators are bounded, linear, and thus continuous.

Definition 6 ([6]) Let $f,g:[x_0,y_0]\longrightarrow\mathbb{R}$, where $0<x_0<y_0$, be functions such that f is positive and integrable over $[x_0,y_0]$ and g is differentiable and strictly increasing. Also, let $\frac{\Psi}{x}$ be an increasing function on $[x_0,\infty)$, and let $\alpha,\xi,\gamma,\zeta\in\mathbb{C}$, $p,\mu,\delta\geq 0$, and $0<k\leq\delta+\mu$. Then for $x\in[x_0,y_0]$, the left and right integral operators are given by

$$\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;p) = \int_{x_0}^{x} G_x^y\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(y)f(y)\,dy, \tag{1.12}$$

$$\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x;p) = \int_{x}^{y_0} G_y^x\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(y)f(y)\,dy, \tag{1.13}$$

where

$$G_x^y\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right) = \frac{\Psi(g(x)-g(y))}{g(x)-g(y)} E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(x)-g(y)\right)^{\mu};p\right). \tag{1.14}$$

For the particular choice of Ψ,g and the parameters involved in Mittag-Leffler functions, several conformable and fractional integrals can be obtained; see [16, Remarks 6 and 7]. In [16], some bounds of the above operators have been proved for convex functions.

Theorem 1 *Let $f:[x_0,y_0]\longrightarrow\mathbb{R}$ be a positive convex function for $0<x_0<y_0$, and let $g:[x_0,y_0]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing. Also, let $\frac{\Psi}{x}$ be an increasing function on $[x_0,y_0]$, and let $\eta,\alpha,\xi,\gamma,\zeta\in\mathbb{C}$, $p,\mu\geq 0$, $\delta\geq 0$, and $0<k\leq\delta+\mu$. Then we have the following bound for $x\in[x_0,y_0]$*:

$$\begin{aligned}
&\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x;p)\\
&\quad\leq E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(x)-g(x_0)\right)^{\mu};p\right)\\
&\qquad\times\Psi\left(g(x)-g(x_0)\right)\left(f(x)+f(x_0)\right) + E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(y_0)-g(x)\right)^{\mu};p\right)\Psi\left(g(y_0)-g(x)\right)\\
&\qquad\times\left(f(x)+f(y_0)\right).
\end{aligned} \tag{1.15}$$

Further, the following bounds hold as a version of the Hadamard inequality.

Theorem 2 *Along with assumptions of the Theorem 1, if f is symmetric about $\frac{x_0+y_0}{2}$, then we have the following inequalities*:

$$f\left(\frac{x_0+y_0}{2}\right)\left(\left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}1\right)(x_0;p) + \left({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}1\right)(y_0;p)\right)$$

$$\begin{aligned}&\le \left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x_0;p)+\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(y_0;p)\\&\le 2\Psi\left(g(y_0)-g(x_0)\right)E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(y_0)-g(x_0)\right)^{\mu};p\right)\left(f(x_0)+f(y_0)\right).\end{aligned}\tag{1.16}$$

Moreover, the following result is produced by defining unified operators for the convolution $f*g$ of functions f and g.

Theorem 3 *Let $f,g:[x_0,y_0]\longrightarrow\mathbb{R}$ be a differentiable functions such that $|f'|$ is convex, $0<x_0<y_0$, and g is a strictly increasing function. Also, let $\frac{\Psi}{x}$ be an increasing function, and let $\eta,\alpha,\xi,\gamma,\zeta\in\mathbb{C}$, $p,\mu,\delta\ge 0$, and $0<k\le\delta+\mu$. Then we have the following modulus inequality for $x\in(x_0,y_0)$:*

$$\begin{aligned}&\left|\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f*g\right)(x;p)+\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f*g\right)(x;p)\right|\\&\quad\le E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(x)-g(x_0)\right)^{\mu};p\right)\Psi\left(g(x)-g(x_0)\right)\left(\left|f'(x)\right|+\left|f'(x_0)\right|\right)\\&\qquad+E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(y_0)-g(x)\right)^{\mu};p\right)\Psi\left(g(y_0)-g(x)\right)\left(\left|f'(x)\right|+\left|f'(y_0)\right|\right),\end{aligned}\tag{1.17}$$

where

$$\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f*g\right)(x;p)=\int_{x_0}^{x}G^{t}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)f'(t)\,d\left(g(t)\right),\tag{1.18}$$

$$\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f*g\right)(x;p)=\int_{x}^{y_0}G^{x}_{t}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)f'(t)\,d\left(g(t)\right).\tag{1.19}$$

In Sect. 2, we use φ-convex functions to obtain bounds of integral operators given in Definition 6. Moreover, we achieve Hadamard-type bounds using the additional condition of symmetry. Also, we get some particular bounds by the φ-convexity of $|f'|$ and defining a convenient integral operator of convolution of two functions. In Sect. 3, we give some applications of the presented results.

2 Main results

Throughout this section, we assume that

$$I(x_0,y_0;g):=\frac{1}{y_0-x_0}\int_{x_0}^{y_0}g(t)\,dt.$$

Theorem 4 *Let $f:[x_0,y_0]\longrightarrow\mathbb{R}$ be a positive φ-convex function, and let $g:[x_0,y_0]\longrightarrow\mathbb{R}$ be a differentiable strictly increasing function. Also, let $\frac{\Psi}{x}$ be an increasing function on $[x_0,y_0]$, and let $\eta,\alpha,\xi,\gamma,\zeta\in\mathbb{C}$, $p,\mu,\nu,\delta\ge 0$, $0<k\le\delta+\mu$, and $0<k\le\delta+\nu$. Then for $x\in[x_0,y_0]$, we have*

$$\begin{aligned}&\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;p)\\&\quad\le E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(x)-g(x_0)\right)^{\mu};p\right)\Psi\left(g(x)-g(x_0)\right)f(x)\\&\qquad+G^{x_0}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\varphi\left(f(x_0),f(x)\right)\left(I(x_0,x,g)-g(x_0)\right),\end{aligned}\tag{2.1}$$

$$\begin{aligned}\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,y_0^-}f\right)(x;p)&\le E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}\left(\eta\left(g(y_0)-g(x)\right)^{\nu};p\right)\Psi\left(g(y_0)-g(x)\right)f(y_0)\\&\quad+G^{x}_{y_0}\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi},g;\Psi\right)\varphi\left(f(x),f(y_0)\right)\left(I(x,y_0;g)-g(x)\right),\end{aligned}\tag{2.2}$$

$$\begin{aligned}
&\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;p) + \left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi,y_0^-}f\right)(x;p) \\
&\quad\le E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(x)-g(x_0)\right)^{\mu};p\right)\Psi\left(g(x)-g(x_0)\right)f(x) \\
&\qquad + G^{x_0}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\varphi\left(f(x_0),f(x)\right)\left(I(x_0,x;g)-g(x_0)\right) \\
&\qquad + E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi}\left(\eta\left(g(y_0)-g(x)\right)^{\nu};p\right)\Psi\left(g(y_0)-g(x)\right)f(y_0) \\
&\qquad + G^{x}_{y_0}\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi},g;\Psi\right)\varphi\left(f(x),f(y_0)\right)\left(I(x,y_0;g)-g(x)\right).
\end{aligned} \tag{2.3}$$

Proof We have the following inequality for the kernel defined in (1.14) and an increasing function g:

$$G^{t}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(t) \le G^{x_0}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(t), \quad t\in[x_0,x], x\in(x_0,y_0). \tag{2.4}$$

Using the φ-convexity of f, we have

$$f(t) \le f(x) + \frac{x-t}{x-x_0}\varphi\left(f(x_0),f(x)\right). \tag{2.5}$$

Inequalities (2.4) and (2.5) constitute the following integral inequality:

$$\begin{aligned}
&\int_{x_0}^{x} G^{t}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)g'(t)f(t)\,dt \\
&\quad\le f(x)G^{x_0}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\int_{x_0}^{x} g'(t)\,dt \\
&\qquad + \frac{\varphi(f(x_0),f(x))}{x-x_0}G^{x_0}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\int_{x_0}^{x}(x-t)g'(t)\,dt.
\end{aligned} \tag{2.6}$$

Using (1.12) of Definition 6 on the left-hand side of inequality (2.6) and integrating the right-hand side, we get

$$\begin{aligned}
&\left({}_{g}F^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;p) \\
&\quad\le E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\left(\eta\left(g(x)-g(x_0)\right)^{\mu};p\right)\Psi\left(g(x)-g(x_0)f(x)\right. \\
&\qquad + G^{x_0}_{x}\left(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\right)\varphi\left(f(x_0),f(x)\right)\left(I(x_0,x;g)-g(x_0)\right).
\end{aligned} \tag{2.7}$$

Now using the same technique for $t\in(x,y_0]$ and $x\in(x_0,y_0)$, we can write

$$G^{x}_{t}\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi},g;\Psi\right)g'(t) \le G^{x}_{y_0}\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi},g;\Psi\right)g'(t). \tag{2.8}$$

Using the φ-convexity of f, we have

$$f(t) \le f(y_0) + \frac{y_0-t}{y_0-x}\varphi\left(f(x),f(y_0)\right). \tag{2.9}$$

Inequalities (2.8) and (2.9) constitute the following inequality:

$$\int_{x}^{y_0} G^{x}_{t}\left(E^{\gamma,\delta,k,\zeta}_{\nu,\alpha,\xi},g;\Psi\right)g'(t)f(t)\,dt$$

$$\leq G_{y_0}^{x}\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)\left(f(y_0)\int_x^{y_0} g'(t)\,dt+\frac{\varphi(f(x),f(y_0))}{y_0-x}\int_x^{y_0}(y_0-t)g'(t)\,dt\right).$$

Using (1.13) of Definition 6 on the left-hand side and integrating by parts the right-hand side, we get

$$\begin{aligned}\left({}_gF_{\nu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}f\right)(x;p)&\leq E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(y_0)-g(x)\right)^{\nu};p\right)\Psi\left(g(y_0)-g(x)\right)f(y_0)\\&\quad+G_{y_0}^{x}\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)\varphi\left(f(x),f(y_0)\right)\left(I(x,y_0;g)-g(x)\right).\end{aligned}\tag{2.10}$$

We obtain inequality (2.3) by summing (2.7) and (2.10). □

Corollary 1 *By setting $\mu=\nu$ in (2.3) we get*

$$\begin{aligned}&\left({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta}f\right)(x;p)+\left({}_gF_{\mu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}f\right)(x;p)\\&\quad\leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(x)-g(x_0)\right)^{\mu};p\right)\Psi\left(g(x)-g(x_0)\right)f(x)\\&\qquad+G_{x}^{x_0}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)\varphi\left(f(x_0),f(x)\right)\left(I(x_0,x;g)-g(x_0)\right)\\&\qquad+E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(y_0)-g(x)\right)^{\mu};p\right)\Psi\left(g(y_0)-g(x)\right)f(y_0)\\&\qquad+G_{y_0}^{x}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\right)\varphi\left(f(x),f(y_0)\right)\left(I(x,y_0;g)-g(x)\right).\end{aligned}\tag{2.11}$$

Remark 1 If we take $\varphi(x,y)=x-y$ in (2.11), we get inequality (1.15) of Theorem 1.

We will need the following lemma in proving the upcoming result.

Lemma 1 *Let $f:[x_0,y_0]\to\mathbb{R}$ be a φ-convex function. If $f(x)=f(x_0+y_0-x)$, $x\in[x_0,y_0]$, then we have the following inequality*:

$$f\left(\frac{x_0+y_0}{2}\right)\leq f(x)+\frac{1}{2}\varphi\left(f(x),f(x)\right).\tag{2.12}$$

Proof Since f is φ-convex, we get

$$\begin{aligned}f\left(\frac{x_0+y_0}{2}\right)&\leq f\left(\frac{x-x_0}{y_0-x_0}x_0+\frac{y_0-x}{y_0-x_0}y_0\right)\\&\quad+\frac{1}{2}\varphi\left(f\left(\frac{x-x_0}{y_0-x_0}y_0+\frac{y_0-x}{y_0-x_0}x_0\right),f\left(\frac{x-x_0}{y_0-x_0}x_0+\frac{y_0-x}{y_0-x_0}y_0\right)\right)\\&\leq f(x_0+y_0-x)+\frac{1}{2}\varphi\left(f(x),f(x_0+y_0-x)\right).\end{aligned}$$

Using $f(x_0+y_0-x)=f(x)$ in this inequality, we get inequality (2.12). □

Remark 2 For $\varphi(x,y)=x-y$, Lemma 1 reduces to [16, Lemma 1].

Theorem 5 *Along with the assumptions of Theorem 4, if $f(x_0+y_0-x)=f(x)$ and $\varphi(x,y)=x+y$, then*

$$\frac{1}{2}f\left(\frac{x_0+y_0}{2}\right)\left(\left({}_gF_{\mu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}1\right)(x_0;p)+\left({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta}1\right)(y_0;p)\right)$$

$$\begin{aligned}
&\leq \big({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\big)(x_0;p) + \big({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\big)(y_0;p) \\
&\leq 2\Psi\big(g(y_0)-g(x_0)\big)E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\big(\eta\big(g(y_0)-g(x_0)\big)^{\mu};p\big)f(y_0) \\
&\quad + 2\big(f(x_0)+f(y_0)\big)G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)\big(I(x_0,y_0;g)-g(x_0)\big).
\end{aligned} \tag{2.13}$$

Proof We have the following inequality for the kernel defined in (1.14) and an increasing function g:

$$G^{x_0}_{x}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)g'(x) \leq G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)g'(x), \quad x\in(x_0,y_0). \tag{2.14}$$

Using the φ-convexity of f over $[x_0,y_0]$, we have

$$f(x) \leq f(y_0) + \frac{y_0-x}{y_0-x_0}\varphi\big(f(x_0),f(y_0)\big). \tag{2.15}$$

Inequalities (2.14) and (2.15) constitute the following integral inequality:

$$\begin{aligned}
&\int_{x_0}^{y_0} G^{x_0}_{x}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)f(x)g'(x)\,dx \\
&\quad\leq f(y_0)G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)\int_{x_0}^{y_0} g'(x)\,dx \\
&\qquad + \frac{\varphi(f(x_0),f(y_0))}{y_0-x_0}G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)\int_{x_0}^{y_0}(y_0-x)g'(x)\,dx.
\end{aligned}$$

Using (1.13) of Definition 6 on left-hand side, integrating the right-hand side, and using $\varphi(x,y)=x+y$, this inequality gives

$$\begin{aligned}
&\big({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\big)(x_0;p) \\
&\quad\leq E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi}\big(\eta\big(g(y_0)-g(x_0)\big)^{\mu};p\big)\Psi\big(g(y_0)-g(x_0)\big)f(y_0) \\
&\qquad + G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)\big(f(x_0)+f(y_0)\big)\big(I(x_0,y_0;g)-g(x_0)\big).
\end{aligned} \tag{2.16}$$

Also, we have the following inequality:

$$G^{x}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)g'(x) \leq G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)g'(x), \quad x\in(x_0,y_0). \tag{2.17}$$

Inequalities (2.15) and (2.17) constitute the following integral inequality:

$$\begin{aligned}
&\int_{x_0}^{y_0} G^{x}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big)g'(x)f(x)\,dx \\
&\quad\leq G^{x_0}_{y_0}\big(E^{\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi},g;\Psi\big) \\
&\qquad\times\left(f(y_0)\int_{x_0}^{y_0} g'(x)\,dx + \frac{\varphi(f(x_0),f(y_0))}{y_0-x_0}\int_{x_0}^{y_0} g'(x)(y_0-x)\,dx\right).
\end{aligned}$$

Using (1.12) of Definition 6 on the left-hand side, integrating the right-hand side, and using $\varphi(x,y)=x+y$, this inequality gives

$$\big({}_gF^{\Psi,\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\big)(y_0;p)$$

$$\begin{aligned}&\leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\big(\eta\big(g(y_0)-g(x_0)\big)^{\mu};p\big)\Psi\big(g(y_0)-g(x_0)\big)f(y_0)\\&\quad+G_{y_0}^{x_0}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)\big(f(x_0)+f(y_0)\big)\big(I(x_0,y_0;g)-g(x_0)\big).\end{aligned}\tag{2.18}$$

Now from (2.12) of Lemma 1 we can write

$$\begin{aligned}&\int_{x_0}^{y_0}f\left(\frac{x_0+y_0}{2}\right)G_x^{x_0}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)g'(x)\,dx\\&\quad\leq\int_{x_0}^{y_0}G_x^{x_0}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)g'(x)f(x)\,dx\\&\qquad+\frac{1}{2}\int_{x_0}^{y_0}G_x^{x_0}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}g;\Psi\big)g'(x)\varphi\big(f(x),f(x)\big)\,dx.\end{aligned}$$

Using (1.13) of Definition 6 and $\varphi(x,y)=x+y$, we get

$$f\left(\frac{x_0+y_0}{2}\right)\big({}_gF_{\mu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}1\big)(x_0;p)\leq 2\big({}_gF_{\mu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}f\big)(x_0;p).\tag{2.19}$$

Again using (2.12) of Lemma 1, we can write

$$\begin{aligned}&\int_{x_0}^{y_0}f\left(\frac{x_0+y_0}{2}\right)G_{y_0}^{x}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)g'(x)\,dx\\&\quad\leq\int_{x_0}^{y_0}G_{y_0}^{x}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)g'(x)f(x)\,dx\\&\qquad+\frac{1}{2}\int_{x_0}^{y_0}G_{y_0}^{x}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)g'(x)\varphi\big(f(x),f(x)\big)\,dx.\end{aligned}\tag{2.20}$$

Using (1.12) of Definition 6 and $\varphi(x,y)=x+y$, we get

$$f\left(\frac{x_0+y_0}{2}\right)\big({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta}1\big)(y_0;p)\leq 2\big({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta}f\big)(y_0;p).\tag{2.21}$$

We obtain inequality (2.13) by using (2.16), (2.18), (2.19), and (2.21). □

Remark 3 By setting $\varphi(x,y)=x-y$ in Theorem 5 we get (1.16) of Theorem 2.

Theorem 6 *Let $f,g:[x_0,y_0]\longrightarrow\mathbb{R}$ be two differentiable functions such that $|f'|$ is φ-convex and g is strictly increasing for $0<x_0<y_0$. Also, let $\frac{\Psi}{x}$ be an increasing function on $[x_0,y_0]$, and let $\alpha,\xi,\gamma,\zeta\in\mathbb{C}$, $p,\mu,\nu,\delta\geq 0$, $0<k\leq\delta+\mu$, and $0<k\leq\delta+\nu$. Then for $x\in(x_0,y_0)$, we have*

$$\begin{aligned}&\big|\big({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta}f*g\big)(x;p)+\big({}_gF_{\nu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}f*g\big)(x;p)\big|\\&\quad\leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\big(\eta\big(g(x)-g(x_0)\big)^{\mu};p\big)\Psi\big(g(x)-g(x_0)\big)\big|f'(x)\big|\\&\qquad+G_x^{x_0}\big(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)\varphi\big(\big|f'(x_0)\big|,\big|f'(x)\big|\big)\big(I(x_0,x;g)-g(x_0)\big)\\&\qquad+E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\big(\eta\big(g(y_0)-g(x)\big)^{\nu};p\big)\Psi\big(g(y_0)-g(x)\big)\big|f'(y_0)\big|\\&\qquad+G_{y_0}^{x}\big(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta},g;\Psi\big)\varphi\big(\big|f'(x)\big|,\big|f'(y_0)\big|\big)\big(I(x,y_0;g)-g(x)\big),\end{aligned}\tag{2.22}$$

where $({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta}f*g)(x;p)$ and $({}_gF_{\nu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta}f*g)(x;p)$ are as in (1.18) and (1.19).

Proof The φ-convexity of $|f'|$ over $[x_0, y_0]$ implies

$$\left|f'(t)\right| \leq \left|f'(x)\right| + \frac{x-t}{x-x_0}\varphi\left(\left|f'(x_0)\right|, \left|f'(x)\right|\right), \quad t \in [x_0, x]. \tag{2.23}$$

By the property of absolute values we can write

$$\begin{aligned} &-\left(\left|f'(x)\right| + \frac{x-t}{x-x_0}\varphi\left(\left|f'(x_0)\right|, \left|f'(x)\right|\right)\right) \\ &\quad \leq f'(t) \leq \left(\left|f'(x)\right| + \frac{x-t}{x-x_0}\varphi\left(\left|f'(x_0)\right|, \left|f'(x)\right|\right)\right). \end{aligned} \tag{2.24}$$

Inequality (2.4) and the second inequality of (2.24) constitute the following inequality:

$$\begin{aligned} &\int_{x_0}^{x} G_x^t\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) g'(t) f'(t)\,dt \\ &\quad \leq \left|f'(x)\right| G_x^{x_0}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) \int_{x_0}^{x} g'(t)\,dt \\ &\qquad + \frac{\varphi(|f'(x_0)|, |f'(x)|)}{x-x_0} G_x^{x_0}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) \int_{x_0}^{x} (x-t) g'(t)\,dt, \end{aligned}$$

from which we get

$$\begin{aligned} &\left({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta} f * g\right)(x;p) \\ &\quad \leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(x)-g(x_0)\right)^{\mu}; p\right) \Psi\left(g(x)-g(x_0)\right) \left|f'(x)\right| \\ &\qquad + G_x^{x_0}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) \varphi\left(\left|f'(x_0)\right|, \left|f'(x)\right|\right)\left(I(x_0, x; g) - g(x_0)\right). \end{aligned} \tag{2.25}$$

Further, inequality (2.4) and the first inequality of (2.24) produce the following inequality:

$$\begin{aligned} &\left({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta} f * g\right)(x;p) \\ &\quad \geq -E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(x)-g(x_0)\right)^{\mu}; p\right) \Psi\left(g(x)-g(x_0)\right) \left|f'(x)\right| \\ &\qquad - G_x^{x_0}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) \varphi\left(\left|f'(x_0)\right|, \left|f'(x)\right|\right)\left(I(x_0, x; g) - g(x_0)\right). \end{aligned} \tag{2.26}$$

Now using the φ-convexity of $|f'|$ over $[x_0, y_0]$, we have

$$\left|f'(t)\right| \leq \left|f'(y_0)\right| + \frac{y_0-t}{y_0-x}\varphi\left(\left|f'(x)\right|, \left|f'(y_0)\right|\right), \quad t \in (x, y_0]. \tag{2.27}$$

Also, we have

$$G_t^x\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) g'(t) \leq G_{y_0}^x\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right) g'(t). \tag{2.28}$$

Proceeding as before, we obtain

$$\begin{aligned} &\left({}_gF_{\nu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta} f * g\right)(x;p) \\ &\quad \leq E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(y_0)-g(x)\right)^{\nu}; p\right) \Psi\left(g(y_0)-g(x)\right) \left|f'(y_0)\right| \end{aligned}$$

$$+ G_{y_0}^{x}\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\varphi\left(\left|f'(x)\right|, \left|f'(y_0)\right|\right)\left(I(x,y_0;g) - g(x)\right) \tag{2.29}$$

and

$$\begin{aligned}
&\left({}_gF_{\nu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta} f * g\right)(x;p) \\
&\quad \geq -E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(y_0) - g(x)\right)^{\nu}; p\right)\Psi\left(g(y_0) - g(x)\right)\left|f'(y_0)\right| \\
&\qquad - G_{y_0}^{x}\left(E_{\nu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\varphi\left(\left|f'(x)\right|, \left|f'(y_0)\right|\right)\left(I(x,y_0;g) - g(x)\right).
\end{aligned} \tag{2.30}$$

We obtain inequality (2.22) by using (2.25), (2.26), (2.29), and (2.30). □

Corollary 2 *By setting $\mu = \nu$ in (2.22) we get the following inequality*:

$$\begin{aligned}
&\left|\left({}_gF_{\mu,\alpha,\xi,x_0^+}^{\Psi,\gamma,\delta,k,\zeta} f * g\right)(x;p) + \left({}_gF_{\mu,\alpha,\xi,y_0^-}^{\Psi,\gamma,\delta,k,\zeta} f * g\right)(x;p)\right| \\
&\quad \leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(x) - g(x_0)\right)^{\mu}; p\right)\Psi\left(g(x) - g(x_0)\right)\left|f'(x)\right| \\
&\qquad + G_{x}^{x_0}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi\right)\varphi\left(\left|f'(x_0)\right|, \left|f'(x)\right|\right)\left(I(x_0,x;g) - g(x)\right) \\
&\qquad + E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}\left(\eta\left(g(y_0) - g(x)\right)^{\mu}; p\right)\Psi\left(g(y_0) - g(x)\right)\left|f'(y_0)\right| \\
&\qquad + G_{y_0}^{x}\left(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,c}, g; \Psi\right)\varphi\left(\left|f'(x)\right|, \left|f'(y_0)\right|\right)\left(I(x,y_0;g) - g(x)\right).
\end{aligned} \tag{2.31}$$

Remark 4 For $\varphi(x,y) = x - y$ in (2.31), we get inequality (1.17) of Theorem 3.

3 Results for fractional and conformable integral operators

In this section, we give bounds of some fractional and conformable fractional integral operators deduced from the results of Sect. 2.

Proposition 1 *Under the assumptions of Theorem 4, we have*

$$\begin{aligned}
&\Gamma(\alpha)\left(\left({}_g^{\alpha}I_{x_0^+}f\right)(x) + \left({}_g^{\alpha}I_{y_0^-}f\right)(x)\right) \\
&\quad \leq \left(g(x) - g(x_0)\right)^{\alpha} f(x) + \left(g(y_0) - g(x)\right)^{\alpha} f(y_0) \\
&\qquad + \left(g(x) - g(x_0)\right)^{\alpha-1}\varphi\left(f(x_0), f(x)\right)\left(I(x_0,x;g) - g(x_0)\right) \\
&\qquad + \left(g(y_0) - g(x)\right)^{\alpha-1}\varphi\left(f(x), f(y_0)\right)\left(I(x,y_0;g) - g(x)\right),
\end{aligned} \tag{3.1}$$

where $({}_g^{\alpha}I_{x_0^+}f)(x)$ and $({}_g^{\alpha}I_{y_0^-}f)(x)$ are defined in [15].

Proof For $\Psi(t) = t^{\alpha}, \alpha > 0$, and $p = \eta = 0$ with $\mu = \nu$ in the proof of Theorem 4, bound (3.1) is satisfied. □

For $\varphi(x,y) = x - y$ in (3.1), we get [16, Proposition 1].

Proposition 2 *Under the assumptions of Theorem 4, we have*

$$\begin{aligned}
&({}_{x_0^+}I_{\Psi}f)(x) + ({}_{y_0^-}I_{\Psi}f)(x) \\
&\quad \leq \Psi(x - x_0)f(x) + \frac{\Psi(x - x_0)}{2}\varphi\left(f(x_0), f(x)\right) + \Psi(y_0 - x)f(y_0)
\end{aligned} \tag{3.2}$$

$$+\frac{\Psi(y_0-x)}{2}\varphi\big(f(x),f(y_0)\big),$$

where $({}_{x_0^+}I_\Psi f)(x)$ *and* $({}_{y_0^-}I_\Psi f)(x)$ *are defined in* [23].

Proof Using $g=I$, $\eta=p=0$, and $\mu=\nu$ in the proof of Theorem 4, bound (3.2) is satisfied. □

For $\varphi(x,y)=x-y$ in (3.2), we get [16, Proposition 2].

Corollary 3 *For* $\Psi(t)=\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $p=\eta=0$, (1.12) *and* (1.13) *reduce to fractional integrals* (1.7) *and* (1.8). *Further, the following bound for* $\alpha\geq k$ *is also satisfied*:

$$\begin{aligned}&\big({}_g^\alpha I_{x_0^+}^k f\big)(x)+\big({}_g^\alpha I_{y_0^-}^k f\big)(x)\\&\quad\leq\frac{1}{k\Gamma_k(\alpha)}\Big[\big(g(x)-g(x_0)\big)^{\frac{\alpha}{k}}f(x)+\big(g(y_0)-g(x)\big)^{\frac{\alpha}{k}}f(y_0)\\&\qquad+\big(g(x)-g(x_0)\big)^{\frac{\alpha}{k}-1}\varphi\big(f(x_0),f(x)\big)\big(I(x_0,x;g)-g(x_0)\big)\\&\qquad+\big(g(y_0)-g(x)\big)^{\frac{\alpha}{k}-1}\varphi\big(f(x),f(y_0)\big)\big(I(x,y_0;g)-g(x)\big)\Big].\end{aligned}$$

For $\varphi(x,y)=x-y$ in this inequality, we get [16, Corollary 12].

Corollary 4 *Using* $\Psi(t)=t^\alpha$ *for* $\alpha\geq1$ *and* $g=I$ *along with* $p=\eta=0$, (1.12) *and* (1.13) *give the fractional integral operators* ${}^\alpha I_{x_0^+}f(x)$ *and* ${}^\alpha I_{y_0^-}f(x)$ *defined in* [15].

Moreover, the following bound is also satisfied:

$$\begin{aligned}&\Gamma(\alpha)\big(\big({}^\alpha I_{x_0^+}f\big)(x)+\big({}^\alpha I_{y_0^-}f\big)(x)\big)\\&\quad\leq\frac{(x-x_0)^\alpha}{2}\big(2f(x)+\varphi\big(f(x_0),f(x)\big)\big)+\frac{(y_0-x)^\alpha}{2}\big(2f(y_0)+\varphi\big(f(x),f(y_0)\big)\big).\end{aligned}$$

For $\varphi(x,y)=x-y$ in this inequality, we get [16, Corollary 13].

Corollary 5 *Using* $\Psi(t)=\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $g=I$ *along with* $p=\eta=0$, (1.12) *and* (1.13) *reduce to the following fractional integral operators given in* [19]:

$$\Big({}_gF_{\mu,\alpha,\xi,x_0^+}^{\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,\zeta}f\Big)(x;0):={}^\alpha I_{x_0^+}^k f(x),\qquad\Big({}_gF_{\mu,\alpha,\xi,y_0^-}^{\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,\zeta}f\Big)(x;0):={}^\alpha I_{y_0^-}^k f(x).$$

Moreover, the following bound is also satisfied for $\alpha\geq k$:

$$\begin{aligned}&\big({}^\alpha I_{x_0^+}^k f\big)(x)+\big({}^\alpha I_{y_0^-}^k f\big)(x)\\&\quad\leq\frac{1}{k\Gamma_k(\alpha)}\bigg[\frac{(x-x_0)^{\frac{\alpha}{k}}}{2}\big(2f(x)+\varphi\big(f(x_0),f(x)\big)\big)+\frac{(y_0-x)^{\frac{\alpha}{k}}}{2}\big(2f(y_0)+\varphi\big(f(x),f(y_0)\big)\big)\bigg].\end{aligned}$$

For $\varphi(x,y)=x-y$ in this inequality, we get [16, Corollary 14].

Corollary 6 *Using $\Psi(t) = t^{\alpha}, \alpha > 0$, and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$, in* (1.12) *and* (1.13), *respectively, with $p = \eta = 0$, we obtain the following fractional integral operators given in* [4]:

$$\left({}_{g}F^{t^{\alpha},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;0) = \left({}^{\rho}I^{\alpha}_{x_0^+}f\right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x_0}^{x}\left(x^{\rho}-t^{\rho}\right)^{\alpha-1}t^{\rho-1}f(t)\,dt, \tag{3.3}$$

$$\left({}_{g}F^{t^{\alpha},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x;0) = \left({}^{\rho}I^{\alpha}_{y_0^-}f\right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{y_0}\left(t^{\rho}-x^{\rho}\right)^{\alpha-1}t^{\rho-1}f(t)\,dt. \tag{3.4}$$

Moreover, the following bound is also satisfied:

$$\begin{aligned}&\left({}^{\rho}I^{\alpha}_{x_0^+}f\right)(x) + \left({}^{\rho}I^{\alpha}_{y_0^-}f\right)(x)\\ &\quad\le \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Bigg[\left(x^{\rho}-x_0^{\rho}\right)^{\alpha}f(x) + \left(y_0^{\rho}-x^{\rho}\right)^{\alpha}f(y_0) + \left(x^{\rho}-x_0^{\rho}\right)^{\alpha-1}\varphi\big(f(x_0),f(x)\big)\\ &\qquad\times\left(\frac{x^{\rho+1}-x_0^{\rho+1}}{(x-x_0)(\rho+1)}-x_0^{\rho}\right) + \left(y_0^{\rho}-x^{\rho}\right)^{\alpha-1}\varphi\big(f(x),f(y_0)\big)\left(\frac{y_0^{\rho+1}-x^{\rho+1}}{(y_0-x)(\rho+1)}-x^{\rho}\right)\Bigg].\end{aligned}$$

For $\varphi(x,y) = x - y$ in this inequality, we get [16, Corollary 15].

Corollary 7 *Using $\Psi(t) = t^{\alpha}, \alpha > 0$, and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0$, in* (1.12) *and* (1.13), *respectively, with $p = \eta = 0$, we obtain the following fractional integral operators*:

$$\begin{aligned}&\left({}_{g}F^{t^{\alpha},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;0)\\ &\quad= \left({}^{s}I^{\alpha}_{x_0^+}f\right)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{x_0}^{x}\left(x^{s+1}-t^{s+1}\right)^{\alpha-1}t^{s}f(t)\,dt,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\left({}_{g}F^{t^{\alpha},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x;0)\\ &\quad= \left({}^{s}I^{\alpha}_{y_0^-}f\right)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{y_0}\left(t^{s+1}-x^{s+1}\right)^{\alpha-1}t^{s}f(t)\,dt.\end{aligned} \tag{3.6}$$

Moreover, the following bound is also satisfied:

$$\begin{aligned}&\left({}^{s}I^{\alpha}_{x_0^+}f\right)(x) + \left({}^{s}I^{\alpha}_{y_0^-}f\right)(x)\\ &\quad\le \frac{1}{(s+1)^{\alpha}\Gamma(\alpha)}\Bigg[\left(x^{s+1}-x_0^{s+1}\right)^{\alpha}f(x) + \left(x^{s+1}-x_0^{s+1}\right)^{\alpha-1}\varphi\big(f(x_0),f(x)\big)\\ &\qquad\times\left(\frac{x^{s+2}-x_0^{s+2}}{(x-x_0)(s+2)}-x_0^{s+1}\right) + \left(y_0^{s+1}-x^{s+1}\right)^{\alpha}f(y_0) + \left(y_0^{s+1}-x^{s+1}\right)^{\alpha-1}\varphi\big(f(x),f(y_0)\big)\\ &\qquad\times\left(\frac{y_0^{s+2}-x^{s+2}}{(y_0-x)(s+2)}-x^{s+1}\right)\Bigg].\end{aligned}$$

For $\varphi(x,y) = x - y$ in this inequality, we get [16, Corollary 16].

Corollary 8 *Using $\Psi(t) = \frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0$, in* (1.12) *and* (1.13), *respectively, with $p = \eta = 0$, we obtain the following fractional integral operators given in* [22]:

$$\left({}_{g}F^{\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;0)$$

$$= \left({}_{k}^{s}I_{x_0^+}^{\alpha}f\right)(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{x_0}^{x}\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s f(t)\,dt, \tag{3.7}$$

$$\left({}_{g}F_{\mu,\alpha,\xi,y_0^-}^{\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,\zeta}f\right)(x;0)$$
$$= \left({}_{k}^{s}I_{y_0^-}^{\alpha}f\right)(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{x}^{y_0}\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}t^s f(t)\,dt. \tag{3.8}$$

Moreover, the following bound is also satisfied:

$$\begin{aligned}
&\left({}_{k}^{s}I_{x_0^+}^{\alpha}f\right)(x) + \left({}_{k}^{s}I_{y_0^-}^{\alpha}f\right)(x) \\
&\quad\leq \frac{1}{(s+1)^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\Bigg[\left(x^{s+1}-x_0^{s+1}\right)^{\frac{\alpha}{k}}f(x) + \left(x^{s+1}-x_0^{s+1}\right)^{\frac{\alpha}{k}-1}\varphi\big(f(x_0),f(x)\big) \\
&\qquad\times\left(\frac{x^{s+2}-x_0^{s+2}}{(x-x_0)(s+2)}-x_0^{s+1}\right) + \left(y_0^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}}f(y_0) + \left(y_0^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}\varphi\big(f(x),f(y_0)\big) \\
&\qquad\times\left(\frac{y_0^{s+2}-x^{s+2}}{(y_0-x)(s+2)}-x^{s+1}\right)\Bigg].
\end{aligned}$$

For $\varphi(x,y) = x-y$ in this inequality, we get [16, Corollary 17].

Corollary 9 *Using $\Psi(t) = t^{\alpha}, \alpha > 0$, and $g(x) = \frac{x^{\beta+s}}{\beta+s}$, $\beta, s > 0$, in* (1.12) *and* (1.13),) *respectively, with $p = \eta = 0$, we obtain the following fractional integral operators given in* [12]:

$$\left({}_{g}F_{\mu,\alpha,\xi,x_0^+}^{t^{\alpha},\gamma,\delta,k,\zeta}f\right)(x;0)$$
$$= \left({}_{\beta}^{s}I_{x_0^+}^{\alpha}f\right)(x) = \frac{(\beta+s)^{1-\alpha}}{\Gamma(\alpha)}\int_{x_0}^{x}\left(x^{\beta+s}-t^{\beta+s}\right)^{\alpha-1}t^s f(t)\,dt, \tag{3.9}$$

$$\left({}_{g}F_{\mu,\alpha,\xi,y_0^-}^{t^{\alpha},\gamma,\delta,k,\zeta}f\right)(x;0)$$
$$= \left({}_{\beta}^{s}I_{y_0^-}^{\alpha}f\right)(x) = \frac{(\beta+s)^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{y_0}\left(t^{\beta+s}-x^{\beta+s}\right)^{\alpha-1}t^s f(t)\,dt. \tag{3.10}$$

Moreover, the following bound is also satisfied:

$$\begin{aligned}
&\left({}_{\beta}^{s}I_{x_0^+}^{\alpha}f\right)(x) + \left({}_{\beta}^{s}I_{y_0^-}^{\alpha}f\right)(x) \\
&\quad\leq \frac{1}{(\beta+s)^{\alpha}\Gamma(\alpha)}\Bigg[\left(x^{\beta+s}-x_0^{\beta+s}\right)^{\alpha}f(x) + \left(x^{\beta+s}-x_0^{\beta+s}\right)^{\alpha-1}\varphi\big(f(x_0),f(x)\big) \\
&\qquad\times\left(\frac{x^{\beta+s+1}-x_0^{\beta+s+1}}{(x-x_0)(\beta+s+1)}-x_0^{\beta+s}\right) + \left(y_0^{\beta+s}-x^{\beta+s}\right)^{\alpha-1}\varphi\big(f(x),f(y_0)\big) \\
&\qquad\times\left(\frac{y_0^{\beta+s+1}-x^{\beta+s+1}}{(y_0-x)(\beta+s+1)}-x^{\beta+s}\right) + \left(y_0^{\beta+s}-x^{\beta+s}\right)^{\alpha}f(y_0)\Bigg].
\end{aligned}$$

For $\varphi(x,y) = x-y$ in this inequality, we get [16, Corollary 18].

Corollary 10 *Using $\Psi(t) = t^{\alpha}, \alpha > 0$, and $g(x) = \frac{(x-x_0)^{\rho}}{\rho}$ in* (1.12) *and $g(x) = \frac{-(y_0-x)^{\rho}}{\rho}$ in* (1.13), *where $\rho > 0$ with $p = \eta = 0$, we obtain the following fractional integral operators* *given in* [10]:

$$\begin{aligned}
&\left({}_{g}F^{t^{\alpha},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;0)\\
&\quad=\left({}^{\rho}I^{\alpha}_{x_0^+}f\right)(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x_0}^{x}\bigl((x-x_0)^{\rho}-(t-x_0)^{\rho}\bigr)^{\alpha-1}(t-x_0)^{\rho-1}f(t)\,dt, \qquad (3.11)\\
&\left({}_{g}F^{t^{\alpha},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x;0)\\
&\quad=\left({}^{\rho}I^{\alpha}_{y_0^-}f\right)(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{y_0}\bigl((y_0-x)^{\rho}-(y_0-t)^{\rho}\bigr)^{\alpha-1}(y_0-t)^{\rho-1}f(t)\,dt. \qquad (3.12)
\end{aligned}$$

Moreover, the following bound is also satisfied:

$$\begin{aligned}
&\left({}^{\rho}I^{\alpha}_{x_0^+}f\right)(x)+\left({}^{\rho}I^{\alpha}_{y_0^-}f\right)(x)\\
&\quad\le\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\biggl[(x-x_0)^{\rho\alpha}f(x)+\varphi\bigl(f(x_0),f(x)\bigr)\frac{(x-x_0)^{\rho\alpha}}{\rho+1}\\
&\qquad+(y_0-x)^{\rho\alpha}f(y_0)+\varphi\bigl(f(x),f(y_0)\bigr)\frac{\rho(y_0-x)^{\rho\alpha}}{\rho+1}\biggr].
\end{aligned}$$

For $\varphi(x,y)=x-y$ in this inequality, we get [16, Corollary 19].

Corollary 11 *For* $\Psi(t)=\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha>k$, *and* $g(x)=\frac{(x-x_0)^{\rho}}{\rho}$ *in* (1.12) *and* $g(x)=\frac{-(y_0-x)^{\rho}}{\rho}$ *in* (1.13), *where* $\rho>0$ *with* $p=\eta=0$, *we obtain the following fractional integral operators given in* [8]:

$$\begin{aligned}
&\left({}_{g}F^{\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,x_0^+}f\right)(x;0)\\
&\quad=\left({}^{\rho}_{k}I^{\alpha}_{x_0^+}f\right)(x)=\frac{\rho^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{x_0}^{x}\bigl((x-x_0)^{\rho}-(t-x_0)^{\rho}\bigr)^{\frac{\alpha}{k}-1}(t-x_0)^{\rho-1}f(t)\,dt, \qquad (3.13)\\
&\left({}_{g}F^{\frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,\zeta}_{\mu,\alpha,\xi,y_0^-}f\right)(x;0)\\
&\quad=\left({}^{\rho}_{k}I^{\alpha}_{y_0^-}f\right)(x)=\frac{\rho^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{x}^{y_0}\bigl((y_0-x)^{\rho}-(y_0-t)^{\rho}\bigr)^{\frac{\alpha}{k}-1}(y_0-t)^{\rho-1}f(t)\,dt. \qquad (3.14)
\end{aligned}$$

Moreover, the following bound is also satisfied:

$$\begin{aligned}
&\left({}^{\rho}_{k}I^{\alpha}_{x_0^+}f\right)(x)+\left({}^{\rho}_{k}I^{\alpha}_{y_0^-}f\right)(x)\\
&\quad\le\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\biggl[(x-x_0)^{\frac{\rho\alpha}{k}}f(x)+\varphi\bigl(f(x_0),f(x)\bigr)\frac{(x-x_0)^{\frac{\rho\alpha}{k}}}{\rho+1}\\
&\qquad+(y_0-x)^{\frac{\rho\alpha}{k}}f(y_0)+\varphi\bigl(f(x),f(y_0)\bigr)\frac{\rho(y_0-x)^{\frac{\rho\alpha}{k}}}{\rho+1}\biggr].
\end{aligned}$$

For $\varphi(x,y)=x-y$ in this inequality, we get [16, Corollary 20].

By applying Theorems 5 and 6 we can obtain results for fractional and conformable fractional integral operators associated with the unified integral operators, which we leave for the reader.

4 Concluding remarks

In this paper, we study the unified integral operators (1.12) and (1.13) for the notion of φ-convex functions. For φ-convex functions, we investigated bounds of these operators in different forms, which lead to bounds of several known fractional and conformable fractional integral operators. We identified some results for fractional integral operators in Sect. 3. Also, we identified connections with the known results.

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