Mlaiki et al. *Advances in Difference Equations* (2020) 2020:289
https://doi.org/10.1186/s13662-020-02752-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Fredholm type integral equation in extended M_b-metric spaces

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Abstract

In this manuscript, we prove some new fixed point results for a self-mapping on extended M_b-metric spaces, under some new types of contractions, which generalizes many results in the literature. Also, we present some interesting examples to illustrate our work.

MSC: 47H10; 54H25

Keywords: Fredholm type integral equation; Extended M_b-metric spaces; Fixed point

1 Introduction and preliminaries

Lately, fixed point theory has become the focus of many researchers and that is due to its applications in many fields, such as engineering and computer sciences. Also fixed point theory can be used to solve differential equations along with integral equations [1–14]. M-metric spaces were introduced by Asadi, Karapinar and Salimi, in [15], they are an extension of a partial metric space. Then some relationships between a partial metric and an M-metric were investigated in [16]. So, first we remind the reader of the definition of a partial metric space and an M-metric space along with some other notations.

Definition 1.1 ([17, 18]) A partial metric on a nonempty set X is a function $p_i : X^2 \to [0,+\infty)$ such that for all $\lambda, \epsilon, z \in X$

(p_i1) $p_i(\lambda,\lambda) = p_i(\epsilon,\epsilon) = p_i(\lambda,\epsilon)$ if and only if $\lambda = \epsilon$,
(p_i2) $p_i(\lambda,\lambda) \le p_i(\lambda,\epsilon)$,
(p_i3) $p_i(\lambda,\epsilon) = p_i(\epsilon,\lambda)$,
(p_i4) $p_i(\lambda,\epsilon) \le p_i(\lambda,z) + p_i(z,\epsilon) - p_i(z,z)$.

A partial metric space is a pair (X,p_i) such that X is a nonempty set and p_i is a partial metric on X.

Notation 1.2 ([15])

1. $k_{\lambda,\epsilon} := \min\{K(\lambda,\lambda),K(\epsilon,\epsilon)\}$.
2. $M_{\lambda,\epsilon} := \max\{K(\lambda,\lambda),K(\epsilon,\epsilon)\}$.

Definition 1.3 ([15]) Let X be a nonempty set. If the function $K : X^2 \to [0, +\infty)$ satisfies the following conditions:

(1) $K(\lambda, \lambda) = K(\epsilon, \epsilon) = K(\lambda, \epsilon)$ if and only if $\lambda = \epsilon$,

(2) $k_{\lambda,\epsilon} \leq K(\lambda, \epsilon)$,

(3) $K(\lambda, \epsilon) = K(\epsilon, \lambda)$,

(4) $(K(\lambda, \epsilon) - k_{\lambda,\epsilon}) \leq (K(\lambda, z) - k_{\lambda,z}) + (K(z, \epsilon) - k_{z,\epsilon})$,

for all $\lambda, \epsilon, z \in X$, then the pair (X, K) is called an M-metric space.

Recently, Mlaiki et al. [19], developed the concept of an M_b-metric space which extends an M-metric space, and some fixed point theorems are established. Also, M_b-metric spaces are a generalization of b-metric spaces; see [20–22]. Now, we remind the reader of some definitions and notations of M_b-metric spaces.

Notation 1.4 ([19])

1. $k_{b\lambda,\epsilon} := \min\{K_b(\lambda, \lambda), K_b(\epsilon, \epsilon)\}$.
2. $M_{b\lambda,\epsilon} := ma\lambda\{K_b(\lambda, \lambda), K_b(\epsilon, \epsilon)\}$.

Definition 1.5 ([19]) An M_b-metric on a nonempty set X is a function $K_b : X^2 \to [0, +\infty)$ that satisfies the following conditions:

(1) $K_b(\lambda, \lambda) = K_b(\epsilon, \epsilon) = K_b(\lambda, \epsilon)$ if and only if $\lambda = \epsilon$,

(2) $k_{b\lambda,\epsilon} \leq K_b(\lambda, \epsilon)$,

(3) $K_b(\lambda, \epsilon) = K_b(\epsilon, \lambda)$,

(4) there exists a real number $s \geq 1$ such that for all $\lambda, \epsilon, z \in X$ we have

$$\bigl(K_b(\lambda, \epsilon) - k_{b\lambda,\epsilon}\bigr) \leq s\bigl[\bigl(K_b(\lambda, z) - k_{b\lambda,z}\bigr) + \bigl(K_b(z, \epsilon) - k_{bz,\epsilon}\bigr)\bigr] - K_b(z, z),$$

for all $\lambda, \epsilon, z \in X$. Then the pair (X, K_b) is called an M_b-metric space and the number s is called the coefficient of the M_b-metric space (X, K_b).

Note that the condition (4) given in Definition 1.5 is equivalent to the following condition:

(4)′ There exists a real number $s \geq 1$ such that for all $\lambda, \epsilon, z \in X$ we have

$$\bigl(K_b(\lambda, \epsilon) - k_{b\lambda,\epsilon}\bigr) \leq s\bigl[\bigl(K_b(\lambda, z) - k_{b\lambda,z}\bigr) + \bigl(K_b(z, \epsilon) - k_{bz,\epsilon}\bigr)\bigr],$$

for all $\lambda, \epsilon, z \in X$.

Indeed, if we take $\lambda = r$ under the condition (4) then we get

$$K_b(\lambda, \lambda) - k_{b\lambda,\lambda} = K_b(\lambda, \lambda) - \min\bigl\{K_b(\lambda, \lambda), K_b(\lambda, \lambda)\bigr\} = 0$$

and so we have

$$0 \leq s\bigl[\bigl(K_b(\lambda, \lambda) - k_{b\lambda,\lambda}\bigr) + \bigl(K_b(\lambda, \lambda) - k_{b\lambda,\lambda}\bigr)\bigr] - K_b(\lambda, \lambda) \leq -K_b(\lambda, \lambda),$$

for $z = \lambda$. Therefore we get $K_b(\lambda, \lambda) = 0$ for all $\lambda \in X$ since $K_b(\lambda, \lambda) \in [0, +\infty)$.

The concept of extended M_b-metric spaces was introduced in [23], which is a generalization of an M_b-metric space which also generalizes extended b-metric spaces [3]. We give basic properties of this new space and its relation with some known metric spaces.

First, we give the following notation.

Notation 1.6
(1) $k_{\alpha\lambda,\epsilon} := \min\{K_\alpha(\lambda,\lambda), K_\alpha(\epsilon,\epsilon)\}$.
(2) $M_{\alpha\lambda,\epsilon} := \max\{K_\alpha(\lambda,\lambda), K_\alpha(\epsilon,\epsilon)\}$.

Definition 1.7 Let $\alpha : X^2 \to [1,+\infty)$ be a function. An extended M_b-metric on a nonempty set X is a function $K_\alpha : X^2 \to [0,+\infty)$ satisfying the following conditions:
(1) $K_\alpha(\lambda,\lambda) = K_\alpha(\epsilon,\epsilon) = K_\alpha(\lambda,\epsilon)$ if and only if $\lambda = \epsilon$,
(2) $k_{\alpha\lambda,\epsilon} \le K_\alpha(\lambda,\epsilon)$,
(3) $K_\alpha(\lambda,\epsilon) = K_\alpha(\epsilon,\lambda)$,
(4) $(K_\alpha(\lambda,\epsilon) - k_{\alpha\lambda,\epsilon}) \le \alpha(\lambda,\epsilon)[(K_\alpha(\lambda,z) - k_{\alpha\lambda,z}) + (K_\alpha(z,\epsilon) - k_{\alpha z,\epsilon})]$,
for all $\lambda,\epsilon,z \in X$. Then the pair (X,K_α) is called an extended M_b-metric space.

We note that if $\alpha(\lambda,\epsilon) = s$ for $s \ge 1$, then we get the definition of an M_b-metric space.

Example 1.8 Let $X = C([a,d],\mathbb{R})$ be the set of all continuous real valued functions on $[a,b]$. We define the functions $K_\alpha : X^2 \to [0,+\infty)$ and $\alpha : X^2 \to [1,+\infty)$ by

$$K_\alpha\bigl(\lambda(t),\epsilon(t)\bigr) = \sup_{t\in[a,b]} \bigl|\lambda(t) - y(t)\bigr|^2$$

and

$$\alpha\bigl(\lambda(t),\epsilon(t)\bigr) = \bigl|\lambda(t)\bigr| + \bigl|y(t)\bigr| + 2.$$

Then (X,K_α) is an extended M_b-metric space with the function α.

Now we give the following proposition.

Proposition 1.9 *Let (X,K_α) be an extended M_b-metric space and $\lambda,\epsilon,z \in X$. Then we have*
(1) $M_{\alpha\lambda,\epsilon} + k_{\alpha\lambda,\epsilon} = K_\alpha(\lambda,\lambda) + K_\alpha(\epsilon,\epsilon) \ge 0$,
(2) $M_{\alpha\lambda,\epsilon} - k_{\alpha\lambda,\epsilon} = |K_\alpha(\lambda,\lambda) - K_\alpha(\epsilon,\epsilon)| \ge 0$,
(3) $M_{\alpha\lambda,\epsilon} - k_{\alpha\lambda,\epsilon} \le \alpha(\lambda,\epsilon)[(M_{\alpha\lambda,z} - k_{\alpha\lambda,z}) + (M_{\alpha z,\epsilon} - k_{\alpha z,\epsilon})]$.

In this section, we give some topological notions on an extended M_b-metric space.

Definition 1.10 Let (X,K_α) be an extended M_b-metric space. Then:
(1) A sequence $\{\lambda_n\}$ in X converges to a point λ if and only if

$$\lim_{n\to+\infty}\bigl(K_\alpha(\lambda_n,\lambda) - k_{\alpha\lambda_n,\lambda}\bigr) = 0.$$

(2) A sequence $\{\lambda_n\}$ in X is said to be a K_α-Cauchy sequence if

$$\lim_{n,m\to+\infty}\bigl(K_\alpha(\lambda_n,\lambda_m) - k_{\alpha\lambda_n,\lambda_m}\bigr)$$

and

$$\lim_{n\to+\infty}(M_{\alpha\lambda_n,\lambda_m} - k_{\alpha\lambda_n,\lambda_m})$$

exist and are finite.

(3) An extended M_b-metric space is said to be K_α-complete if every K_α-Cauchy sequence $\{\lambda_n\}$ converges to a point λ such that

$$\lim_{n\to+\infty}\left(K_\alpha(\lambda_n,\lambda) - k_{\alpha\lambda_n,\lambda}\right) = 0$$

and

$$\lim_{n\to+\infty}(M_{\alpha\lambda_n,\lambda} - k_{\alpha\lambda_n,\lambda}) = 0.$$

Remark 1.11 If we consider Example 1.8, then it is not difficult to see that (X, K_α) is a complete extended M_b-metric space.

Lemma 1.12 *Let (X, K_α) be an extended M_b-metric space. Then we get:*

(1) *$\{\lambda_n\}$ is an K_α-Cauchy sequence in (X, K_α) if and only if $\{\lambda_n\}$ is a Cauchy sequence in (X, K_α^b).*

(2) *(X, K_α) is complete if and only if (X, K_α^b) is complete.*

2 Main result

First, we start this section by proving the following theorem, which we consider our main result.

Theorem 2.1 *Let (X, K_α) be a complete extended M_b-metric space and f be a continuous self-mapping on X. Suppose that there exists $p \in [0, 1)$ such that for all $\lambda, \epsilon \in X$ we have*

$$K_\alpha(f\lambda, f\epsilon) \leq p\alpha(\lambda,\epsilon)K_\alpha(\lambda,\epsilon). \tag{2.1}$$

Also, fix $\lambda_0 \in X$ and define the sequence (λ_n) defined by $\lambda_i = f\lambda_{i-1}$. If

$$\sup_{m\geq 1}\lim_{n}\alpha(\lambda_m,\lambda_n)\alpha(\lambda_n,\lambda_{n+1}) < \frac{1}{p},$$

and for every $\lambda \in X$ we have $\{\alpha(\lambda,\lambda_n)\}_n$ and $\{\alpha(\lambda_n,\lambda)\}_n$ are bounded. Then f has a fixed point on X. Moreover, if for every two fixed points $r, s \in X$ we have $\alpha(r, s) < \frac{1}{p}$, then the fixed point is unique.

Proof Using the sequence as defined in the hypotheses of the theorem and (2.1)

$$\begin{aligned}
K_\alpha(\lambda_n,\lambda_{n+1}) &= K_\alpha(f\lambda_{n-1}, f\lambda_n) \\
&\leq p\alpha(\lambda_{n-1},\lambda_n)K_\alpha(\lambda_{n-1},\lambda_n) \\
&= p\alpha(\lambda_{n-1},\lambda_n)K_\alpha(f\lambda_{n-2},\lambda_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&\leq p^2\alpha(\lambda_{n-1},\lambda_n)\alpha(\lambda_{n-2},\lambda_{n-1})K_\alpha(\lambda_{n-2},\lambda_{n-1})\\
&\ \ \vdots\\
&\leq p^n\prod_{i=1}^{n}\alpha(\lambda_{i-1},\lambda_i)K_\alpha(\lambda_0,\lambda_1).
\end{aligned}
$$

Now, consider $n,m\in\mathbb{N}$ where $m>n$. Then

$$
\begin{aligned}
K_\alpha(\lambda_n,\lambda_m)-k_{\alpha\lambda_n,\lambda_m} &\leq \alpha(\lambda_n,\lambda_m)\big[\big(K_\alpha(\lambda_n,\lambda_{n+1})-k_{\alpha\lambda_n,\lambda_{n+1}}\big)\\
&\quad+\big(K_\alpha(\lambda_{n+1},\lambda_m)-k_{\alpha\lambda_{n+1},\lambda_m}\big)\big]\\
&\leq \alpha(\lambda_n,\lambda_m)\big[K_\alpha(\lambda_n,\lambda_{n+1})-k_{\alpha\lambda_n,\lambda_{n+1}}\big]\\
&\quad+\alpha(\lambda_n,\lambda_m)\alpha(\lambda_{n+1},\lambda_m)\big[\big(K_\alpha(\lambda_{n+1},\lambda_{n+2})-k_{\alpha\lambda_{n+1},\lambda_{n+2}}\big)\\
&\quad+\big(K_\alpha(\lambda_{n+2},\lambda_m)-k_{\alpha\lambda_{n+2},\lambda_m}\big)\big]\\
&\ \ \vdots\\
&\leq \sum_{i=n}^{m-1}\prod_{j=n}^{i}\alpha(\lambda_j,\lambda_m)\big[K_\alpha(\lambda_i,\lambda_{i+1})-k_{\alpha\lambda_i,\lambda_{i+1}}\big]\\
&\leq \sum_{i=n}^{m-1}\prod_{j=n}^{i}\alpha(\lambda_j,\lambda_m)\big[K_\alpha(\lambda_i,\lambda_{i+1})\big]\\
&\leq \sum_{i=n}^{m-1}\prod_{j=n}^{i}\alpha(\lambda_j,\lambda_m)p^i\prod_{s=1}^{i}\alpha(\lambda_{s-1},\lambda_s)K_\alpha(\lambda_0,\lambda_1).
\end{aligned}
$$

Now, let

$$
\Gamma_i := \prod_{j=n}^{i}\alpha(\lambda_j,\lambda_m)p^i\prod_{s=1}^{i}\alpha(\lambda_{s-1},\lambda_s)K_\alpha(\lambda_0,\lambda_1),
$$

then

$$
\Gamma_{i+1} := \prod_{j=n}^{i+1}\alpha(\lambda_j,\lambda_m)p^i\prod_{s=1}^{i+1}\alpha(\lambda_{s-1},\lambda_s)K_\alpha(\lambda_0,\lambda_1).
$$

Thus,

$$
\frac{\Gamma_{i+1}}{\Gamma_i}=\alpha(\lambda_{i+1},\lambda_m)\alpha(\lambda_{i+1},\lambda_i)p.
$$

Therefore

$$
\sup_{m\geq 1}\lim_i\frac{\Gamma_{i+1}}{\Gamma_i}=p\sup_{m\geq 1}\lim_i\alpha(\lambda_m,\lambda_i)\alpha(\lambda_i,\lambda_{i+1})<1,
$$

which leads us to conclude that (λ_n) is K_α-Cauchy sequence. Since (X,K_α) is a complete extended M_b-metric space, we deduce that (λ_n) is convergent in X to some $u\in X$. Note that $k_{\alpha u,fu} \le K_\alpha(u,fu)$ and

$$K_\alpha(u,fu) - k_{\alpha u,fu} \le \alpha(u,fu)\big[\big(K_\alpha(u,\lambda_n) - k_{\alpha u,\lambda_n}\big) + \big(K_\alpha(\lambda_n,fu) - k_{\alpha\lambda_n,fu}\big)\big].$$

Since f is continuous and taking the limit in the above inequality we deduce that

$$K_\alpha(u,fu) = k_{\alpha u,fu}.$$

Now, without loss of generality we can suppose that $M_{\alpha u,fu} = K_\alpha(u,u)$.

$$\begin{aligned}
M_{\alpha\lambda_n,f\lambda_n} = K_\alpha(\lambda_n,\lambda_n) &\le p\alpha(\lambda_{n-1},\lambda_{n-1})K_\alpha(\lambda_{n-1},\lambda_{n-1}) \\
&\le p^2\alpha(\lambda_{n-1},\lambda_{n-1})\alpha(\lambda_{n-2},\lambda_{n-2})K_\alpha(\lambda_{n-2},\lambda_{n-2}) \\
&\ \vdots \\
&\le p^n\prod_{i=0}^{n-1}\alpha(\lambda_i,\lambda_i)K_\alpha(\lambda_0,\lambda_0).
\end{aligned}$$

Taking the limit on both sides as $n \to +\infty$ we have

$$M_{\alpha u,fu} = 0.$$

Finally, since $K_\alpha(u,fu) = k_{\alpha u,fu} \le M_{\alpha u,fu} = 0$ and since $K_\alpha(fu,fu) = k_{\alpha u,fu}$, it is easy to conclude that $fu = u$. That is, f has a fixed point. Now,assume that f has two fixed points say $s,r \in X$, that is, $fs = s$ and $fr = r$. Thus,

$$\begin{aligned}
K_\alpha(s,r) &= K_\alpha(fs,fr) \\
&\le p\alpha(s,r)K_\alpha(s,r) \\
&< p.\frac{1}{p}K_\alpha(s,r) \\
&= K_\alpha(s,r),
\end{aligned}$$

which implies that $K_\alpha(s,r) = 0$, therefore $K_\alpha(s,r) = k_{\alpha_{r,s}} = 0$. Now, we may assume that $M_{\alpha_{r,s}} = K_\alpha(s,s)$, hence $K_\alpha(s,s) = K_\alpha(fs,fs) \le p\alpha(s,s)K_\alpha(s,s) < K_\alpha(s,s)$. Hence, $K_\alpha(s,s) = 0$, which leads us to conclude that

$$K_\alpha(s,r) = k_{\alpha_{r,s}} = M_{\alpha_{r,s}} = 0$$

and that $r = s$ as required. □

Example 2.2 Let $X = [0,1]$ and let $f: X \to X$ defined by

$$f(\lambda) = \frac{\lambda}{2+2\lambda}.$$

Then f has a unique fixed point.

Proof For all $\lambda, \epsilon \in X$, let $K_\alpha(\lambda, \epsilon) = \frac{(\lambda+\epsilon)^2}{2}$ and $\alpha(\lambda, \epsilon) = 1 + \lambda + \epsilon$. An easy argument shows that (X, K_α) is a K_α-complete extended M_b-metric space. Also we have

$$\begin{aligned} K_\alpha(f\lambda, f\epsilon) &= \frac{(f\lambda + f\epsilon)^2}{2} = \frac{(\frac{\lambda}{2+2\lambda} + \frac{\epsilon}{2+2\epsilon})^2}{2} \\ &\leq \frac{1}{4}\frac{(\lambda+\epsilon)^2}{2} \\ &\leq \frac{1}{4}(1+\lambda+\epsilon)\frac{(\lambda+\epsilon)^2}{2} \\ &= \frac{1}{4}\alpha(\lambda, \epsilon)K_\alpha(\lambda, \epsilon). \end{aligned}$$

Hence,

$$K_\alpha(f\lambda, f\epsilon) \leq p\alpha(\lambda, \epsilon)K_\alpha(\lambda, \epsilon), \quad \text{where } p = \frac{1}{4}.$$

Now, by induction it is not difficult to deduce that

$$\lambda_n = f^n(\lambda) = \frac{\lambda}{2^n + (\sum_{k=1}^n 2^k)\lambda}$$

for all $n \in \mathbb{N}$. Thus,

$$\lim_{n\to+\infty} \alpha(\lambda, \lambda_n) = \lim_{n\to+\infty} \alpha(\lambda_n, \lambda) = 1 + \lambda.$$

On the other hand,

$$\begin{aligned} \sup_{m\geq 1} \lim_{n\to+\infty} \alpha(\lambda_n, \lambda_m)\alpha(\lambda_{n+1}, \lambda_n) &= \sup_{m\geq 1}\left(1 + \frac{\lambda}{2^m + (\sum_{k=1}^m 2^k)\lambda}\right) \\ &= 1 + \frac{\lambda}{2+2\lambda} \\ &\leq 2 < 4 = \frac{1}{p}. \end{aligned}$$

It is not difficult to check that $f : (X, K_\alpha) \to (X, K_\alpha)$ is continuous. Finally, note that f satisfies all the hypotheses of Theorem 2.1. Therefore, f has a unique fixed point in X. □

Theorem 2.3 *Let (X, K_α) be a complete extended M_b-metric space, and let f be a continuous self-mapping on X. Assume that there exist $a, b \in [0, +\infty)$ with*

$$\lim_n \frac{a\alpha(\lambda_n, \lambda_{n-1})}{1 - b\alpha(\lambda_n, \lambda_{n+1})} < 1$$

and

$$\alpha(\lambda_n, \lambda_{n+1}) < \frac{1}{a+b}, \quad \textit{where } \lambda_n = f^n\lambda_0.$$

If $K_\alpha(f\lambda, fy) \leq a\alpha(\lambda, f\lambda K_\alpha(\lambda, f\lambda + b\alpha(\epsilon, fy)K_\alpha(\epsilon, fy)$, then f has a unique fixed point in X.

Proof Let $\lambda_0 \in X$ and define the sequence $\{\lambda_n\}$ as follows:

$$\lambda_1 = f\lambda_0, \qquad \lambda_2 = f\lambda_1 = f^2\lambda_0, \qquad \dots, \qquad \lambda_n = f\lambda_{n-1} = f^n\lambda_0, \qquad \dots.$$

We prove first that

$$K_\alpha(\lambda_n, \lambda_{n+1}) \le a^n \prod_{i=1}^{n} \left[\frac{\alpha(\lambda_i, \lambda_{i-1})}{1 - b\alpha(\lambda_i, \lambda_{i+1})}\right] K_\alpha(\lambda_0, \lambda_1).$$

To this end, let $n \in \mathbb{N}^\star$, then

$$\begin{aligned} K_\alpha(\lambda_n, \lambda_{n+1}) &= K_\alpha(f\lambda_{n-1}, f\lambda_n) \\ &\le a\alpha(\lambda_{n-1}, f\lambda_{n-1})K_\alpha(\lambda_{n-1}, f\lambda_{n-1}) + b\alpha(\lambda_n, f\lambda_n)K_\alpha(\lambda_n, f\lambda_n) \\ &= a\alpha(\lambda_{n-1}, \lambda_n)K_\alpha(\lambda_{n-1}, \lambda_n) + b\alpha(\lambda_n, \lambda_{n+1})K_\alpha(\lambda_n, \lambda_{n+1}). \end{aligned}$$

Hence,

$$\begin{aligned} K_\alpha(\lambda_n, \lambda_{n+1}) &\le a\frac{\alpha(\lambda_n, \lambda_{n-1})}{1 - b\alpha(\lambda_n, \lambda_{n+1})}K_\alpha(\lambda_n, \lambda_{n-1}) \\ &= a\frac{\alpha(\lambda_n, \lambda_{n-1})}{1 - b\alpha(\lambda_n, \lambda_{n+1})}K_\alpha(f\lambda_{n-1}, f\lambda_{n-2}) \\ &\le a^2\frac{\alpha(\lambda_n, \lambda_{n-1}\alpha(\lambda_{n-1}, \lambda_{n-2})}{(1 - b\alpha(\lambda_n, \lambda_{n+1}))(1 - b\alpha(\lambda_{n-1}, \lambda_n))}K_\alpha(\lambda_{n-1}, \lambda_{n-2}) \\ &\le \cdots \\ &\le a^n \prod_{i=1}^{n} \left[\frac{\alpha(\lambda_i, \lambda_{i-1})}{1 - b\alpha(\lambda_i, \lambda_{i+1})}\right] K_\alpha(\lambda_0, \lambda_1). \end{aligned}$$

Since $\lim_n \frac{a\alpha(\lambda_n, \lambda_{n-1})}{1 - b\alpha(\lambda_n, \lambda_{n+1})} < 1$, it follows from the ratio test that

$$\sum_{n=1}^{\infty} a^n \prod_{i=1}^{n} \frac{\alpha(\lambda_i, \lambda_{i-1})}{1 - b\alpha(\lambda_i, \lambda_{i+1})}$$

converges, which implies that $K_\alpha(\lambda_n, \lambda_{n+1})$converges to 0.

Next, let $n, m \in \mathbb{N}^\star$, then

$$\begin{aligned} K_\alpha(\lambda_n, \lambda_m) &= K_\alpha(f\lambda_{n-1}, f\lambda_{m-1}) \\ &\le a\alpha(\lambda_{n-1}, \lambda_n)K_\alpha(\lambda_{n-1}, \lambda_n) + b\alpha(\lambda_{m-1}, f\lambda_m)K_\alpha(\lambda_{m-1}, \lambda_m). \end{aligned}$$

By the above inequality, we deduce that $K_\alpha(\lambda_n, \lambda_m)$converges to 0. Since

$$k_{\alpha\lambda_m, \lambda_n} := \min\bigl(K_\alpha(\lambda_n, \lambda_n), K_\alpha(\lambda_m, \lambda_m)\bigr) \le K_\alpha(\lambda_n, \lambda_m),$$

we conclude that

$$\lim_{n,m \to +\infty} \bigl(K_\alpha(\lambda_n, \lambda_m) - k_{\alpha\lambda_m, \lambda_n}\bigr) = 0.$$

Now, without loss of generality we may assume that

$$M_{\alpha\lambda_m,\lambda_n} := \max\bigl(K_\alpha(\lambda_n,\lambda_n),K_\alpha(\lambda_m,\lambda_m)\bigr) = K_\alpha(\lambda_n,\lambda_n).$$

Hence, we obtain

$$\begin{aligned} M_{\alpha\lambda_m,\lambda_n} - k_{\alpha\lambda_m,\lambda_n} &\le M_{\alpha\lambda_m,\lambda_n} \\ &= K_\alpha(\lambda_n,\lambda_n) \\ &\le (a+b)\alpha(\lambda_{n-1},\lambda_n)K_\alpha(\lambda_{n-1},\lambda_n). \end{aligned}$$

Taking the limit of the above inequality as $n \to +\infty$ we deduce that

$$\lim_{n,m\to+\infty}(M_{\alpha\lambda_m,\lambda_n} - k_{\alpha\lambda_m,\lambda_n}) = 0.$$

Thus, the sequence $\{\lambda_n\}$ is a K_α-Cauchy sequence. Since (X,K_α) is a K_α-complete extended b-metric space, we conclude that $\{\lambda_n\}$ converges to some $\omega \in X$, and so $\{f\lambda_n = \lambda_{n+1}\}$ converges to $\omega \in X$. On the other hand, by the hypotheses of the theorem ($f : (X,K_\alpha) \to (X,K_\alpha)$ is continuous) it is not difficult to conclude that $\{f\lambda_n\}$ converges to $f\omega \in X$. From Lemma 3.3 in [23], we have

$$K_\alpha(\omega,f\omega) - k_{\alpha\omega,f\omega} = 0.$$

Then

$$\begin{aligned} K_\alpha(\omega,f\omega) &= K_\alpha(f\omega,f\omega) \\ &\le a\alpha(\omega,f\omega)K_\alpha(\omega,f\omega) + b\alpha(\omega,f\omega)K_\alpha(\omega,f\omega) \\ &= (a+b)\alpha(\omega,f\omega)K_\alpha(\omega,f\omega) \\ &< K_\alpha(\omega,f\omega). \end{aligned}$$

Hence

$$K_\alpha(\omega,f\omega) = K_\alpha(f\omega,f\omega) = 0.$$

Similarly to the above we have

$$K_\alpha\bigl(f\omega,f^2\omega\bigr) = K_\alpha\bigl(f^2\omega,f^2\omega\bigr) = 0. \tag{2.2}$$

Since (X,K_α) is an extended b-metric space, it follows that

$$ff\omega = f\omega.$$

We deduce that $\omega' = f\omega$ is a fixed point of f. Finally, to show uniqueness assume that there exists another fixed point of f, say u. By the contractive property of f we have

$$\begin{aligned} K_\alpha\bigl(u,\omega'\bigr) &= K_\alpha\bigl(fu,f\omega'\bigr) \\ &\le a\alpha(u,fu)K_\alpha(u,fu) + b\alpha\bigl(\omega',f\omega'\bigr)K_\alpha\bigl(\omega',f\omega'\bigr). \end{aligned}$$

From (2.2) we get

$$K_\alpha\big(u,\omega'\big) \le a\alpha(u,fu)K_\alpha(u,fu).$$

Hence,

$$\begin{aligned}K_\alpha\big(u,\omega'\big) &\le a\alpha(u,fu)K_\alpha(u,fu)\\ &= a\alpha(u,fu)K_\alpha(fu,fu)\\ &\le a\alpha(u,fu)\big[(a+b)\alpha(u,fu)\big]K_\alpha(u,fu)\\ &\le \cdots\\ &\le a\alpha(u,fu)\big[(a+b)\alpha(u,fu)\big]^n K_\alpha(u,fu).\end{aligned}$$

Since $(a+b)\alpha(u,fu) = (a+b)\alpha(fu,f^2u) < 1$, it follows that $[(a+b)\alpha(u,fu)]^n$ converges to 0. So $K_\alpha(u,\omega') = K_\alpha(u,u) = 0$. By (2.2) we have

$$K_\alpha\big(u,\omega'\big) = K_\alpha(u,u) = K_\alpha\big(\omega',\omega'\big) = 0.$$

Thus, f has a unique fixed point as required. □

3 Application

Consider the set $X = C([0,1],\mathbb{R})$ and the following Fredholm type integral equation:

$$x'(t) = \int_0^1 \mathbb{G}\big(t,s,x'(t)\big)\,ds, \quad \text{for } t,s \in [0,1], \tag{3.1}$$

where $\mathbb{G}(t,s,x'(t))$ is a continuous function from $[0,1]^2$ into $\mathbb{R}$. Now, define

$$\begin{aligned}&K_\alpha : X\times X \longrightarrow \mathbb{R}\\ &\quad (x,y) \mapsto \sup_{t\in[0,1]}\left(\frac{|x'(t)|+|y(t)|}{2}\right).\end{aligned}$$

Note that (X,K_α) is a K_α-complete extended M_b-metric space, where

$$\alpha(x,y) = 1 + \sup_{t\in[0,1]}\big(\big|x'(t)\big|\big|y(t)\big|\big).$$

Theorem 3.1 *Assume that for all $x,y \in X$:*

(1) $|\mathbb{G}(t,s,x'(t))| + |\mathbb{G}(t,s,y(t))| \le p(1+\sup_{t\in[0,1]}\{|x'(t)||y(t)|\})(|x'(t)|+|y(t)|)$, *for some* $p \in [0, \frac{1}{(1+\sup_{t,s}|\mathbb{G}(t,s,x'(t))||\mathbb{G}(t,s,y(t))|)^2})$.

(2) $\mathbb{G}(t,s,\int_0^1 \mathbb{G}(t,s,x'(t))\,ds) < \mathbb{G}(t,s,x'(t))$ *for all t, s.*

Then the above integral equation has a unique solution.

Proof Let $f : X \longrightarrow X$ be defined by $fx'(t) = \int_0^1 \mathbb{G}(t,s,x'(t))\,ds$, then

$$K_\alpha(fx,fy) = \sup_{t\in[0,1]}\left(\frac{|fx'(t)|+|fy(t)|}{2}\right).$$

Now we have

$$\begin{aligned}
\frac{|fx'(t)| + |fy(t)|}{2} &= \frac{|\int_0^1 \mathbb{G}(t,s,x'(t))\,ds| + |\int_0^1 \mathbb{G}(t,s,y(t))\,ds|}{2} \\
&\leq \frac{\int_0^1 |\mathbb{G}(t,s,x'(t))|\,ds + \int_0^1 |\mathbb{G}(t,s,y(t))|\,ds}{2} \\
&= \frac{\int_0^1 (|\mathbb{G}(t,s,x'(t))| + |\mathbb{G}(t,s,y(t))|)\,ds}{2} \\
&\leq \frac{\int_0^1 p(1+\sup_{t\in[0,1]}\{|x'(t)||y(t)|\})(|x'(t)| + |y(t)|)\,ds}{2} \\
&\leq p\alpha(x,y)K_\alpha(x,y).
\end{aligned}$$

Consequently, $K_\alpha(fx,fy) \leq \alpha(x,y)K_\alpha(x,y)$. On the other hand, let $n \in \mathbb{N}^\star$ and $x \in X$, then

$$\begin{aligned}
\big(f^n x\big)(t) = f\big(f^{n-1}x'(t)\big) &= \int_0^1 \mathbb{G}\big(t,s,f^{n-1}x'(t)\big)\,ds \\
&= \int_0^1 \mathbb{G}\big(t,s,f\big(f^{n-2}x\big)(t)\big)\,ds \\
&= \int_0^1 \mathbb{G}\left(t,s,\int_0^1 \mathbb{G}\big(t,s,\big(f^{n-2}x'(t)\big)\big)\right)ds \\
&< \int_0^1 \mathbb{G}\big(t,s,\big(f^{n-2}x'(t)\big)\big)\,ds = \big(f^{n-1}x'(t)\big).
\end{aligned}$$

Thus, for all $t \in [0,1]$ we find that $(f^n x'(t))_n$ is strictly decreasing and a sequence bounded below, and so it converges to some l. Since $(f_n)_n$ is a monotone sequence, it follows from the Dini theorem that $\sup_t |f^n x'(t)|$ converges to some $l' \leq \sup_{t,s} |\mathbb{G}(t,s,x'(t))|$. Observe that $\alpha(f^n x, f^m(x)) = 1 + \sup_t |f^n x'(t)||f^m x'(t)|$ converges to $1 + l'^2 \leq 1 + (\sup_{t,s}|\mathbb{G}(t,s,x'(t))|)^2$. So

$$\sup_m \lim_n \alpha\big(f^n x, f^m(x)\big)\alpha\big(f^n x, f^{n+1}x'(t)\big) \leq \left(1 + \left(\sup_{t,s}\big|\mathbb{G}\big(t,s,x'(t)\big)\big|\right)^2\right)^2 < \frac{1}{p}.$$

Now, note that all the hypotheses of Theorem 2.1, are satisfied and thus Eq. (3.1) has a unique solution. □

4 Conclusion

In closing, note that in this manuscript we proved fixed point results for mappings that satisfy more general contractions, which generalizes many results obtained for mapping satisfying Banach contraction and by taking $\alpha(\lambda,\epsilon) = 1$ for all $\lambda,\epsilon \in X$ in Theorem 2.1.

Acknowledgements

The first and thrid authors would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Funding

Not applicable.

Abbreviations

Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare to have no conflict of interest.

Authors' contributions
All the authors have contributed equally in all parts. All authors read and approved the final manuscript.

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Publisher's Note

Received: 9 May 2020 Accepted: 4 June 2020 Published online: 13 June 2020

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