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Positive solutions for a system of Riemann–Liouville fractional boundary value problems with p -Laplacian operators

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Abstract

We study the existence and nonexistence of positive solutions for a system of Riemann–Liouville fractional differential equations with p -Laplacian operators, nonnegative nonlinearities and positive parameters, subject to coupled nonlocal boundary conditions which contain Riemann–Stieltjes integrals and various fractional derivatives. We use the Guo–Krasnosel'skii fixed point theorem in the proof of the main existence results.

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1 Introduction

In this paper we consider the system of nonlinear ordinary fractional differential equations with ϱ_1 -Laplacian and ϱ_2 -Laplacian operators

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{\varrho_1}(D_{0+}^{\beta_1}x(t))) + \lambda f(t, x(t), y(t)) = 0, & t \in (0, 1), \\ D_{0+}^{\alpha_2}(\varphi_{\varrho_2}(D_{0+}^{\beta_2}y(t))) + \mu g(t, x(t), y(t)) = 0, & t \in (0, 1), \end{cases} \quad (1)$$

with the coupled nonlocal boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, & j = 0, \dots, n-2; & D_{0+}^{\beta_1}x(0) = 0, \\ D_{0+}^{\gamma_0}x(1) = \sum_{i=1}^p \int_0^1 D_{0+}^{\gamma_i}y(t) dH_i(t), \\ y^{(j)}(0) = 0, & j = 0, \dots, m-2; & D_{0+}^{\beta_2}y(0) = 0, \\ D_{0+}^{\delta_0}y(1) = \sum_{i=1}^q \int_0^1 D_{0+}^{\delta_i}x(t) dK_i(t), \end{cases} \quad (2)$$

where $\alpha_1, \alpha_2 \in (0, 1]$, $\beta_1 \in (n-1, n]$, $\beta_2 \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 3$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, 1, \dots, p$, $0 \leq \gamma_1 < \gamma_2 < \dots < \gamma_p \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $\delta_i \in \mathbb{R}$ for all $i = 0, 1, \dots, q$, $0 \leq \delta_1 < \delta_2 < \dots < \delta_q \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $\varrho_1, \varrho_2 > 1$, $\varphi_{\varrho_i}(s) = |s|^{q_i-2}s$, $\varphi_{\varrho_i}^{-1} = \varphi_{\rho_i}$, $\frac{1}{\varrho_i} + \frac{1}{\rho_i} = 1$,

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$i = 1, 2$, $\lambda, \mu > 0$, the functions $f, g \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ ($\mathbb{R}_+ = [0, \infty)$), the integrals from (2) are Riemann–Stieltjes integrals with H_i , $i = 1, \dots, p$, and K_i , $i = 1, \dots, q$, functions of bounded variation, and D_{0+}^k denotes the Riemann–Liouville derivative of order k (for $k = \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_i$ for $i = 0, 1, \dots, p$, δ_i for $i = 0, 1, \dots, q$).

We present sufficient conditions on the functions f and g , and intervals for the parameters λ and μ such that problem (1), (2) has positive solutions. A positive solution of problem (1), (2) is a pair of functions $(x, y) \in (C([0, 1], \mathbb{R}_+))^2$, satisfying (1) and (2) with $x(t) > 0$ for all $t \in (0, 1]$ or $y(t) > 0$ for all $t \in (0, 1]$. We also investigate the nonexistence of positive solutions for this problem. The problem (1), (2) is a generalization of the problem studied in [30]. Indeed, if $p = 1$, $q = 1$, $\gamma_0 = p_1$, $\gamma_1 = q_1$, $\delta_0 = p_2$, $\delta_1 = q_2$, H_1 is a step function given by $H_1(t) = \{0, t \in [0, \xi_1); a_1, t \in [\xi_1, \xi_2); a_1 + a_2, t \in [\xi_2, \xi_3); \dots; \sum_{i=1}^N a_i, t \in [\xi_N, 1]\}$, and K_1 is a step function given by $K_1(t) = \{0, t \in [0, \eta_1); b_1, t \in [\eta_1, \eta_2); b_1 + b_2, t \in [\eta_2, \eta_3); \dots; \sum_{i=1}^M b_i, t \in [\eta_M, 1]\}$, then the boundary conditions (2) become the multi-point boundary conditions (BC) from [30]. The system (1) with uncoupled multi-point boundary conditions was studied in [29]. Fractional differential equations and systems of fractional differential equations, subject to various multi-point or Riemann–Stieltjes integral boundary conditions, were studied in the last years in [1–8, 17–25, 28, 31, 37, 39–41]. For various applications of the fractional calculus in several scientific and engineering fields, the reader may consult the books [10, 11, 26, 27, 34–36], and the papers [9, 12–15, 32, 33, 38].

In Sect. 2, we study a nonlocal boundary value problem for fractional differential equations with p -Laplacian operators, and we present some properties of the associated Green functions. Section 3 is devoted to the main existence theorems for the positive solutions with respect to a cone for our problem (1), (2), which are based on the Guo–Krasnosel'skii fixed point theorem (see [16]). In Sect. 4, we present some nonexistence results for the positive solutions of (1), (2), and in Sect. 5, we give an example which illustrates our results.

2 Preliminary results

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{\varrho_1}(D_{0+}^{\beta_1}x(t))) + \psi(t) = 0, & t \in (0, 1), \\ D_{0+}^{\alpha_2}(\varphi_{\varrho_2}(D_{0+}^{\beta_2}y(t))) + \chi(t) = 0, & t \in (0, 1), \end{cases} \quad (3)$$

with the coupled boundary conditions (2), where $\psi, \chi \in C(0, 1) \cap L^1(0, 1)$.

We denote $\varphi_{\varrho_1}(D_{0+}^{\beta_1}x(t)) = u(t)$ and $\varphi_{\varrho_2}(D_{0+}^{\beta_2}y(t)) = v(t)$. Then problem (3), (2) is equivalent to the following three problems:

$$\begin{cases} D_{0+}^{\alpha_1}u(t) + \psi(t) = 0, & 0 < t < 1, \\ u(0) = 0, \end{cases} \quad (4)$$

$$\begin{cases} D_{0+}^{\alpha_2}v(t) + \chi(t) = 0, & 0 < t < 1, \\ v(0) = 0, \end{cases} \quad (5)$$

and

$$\begin{cases} D_{0+}^{\beta_1} x(t) = \varphi_{\rho_1}(u(t)), & t \in (0, 1), \\ D_{0+}^{\beta_2} y(t) = \varphi_{\rho_2}(v(t)), & t \in (0, 1), \end{cases} \quad (6)$$

with the boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, & j = 0, \dots, n-2; & D_{0+}^{\gamma_0} x(1) = \sum_{i=1}^p \int_0^1 D_{0+}^{\gamma_i} y(t) dH_i(t), \\ y^{(j)}(0) = 0, & j = 0, \dots, m-2; & D_{0+}^{\delta_0} y(1) = \sum_{i=1}^q \int_0^1 D_{0+}^{\delta_i} x(t) dK_i(t). \end{cases} \quad (7)$$

The first two problems (4) and (5) have the unique solutions $u \in C[0, 1]$ and $v \in C[0, 1]$, respectively, of the form

$$u(t) = -I_{0+}^{\alpha_1} \psi(t) = -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-v)^{\alpha_1-1} \psi(v) dv, \quad t \in [0, 1], \quad (8)$$

and

$$v(t) = -I_{0+}^{\alpha_2} \chi(t) = -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-v)^{\alpha_2-1} \chi(v) dv, \quad t \in [0, 1], \quad (9)$$

respectively.

For the third problem (6), (7), we denote

$$\begin{aligned} \Delta = & \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1-\gamma_0)\Gamma(\beta_2-\delta_0)} - \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\gamma_i)} \int_0^1 \tau^{\beta_2-\gamma_i-1} dH_i(\tau) \right) \\ & \times \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\delta_i)} \int_0^1 \tau^{\beta_1-\delta_i-1} dK_i(\tau) \right). \end{aligned} \quad (10)$$

Lemma 2.1 *If $\Delta \neq 0$, then the unique solution $(x, y) \in C[0, 1] \times C[0, 1]$ of problem (6), (7) is given by*

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\beta_1)} \int_0^t (t-v)^{\beta_1-1} \varphi_{\rho_1}(u(v)) dv \\ & + \frac{t^{\beta_1-1}}{\Delta} \left[-\frac{\Gamma(\beta_2)}{\Gamma(\beta_1-\gamma_0)\Gamma(\beta_2-\delta_0)} \int_0^1 (1-v)^{\beta_1-\gamma_0-1} \varphi_{\rho_1}(u(v)) dv \right. \\ & + \frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\delta_0)} \sum_{i=1}^p \frac{1}{\Gamma(\beta_2-\gamma_i)} \int_0^1 \left(\int_0^v (v-\tau)^{\beta_2-\gamma_i-1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(v) \\ & - \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\gamma_i)} \int_0^1 v^{\beta_2-\gamma_i-1} dH_i(v) \right) \\ & \times \left(\frac{1}{\Gamma(\beta_2-\delta_0)} \int_0^1 (1-v)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(v)) dv \right) \\ & \left. + \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\gamma_i)} \int_0^1 v^{\beta_2-\gamma_i-1} dH_i(v) \right) \right] \end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^v (v - \tau)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(v) \right) \Bigg], \\
& \forall t \in [0, 1], \\
& y(t) = \frac{1}{\Gamma(\beta_2)} \int_0^t (t - v)^{\beta_2 - 1} \varphi_{\rho_2}(v(v)) dv \\
& + \frac{t^{\beta_2 - 1}}{\Delta} \left[-\frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1 - v)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(v)) dv \right. \\
& + \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^v (v - \tau)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(v) \\
& - \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 v^{\beta_1 - \delta_i - 1} dK_i(v) \right) \\
& \times \left(\frac{1}{\Gamma(\beta_1 - \gamma_0)} \int_0^1 (1 - v)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(v)) dv \right) \\
& + \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 v^{\beta_1 - \delta_i - 1} dK_i(v) \right) \\
& \times \left. \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^v (v - \tau)^{\beta_2 - \gamma_i - 1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(v) \right) \right], \\
& \forall t \in [0, 1].
\end{aligned} \tag{11}$$

Proof The solutions $(x, y) \in (C(0, 1) \cap L^1(0, 1))^2$ of system (6) are

$$\begin{aligned}
x(t) &= I_{0+}^{\beta_1} \varphi_{\rho_1}(u(t)) + a_1 t^{\beta_1 - 1} + a_2 t^{\beta_1 - 2} + \dots + a_n t^{\beta_1 - n}, \\
y(t) &= I_{0+}^{\beta_2} \varphi_{\rho_2}(v(t)) + b_1 t^{\beta_2 - 1} + b_2 t^{\beta_2 - 2} + \dots + b_m t^{\beta_2 - m},
\end{aligned}$$

with $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in \mathbb{R}$. By using the boundary conditions $x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0$ and $y(0) = y'(0) = \dots = y^{(m-2)}(0) = 0$, we obtain $a_2 = \dots = a_n = 0$ and $b_2 = \dots = b_m = 0$. So the above solutions become

$$\begin{aligned}
x(t) &= I_{0+}^{\beta_1} \varphi_{\rho_1}(u(t)) + a_1 t^{\beta_1 - 1} = \frac{1}{\Gamma(\beta_1)} \int_0^t (t - v)^{\beta_1 - 1} \varphi_{\rho_1}(u(v)) dv + a_1 t^{\beta_1 - 1}, \\
y(t) &= I_{0+}^{\beta_2} \varphi_{\rho_2}(v(t)) + b_1 t^{\beta_2 - 1} = \frac{1}{\Gamma(\beta_2)} \int_0^t (t - v)^{\beta_2 - 1} \varphi_{\rho_2}(v(v)) dv + b_1 t^{\beta_2 - 1}.
\end{aligned} \tag{12}$$

For the obtained functions x and y we have

$$\begin{aligned}
D_{0+}^{\gamma_i} y(t) &= b_1 \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} t^{\beta_2 - \gamma_i - 1} + I_{0+}^{\beta_1 - \gamma_i} \varphi_{\rho_2}(v(t)), \quad i = 1, \dots, p, \\
D_{0+}^{\gamma_0} x(t) &= a_1 \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0)} t^{\beta_1 - \gamma_0 - 1} + I_{0+}^{\beta_1 - \gamma_0} \varphi_{\rho_1}(u(t)), \\
D_{0+}^{\delta_i} x(t) &= a_1 \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} t^{\beta_1 - \delta_i - 1} + I_{0+}^{\beta_1 - \delta_i} \varphi_{\rho_1}(u(t)), \quad i = 1, \dots, q, \\
D_{0+}^{\delta_0} y(t) &= b_1 \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \delta_0)} t^{\beta_2 - \delta_0 - 1} + I_{0+}^{\beta_2 - \delta_0} \varphi_{\rho_2}(v(t)),
\end{aligned}$$

$$\begin{aligned}
D_{0+}^{\gamma_0} x(1) &= a_1 \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0)} + I_{0+}^{\beta_1 - \gamma_0} \varphi_{\rho_1}(u(t))|_{t=1} \\
&= a_1 \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0)} + \frac{1}{\Gamma(\beta_1 - \gamma_0)} \int_0^1 (1-v)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(v)) dv, \\
D_{0+}^{\delta_0} y(1) &= b_1 \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \delta_0)} + I_{0+}^{\beta_2 - \delta_0} \varphi_{\rho_2}(v(t))|_{t=1} \\
&= b_1 \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \delta_0)} + \frac{1}{\Gamma(\beta_2 - \delta_0)} \int_0^1 (1-v)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(v)) dv.
\end{aligned}$$

By imposing the boundary conditions $D_{0+}^{\gamma_0} x(1) = \sum_{i=1}^p \int_0^1 D_{0+}^{\gamma_i} y(s) dH_i(s)$ and $D_{0+}^{\delta_0} y(1) = \sum_{i=1}^q \int_0^1 D_{0+}^{\delta_i} x(s) dK_i(s)$, we deduce the following system in a_1 and b_1 :

$$\begin{cases}
a_1 \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0)} - b_1 \sum_{i=1}^p \int_0^1 \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} t^{\beta_2 - \gamma_i - 1} dH_i(t) \\
= -\frac{1}{\Gamma(\beta_1 - \gamma_0)} \int_0^1 (1-v)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(v)) dv \\
+ \sum_{i=1}^p \int_0^1 \frac{1}{\Gamma(\beta_2 - \gamma_i)} \left(\int_0^t (t-v)^{\beta_2 - \gamma_i - 1} \varphi_{\rho_2}(v(v)) dv \right) dH_i(t), \\
b_1 \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \delta_0)} - a_1 \sum_{i=1}^q \int_0^1 \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} t^{\beta_1 - \delta_i - 1} dK_i(t) \\
= -\frac{1}{\Gamma(\beta_2 - \delta_0)} \int_0^1 (1-v)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(v)) dv \\
+ \sum_{i=1}^q \int_0^1 \frac{1}{\Gamma(\beta_1 - \delta_i)} \left(\int_0^t (t-v)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(v)) dv \right) dK_i(t).
\end{cases} \quad (13)$$

The above system in the unknowns a_1 and b_1 has the determinant Δ , which, by the assumption of the lemma, is nonzero. So the system (13) has the unique solution

$$\begin{aligned}
a_1 &= -\frac{\Gamma(\beta_2)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-v)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(v)) dv \\
&+ \frac{\Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^t (t-v)^{\beta_2 - \gamma_i - 1} \varphi_{\rho_2}(v(v)) dv \right) dH_i(t) \\
&- \frac{1}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 t^{\beta_2 - \gamma_i - 1} dH_i(t) \right) \\
&\times \left(\frac{1}{\Gamma(\beta_2 - \delta_0)} \int_0^1 (1-v)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(v)) dv \right) \\
&+ \frac{1}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 t^{\beta_2 - \gamma_i - 1} dH_i(t) \right) \\
&\times \left(\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^t (t-v)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(v)) dv \right) dK_i(t) \right), \\
b_1 &= -\frac{\Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-v)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(v)) dv \\
&+ \frac{\Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^t (t-v)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(v)) dv \right) dK_i(t) \\
&- \frac{1}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 t^{\beta_1 - \delta_i - 1} dK_i(t) \right)
\end{aligned} \quad (14)$$

$$\begin{aligned} & \times \left(\frac{1}{\Gamma(\beta_1 - \gamma_0)} \int_0^1 (1-v)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(v)) dv \right) \\ & + \frac{1}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 t^{\beta_1 - \delta_i - 1} dK_i(t) \right) \\ & \times \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^t (t-v)^{\beta_2 - \gamma_i - 1} \varphi_{\rho_2}(v(v)) dv \right) dH_i(t) \right). \end{aligned}$$

Now replacing the constants a_1 and b_1 given by (14) in (12) we find the solution $(x, y) \in C[0, 1] \times C[0, 1]$ of problem (6), (7) presented in (11). Conversely, we can easily prove that the functions x, y given by (11) satisfy the problem (6), (7). $\square$

Lemma 2.2 *If $\Delta \neq 0$, then the solution (x, y) of problem (6), (7) given by (11) can be written as*

$$\begin{cases} x(t) = - \int_0^1 \mathcal{G}_1(t, v) \varphi_{\rho_1}(u(v)) dv - \int_0^1 \mathcal{G}_2(t, v) \varphi_{\rho_2}(v(v)) dv, & \forall t \in [0, 1], \\ y(t) = - \int_0^1 \mathcal{G}_3(t, v) \varphi_{\rho_1}(u(v)) dv - \int_0^1 \mathcal{G}_4(t, v) \varphi_{\rho_2}(v(v)) dv, & \forall t \in [0, 1], \end{cases} \quad (15)$$

where

$$\begin{aligned} \mathcal{G}_1(t, v) &= g_1(t, v) + \frac{t^{\beta_1 - 1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2 - \gamma_i - 1} dH_i(s) \right) \\ &\quad \times \left(\sum_{i=1}^q \int_0^1 g_{1i}(\tau, v) dK_i(\tau) \right), \\ \mathcal{G}_2(t, v) &= \frac{t^{\beta_1 - 1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \int_0^1 g_{2i}(\tau, v) dH_i(\tau), \\ \mathcal{G}_3(t, v) &= \frac{t^{\beta_2 - 1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \int_0^1 g_{1i}(\tau, v) dK_i(\tau), \\ \mathcal{G}_4(t, v) &= g_2(t, v) + \frac{t^{\beta_2 - 1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1 - \delta_i - 1} dK_i(s) \right) \\ &\quad \times \left(\sum_{i=1}^p \int_0^1 g_{2i}(\tau, v) dH_i(\tau) \right), \end{aligned} \quad (16)$$

for all $(t, v) \in [0, 1] \times [0, 1]$ and

$$\begin{aligned} g_1(t, s) &= \frac{1}{\Gamma(\beta_1)} \begin{cases} t^{\beta_1 - 1} (1-s)^{\beta_1 - \gamma_0 - 1} - (t-s)^{\beta_1 - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta_1 - 1} (1-s)^{\beta_1 - \gamma_0 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ g_{1i}(\tau, s) &= \frac{1}{\Gamma(\beta_1 - \delta_i)} \begin{cases} \tau^{\beta_1 - \delta_i - 1} (1-s)^{\beta_1 - \gamma_0 - 1} - (\tau-s)^{\beta_1 - \delta_i - 1}, & 0 \leq s \leq \tau \leq 1, \\ \tau^{\beta_1 - \delta_i - 1} (1-s)^{\beta_1 - \gamma_0 - 1}, & 0 \leq \tau \leq s \leq 1, \end{cases} \\ g_2(t, s) &= \frac{1}{\Gamma(\beta_2)} \begin{cases} t^{\beta_2 - 1} (1-s)^{\beta_2 - \delta_0 - 1} - (t-s)^{\beta_2 - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta_2 - 1} (1-s)^{\beta_2 - \delta_0 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \end{aligned} \quad (17)$$

$$g_{2j}(\tau, s) = \frac{1}{\Gamma(\beta_2 - \gamma_j)} \begin{cases} \tau^{\beta_2 - \gamma_j - 1} (1 - s)^{\beta_2 - \delta_0 - 1} - (\tau - s)^{\beta_2 - \gamma_j - 1}, & 0 \leq s \leq \tau \leq 1, \\ \tau^{\beta_2 - \gamma_j - 1} (1 - s)^{\beta_2 - \delta_0 - 1}, & 0 \leq \tau \leq s \leq 1, \end{cases}$$

for all $i = 1, \dots, q$ and $j = 1, \dots, p$.

Proof For $x(t)$ from (11) we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1-1} \varphi_{\rho_1}(u(s)) ds \\ &\quad - \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-s)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(s)) ds \\ &\quad + \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2 - \gamma_i - 1} dH_i(s) \right) \\ &\quad \times \left(\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(s) \right) \\ &\quad + \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_2 - \gamma_i - 1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(s) \\ &\quad - \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2 - \gamma_i - 1} dH_i(s) \right) \\ &\quad \times \left(\frac{1}{\Gamma(\beta_2 - \delta_0)} \int_0^1 (1-s)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(s)) ds \right) \\ &= -\frac{1}{\Gamma(\beta_1)} \int_0^t [t^{\beta_1-1} (1-s)^{\beta_1 - \gamma_0 - 1} - (t-s)^{\beta_1-1}] \varphi_{\rho_1}(u(s)) ds \\ &\quad - \frac{1}{\Gamma(\beta_1)} \int_t^1 t^{\beta_1-1} (1-s)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(s)) ds \\ &\quad + \frac{1}{\Gamma(\beta_1)} \int_0^1 t^{\beta_1-1} (1-s)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(s)) ds \\ &\quad - \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-s)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(s)) ds \\ &\quad + \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2 - \gamma_i - 1} dH_i(s) \right) \\ &\quad \times \left(\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_1 - \delta_i - 1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(s) \right) \\ &\quad + \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_\tau^1 (s-\tau)^{\beta_2 - \gamma_i - 1} dH_i(s) \right) \varphi_{\rho_2}(v(\tau)) d\tau \\ &\quad - \frac{t^{\beta_1-1}}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^1 \tau^{\beta_2 - \gamma_i - 1} dH_i(\tau) \right) \\ &\quad \times (1-s)^{\beta_2 - \delta_0 - 1} \varphi_{\rho_2}(v(s)) ds \\ &= -\int_0^1 g_1(t, s) \varphi_{\rho_1}(u(s)) ds + \frac{1}{\Gamma(\beta_1)} \int_0^1 t^{\beta_1-1} (1-s)^{\beta_1 - \gamma_0 - 1} \varphi_{\rho_1}(u(s)) ds \end{aligned}$$

$$\begin{aligned}
& - \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-s)^{\beta_1-\gamma_0-1} \varphi_{\rho_1}(u(s)) ds \\
& + \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \\
& \times \left(\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_1-\delta_i-1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(s) \right) \\
& - \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \left[- \int_0^1 \left(\int_s^1 (\tau-s)^{\beta_2-\gamma_i-1} dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \right. \\
& \left. + \int_0^1 \left(\int_0^1 \tau^{\beta_2-\gamma_i-1} (1-s)^{\beta_2-\delta_0-1} dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \right], \quad \forall t \in [0, 1].
\end{aligned}$$

Then we deduce

$$\begin{aligned}
x(t) &= - \int_0^1 g_1(t, s) \varphi_{\rho_1}(u(s)) ds \\
& - \frac{1}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \times \frac{1}{\Gamma(\beta_1)} \int_0^1 t^{\beta_1-1} (1-s)^{\beta_1-\gamma_0-1} \varphi_{\rho_1}(u(s)) ds \\
& + \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \\
& \times \left(\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_1-\delta_i-1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(s) \right) \\
& - \frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \int_0^1 \left(\int_0^1 g_{2i}(\tau, s) dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \\
& = - \int_0^1 g_1(t, s) \varphi_{\rho_1}(u(s)) ds - \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \\
& \times \left[\sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \left(\int_0^1 \tau^{\beta_1-\delta_i-1} dK_i(\tau) \right) \left(\int_0^1 (1-s)^{\beta_1-\gamma_0-1} \varphi_{\rho_1}(u(s)) ds \right) \right. \\
& \left. - \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_1-\delta_i-1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(s) \right] \\
& - \int_0^1 \left(\frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \int_0^1 g_{2i}(\tau, s) dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \\
& = - \int_0^1 g_1(t, s) \varphi_{\rho_1}(u(s)) ds - \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \\
& \times \left\{ \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \left[\int_0^1 \left(\int_0^1 \tau^{\beta_1-\delta_i-1} (1-s)^{\beta_1-\gamma_0-1} dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \right. \right. \\
& \left. \left. - \int_0^1 \left(\int_s^1 (\tau-s)^{\beta_1-\delta_i-1} dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& - \int_0^1 \left(\frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \int_0^1 g_{2i}(\tau, s) dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \\
& = - \int_0^1 g_1(t, s) \varphi_{\rho_1}(u(s)) ds - \frac{t^{\beta_1-1}}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \\
& \quad \times \left[\sum_{i=1}^q \int_0^1 \left(\int_0^1 g_{1i}(\tau, s) dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \right] \\
& \quad - \int_0^1 \left(\frac{t^{\beta_1-1} \Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \int_0^1 g_{2i}(\tau, s) dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \\
& = - \int_0^1 \mathcal{G}_1(t, s) \varphi_{\rho_1}(u(s)) ds - \int_0^1 \mathcal{G}_2(t, s) \varphi_{\rho_2}(v(s)) ds, \quad \forall t \in [0, 1],
\end{aligned}$$

where $g_1, g_{1i}, i = 1, \dots, q, g_{2i}, i = 1, \dots, p$, are given by (17), and $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by (16).

For $y(t)$ we find

$$\begin{aligned}
y(t) &= \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_0^s (s - \tau)^{\beta_1-\delta_i-1} \varphi_{\rho_1}(u(\tau)) d\tau \right) dK_i(s) \\
& \quad - \frac{t^{\beta_2-1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \quad \times \left(\frac{1}{\Gamma(\beta_1 - \gamma_0)} \int_0^1 (1 - s)^{\beta_1-\gamma_0-1} \varphi_{\rho_1}(u(s)) ds \right) \\
& \quad + \frac{1}{\Gamma(\beta_2)} \int_0^t (t - s)^{\beta_2-1} \varphi_{\rho_2}(v(s)) ds \\
& \quad - \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1 - s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds \\
& \quad + \frac{t^{\beta_2-1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \quad \times \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^s (s - \tau)^{\beta_2-\gamma_i-1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(s) \right) \\
& = \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_\tau^1 (s - \tau)^{\beta_1-\delta_i-1} dK_i(s) \right) \varphi_{\rho_1}(u(\tau)) d\tau \\
& \quad - \frac{t^{\beta_2-1}}{\Delta} \sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i) \Gamma(\beta_1 - \gamma_0)} \\
& \quad \times \int_0^1 \left(\int_0^1 \tau^{\beta_1-\delta_i-1} (1 - s)^{\beta_1-\gamma_0-1} dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& \quad - \frac{1}{\Gamma(\beta_2)} \int_0^t [t^{\beta_2-1} (1 - s)^{\beta_2-\delta_0-1} - (t - s)^{\beta_2-1}] \varphi_{\rho_2}(v(s)) ds \\
& \quad - \frac{1}{\Gamma(\beta_2)} \int_t^1 t^{\beta_2-1} (1 - s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds \\
& \quad + \frac{1}{\Gamma(\beta_2)} \int_0^1 t^{\beta_2-1} (1 - s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds
\end{aligned}$$

$$\begin{aligned}
& - \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds \\
& + \frac{t^{\beta_2-1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \times \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_2-\gamma_i-1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(s) \right) \\
& = \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \left(\int_s^1 (\tau-s)^{\beta_1-\delta_i-1} dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& - \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\beta_1 - \delta_i)} \\
& \times \int_0^1 \left(\int_0^1 \tau^{\beta_1-\delta_i-1} (1-s)^{\beta_1-\gamma_0-1} dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& - \int_0^1 g_2(t,s) \varphi_{\rho_2}(v(s)) ds + \frac{1}{\Gamma(\beta_2)} \int_0^1 t^{\beta_2-1} (1-s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds \\
& - \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} \int_0^1 (1-s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds \\
& + \frac{t^{\beta_2-1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \times \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_2-\gamma_i-1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(s) \right), \quad \forall t \in [0, 1].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
y(t) & = - \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \int_0^1 \left(\int_0^1 g_{1i}(\tau, s) dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& - \int_0^1 g_2(t,s) \varphi_{\rho_2}(v(s)) ds \\
& - \frac{1}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \times \frac{1}{\Gamma(\beta_2)} \int_0^1 t^{\beta_2-1} (1-s)^{\beta_2-\delta_0-1} \varphi_{\rho_2}(v(s)) ds \\
& + \frac{t^{\beta_2-1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\
& \times \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta_2-\gamma_i-1} \varphi_{\rho_2}(v(\tau)) d\tau \right) dH_i(s) \right) \\
& = - \frac{t^{\beta_2-1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \int_0^1 \left(\int_0^1 g_{1i}(\tau, s) dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& - \int_0^1 g_2(t,s) \varphi_{\rho_2}(v(s)) ds - \frac{t^{\beta_2-1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_0^1 \tau^{\beta_2 - \gamma_i - 1} (1-s)^{\beta_2 - \delta_0 - 1} dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \right. \\
& \left. - \sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \left(\int_\tau^1 (s-\tau)^{\beta_2 - \gamma_i - 1} dH_i(s) \right) \varphi_{\rho_2}(v(\tau)) d\tau \right] \\
& = - \frac{t^{\beta_2 - 1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \int_0^1 \left(\int_0^1 g_{1i}(\tau, s) dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& \quad - \int_0^1 g_2(t, s) \varphi_{\rho_2}(v(s)) ds - \frac{t^{\beta_2 - 1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1 - \delta_i - 1} dK_i(s) \right) \\
& \quad \times \left\{ \sum_{i=1}^p \frac{1}{\Gamma(\beta_2 - \gamma_i)} \left[\int_0^1 \left(\int_0^1 \tau^{\beta_2 - \gamma_i - 1} (1-s)^{\beta_2 - \delta_0 - 1} dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \right. \right. \\
& \quad \left. \left. - \int_0^1 \left(\int_s^1 (\tau-s)^{\beta_2 - \gamma_i - 1} dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \right] \right\} \\
& = - \frac{t^{\beta_2 - 1} \Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \int_0^1 \left(\int_0^1 g_{1i}(\tau, s) dK_i(\tau) \right) \varphi_{\rho_1}(u(s)) ds \\
& \quad - \int_0^1 g_2(t, s) \varphi_{\rho_2}(v(s)) ds - \frac{t^{\beta_2 - 1}}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1 - \delta_i - 1} dK_i(s) \right) \\
& \quad \times \left[\sum_{i=1}^p \int_0^1 \left(\int_0^1 g_{2i}(\tau, s) dH_i(\tau) \right) \varphi_{\rho_2}(v(s)) ds \right] \\
& = - \int_0^1 \mathcal{G}_3(t, s) \varphi_{\rho_1}(u(s)) ds - \int_0^1 \mathcal{G}_4(t, s) \varphi_{\rho_2}(v(s)) ds, \quad \forall t \in [0, 1],
\end{aligned}$$

where g_{1i} , $i = 1, \dots, q$, g_2 , g_{2i} , $i = 1, \dots, p$, are given by (17), and $\mathcal{G}_3$ and $\mathcal{G}_4$ are given by (16). $\square$

Therefore, by (8), (9) and (15) we deduce the following lemma.

Lemma 2.3 *If $\Delta \neq 0$, then the unique solution $(x, y) \in C[0, 1] \times C[0, 1]$ of problem (3), (2) is given by*

$$\begin{cases} x(t) = \int_0^1 \mathcal{G}_1(t, v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} \psi(v)) dv + \int_0^1 \mathcal{G}_2(t, v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} \chi(v)) dv, & \forall t \in [0, 1], \\ y(t) = \int_0^1 \mathcal{G}_3(t, v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} \psi(v)) dv + \int_0^1 \mathcal{G}_4(t, v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} \chi(v)) dv, & \forall t \in [0, 1]. \end{cases} \quad (18)$$

In the next lemma, we present some properties of the functions g_1 , g_2 , g_{1i} , $i = 1, \dots, q$, and g_{2i} , $i = 1, \dots, p$.

Lemma 2.4 ([19]) *The functions g_1 , g_2 , g_{1i} , $i = 1, \dots, q$, g_{2i} , $i = 1, \dots, p$, given by (17) have the properties:*

- The functions g_1, g_2, g_{1i} , $i = 1, \dots, q$, g_{2i} , $i = 1, \dots, p$, are continuous on $[0, 1] \times [0, 1]$; $g_1(t, v) \geq 0$, $g_2(t, v) \geq 0$, $g_{1i}(t, v) \geq 0$, $i = 1, \dots, q$, $g_{2i}(t, v) \geq 0$, $i = 1, \dots, p$, for all $(t, v) \in [0, 1] \times [0, 1]$; $g_1(t, v) > 0$, $g_2(t, v) > 0$, $g_{1i}(t, v) > 0$, $i = 1, \dots, q$, $g_{2i}(t, v) > 0$, $i = 1, \dots, p$, for all $(t, v) \in (0, 1) \times (0, 1)$;
- $g_1(t, v) \leq h_1(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where $h_1(v) = \frac{1}{\Gamma(\beta_1)}(1-v)^{\beta_1 - \gamma_0 - 1}(1 - (1-v)^{\gamma_0})$, $\forall v \in [0, 1]$;

- (c) $g_1(t, v) \geq t^{\beta_1-1} h_1(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$;
- (d) $g_2(t, v) \leq h_2(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$h_2(v) = \frac{1}{\Gamma(\beta_2)} (1-v)^{\beta_2-\delta_0-1} (1-(1-v)^{\delta_0}), \forall v \in [0, 1];$$
- (e) $g_2(t, v) \geq t^{\beta_1-1} h_2(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$;
- (f) $g_{1i}(t, v) \leq \frac{1}{\Gamma(\beta_1-\delta_i)} t^{\beta_1-\delta_i-1} (1-v)^{\beta_1-\gamma_0-1}$ for all $(t, v) \in [0, 1] \times [0, 1]$, $i = 1, \dots, q$;
- (g) $g_{1i}(t, v) \geq t^{\beta_1-\delta_i-1} h_{1i}(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$h_{1i}(v) = \frac{1}{\Gamma(\beta_1-\delta_i)} (1-v)^{\beta_1-\gamma_0-1} (1-(1-v)^{\gamma_0-\delta_i}), \forall v \in [0, 1], i = 1, \dots, q;$$
- (h) $g_{2i}(t, v) \leq \frac{1}{\Gamma(\beta_2-\gamma_i)} t^{\beta_2-\gamma_i-1} (1-v)^{\beta_2-\delta_0-1}$ for all $(t, v) \in [0, 1] \times [0, 1]$, $i = 1, \dots, p$;
- (i) $g_{2i}(t, v) \geq t^{\beta_2-\gamma_i-1} h_{2i}(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$h_{2i}(v) = \frac{1}{\Gamma(\beta_2-\gamma_i)} (1-v)^{\beta_2-\delta_0-1} (1-(1-v)^{\delta_0-\gamma_i}), \forall v \in [0, 1], i = 1, \dots, p.$$

By using Lemma 2.4, we obtain the following properties for the functions $\mathcal{G}_i$, $i = 1, \dots, 4$.

Lemma 2.5 Assume that $\Delta > 0$, H_i , $i = 1, \dots, p$, K_i , $i = 1, \dots, q$, are nondecreasing functions. Then the functions $\mathcal{G}_i$, $i = 1, \dots, 4$, given by (16) have the properties:

- (a) $\mathcal{G}_i : [0, 1] \times [0, 1] \rightarrow [0, \infty)$, $i = 1, \dots, 4$, are continuous functions;
- (b) $\mathcal{G}_1(t, v) \leq \mathcal{J}_1(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$\begin{aligned} \mathcal{J}_1(v) = & h_1(v) + \frac{1}{\Delta} \left(\sum_{i=1}^p \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 s^{\beta_2-\gamma_i-1} dH_i(s) \right) \\ & \times \left(\sum_{i=1}^q \int_0^1 g_{1i}(\tau, v) dK_i(\tau) \right), \quad \forall v \in [0, 1]; \end{aligned}$$

- (c) $\mathcal{G}_1(t, v) \geq t^{\beta_1-1} \mathcal{J}_1(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$;
- (d) $\mathcal{G}_2(t, v) \leq \mathcal{J}_2(v)$, for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$\mathcal{J}_2(v) = \frac{\Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^p \int_0^1 g_{2i}(\tau, v) dH_i(\tau), \quad \forall v \in [0, 1];$$

- (e) $\mathcal{G}_2(t, v) = t^{\beta_1-1} \mathcal{J}_2(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$;
- (f) $\mathcal{G}_3(t, v) \leq \mathcal{J}_3(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$\mathcal{J}_3(v) = \frac{\Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^q \int_0^1 g_{1i}(\tau, v) dK_i(\tau), \quad \forall v \in [0, 1];$$

- (g) $\mathcal{G}_3(t, v) = t^{\beta_2-1} \mathcal{J}_3(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$;
- (h) $\mathcal{G}_4(t, v) \leq \mathcal{J}_4(v)$ for all $(t, v) \in [0, 1] \times [0, 1]$, where

$$\begin{aligned} \mathcal{J}_4(v) = & h_2(v) + \frac{1}{\Delta} \left(\sum_{i=1}^q \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \delta_i)} \int_0^1 s^{\beta_1-\delta_i-1} dK_i(s) \right) \\ & \times \left(\sum_{i=1}^p \int_0^1 g_{2i}(\tau, v) dH_i(\tau) \right), \quad \forall v \in [0, 1]; \end{aligned}$$

- (i) $\mathcal{G}_4(t, v) \geq t^{\beta_2-1} \mathcal{J}_4(v)$, for all $(t, v) \in [0, 1] \times [0, 1]$.

3 Existence of positive solutions

In this section we present sufficient conditions for the functions f and g , and intervals for the parameters λ and μ such that problem (1), (2) has at least one positive solution.

We give now the basic assumptions that we will use in the main results.

(I1) $\alpha_1, \alpha_2 \in (0, 1]$, $\beta_1 \in (n-1, n]$, $\beta_2 \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 3$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, 1, \dots, p$, $0 \leq \gamma_1 < \gamma_2 < \dots < \gamma_p \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $\delta_i \in \mathbb{R}$ for all $i = 0, 1, \dots, q$, $0 \leq \delta_1 < \delta_2 < \dots < \delta_q \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $H_i : [0, 1] \rightarrow \mathbb{R}$, $i = 1, \dots, p$, and $K_j : [0, 1] \rightarrow \mathbb{R}$, $j = 1, \dots, q$, are nondecreasing functions, $\exists i_0 \in \{1, \dots, p\}$, such that $H_{i_0}(1) > H_{i_0}(0)$, $\exists j_0 \in \{1, \dots, q\}$, such that $K_{j_0}(1) > K_{j_0}(0)$, $\lambda > 0$, $\mu > 0$, $\Delta > 0$ (Δ is given by (10)), $\varrho_i > 1$, $\varphi_{\varrho_i}(s) = |s|^{\varrho_i-2}s$, $\varphi_{\varrho_i}^{-1} = \varphi_{\rho_i}$, $\rho_i = \frac{\varrho_i}{\varrho_i-1}$, $i = 1, 2$.

(I2) The functions $f, g : [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous.

For $[\theta_1, \theta_2] \subset [0, 1]$ with $0 < \theta_1 < \theta_2 \leq 1$, we introduce the following extreme limits:

$$\begin{aligned} f_0^s &= \limsup_{\substack{x+y \rightarrow 0^+ \\ x, y \geq 0}} \max_{t \in [0, 1]} \frac{f(t, x, y)}{\varphi_{\varrho_1}(x+y)}, & g_0^s &= \limsup_{\substack{x+y \rightarrow 0^+ \\ x, y \geq 0}} \max_{t \in [0, 1]} \frac{g(t, x, y)}{\varphi_{\varrho_2}(x+y)}, \\ f_0^i &= \liminf_{\substack{x+y \rightarrow 0^+ \\ x, y \geq 0}} \min_{t \in [\theta_1, \theta_2]} \frac{f(t, x, y)}{\varphi_{\varrho_1}(x+y)}, & g_0^i &= \liminf_{\substack{x+y \rightarrow 0^+ \\ x, y \geq 0}} \min_{t \in [\theta_1, \theta_2]} \frac{g(t, x, y)}{\varphi_{\varrho_2}(x+y)}, \\ f_\infty^s &= \limsup_{\substack{x+y \rightarrow \infty \\ x, y \geq 0}} \max_{t \in [0, 1]} \frac{f(t, x, y)}{\varphi_{\varrho_1}(x+y)}, & g_\infty^s &= \limsup_{\substack{x+y \rightarrow \infty \\ x, y \geq 0}} \max_{t \in [0, 1]} \frac{g(t, x, y)}{\varphi_{\varrho_2}(x+y)}, \\ f_\infty^i &= \liminf_{\substack{x+y \rightarrow \infty \\ x, y \geq 0}} \min_{t \in [\theta_1, \theta_2]} \frac{f(t, x, y)}{\varphi_{\varrho_1}(x+y)}, & g_\infty^i &= \liminf_{\substack{x+y \rightarrow \infty \\ x, y \geq 0}} \min_{t \in [\theta_1, \theta_2]} \frac{g(t, x, y)}{\varphi_{\varrho_2}(x+y)}. \end{aligned}$$

By using Lemma 2.3 (Eqs. (18)), (x, y) is a solution of problem (1), (2) if and only if (x, y) is a solution of the following nonlinear system of integral equations:

$$\begin{cases} x(t) = \lambda^{\rho_1-1} \int_0^1 \mathcal{G}_1(t, v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ \quad + \mu^{\rho_2-1} \int_0^1 \mathcal{G}_2(t, v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, & t \in [0, 1], \\ y(t) = \lambda^{\rho_1-1} \int_0^1 \mathcal{G}_3(t, v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ \quad + \mu^{\rho_2-1} \int_0^1 \mathcal{G}_4(t, v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, & t \in [0, 1]. \end{cases}$$

We consider the Banach space $\mathcal{X} = C[0, 1]$ with the supremum norm

$$\|x\| = \sup_{t \in [0, 1]} |x(t)|,$$

and the Banach space $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ with the norm $\|(x, y)\|_{\mathcal{Y}} = \|x\| + \|y\|$. We define the cones

$$\begin{aligned} \mathcal{P}_1 &= \{x \in \mathcal{X}, x(t) \geq t^{\beta_1-1} \|x\|, \forall t \in [0, 1]\} \subset \mathcal{X}, \\ \mathcal{P}_2 &= \{y \in \mathcal{X}, y(t) \geq t^{\beta_2-1} \|y\|, \forall t \in [0, 1]\} \subset \mathcal{X}, \end{aligned}$$

and $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \subset \mathcal{Y}$.

We also define the operators $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{Y} \rightarrow \mathcal{X}$ and $\mathcal{A} : \mathcal{Y} \rightarrow \mathcal{Y}$ by

$$\begin{aligned}\mathcal{A}_1(x, y)(t) &= \lambda^{\rho_1-1} \int_0^1 \mathcal{G}_1(t, v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{G}_2(t, v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, \quad t \in [0, 1], \\ \mathcal{A}_2(x, y)(t) &= \lambda^{\rho_1-1} \int_0^1 \mathcal{G}_3(t, v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{G}_4(t, v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, \quad t \in [0, 1],\end{aligned}$$

and $\mathcal{A}(x, y) = (\mathcal{A}_1(x, y), \mathcal{A}_2(x, y))$, $(x, y) \in \mathcal{Y}$. The pair (x, y) is a solution of problem (1), (2) if and only if (x, y) is a fixed point of the operator $\mathcal{A}$.

Lemma 3.1 *If (I1) and (I2) hold, then $\mathcal{A} : \mathcal{P} \rightarrow \mathcal{P}$ is a completely continuous operator.*

Proof Let $(x, y) \in \mathcal{P}$ be an arbitrary element. Because $\mathcal{A}_1(x, y)$ and $\mathcal{A}_2(x, y)$ satisfy the problem (3), (2) for $\psi(t) = \lambda f(t, x(t), y(t))$ and $\chi(t) = \mu g(t, x(t), y(t))$, $t \in [0, 1]$, then by Lemma 2.5 we obtain

$$\begin{aligned}\mathcal{A}_1(x, y)(t) &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, \quad \forall t \in [0, 1], \\ \mathcal{A}_2(x, y)(t) &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_3(v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_4(v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, \quad \forall t \in [0, 1],\end{aligned}$$

and then

$$\begin{aligned}\|\mathcal{A}_1(x, y)\| &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv, \\ \|\mathcal{A}_2(x, y)\| &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_3(v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_4(v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv.\end{aligned}$$

Hence by using again Lemma 2.5, we deduce

$$\begin{aligned}\mathcal{A}_1(x, y)(t) &\geq \lambda^{\rho_1-1} \int_0^1 t^{\beta_1-1} \mathcal{J}_1(v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 t^{\beta_1-1} \mathcal{J}_2(v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv \\ &\geq t^{\beta_1-1} \|\mathcal{A}_1(x, y)\|, \quad \forall t \in [0, 1],\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_2(x, y)(t) &\geq \lambda^{\varrho_1-1} \int_0^1 t^{\beta_2-1} \mathcal{J}_3(v) \varphi_{\rho_1} \left(I_{0+}^{\alpha_1} f(v, x(v), y(v)) \right) dv \\
&\quad + \mu^{\rho_2-1} \int_0^1 t^{\beta_2-1} \mathcal{J}_4(v) \varphi_{\rho_2} \left(I_{0+}^{\alpha_2} g(v, x(v), y(v)) \right) dv \\
&\geq t^{\beta_2-1} \|\mathcal{A}_2(x, y)\|, \quad \forall t \in [0, 1].
\end{aligned}$$

Therefore $\mathcal{A}(x, y) = (\mathcal{A}_1(x, y), \mathcal{A}_2(x, y)) \in \mathcal{P}$, and then $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. By using the continuity of the functions $f, g, \mathcal{G}_i, i = 1, \dots, 4$, and the Ascoli–Arzela theorem, we can prove that $\mathcal{A}_1$ and $\mathcal{A}_2$ are completely continuous operators (compact operators, that is, they map bounded sets into relatively compact sets, and continuous), and then $\mathcal{A}$ is a completely continuous operator. $\square$

For $[\theta_1, \theta_2] \subset [0, 1]$ with $0 < \theta_1 < \theta_2 \leq 1$, we denote

$$\begin{aligned}
L_1 &= \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} \int_0^1 v^{\alpha_1(\rho_1-1)} \mathcal{J}_1(v) dv, \\
L_2 &= \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} \int_0^1 v^{\alpha_2(\rho_2-1)} \mathcal{J}_2(v) dv, \\
L_3 &= \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} \int_0^1 v^{\alpha_1(\rho_1-1)} \mathcal{J}_3(v) dv, \\
L_4 &= \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} \int_0^1 v^{\alpha_2(\rho_2-1)} \mathcal{J}_4(v) dv, \\
\tilde{L}_1 &= \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} \int_{\theta_1}^{\theta_2} (v - \theta_1)^{\alpha_1(\rho_1-1)} \mathcal{J}_1(v) dv, \\
\tilde{L}_2 &= \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} \int_{\theta_1}^{\theta_2} (v - \theta_1)^{\alpha_2(\rho_2-1)} \mathcal{J}_2(v) dv, \\
\tilde{L}_3 &= \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} \int_{\theta_1}^{\theta_2} (v - \theta_1)^{\alpha_1(\rho_1-1)} \mathcal{J}_3(v) dv, \\
\tilde{L}_4 &= \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} \int_{\theta_1}^{\theta_2} (v - \theta_1)^{\alpha_2(\rho_2-1)} \mathcal{J}_4(v) dv,
\end{aligned} \tag{19}$$

where $\mathcal{J}_i, i = 1, \dots, 4$, are defined in Lemma 2.5.

For $\tilde{f}_0^s, \tilde{g}_0^s, \tilde{f}_\infty^i, \tilde{g}_\infty^i \in (0, \infty)$ and numbers $c_1, c_2 \in [0, 1]$, $c_3, c_4 \in (0, 1)$, $a \in [0, 1]$ and $b \in (0, 1)$, we introduce the numbers

$$\begin{aligned}
A &= \max \left\{ \frac{1}{\tilde{f}_\infty^i} \varphi_{\varrho_1} \left(\frac{ac_1}{\zeta \tilde{\zeta}_1 \tilde{L}_1} \right), \frac{1}{\tilde{f}_\infty^i} \varphi_{\varrho_1} \left(\frac{(1-a)c_2}{\zeta \tilde{\zeta}_2 \tilde{L}_3} \right) \right\}, \\
B &= \min \left\{ \frac{1}{\tilde{f}_0^s} \varphi_{\varrho_1} \left(\frac{bc_3}{L_1} \right), \frac{1}{\tilde{f}_0^s} \varphi_{\varrho_1} \left(\frac{(1-b)c_4}{L_3} \right) \right\}, \\
C &= \max \left\{ \frac{1}{\tilde{g}_\infty^i} \varphi_{\varrho_2} \left(\frac{a(1-c_1)}{\zeta \tilde{\zeta}_1 \tilde{L}_2} \right), \frac{1}{\tilde{g}_\infty^i} \varphi_{\varrho_2} \left(\frac{(1-a)(1-c_2)}{\zeta \tilde{\zeta}_2 \tilde{L}_4} \right) \right\}, \\
D &= \min \left\{ \frac{1}{\tilde{g}_0^s} \varphi_{\varrho_2} \left(\frac{b(1-c_3)}{L_2} \right), \frac{1}{\tilde{g}_0^s} \varphi_{\varrho_2} \left(\frac{(1-b)(1-c_4)}{L_4} \right) \right\}, \\
E &= \min \left\{ \frac{1}{\tilde{f}_0^s} \varphi_{\varrho_1} \left(\frac{b}{L_1} \right), \frac{1}{\tilde{f}_0^s} \varphi_{\varrho_1} \left(\frac{1-b}{L_3} \right) \right\},
\end{aligned}$$

$$F = \min \left\{ \frac{1}{g_0^s} \varphi_{\varrho_2} \left(\frac{b}{L_2} \right), \frac{1}{g_0^s} \varphi_{\varrho_2} \left(\frac{1-b}{L_4} \right) \right\},$$

where $\zeta_1 = \theta_1^{\beta_1-1}$, $\zeta_2 = \theta_1^{\beta_2-1}$, $\zeta = \min\{\zeta_1, \zeta_2\}$.

Theorem 3.1 Assume that (I1) and (I2) hold, $[\theta_1, \theta_2] \subset [0, 1]$ with $0 < \theta_1 < \theta_2 \leq 1$, $c_1, c_2 \in [0, 1]$, $c_3, c_4 \in (0, 1)$, $a \in [0, 1]$ and $b \in (0, 1)$.

- (1) If $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0, \infty)$, $A < B$ and $C < D$, then, for each $\lambda \in (A, B)$ and $\mu \in (C, D)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (2) If $f_0^s = 0$, $g_0^s, f_\infty^i, g_\infty^i \in (0, \infty)$ and $C < F$, then, for each $\lambda \in (A, \infty)$ and $\mu \in (C, F)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (3) If $g_0^s = 0$, $f_0^s, f_\infty^i, g_\infty^i \in (0, \infty)$ and $A < E$, then, for each $\lambda \in (A, E)$ and $\mu \in (C, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (4) If $f_0^s = g_0^s = 0$, $f_\infty^i, g_\infty^i \in (0, \infty)$, then, for each $\lambda \in (A, \infty)$ and $\mu \in (C, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (5) If $f_0^s, g_0^s \in (0, \infty)$ and at least one of f_∞^i, g_∞^i is ∞ , then, for each $\lambda \in (0, B)$ and $\mu \in (0, D)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (6) If $f_0^s = 0$, $g_0^s \in (0, \infty)$ and at least one of f_∞^i, g_∞^i is ∞ , then, for each $\lambda \in (0, \infty)$ and $\mu \in (0, F)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (7) If $f_0^s \in (0, \infty)$, $g_0^s = 0$ and at least one of f_∞^i, g_∞^i is ∞ , then, for each $\lambda \in (0, E)$ and $\mu \in (0, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (8) If $f_0^s = g_0^s = 0$ and at least one of f_∞^i, g_∞^i is ∞ , then, for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.

Proof We consider the cone $\mathcal{P} \subset \mathcal{Y}$ and the operators $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$. The proofs of the above cases are similar, and for this reason we will prove only two cases, namely (1) and (6).

Case (1). We have $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0, \infty)$, $A < B$ and $C < D$. Let $\lambda \in (A, B)$ and $\mu \in (C, D)$. We consider $\epsilon > 0$ such that $\epsilon < f_\infty^i$, $\epsilon < g_\infty^i$ and

$$\begin{aligned} & \max \left\{ \frac{1}{f_\infty^i - \epsilon} \varphi_{\varrho_1} \left(\frac{ac_1}{\zeta \zeta_1 \tilde{L}_1} \right), \frac{1}{f_\infty^i - \epsilon} \varphi_{\varrho_1} \left(\frac{(1-a)c_2}{\zeta \zeta_2 \tilde{L}_3} \right) \right\} \\ & \leq \lambda \leq \min \left\{ \frac{1}{f_0^s + \epsilon} \varphi_{\varrho_1} \left(\frac{bc_3}{L_1} \right), \frac{1}{f_0^s + \epsilon} \varphi_{\varrho_1} \left(\frac{(1-b)c_4}{L_3} \right) \right\}, \\ & \max \left\{ \frac{1}{g_\infty^i - \epsilon} \varphi_{\varrho_2} \left(\frac{a(1-c_1)}{\zeta \zeta_1 \tilde{L}_2} \right), \frac{1}{g_\infty^i - \epsilon} \varphi_{\varrho_2} \left(\frac{(1-a)(1-c_2)}{\zeta \zeta_2 \tilde{L}_4} \right) \right\} \\ & \leq \mu \leq \min \left\{ \frac{1}{g_0^s + \epsilon} \varphi_{\varrho_2} \left(\frac{b(1-c_3)}{L_2} \right), \frac{1}{g_0^s + \epsilon} \varphi_{\varrho_2} \left(\frac{(1-b)(1-c_4)}{L_4} \right) \right\}. \end{aligned}$$

By using (I2) and the definitions of f_0^s and g_0^s , we find that there exists $r_1 > 0$ such that

$$f(t, x, y) \leq (f_0^s + \epsilon) \varphi_{\varrho_1}(x + y), \quad g(t, x, y) \leq (g_0^s + \epsilon) \varphi_{\varrho_2}(x + y),$$

for all $t \in [0, 1]$ and $x, y \geq 0$, $x + y \leq r_1$.

We introduce the set $\Lambda_1 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_1\}$. Let $(x, y) \in \mathcal{P} \cap \partial \Lambda_1$, that is $(x, y) \in \mathcal{P}$ with $\|(x, y)\|_{\mathcal{Y}} = r_1$ or $\|x\| + \|y\| = r_1$. Then $x(t) + y(t) \leq r_1$ for all $t \in [0, 1]$, and by

Lemma 2.5, we deduce

$$\begin{aligned}
 & \mathcal{A}_1(x, y)(t) \\
 & \leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^v (v-s)^{\alpha_1-1} f(s, x(s), y(s)) ds \right) dv \\
 & \quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_0^v (v-s)^{\alpha_2-1} g(s, x(s), y(s)) ds \right) dv \\
 & \leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^v (v-s)^{\alpha_1-1} (f_0^s + \epsilon) \varphi_{\rho_1}(x(s) + y(s)) ds \right) dv \\
 & \quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_0^v (v-s)^{\alpha_2-1} (g_0^s + \epsilon) \varphi_{\rho_2}(x(s) + y(s)) ds \right) dv \\
 & \leq \lambda^{\rho_1-1} \varphi_{\rho_1}(f_0^s + \epsilon) \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^v (v-s)^{\alpha_1-1} \varphi_{\rho_1}(\|x\| + \|y\|) ds \right) dv \\
 & \quad + \mu^{\rho_2-1} \varphi_{\rho_2}(g_0^s + \epsilon) \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_0^v (v-s)^{\alpha_2-1} \varphi_{\rho_2}(\|x\| + \|y\|) ds \right) dv \\
 & = \lambda^{\rho_1-1} \varphi_{\rho_1}(f_0^s + \epsilon) \| (x, y) \|_{\mathcal{Y}} \int_0^1 \mathcal{J}_1(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} v^{\alpha_1(\rho_1-1)} dv \\
 & \quad + \mu^{\rho_2-1} \varphi_{\rho_2}(g_0^s + \epsilon) \| (x, y) \|_{\mathcal{Y}} \int_0^1 \mathcal{J}_2(v) \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} v^{\alpha_2(\rho_2-1)} dv \\
 & = [\varphi_{\rho_1}(\lambda(f_0^s + \epsilon))L_1 + \varphi_{\rho_2}(\mu(g_0^s + \epsilon))L_2] \| (x, y) \|_{\mathcal{Y}} \\
 & \leq [bc_3 + b(1 - c_3)] \| (x, y) \|_{\mathcal{Y}} = b \| (x, y) \|_{\mathcal{Y}}, \quad \forall t \in [0, 1].
 \end{aligned}$$

Therefore $\|\mathcal{A}_1(x, y)\| \leq b \| (x, y) \|_{\mathcal{Y}}$.

In a similar manner we obtain

$$\begin{aligned}
 \mathcal{A}_2(x, y)(t) & \leq \lambda^{\rho_1-1} \varphi_{\rho_1}(f_0^s + \epsilon) \| (x, y) \|_{\mathcal{Y}} \int_0^1 \mathcal{J}_3(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} v^{\alpha_1(\rho_1-1)} dv \\
 & \quad + \mu^{\rho_2-1} \varphi_{\rho_2}(g_0^s + \epsilon) \| (x, y) \|_{\mathcal{Y}} \int_0^1 \mathcal{J}_4(v) \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} v^{\alpha_2(\rho_2-1)} dv \\
 & = [\varphi_{\rho_1}(\lambda(f_0^s + \epsilon))L_3 + \varphi_{\rho_2}(\mu(g_0^s + \epsilon))L_4] \| (x, y) \|_{\mathcal{Y}} \\
 & \leq [(1 - b)c_4 + (1 - b)(1 - c_4)] \| (x, y) \|_{\mathcal{Y}} = (1 - b) \| (x, y) \|_{\mathcal{Y}}, \quad \forall t \in [0, 1].
 \end{aligned}$$

Then $\|\mathcal{A}_2(x, y)\| \leq (1 - b) \| (x, y) \|_{\mathcal{Y}}$.

Therefore, for $(x, y) \in \mathcal{P} \cap \partial \mathcal{A}_1$, we conclude

$$\|\mathcal{A}(x, y)\|_{\mathcal{Y}} = \|\mathcal{A}_1(x, y)\| + \|\mathcal{A}_2(x, y)\| \leq b \| (x, y) \|_{\mathcal{Y}} + (1 - b) \| (x, y) \|_{\mathcal{Y}} = \| (x, y) \|_{\mathcal{Y}}. \quad (20)$$

By the definitions of f_{∞}^i and g_{∞}^i , we find that there exists $r'_2 > 0$ such that

$$f(t, x, y) \geq (f_{\infty}^i - \epsilon) \varphi_{\rho_1}(x + y), \quad g(t, x, y) \geq (g_{\infty}^i - \epsilon) \varphi_{\rho_2}(x + y),$$

for all $t \in [\theta_1, \theta_2]$ and $x, y \geq 0, x + y \geq r'_2$.

We consider $r_2 = \max\{2r_1, r'_2/\zeta\}$ and we introduce the set $\Lambda_2 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_2\}$. Then, for $(x, y) \in \mathcal{P} \cap \partial \Lambda_2$, we deduce

$$\begin{aligned} x(t) + y(t) &\geq \min_{t \in [\theta_1, \theta_2]} t^{\beta_1-1} \|x\| + \min_{t \in [\theta_1, \theta_2]} t^{\beta_2-1} \|y\| = \theta_1^{\beta_1-1} \|x\| + \theta_1^{\beta_2-1} \|y\| \\ &= \zeta_1 \|x\| + \zeta_2 \|y\| \geq \zeta \|(x, y)\|_{\mathcal{Y}} = \zeta r_2 \geq r'_2, \quad \forall t \in [\theta_1, \theta_2]. \end{aligned}$$

Therefore, by Lemma 2.5, we obtain

$$\begin{aligned} &\mathcal{A}_1(x, y)(\theta_1) \\ &\geq \lambda^{\rho_1-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_1(v) \varphi_{\rho_1} (I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_2(v) \varphi_{\rho_2} (I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv \\ &\geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} f(s, x(s), y(s)) ds \right) dv \\ &\quad + \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} g(s, x(s), y(s)) ds \right) dv \\ &\geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} (\mathfrak{f}_{\infty}^i - \epsilon) \varphi_{\varrho_1}(x(s) + y(s)) ds \right) dv \\ &\quad + \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} (\mathfrak{g}_{\infty}^i - \epsilon) \varphi_{\varrho_1}(x(s) + y(s)) ds \right) dv \\ &\geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} (\mathfrak{f}_{\infty}^i - \epsilon) \varphi_{\varrho_1}(\zeta \|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &\quad + \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} (\mathfrak{g}_{\infty}^i - \epsilon) \varphi_{\varrho_1}(\zeta \|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &= \zeta \zeta_1 \varphi_{\rho_1}(\lambda(\mathfrak{f}_{\infty}^i - \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} (v - \theta_1)^{\alpha_1(\rho_1-1)} dv \\ &\quad + \zeta \zeta_1 \varphi_{\rho_2}(\mu(\mathfrak{g}_{\infty}^i - \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} (v - \theta_1)^{\alpha_2(\rho_2-1)} dv \\ &= [\zeta \zeta_1 \varphi_{\rho_1}(\lambda(\mathfrak{f}_{\infty}^i - \epsilon)) \tilde{L}_1 + \zeta \zeta_1 \varphi_{\rho_2}(\mu(\mathfrak{g}_{\infty}^i - \epsilon)) \tilde{L}_2] \|(x, y)\|_{\mathcal{Y}} \\ &\geq [ac_1 + a(1 - c_1)] \|(x, y)\|_{\mathcal{Y}} = a \|(x, y)\|_{\mathcal{Y}}. \end{aligned}$$

Then $\|\mathcal{A}_1(x, y)\| \geq \mathcal{A}_1(x, y)(\theta_1) \geq a \|(x, y)\|_{\mathcal{Y}}$.

In a similar manner, we deduce

$$\begin{aligned} &\mathcal{A}_2(x, y)(\theta_1) \\ &\geq \zeta \zeta_2 \varphi_{\rho_1}(\lambda(\mathfrak{f}_{\infty}^i - \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_3(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} (v - \theta_1)^{\alpha_1(\rho_1-1)} dv \\ &\quad + \zeta \zeta_2 \varphi_{\rho_2}(\mu(\mathfrak{g}_{\infty}^i - \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_4(v) \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} (v - \theta_1)^{\alpha_2(\rho_2-1)} dv \end{aligned}$$

$$\begin{aligned}
&= [\zeta \zeta_2 \varphi_{\rho_1}(\lambda(\mathfrak{f}_\infty^i - \epsilon))\tilde{L}_3 + \zeta \zeta_2 \varphi_{\rho_2}(\mu(\mathfrak{g}_\infty^i - \epsilon))\tilde{L}_4] \|(x, y)\|_{\mathcal{Y}} \\
&\geq [(1-a)c_2 + (1-a)(1-c_2)] \|(x, y)\|_{\mathcal{Y}} = (1-a) \|(x, y)\|_{\mathcal{Y}}.
\end{aligned}$$

Hence $\|\mathcal{A}_2(x, y)\| \geq \mathcal{A}_2(x, y)(\theta_1) \geq (1-a) \|(x, y)\|_{\mathcal{Y}}$.

Therefore for $(x, y) \in \mathcal{P} \cap \partial \Lambda_2$ we find

$$\|\mathcal{A}(x, y)\|_{\mathcal{Y}} = \|\mathcal{A}_1(x, y)\| + \|\mathcal{A}_2(x, y)\| \geq a \|(x, y)\|_{\mathcal{Y}} + (1-a) \|(x, y)\|_{\mathcal{Y}} = \|(x, y)\|_{\mathcal{Y}}. \quad (21)$$

By Lemma 3.1, Eqs. (20), (21) and the Guo–Krasnosel'skii fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $(x, y) \in \mathcal{P} \cap (\bar{\Lambda}_2 \setminus \Lambda_1)$ such that $r_1 \leq \|x\| + \|y\| \leq r_2$, $x(t) \geq t^{\beta_1-1} \|x\|$, $y(t) \geq t^{\beta_2-1} \|y\|$ for all $t \in [0, 1]$. If $\|x\| > 0$ then $x(t) > 0$ for all $t \in (0, 1]$ and if $\|y\| > 0$ then $y(t) > 0$ for all $t \in (0, 1]$. Therefore (x, y) is a positive solution for problem (1), (2).

Case (6). We have $\mathfrak{f}_0^s = 0$, $\mathfrak{g}_0^s \in (0, \infty)$ and $\mathfrak{f}_\infty^i = \infty$. Let $\lambda \in (0, \infty)$ and $\mu \in (0, F)$. We choose $\tilde{c}_3 \in (0, 1 - \varphi_{\rho_2}(\mu \mathfrak{g}_0^s) \frac{L_2}{b})$ and $\tilde{c}_4 \in (0, 1 - \varphi_{\rho_2}(\mu \mathfrak{g}_0^s) \frac{L_4}{1-b})$. The choice of $\tilde{c}_3$ and $\tilde{c}_4$ is possible because $\mu < \frac{1}{\mathfrak{g}_0^s} \varphi_{\rho_2}(\frac{b}{L_2})$ and $\mu < \frac{1}{\mathfrak{g}_0^s} \varphi_{\rho_2}(\frac{1-b}{L_4})$. Let $\epsilon > 0$ be such that

$$\begin{aligned}
\epsilon \varphi_{\rho_1}\left(\frac{1}{\zeta \zeta_1 \tilde{L}_1}\right) &\leq \lambda \leq \min\left\{\frac{1}{\epsilon} \varphi_{\rho_1}\left(\frac{b \tilde{c}_3}{L_1}\right), \frac{1}{\epsilon} \varphi_{\rho_1}\left(\frac{(1-b) \tilde{c}_4}{L_3}\right)\right\}, \\
\mu &\leq \min\left\{\frac{1}{\mathfrak{g}_0^s + \epsilon} \varphi_{\rho_2}\left(\frac{b(1-\tilde{c}_3)}{L_2}\right), \frac{1}{\mathfrak{g}_0^s + \epsilon} \varphi_{\rho_2}\left(\frac{(1-b)(1-\tilde{c}_4)}{L_4}\right)\right\}.
\end{aligned}$$

By using (I2) and the definitions of $\mathfrak{f}_0^s$ and $\mathfrak{g}_0^s$, we find that there exists $r_1 > 0$ such that

$$f(t, x, y) \leq \epsilon \varphi_{\rho_1}(x + y), \quad g(t, x, y) \leq (\mathfrak{g}_0^s + \epsilon) \varphi_{\rho_2}(x + y)$$

for all $t \in [0, 1]$ and $x, y \geq 0$, $x + y \leq r_1$.

We introduce the set $\Lambda_1 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_1\}$. As in the proof of Case (1), for any $(x, y) \in \mathcal{P} \cap \partial \Lambda_1$, we deduce

$$\begin{aligned}
\mathcal{A}_1(x, y)(t) &\leq [\varphi_{\rho_1}(\lambda \epsilon) L_1 + \varphi_{\rho_2}(\mu(\mathfrak{g}_0^s + \epsilon)) L_2] \|(x, y)\|_{\mathcal{Y}} \\
&\leq [b \tilde{c}_3 + b(1 - \tilde{c}_3)] \|(x, y)\|_{\mathcal{Y}} = b \|(x, y)\|_{\mathcal{Y}}, \quad \forall t \in [0, 1], \\
\mathcal{A}_2(x, y)(t) &\leq [\varphi_{\rho_1}(\lambda \epsilon) L_3 + \varphi_{\rho_2}(\mu(\mathfrak{g}_0^s + \epsilon)) L_4] \|(x, y)\|_{\mathcal{Y}} \\
&\leq [(1-b) \tilde{c}_4 + (1-b)(1 - \tilde{c}_4)] \|(x, y)\|_{\mathcal{Y}} = (1-b) \|(x, y)\|_{\mathcal{Y}}, \quad \forall t \in [0, 1],
\end{aligned}$$

and so $\|\mathcal{A}(x, y)\|_{\mathcal{Y}} \leq \|(x, y)\|_{\mathcal{Y}}$.

For the second part, by the definition of $\mathfrak{f}_\infty^i$, we find that there exists $r'_2 > 0$ such that

$$f(t, x, y) \geq \frac{1}{\epsilon} \varphi_{\rho_1}(x + y), \quad \forall t \in [\theta_1, \theta_2], x, y \geq 0, x + y \geq r'_2.$$

We consider $r_2 = \max\{2r_1, r'_2/\zeta\}$ and we introduce the set $\Lambda_2 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_2\}$. Then, for any $(x, y) \in \mathcal{P} \cap \partial \Lambda_2$, we obtain as in Case (1) that $x(t) + y(t) \geq \zeta \|(x, y)\|_{\mathcal{Y}} = \zeta r_2 \geq r'_2$ for all $t \in [\theta_1, \theta_2]$.

Hence by Lemma 2.5 we deduce

$$\begin{aligned}
 & \mathcal{A}_1(x, y)(\theta_1) \\
 & \geq \lambda^{\rho_1-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_1(v) \varphi_{\rho_1} \left(I_{0+}^{\alpha_1} f(v, x(v), y(v)) \right) dv \\
 & \quad + \mu^{\rho_2-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_2(v) \varphi_{\rho_2} \left(I_{0+}^{\alpha_2} g(v, x(v), y(v)) \right) dv \\
 & \geq \lambda^{\rho_1-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_1(v) \varphi_{\rho_1} \left(I_{0+}^{\alpha_1} f(v, x(v), y(v)) \right) dv \\
 & \geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} f(s, x(s), y(s)) ds \right) dv \\
 & \geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} \frac{1}{\epsilon} \varphi_{\rho_1}(x(s) + y(s)) ds \right) dv \\
 & \geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} \frac{1}{\epsilon} \varphi_{\rho_1}(\zeta \| (x, y) \|_Y) ds \right) dv \\
 & = \zeta \zeta_1 \varphi_{\rho_1} \left(\frac{\lambda}{\epsilon} \right) \| (x, y) \|_Y \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} (v - \theta_1)^{\alpha_1(\rho_1-1)} dv \\
 & = \zeta \zeta_1 \varphi_{\rho_1} \left(\frac{\lambda}{\epsilon} \right) \| (x, y) \|_Y \tilde{L}_1 \geq \| (x, y) \|_Y.
 \end{aligned}$$

Then we conclude that $\| \mathcal{A}_1(x, y) \| \geq \mathcal{A}_1(x, y)(\theta_1) \geq \| (x, y) \|_Y$ and $\| \mathcal{A}(x, y) \|_Y \geq \| \mathcal{A}_1(x, y) \| \geq \| (x, y) \|_Y$. Therefore we obtain the conclusion of the theorem. $\square$

Next for $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0, \infty)$ and numbers $c_1, c_2 \in [0, 1]$, $c_3, c_4 \in (0, 1)$, $a \in [0, 1]$ and $b \in (0, 1)$, we introduce the numbers

$$\begin{aligned}
 \tilde{A} &= \max \left\{ \frac{1}{f_0^i} \varphi_{\rho_1} \left(\frac{ac_1}{\zeta \zeta_1 \tilde{L}_1} \right), \frac{1}{f_0^i} \varphi_{\rho_1} \left(\frac{(1-a)c_2}{\zeta \zeta_2 \tilde{L}_3} \right) \right\}, \\
 \tilde{B} &= \min \left\{ \frac{1}{f_\infty^s} \varphi_{\rho_1} \left(\frac{bc_3}{L_1} \right), \frac{1}{f_\infty^s} \varphi_{\rho_1} \left(\frac{(1-b)c_4}{L_3} \right) \right\}, \\
 \tilde{C} &= \max \left\{ \frac{1}{g_0^i} \varphi_{\rho_2} \left(\frac{a(1-c_1)}{\zeta \zeta_1 \tilde{L}_2} \right), \frac{1}{g_0^i} \varphi_{\rho_2} \left(\frac{(1-a)(1-c_2)}{\zeta \zeta_2 \tilde{L}_4} \right) \right\}, \\
 \tilde{D} &= \min \left\{ \frac{1}{g_\infty^s} \varphi_{\rho_2} \left(\frac{b(1-c_3)}{L_2} \right), \frac{1}{g_\infty^s} \varphi_{\rho_2} \left(\frac{(1-b)(1-c_4)}{L_4} \right) \right\}, \\
 \tilde{E} &= \min \left\{ \frac{1}{f_\infty^s} \varphi_{\rho_1} \left(\frac{b}{L_1} \right), \frac{1}{f_\infty^s} \varphi_{\rho_1} \left(\frac{1-b}{L_3} \right) \right\}, \\
 \tilde{F} &= \min \left\{ \frac{1}{g_\infty^s} \varphi_{\rho_2} \left(\frac{b}{L_2} \right), \frac{1}{g_\infty^s} \varphi_{\rho_2} \left(\frac{1-b}{L_4} \right) \right\}.
 \end{aligned}$$

Theorem 3.2 Assume that (I1) and (I2) hold, $[\theta_1, \theta_2] \subset [0, 1]$ with $0 < \theta_1 < \theta_2 \leq 1$, $c_1, c_2 \in [0, 1]$, $c_3, c_4 \in (0, 1)$, $a \in [0, 1]$ and $b \in (0, 1)$.

- (1) If $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0, \infty)$, $\tilde{A} < \tilde{B}$ and $\tilde{C} < \tilde{D}$, then, for each $\lambda \in (\tilde{A}, \tilde{B})$ and $\mu \in (\tilde{C}, \tilde{D})$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (2) If $f_0^i, g_0^i, f_\infty^s \in (0, \infty)$, $g_\infty^s = 0$ and $\tilde{A} < \tilde{E}$, then, for each $\lambda \in (\tilde{A}, \tilde{E})$ and $\mu \in (\tilde{C}, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.

- (3) If $f_0^i, g_0^i, g_\infty^s \in (0, \infty)$, $f_\infty^s = 0$ and $\tilde{C} < \tilde{F}$, then, for each $\lambda \in (\tilde{A}, \infty)$ and $\mu \in (\tilde{C}, \tilde{F})$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (4) If $f_0^i, g_0^i \in (0, \infty)$, $f_\infty^s = g_\infty^s = 0$, then, for each $\lambda \in (\tilde{A}, \infty)$ and $\mu \in (\tilde{C}, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (5) If $f_\infty^s, g_\infty^s \in (0, \infty)$ and at least one of f_0^i, g_0^i is ∞ , then, for each $\lambda \in (0, \tilde{B})$ and $\mu \in (0, \tilde{D})$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (6) If $f_\infty^s \in (0, \infty)$, $g_\infty^s = 0$ and at least one of f_0^i, g_0^i is ∞ , then, for each $\lambda \in (0, \tilde{E})$ and $\mu \in (0, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (7) If $f_\infty^s = 0$, $g_\infty^s \in (0, \infty)$ and at least one of f_0^i, g_0^i is ∞ , then, for each $\lambda \in (0, \infty)$ and $\mu \in (0, \tilde{F})$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.
- (8) If $f_\infty^s = g_\infty^s = 0$ and at least one of f_0^i, g_0^i is ∞ , then, for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, the problem (1), (2) has at least one positive solution $(x(t), y(t))$, $t \in [0, 1]$.

Proof We consider again the cone $\mathcal{P} \subset \mathcal{Y}$ and the operators $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$. The proofs of the above cases are similar, and for this reason we will prove only two cases, namely (1) and (6).

Case (1). We have $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0, \infty)$, $\tilde{A} < \tilde{B}$ and $\tilde{C} < \tilde{D}$. Let $\lambda \in (\tilde{A}, \tilde{B})$ and $\mu \in (\tilde{C}, \tilde{D})$. We consider $\epsilon > 0$ such that $\epsilon < f_0^i$, $\epsilon < g_0^i$ and

$$\begin{aligned} & \max \left\{ \frac{1}{f_0^i - \epsilon} \varphi_{e1} \left(\frac{ac_1}{\zeta \zeta_1 \tilde{L}_1} \right), \frac{1}{f_0^i - \epsilon} \varphi_{e1} \left(\frac{(1-a)c_2}{\zeta \zeta_2 \tilde{L}_3} \right) \right\} \\ & \leq \lambda \leq \min \left\{ \frac{1}{f_\infty^s + \epsilon} \varphi_{e1} \left(\frac{bc_3}{L_1} \right), \frac{1}{f_\infty^s + \epsilon} \varphi_{e1} \left(\frac{(1-b)c_4}{L_3} \right) \right\}, \\ & \max \left\{ \frac{1}{g_0^i - \epsilon} \varphi_{e2} \left(\frac{a(1-c_1)}{\zeta \zeta_1 \tilde{L}_2} \right), \frac{1}{g_0^i - \epsilon} \varphi_{e2} \left(\frac{(1-a)(1-c_2)}{\zeta \zeta_2 \tilde{L}_4} \right) \right\} \\ & \leq \mu \leq \min \left\{ \frac{1}{g_\infty^s + \epsilon} \varphi_{e2} \left(\frac{b(1-c_3)}{L_2} \right), \frac{1}{g_\infty^s + \epsilon} \varphi_{e2} \left(\frac{(1-b)(1-c_4)}{L_4} \right) \right\}. \end{aligned}$$

By using (I2) and the definitions of f_0^i and g_0^i we deduce that there exists $r_3 > 0$ such that

$$f(t, x, y) \geq (f_0^i - \epsilon) \varphi_{e1}(x + y), \quad g(t, x, y) \geq (g_0^i - \epsilon) \varphi_{e2}(x + y),$$

for all $t \in [\theta_1, \theta_2]$, $x, y \geq 0$, $x + y \leq r_3$.

We introduce the set $\Lambda_3 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_3\}$. Let $(x, y) \in \mathcal{P} \cap \partial \Lambda_3$, that is $(x, y) \in \mathcal{P}$ with $\|(x, y)\|_{\mathcal{Y}} = r_3$ or $\|x\| + \|y\| = r_3$. Then $x(t) + y(t) \leq r_3$ for all $t \in [0, 1]$, and $x(t) + y(t) \geq \zeta \|(x, y)\|_{\mathcal{Y}}$ for all $t \in [\theta_1, \theta_2]$ (see the proof of Case (1) in Theorem 3.1). So, by Lemma 2.5, we obtain

$$\begin{aligned} & \mathcal{A}_1(x, y)(\theta_1) \\ & \geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} f(s, x(s), y(s)) ds \right) dv \\ & \quad + \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} g(s, x(s), y(s)) ds \right) dv \\ & \geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} (f_0^i - \epsilon) \varphi_{e1}(x(s) + y(s)) ds \right) dv \end{aligned}$$

$$\begin{aligned}
& + \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} (\mathfrak{g}_0^i - \epsilon) \varphi_{\rho_2}(x(s) + y(s)) ds \right) dv \\
& \geq \lambda^{\rho_1-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^v (v-s)^{\alpha_1-1} (\mathfrak{f}_0^i - \epsilon) \varphi_{\rho_1}(\zeta \| (x, y) \|_{\mathcal{Y}}) ds \right) dv \\
& \quad + \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} (\mathfrak{g}_0^i - \epsilon) \varphi_{\rho_2}(\zeta \| (x, y) \|_{\mathcal{Y}}) ds \right) dv \\
& = \zeta \zeta_1 \varphi_{\rho_1}(\lambda(\mathfrak{f}_0^i - \epsilon)) \| (x, y) \|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_1(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} (v - \theta_1)^{\alpha_1(\rho_1-1)} dv \\
& \quad + \zeta \zeta_1 \varphi_{\rho_2}(\mu(\mathfrak{g}_0^i - \epsilon)) \| (x, y) \|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} (v - \theta_1)^{\alpha_2(\rho_2-1)} dv \\
& = [\zeta \zeta_1 \varphi_{\rho_1}(\lambda(\mathfrak{f}_0^i - \epsilon)) \tilde{L}_1 + \zeta \zeta_1 \varphi_{\rho_2}(\mu(\mathfrak{g}_0^i - \epsilon)) \tilde{L}_2] \| (x, y) \|_{\mathcal{Y}} \\
& \geq [ac_1 + a(1 - c_1)] \| (x, y) \|_{\mathcal{Y}} = a \| (x, y) \|_{\mathcal{Y}}.
\end{aligned}$$

Then $\|\mathcal{A}_1(x, y)\| \geq \mathcal{A}_1(x, y)(\theta_1) \geq a \| (x, y) \|_{\mathcal{Y}}$.

In a similar manner we find

$$\begin{aligned}
& \mathcal{A}_2(x, y)(\theta_1) \\
& \geq \zeta \zeta_2 \varphi_{\rho_1}(\lambda(\mathfrak{f}_0^i - \epsilon)) \| (x, y) \|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_3(v) \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} (v - \theta_1)^{\alpha_1(\rho_1-1)} dv \\
& \quad + \zeta \zeta_2 \varphi_{\rho_2}(\mu(\mathfrak{g}_0^i - \epsilon)) \| (x, y) \|_{\mathcal{Y}} \int_{\theta_1}^{\theta_2} \mathcal{J}_4(v) \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} (v - \theta_1)^{\alpha_2(\rho_2-1)} dv \\
& = [\zeta \zeta_2 \varphi_{\rho_1}(\lambda(\mathfrak{f}_0^i - \epsilon)) \tilde{L}_3 + \zeta \zeta_2 \varphi_{\rho_2}(\mu(\mathfrak{g}_0^i - \epsilon)) \tilde{L}_4] \| (x, y) \|_{\mathcal{Y}} \\
& \geq [(1 - a)c_2 + (1 - a)(1 - c_2)] \| (x, y) \|_{\mathcal{Y}} = (1 - a) \| (x, y) \|_{\mathcal{Y}}.
\end{aligned}$$

So, $\|\mathcal{A}_2(x, y)\| \geq \mathcal{A}_2(x, y)(\theta_1) \geq (1 - a) \| (x, y) \|_{\mathcal{Y}}$.

Then, for $(x, y) \in \mathcal{P} \cap \partial \Lambda_3$, we conclude

$$\|\mathcal{A}(x, y)\|_{\mathcal{Y}} = \|\mathcal{A}_1(x, y)\| + \|\mathcal{A}_2(x, y)\| \geq a \| (x, y) \|_{\mathcal{Y}} + (1 - a) \| (x, y) \|_{\mathcal{Y}} = \| (x, y) \|_{\mathcal{Y}}. \quad (22)$$

We introduce now the functions $\mathfrak{f}^*, \mathfrak{g}^* : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\mathfrak{f}^*(t, u) = \max_{0 \leq x+y \leq u} f(t, x, y)$, $\mathfrak{g}^*(t, u) = \max_{0 \leq x+y \leq u} g(t, x, y)$ for all $t \in [0, 1]$ and $u \in \mathbb{R}_+$. Then

$$f(t, x, y) \leq \mathfrak{f}^*(t, u), \quad g(t, x, y) \leq \mathfrak{g}^*(t, u), \quad \forall t \in [0, 1], x, y \geq 0, x + y \leq u.$$

The functions $\mathfrak{f}^*(t, \cdot)$, $\mathfrak{g}^*(t, \cdot)$ are nondecreasing for every $t \in [0, 1]$ and they satisfy the conditions

$$\limsup_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{\mathfrak{f}^*(t, u)}{\varphi_{\rho_1}(u)} = \mathfrak{f}_{\infty}^s, \quad \limsup_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{\mathfrak{g}^*(t, u)}{\varphi_{\rho_2}(u)} = \mathfrak{g}_{\infty}^s.$$

Then for $\epsilon > 0$ there exists $r'_4 > 0$ such that for all $u \geq r'_4$ and $t \in [0, 1]$ we obtain

$$\frac{\mathfrak{f}^*(t, u)}{\varphi_{\rho_1}(u)} \leq \limsup_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{f^*(t, u)}{\varphi_{\rho_1}(u)} + \epsilon = \mathfrak{f}_{\infty}^s + \epsilon,$$

$$\frac{g^*(t, u)}{\varphi_{\rho_2}(u)} \leq \limsup_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{g^*(t, u)}{\varphi_{\rho_2}(u)} + \epsilon = g_{\infty}^s + \epsilon,$$

and hence $f^*(t, u) \leq (f_{\infty}^s + \epsilon)\varphi_{\rho_1}(u)$ and $g^*(t, u) \leq (g_{\infty}^s + \epsilon)\varphi_{\rho_2}(u)$.

We consider $r_4 = \max\{2r_3, r'_4\}$ and we introduce the set $\Lambda_4 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_4\}$. Let $(x, y) \in \mathcal{P} \cap \partial \Lambda_4$. By the definitions of f^* and g^* we deduce

$$\begin{aligned} f(t, x(t), y(t)) &\leq f^*(t, \|(x, y)\|_{\mathcal{Y}}), \\ g(t, x(t), y(t)) &\leq g^*(t, \|(x, y)\|_{\mathcal{Y}}), \quad \forall t \in [0, 1]. \end{aligned} \quad (23)$$

Then for all $t \in [0, 1]$ we find

$$\begin{aligned} \mathcal{A}_1(x, y)(t) &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^v (v-s)^{\alpha_1-1} f(s, x(s), y(s)) ds \right) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_0^v (v-s)^{\alpha_2-1} g(s, x(s), y(s)) ds \right) dv \\ &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^v (v-s)^{\alpha_1-1} f^*(s, \|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_0^v (v-s)^{\alpha_2-1} g^*(s, \|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &\leq \lambda^{\rho_1-1} \int_0^1 \mathcal{J}_1(v) \varphi_{\rho_1} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^v (v-s)^{\alpha_1-1} (f_{\infty}^s + \epsilon) \varphi_{\rho_1}(\|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_0^v (v-s)^{\alpha_2-1} (g_{\infty}^s + \epsilon) \varphi_{\rho_2}(\|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &= \varphi_{\rho_1}(\lambda(f_{\infty}^s + \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_0^1 \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} v^{\alpha_1(\rho_1-1)} \mathcal{J}_1(v) dv \\ &\quad + \varphi_{\rho_2}(\mu(g_{\infty}^s + \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_0^1 \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} v^{\alpha_2(\rho_2-1)} \mathcal{J}_2(v) dv \\ &= [\varphi_{\rho_1}(\lambda(f_{\infty}^s + \epsilon))L_1 + \varphi_{\rho_2}(\mu(g_{\infty}^s + \epsilon))L_2] \|(x, y)\|_{\mathcal{Y}} \\ &\leq [bc_3 + b(1 - c_3)] \|(x, y)\|_{\mathcal{Y}} = b \|(x, y)\|_{\mathcal{Y}}, \end{aligned}$$

and so $\|\mathcal{A}_1(x, y)\| \leq b \|(x, y)\|_{\mathcal{Y}}$.

In a similar manner, we obtain

$$\begin{aligned} \mathcal{A}_2(x, y)(t) &\leq \varphi_{\rho_1}(\lambda(f_{\infty}^s + \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_0^1 \frac{1}{(\Gamma(\alpha_1 + 1))^{\rho_1-1}} v^{\alpha_1(\rho_1-1)} \mathcal{J}_3(v) dv \\ &\quad + \varphi_{\rho_2}(\mu(g_{\infty}^s + \epsilon)) \|(x, y)\|_{\mathcal{Y}} \int_0^1 \frac{1}{(\Gamma(\alpha_2 + 1))^{\rho_2-1}} v^{\alpha_2(\rho_2-1)} \mathcal{J}_4(v) dv \\ &= [\varphi_{\rho_1}(\lambda(f_{\infty}^s + \epsilon))L_3 + \varphi_{\rho_2}(\mu(g_{\infty}^s + \epsilon))L_4] \|(x, y)\|_{\mathcal{Y}} \\ &\leq [(1 - b)c_4 + (1 - b)(1 - c_4)] \|(x, y)\|_{\mathcal{Y}} = (1 - b) \|(x, y)\|_{\mathcal{Y}}, \end{aligned}$$

and then $\|\mathcal{A}_2(x, y)\| \leq (1 - b) \|(x, y)\|_{\mathcal{Y}}$.

Therefore, for $(x, y) \in \mathcal{P} \cap \partial \Lambda_4$, we deduce

$$\|\mathcal{A}(x, y)\|_{\mathcal{Y}} = \|\mathcal{A}_1(x, y)\| + \|\mathcal{A}_2(x, y)\| \leq b\|(x, y)\|_{\mathcal{Y}} + (1-b)\|(x, y)\|_{\mathcal{Y}} = \|(x, y)\|_{\mathcal{Y}}. \quad (24)$$

By using Lemma 3.1, Eqs. (23), (24) and the Guo–Krasnosel'skii fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $(x, y) \in \mathcal{P} \cap (\overline{\Lambda}_4 \setminus \Lambda_3)$, which is a positive solution of problem (1), (2).

Case (6). We have $f_{\infty}^s \in (0, \infty)$, $g_{\infty}^s = 0$ and $g_0^i = \infty$. Let $\lambda \in (0, \tilde{E})$ and $\mu \in (0, \infty)$. We choose $\tilde{c}_3 \in (\varphi_{\rho_1}(\lambda f_{\infty}^s) \frac{L_1}{b}, 1)$ and $\tilde{c}_4 \in (\varphi_{\rho_1}(\lambda f_{\infty}^s) \frac{L_3}{1-b}, 1)$. The choice of $\tilde{c}_3$ and $\tilde{c}_4$ is possible because $\lambda < \frac{1}{f_{\infty}^s} \varphi_{\rho_1}(\frac{b}{L_1})$ and $\lambda < \frac{1}{f_{\infty}^s} \varphi_{\rho_1}(\frac{1-b}{L_3})$. Let $\epsilon > 0$ be such that

$$\begin{aligned} \lambda &\leq \min \left\{ \frac{1}{f_{\infty}^s + \epsilon} \varphi_{\rho_1} \left(\frac{b\tilde{c}_3}{L_1} \right), \frac{1}{f_{\infty}^s + \epsilon} \varphi_{\rho_1} \left(\frac{(1-b)\tilde{c}_4}{L_3} \right) \right\}, \\ \epsilon \varphi_{\rho_2} \left(\frac{1}{\zeta \zeta_1 \tilde{L}_2} \right) &\leq \mu \leq \min \left\{ \frac{1}{\epsilon} \varphi_{\rho_2} \left(\frac{b(1-\tilde{c}_3)}{L_2} \right), \frac{1}{\epsilon} \varphi_{\rho_2} \left(\frac{(1-b)(1-\tilde{c}_4)}{L_4} \right) \right\}. \end{aligned}$$

By (I2) and the definition of g_0^i we find that there exists $r_3 > 0$ such that

$$g(t, x, y) \geq \frac{1}{\epsilon} \varphi_{\rho_2}(x + y), \quad \forall t \in [\theta_1, \theta_2], x, y \geq 0, x + y \leq r_3.$$

We define the set $\Lambda_3 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_3\}$. Let $(x, y) \in \mathcal{P} \cap \partial \Lambda_3$, that is $\|x\| + \|y\| = r_3$. Because $x(t) + y(t) \leq \|x\| + \|y\| = r_3$ for all $t \in [0, 1]$, then by Lemma 2.5, we find

$$\begin{aligned} &\mathcal{A}_1(x, y)(\theta_1) \\ &\geq \lambda^{\rho_1-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_1(v) \varphi_{\rho_1}(I_{0+}^{\alpha_1} f(v, x(v), y(v))) dv \\ &\quad + \mu^{\rho_2-1} \int_0^1 \theta_1^{\beta_1-1} \mathcal{J}_2(v) \varphi_{\rho_2}(I_{0+}^{\alpha_2} g(v, x(v), y(v))) dv \\ &\geq \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(s) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} g(s, x(s), y(s)) ds \right) dv \\ &\geq \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} \frac{1}{\epsilon} \varphi_{\rho_2}(x(s) + y(s)) ds \right) dv \\ &\geq \mu^{\rho_2-1} \theta_1^{\beta_1-1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(v) \varphi_{\rho_2} \left(\frac{1}{\Gamma(\alpha_2)} \int_{\theta_1}^v (v-s)^{\alpha_2-1} \frac{1}{\epsilon} \varphi_{\rho_2}(\zeta \|(x, y)\|_{\mathcal{Y}}) ds \right) dv \\ &= \zeta \zeta_1 \varphi_{\rho_2} \left(\frac{\mu}{\epsilon} \right) \|(x, y)\|_{\mathcal{Y}} \tilde{L}_2 \geq \|(x, y)\|_{\mathcal{Y}}. \end{aligned}$$

Hence we deduce $\|\mathcal{A}_1(x, y)\| \geq \mathcal{A}_1(x, y)(\theta_1) \geq \|(x, y)\|_{\mathcal{Y}}$ and $\|\mathcal{A}(x, y)\|_{\mathcal{Y}} \geq \|\mathcal{A}_1(x, y)\| \geq \|(x, y)\|_{\mathcal{Y}}$.

For the second part of the proof, we use the functions f^* and g^* from Case (1), which satisfy here the conditions

$$\limsup_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{f^*(t, u)}{\varphi_{\rho_1}(u)} = f_{\infty}^s, \quad \lim_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{g^*(t, u)}{\varphi_{\rho_2}(u)} = 0.$$

Then for $\epsilon > 0$ there exists $r'_4 > 0$ such that for all $u \geq r'_4$ and $t \in [0, 1]$ we obtain

$$\begin{aligned}\frac{f^*(t, u)}{\varphi_{\varrho_1}(u)} &\leq \limsup_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{f^*(t, u)}{\varphi_{\varrho_1}(u)} + \epsilon = f_\infty^s + \epsilon, \\ \frac{g^*(t, u)}{\varphi_{\varrho_2}(u)} &\leq \lim_{u \rightarrow \infty} \max_{t \in [0, 1]} \frac{g^*(t, u)}{\varphi_{\varrho_2}(u)} + \epsilon = \epsilon,\end{aligned}$$

and hence $f^*(t, u) \leq (f_\infty^s + \epsilon)\varphi_{\varrho_1}(u)$ and $g^*(t, u) \leq \epsilon\varphi_{\varrho_2}(u)$.

We define $r_4 = \max\{2r_3, r'_4\}$ and we introduce the set $\Lambda_4 = \{(x, y) \in \mathcal{Y}, \|(x, y)\|_{\mathcal{Y}} < r_4\}$. Let $(x, y) \in \mathcal{P} \cap \partial \Lambda_4$. By the definitions of f^* and g^* we deduce Eq. (23). Besides, in a similar manner to that used in the proof of Case (1), we find

$$\begin{aligned}\mathcal{A}_1(x, y)(t) &\leq \varphi_{\varrho_1}(\lambda(f_\infty^s + \epsilon))L_1\|(x, y)\|_{\mathcal{Y}} + \varphi_{\varrho_2}(\mu\epsilon)L_2\|(x, y)\|_{\mathcal{Y}} \\ &\leq [b\tilde{c}_3 + b(1 - \tilde{c}_3)]\|(x, y)\|_{\mathcal{Y}} = b\|(x, y)\|_{\mathcal{Y}}, \quad \forall t \in [0, 1], \\ \mathcal{A}_2(x, y)(t) &\leq \varphi_{\varrho_1}(\lambda(f_\infty^s + \epsilon))L_3\|(x, y)\|_{\mathcal{Y}} + \varphi_{\varrho_2}(\mu\epsilon)L_4\|(x, y)\|_{\mathcal{Y}} \\ &\leq [(1 - b)\tilde{c}_4 + (1 - b)(1 - \tilde{c}_4)]\|(x, y)\|_{\mathcal{Y}} = (1 - b)\|(x, y)\|_{\mathcal{Y}}, \quad \forall t \in [0, 1],\end{aligned}$$

and then $\|\mathcal{A}(x, y)\|_{\mathcal{Y}} \leq b\|(x, y)\|_{\mathcal{Y}} + (1 - b)\|(x, y)\|_{\mathcal{Y}} = \|(x, y)\|_{\mathcal{Y}}$.

Therefore we obtain the conclusion of the theorem. $\square$

4 Nonexistence of the positive solutions

In this section we give intervals for λ and μ for which there exist no positive solutions of problem (1), (2). By using similar arguments to those used in the proofs of Theorems 4.1–4.4 from [30], we obtain the following theorems for problem (1), (2).

Theorem 4.1 *Assume that (I1) and (I2) hold. If there exist positive numbers T_1, T_2 such that*

$$f(t, x, y) \leq T_1\varphi_{\varrho_1}(x + y), \quad g(t, x, y) \leq T_2\varphi_{\varrho_2}(x + y), \quad \forall t \in [0, 1], x, y \geq 0, \quad (25)$$

then there exist positive constants λ_0 and μ_0 such that for every $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$ the boundary value problem (1), (2) has no positive solution.

In the proof of Theorem 4.1 we define $\lambda_0 = \min\{(T_1\varphi_{\varrho_1}(4L_1))^{-1}, (T_1\varphi_{\varrho_1}(4L_3))^{-1}\}$ and $\mu_0 = \min\{(T_2\varphi_{\varrho_2}(4L_2))^{-1}, (T_2\varphi_{\varrho_2}(4L_4))^{-1}\}$, where $L_i, i = 1, \dots, 4$, are given by (19).

Remark 4.1 If $f_0^s, g_0^s, f_\infty^s, g_\infty^s < \infty$, then there exist positive constants T_1, T_2 such that relation (25) holds, and so we obtain the conclusion of Theorem 4.1.

Theorem 4.2 *Assume that (I1) and (I2) hold. If there exist positive numbers θ_1, θ_2 with $0 < \theta_1 < \theta_2 \leq 1$ and $q_1 > 0$ such that*

$$f(t, x, y) \geq q_1\varphi_{\varrho_1}(x + y), \quad \forall t \in [\theta_1, \theta_2], x, y \geq 0, \quad (26)$$

then there exists a positive constant λ'_0 such that for every $\lambda > \lambda'_0$ and $\mu > 0$, the boundary value problem (1), (2) has no positive solution.

In the proof of Theorem 4.2 we define $\lambda'_0 = \min\{(q_1\varphi_{\varrho_1}(\zeta\zeta_1\tilde{L}_1))^{-1}, (q_1\varphi_{\varrho_1}(\zeta\zeta_2\tilde{L}_3))^{-1}\}$, where $\tilde{L}_1$ and $\tilde{L}_3$ are given by (19).

Remark 4.2 If for θ_1, θ_2 with $0 < \theta_1 < \theta_2 \leq 1$, we have $f_0^i, f_\infty^i > 0$, and $f(t, x, y) > 0$ for all $t \in [\theta_1, \theta_2]$ and $x, y \geq 0$ with $x + y > 0$, then Eq. (26) holds, and therefore we obtain the conclusion of Theorem 4.2.

Theorem 4.3 Assume that (I1) and (I2) hold. If there exist positive numbers θ_1, θ_2 with $0 < \theta_1 < \theta_2 \leq 1$ and $q_2 > 0$ such that

$$g(t, x, y) \geq q_2\varphi_{\varrho_2}(x + y), \quad \forall t \in [\theta_1, \theta_2], x, y \geq 0, \quad (27)$$

then there exists a positive constant μ'_0 such that for every $\mu > \mu'_0$ and $\lambda > 0$ the boundary value problem (1), (2) has no positive solution.

In the proof of Theorem 4.3 we define $\mu'_0 = \min\{(q_2\varphi_{\varrho_2}(\zeta\zeta_1\tilde{L}_2))^{-1}, (q_2\varphi_{\varrho_2}(\zeta\zeta_2\tilde{L}_4))^{-1}\}$, where $\tilde{L}_2$ and $\tilde{L}_4$ are given by (19).

Remark 4.3 If for θ_1, θ_2 with $0 < \theta_1 < \theta_2 \leq 1$, we have $g_0^i, g_\infty^i > 0$, and $g(t, x, y) > 0$ for all $t \in [\theta_1, \theta_2]$ and $x, y \geq 0$ with $x + y > 0$, then Eq. (27) holds, and hence we deduce the conclusion of Theorem 4.3.

Theorem 4.4 Assume that (I1) and (I2) hold. If there exist positive numbers θ_1, θ_2 with $0 < \theta_1 < \theta_2 \leq 1$ and $q_1, q_2 > 0$ such that

$$f(t, x, y) \geq q_1\varphi_{\varrho_1}(x + y), \quad g(t, x, y) \geq q_2\varphi_{\varrho_2}(x + y), \quad \forall t \in [\theta_1, \theta_2], x, y \geq 0, \quad (28)$$

then there exist positive constants $\tilde{\lambda}_0$ and $\tilde{\mu}_0$ such that for every $\lambda > \tilde{\lambda}_0$ and $\mu > \tilde{\mu}_0$ the boundary value problem (1), (2) has no positive solution.

In the proof of Theorem 4.4 we define $\tilde{\lambda}_0 = (q_1\varphi_{\varrho_1}(2\zeta\zeta_1\tilde{L}_1))^{-1}$ and $\tilde{\mu}_0 = (q_2\varphi_{\varrho_2}(2\zeta\zeta_2\tilde{L}_4))^{-1}$.

Remark 4.4 If for θ_1, θ_2 with $0 < \theta_1 < \theta_2 \leq 1$, we have $f_0^i, f_\infty^i, g_0^i, g_\infty^i > 0$, and $f(t, x, y) > 0$ and $g(t, x, y) > 0$ for all $t \in [\theta_1, \theta_2]$ and $x, y \geq 0$ with $x + y > 0$, then Eq. (28) holds, and so we obtain the conclusion of Theorem 4.4.

5 An example

Let $\alpha_1 = 1/2$, $\alpha_2 = 1/3$, $\beta_1 = 10/3$, $n = 4$, $\beta_2 = 17/4$, $m = 5$, $p = 2$, $q = 1$, $\delta_0 = 11/5$, $\gamma_1 = 5/6$, $\gamma_2 = 3/2$, $\gamma_0 = 9/8$, $\delta_1 = 3/7$, $H_1(t) = 4t$ for all $t \in [0, 1]$, $H_2(t) = \{1, t \in [0, 1/4]; 8, t \in [1/4, 1]\}$, $K_1(t) = \{1, t \in [0, 1/2]; 3, t \in [1/2, 1]\}$, $\varrho_1 = 5$, $\rho_1 = 5/4$, $\varrho_2 = 3$, $\rho_2 = 3/2$, $\varphi_{\varrho_1}(s) = |s|^3s$, $\varphi_{\rho_1}(s) = |s|^{-3/4}s$, $\varphi_{\varrho_2}(s) = |s|s$, $\varphi_{\rho_2}(s) = |s|^{-1/2}s$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{1/2}(\varphi_5(D_{0+}^{10/3}x(t))) + \lambda(3-t)^{a_0}(e^{(x(t)+y(t))^4} - 1) = 0, & t \in (0, 1), \\ D_{0+}^{1/3}(\varphi_3(D_{0+}^{17/4}y(t))) + \mu(t+2)^{b_0}(x^3(t) + y^3(t)) = 0, & t \in (0, 1), \end{cases} \quad (29)$$

with the coupled nonlocal boundary conditions

$$\begin{cases} x(0) = x'(0) = x''(0) = 0, & D_{0+}^{10/3}x(0) = 0, \\ D_{0+}^{9/8}x(1) = 4 \int_0^1 D_{0+}^{5/6}y(t) dt + 7D_{0+}^{3/2}y\left(\frac{1}{4}\right), \\ y(0) = y'(0) = y''(0) = y'''(0) = 0, & D_{0+}^{17/4}y(0) = 0, \\ D_{0+}^{11/5}y(1) = 2D_{0+}^{3/7}x\left(\frac{1}{2}\right), \end{cases} \quad (30)$$

where $a_0, b_0 > 0$.

Here we have $f(t, x, y) = (3 - t)^{a_0}(e^{(x+y)^4} - 1)$, $g(t, x, y) = (t + 2)^{b_0}(x^3 + y^3)$ for all $t \in [0, 1]$ and $x, y \geq 0$. We obtain $\Delta \approx 15.18283383 > 0$, and then the assumptions (I1) and (I2) are satisfied. Besides, we find

$$\begin{aligned} g_1(t, s) &= \frac{1}{\Gamma(10/3)} \begin{cases} t^{7/3}(1-s)^{29/24} - (t-s)^{7/3}, & 0 \leq s \leq t \leq 1, \\ t^{7/3}(1-s)^{29/24}, & 0 \leq t \leq s \leq 1, \end{cases} \\ g_{11}(t, s) &= \frac{1}{\Gamma(61/21)} \begin{cases} t^{40/21}(1-s)^{29/24} - (t-s)^{40/21}, & 0 \leq s \leq t \leq 1, \\ t^{40/21}(1-s)^{29/24}, & 0 \leq t \leq s \leq 1, \end{cases} \\ g_2(t, s) &= \frac{1}{\Gamma(17/4)} \begin{cases} t^{13/4}(1-s)^{21/20} - (t-s)^{13/4}, & 0 \leq s \leq t \leq 1, \\ t^{13/4}(1-s)^{21/20}, & 0 \leq t \leq s \leq 1, \end{cases} \\ g_{21}(t, s) &= \frac{1}{\Gamma(41/12)} \begin{cases} t^{29/12}(1-s)^{21/20} - (t-s)^{29/12}, & 0 \leq s \leq t \leq 1, \\ t^{29/12}(1-s)^{21/20}, & 0 \leq t \leq s \leq 1, \end{cases} \\ g_{22}(t, s) &= \frac{1}{\Gamma(11/4)} \begin{cases} t^{7/4}(1-s)^{21/20} - (t-s)^{7/4}, & 0 \leq s \leq t \leq 1, \\ t^{7/4}(1-s)^{21/20}, & 0 \leq t \leq s \leq 1, \end{cases} \\ \mathcal{G}_1(t, s) &= g_1(t, s) + \frac{2t^{7/3}}{\Delta} \left[\frac{48\Gamma(17/4)}{41\Gamma(41/12)} + 7\left(\frac{1}{4}\right)^{7/4} \frac{\Gamma(17/4)}{\Gamma(11/4)} \right] g_{11}\left(\frac{1}{2}, s\right), \\ \mathcal{G}_2(t, s) &= \frac{t^{7/3}\Gamma(17/4)}{\Delta\Gamma(41/20)} \left[4 \int_0^1 g_{21}(\tau, s) d\tau + 7g_{22}\left(\frac{1}{4}, s\right) \right], \\ \mathcal{G}_3(t, s) &= \frac{2t^{13/4}\Gamma(10/3)}{\Delta\Gamma(53/24)} g_{11}\left(\frac{1}{2}, s\right), \\ \mathcal{G}_4(t, s) &= g_2(t, s) + \frac{t^{13/4}\Gamma(10/3)}{\Delta\Gamma(61/21)2^{19/21}} \left[4 \int_0^1 g_{21}(\tau, s) d\tau + 7g_{22}\left(\frac{1}{4}, s\right) \right], \end{aligned}$$

for all $(t, s) \in [0, 1] \times [0, 1]$. For the functions h_1, h_2 and $\mathcal{J}_i, i = 1, \dots, 4$, we obtain

$$\begin{aligned} h_1(s) &= \frac{1}{\Gamma(10/3)} [(1-s)^{29/24} - (1-s)^{7/3}], \\ h_2(s) &= \frac{1}{\Gamma(17/4)} [(1-s)^{21/20} - (1-s)^{13/4}], \\ &\text{for all } s \in [0, 1], \end{aligned}$$

$$\begin{aligned}
\mathcal{J}_1(s) &= \begin{cases} \frac{1}{\Gamma(10/3)} [(1-s)^{29/24} - (1-s)^{7/3}] + \frac{1}{\Delta} \left[\frac{48\Gamma(17/4)}{41\Gamma(41/12)} + 7\left(\frac{1}{4}\right)^{7/4} \frac{\Gamma(17/4)}{\Gamma(11/4)} \right] \\ \quad \times \frac{2}{\Gamma(61/21)} \left[\left(\frac{1}{2}\right)^{40/21} (1-s)^{29/24} - \left(\frac{1}{2} - s\right)^{40/21} \right], & 0 \leq s < \frac{1}{2}, \\ \frac{1}{\Gamma(10/3)} [(1-s)^{29/24} - (1-s)^{7/3}] + \frac{1}{\Delta} \left[\frac{48\Gamma(17/4)}{41\Gamma(41/12)} + 7\left(\frac{1}{4}\right)^{7/4} \frac{\Gamma(17/4)}{\Gamma(11/4)} \right] \\ \quad \times \frac{2}{\Gamma(61/21)} \left(\frac{1}{2}\right)^{40/21} (1-s)^{29/24}, & \frac{1}{2} \leq s \leq 1, \end{cases} \\
\mathcal{J}_2(s) &= \begin{cases} \frac{\Gamma(17/4)}{\Delta\Gamma(41/20)} \left\{ \frac{48}{41\Gamma(41/12)} [(1-s)^{21/20} - (1-s)^{41/12}] \right. \\ \quad \left. + \frac{7}{\Gamma(11/4)} \left[\left(\frac{1}{4}\right)^{7/4} (1-s)^{21/20} - \left(\frac{1}{4} - s\right)^{7/4} \right] \right\}, & 0 \leq s < \frac{1}{4}, \\ \frac{\Gamma(17/4)}{\Delta\Gamma(41/20)} \left\{ \frac{48}{41\Gamma(41/12)} [(1-s)^{21/20} - (1-s)^{41/12}] \right. \\ \quad \left. + \frac{7}{\Gamma(11/4)} \left(\frac{1}{4}\right)^{7/4} (1-s)^{21/20} \right\}, & \frac{1}{4} \leq s \leq 1, \end{cases} \\
\mathcal{J}_3(s) &= \begin{cases} \frac{2\Gamma(10/3)}{\Delta\Gamma(53/24)\Gamma(61/21)} \left[\left(\frac{1}{2}\right)^{40/21} (1-s)^{29/24} - \left(\frac{1}{2} - s\right)^{40/21} \right], & 0 \leq s < \frac{1}{2}, \\ \frac{2\Gamma(10/3)}{\Delta\Gamma(53/24)\Gamma(61/21)} \left(\frac{1}{2}\right)^{40/21} (1-s)^{29/24}, & \frac{1}{2} \leq s \leq 1, \end{cases} \\
\mathcal{J}_4(s) &= \begin{cases} \frac{1}{\Gamma(17/4)} [(1-s)^{21/20} - (1-s)^{13/4}] \\ \quad + \frac{\Gamma(10/3)}{\Delta\Gamma(61/21)2^{19/21}} \left\{ \frac{48}{41\Gamma(41/12)} [(1-s)^{21/20} \right. \\ \quad \left. - (1-s)^{41/12}] + \frac{7}{\Gamma(11/4)} \left[\left(\frac{1}{4}\right)^{7/4} (1-s)^{21/20} - \left(\frac{1}{4} - s\right)^{7/4} \right] \right\}, & 0 \leq s < \frac{1}{4}, \\ \frac{1}{\Gamma(17/4)} [(1-s)^{21/20} - (1-s)^{13/4}] \\ \quad + \frac{\Gamma(10/3)}{\Delta\Gamma(61/21)2^{19/21}} \left\{ \frac{48}{41\Gamma(41/12)} [(1-s)^{21/20} \right. \\ \quad \left. - (1-s)^{41/12}] + \frac{7}{\Gamma(11/4)} \left(\frac{1}{4}\right)^{7/4} (1-s)^{21/20} \right\}, & \frac{1}{4} \leq s \leq 1. \end{cases}
\end{aligned}$$

We choose $\theta_1 = 1/4$ and $\theta_2 = 3/4$, and hence we obtain $\zeta_1 = (1/4)^{7/3}$, $\zeta_2 = (1/4)^{13/4}$ and $\zeta = \zeta_2$. Besides, we deduce $f_0^s = 3^{a_0}$, $f_\infty^i = \infty$, $g_0^s = 0$, $g_\infty^i = \infty$, $L_1 \approx 0.08207184$, $L_3 \approx 0.01229905$, $\tilde{L}_1 \approx 0.05185073$, $\tilde{L}_3 \approx 0.00775575$.

By Theorem 3.1(7), if we consider $b = 1/2$, then, for any $\lambda \in (0, E)$ and $\mu \in (0, \infty)$ with $E = 3^{-a_0}(2L_1)^{-4}$, the problem (29), (30) has a positive solution $(x(t), y(t))$, $t \in [0, 1]$. For example, if $a_0 = 1$ we find $E \approx 459.179$.

We can also use Theorem 4.2, because $f(t, x, y) \geq q_1(x + y)^4$ for all $t \in [1/4, 3/4]$ and $x, y \geq 0$, with $q_1 = (9/4)^{a_0}$. If $a_0 = 1$, we obtain $\lambda'_0 = (q_1(\zeta \zeta_1 \tilde{L}_1)^4)^{-1} \approx 1.71714 \times 10^{18}$, and then we deduce then, for every $\lambda > \lambda'_0$ and $\mu > 0$, that the boundary value problem (29), (30) has no positive solution.

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