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Weakly compatible and quasi-contraction results in fuzzy cone metric spaces with application to the Urysohn type integral equations

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Abstract

In this paper, we present some weakly compatible and quasi-contraction results for self-mappings in fuzzy cone metric spaces and prove some coincidence point and common fixed point theorems in the said space. Moreover, we use two Urysohn type integral equations to get the existence theorem for common solution to support our results. The two Urysohn type integral equations are as follows:

$$x(l) = \int_0^1 K_1\big(l, v, x(v)\big)\,dv + g(l),$$

$$y(l) = \int_0^1 K_2\big(l, v, y(v)\big)\,dv + g(l),$$

where $l \in [0,1]$ and $x, y, g \in \mathbf{E}$, where $\mathbf{E}$ is a real Banach space and $K_1, K_2 : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$.

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1 Introduction

In 2007, Huang et al. [1] introduced the concept of cone metric space and proved some fixed point theorems for the underlying cone. In [2] Abbas et al. presented some noncommuting mapping results in cone metric spaces without continuity. After that, a series of authors (see [3–11]) contributed their ideas to the problems on cone metric spaces.

The initial version of fuzzy set theory was given by Zadeh [12], while Kramosil et al. in [13] introduced the fuzzy metric space or (shortly FM-space). Later on, a stronger form of the metric fuzziness was given by George et al. [14]. Some more related results in the context of fuzzy metric space can be found (e.g., see [15–19]).

Recently, Oner et al. in [20] introduced the concept of fuzzy cone metric space or shortly FCM-space. They presented some basic properties and a fuzzy cone Banach contraction theorem in a fuzzy cone metric space with the assumption that fuzzy cone contractive sequences are Cauchy. Some more properties and fixed point results in FCM-spaces can be found (e.g., see[21–26] and the references therein).

The aim of this paper is to obtain some coincidence point and common fixed point results for weakly compatible self-mappings in FCM-spaces. We also give the concept of quasi-contraction for weakly compatible self-mappings and establish some common fixed point theorems. Moreover, we present an integral type application from which we obtained the existence of fixed point results. The application of integral equations in fuzzy cone metric spaces is the new direction in the theory of fixed point. This new concept of application will be very fruitful for finding the existence solution of integral value problems on FCM-spaces. For this purpose, we use the two Urysohn integral type equations for common solution to support our results. We also present some illustrative examples to support our work.

2 Preliminaries

In this section, we present some basic definitions and a helpful concept for our main results.

Definition 2.1 ([27]) An operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous s-norm if the following hold:

(1) $*$ is commutative, associative, and continuous.

(2) $1*\beta=\beta$ and $\beta*\beta_1\leq\delta*\delta_1$, whenever, $\beta\leq\delta$ and $\beta_1\leq\delta_1$, for each $\beta,\beta_1,\delta,\delta_1\in[0,1]$.

The basic continuous s-norms of product, Lukasiewicz, and minimum are defined respectively as follows (see [27]):

$$\beta*\delta=\beta\delta,\qquad \beta*\delta=\max\{\beta+\delta-1,0\},\quad\text{and}\quad \beta*\delta=\min\{\beta,\delta\}.$$

Definition 2.2 ([14]) A three-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous s-norm, and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions:

(i) $M(x,y,s)>0$ and $M(x,y,s)=1\Leftrightarrow x=y$;

(ii) $M(x,y,s)=M(y,x,s)$;

(iii) $M(x,y,t+s)\geq M(x,z,t)*M(z,y,s)$;

(iv) $M(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous;

$\forall x,y,z\in X$ and $s,t>0$.

For more details, we shall refer the readers to study [14].

Definition 2.3 ([1]) A subset P of a real Banach space $\mathbf{E}$ is called a cone if

(1) P is closed, nonempty and $P\neq\{\vartheta\}$, where ϑ is the zero element of $\mathbf{E}$.

(2) If $x,y\in P$ and $\beta,\delta\in[0,\infty)$, then $\beta x+\delta y\in P$.

(3) If both $x,-x\in P$, then $x=\vartheta$.

A partial ordering "$\preceq$" for a given cone P on $\mathbf{E}$ is defined as $y \preceq x$ iff $x - y \in P$. $y \prec x$ stands for $y \preceq x$ and $y \neq x$, while $y \ll x$ stands for $x - y \in \operatorname{int}(P)$. Throughout this paper, all the cones have nonempty interior.

Definition 2.4 ([20]) A three-tuple $(X, M, *)$ is known as a fuzzy cone metric space (*FCM*-space) if P is a cone of $\mathbf{E}$, X is an arbitrary set, $*$ is a continuous s-norm, and a fuzzy set M on $X^2 \times \operatorname{int}(P)$ satisfies the following:

(1) $M(x,y,s) > 0$ and $M(x,y,s) = 1 \Leftrightarrow x = y$;
(2) $M(x,y,s) = M(y,x,s)$;
(3) $M(x,y,s) * M(y,z,t) \preceq M(x,z,s+t)$;
(4) $M(x,y,\cdot) : \operatorname{int}(P) \to [0,1]$ is continuous;

$\forall x,y,z \in X$ and $s,t \in \operatorname{int}(P)$.

Remark 2.5 Every *FM*-space becomes an *FCM*-space if $\mathbf{E} = \mathbb{R}$, $P = [0,\infty)$, and $\beta * \delta = \beta\delta$ [20–22].

Definition 2.6 ([20]) Let $(X, M, *)$ be an *FCM*-space, $x \in X$, and (x_i) be a sequence in X. Then,

(i) (x_i) converges to x, if for $s \gg \vartheta$ and $0 < r < 1$, $\exists i_1 \in \mathbf{N}$, s.t. $M(x_i, x, s) > 1 - r$, $\forall i \geq i_1$. We denote this by $\lim_{i\to\infty} x_i = x$ or $x_i \to x$ as $i \to \infty$.
(ii) (x_i) is said to be a Cauchy sequence if, for $0 < r < 1$ and $s \gg \vartheta$, $\exists i_1 \in \mathbf{N}$, s.t. $M(x_i, x_j, s) > 1 - r$, $\forall i,j \geq i_1$.
(iii) $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent in X.
(iv) (x_i) is said to be a fuzzy cone contractive if $\exists \beta \in (0,1)$ such that

$$\frac{1}{M(x_i, x_{i+1}, s)} - 1 \leq \beta\left(\frac{1}{M(x_{i-1}, x_i, s)} - 1\right)$$

for $s \gg \vartheta$, $i \geq 1$.

Definition 2.7 ([25]) Let $(X, M, *)$ be an *FCM*-space. The fuzzy cone metric M is triangular if

$$\frac{1}{M(x,z,s)} - 1 \leq \left(\frac{1}{M(x,y,s)} - 1\right) + \left(\frac{1}{M(y,z,s)} - 1\right),$$

$\forall x,y,z \in X$ and each $s \gg \vartheta$.

Lemma 2.8 ([20]) *Let $x \in X$ in an FCM-space $(X, M, *)$ and (x_i) be a sequence in X. Then (x_i) converges to x if and only if $M(x_i, x, s) \to 1$ as $i \to \infty$ for each $s \gg \vartheta$.*

Definition 2.9 ([20]) Let $(X, M, *)$ be an *FCM*-space, and a mapping $F_1 : X \to X$ is said to be fuzzy cone contractive if $\exists \beta \in (0,1)$ such that

$$\frac{1}{M(F_1x, F_1y, s)} - 1 \leq \beta\left(\frac{1}{M(x,y,s)} - 1\right) \tag{2.1}$$

for each $x, y \in X$ and $s \gg \vartheta$. β is called the contraction constant of F_1.

Definition 2.10 ([2]) Let F_1 and ℓ be two self-mappings on a set X (i.e., $F_1, \ell : X \to X$). If $u = F_1\nu = \ell\nu$ for some $\nu \in X$, then ν is called a coincidence point of F_1 and ℓ, and u is called a point of coincidence of F_1 and ℓ. The self-mappings F_1 and ℓ are said to be weakly compatible if they commute at their coincidence point, i.e., $F_1\nu = \ell\nu$ for some $\nu \in X$, then $F_1\ell\nu = \ell F_1\nu$.

Proposition 2.11 ([2]) *Let F_1 and ℓ be weakly compatible self-maps of a set X. If F_1 and ℓ have a unique point of coincidence $u = F_1\nu = \ell\nu$, then u is the unique common fixed point of F_1 and ℓ.*

Definition 2.12 ([28]) A pair (ℓ, F_1) of self-maps on X is called occasionally weakly compatible if $\exists \nu \in X$ such that $\ell\nu = F_1\nu$ and $F_1\ell\nu = \ell F_1\nu$.

Lemma 2.13 ([28]) *Let F_1 and ℓ be occasionally weakly compatible self-maps of a set X. If F_1 and ℓ have a unique point of coincidence, $F_1\nu = \ell\nu = u$, then u is a unique common fixed point of ℓ and F_1.*

"A self-mapping F_1 in a complete FCM-space in which the contractive sequences are Cauchy and hold (2.1), then F_1 has a unique fixed point in X" is a Banach contraction principle, which has been obtained in [20].

We note that fuzzy cone contractive sequences can be proved to be Cauchy sequences for weakly compatible self-mappings in FCM-spaces (see the proof of Theorem 3.1). In this paper we use the concept of complete FCM-spaces given by Rehman and Li [25] and prove some coincidence point and common fixed point theorems for weakly compatible three self-mappings and some quasi-contraction results in FCM-spaces. Moreover, we present some illustrative examples and the application of two Urysohn's integral type equations for the existence of common solution to support our work.

3 Weakly compatible mapping results in *FCM*-space

Theorem 3.1 *Let $F_1, F_2, \ell : X \to X$ be three self-maps and M be triangular in a complete FCM-space $(X, M, *)$ satisfying $\forall x, y \in X$,*

$$\begin{aligned} \frac{1}{M(F_1x, F_2y, s)} - 1 \leq \beta\left(\frac{1}{M(\ell x, \ell y, s)} - 1\right) + \gamma\left(\frac{1}{M(\ell x, F_1x, s)} - 1 + \frac{1}{M(\ell y, F_2y, s)} - 1\right) \\ + \delta\left(\frac{1}{M(\ell y, F_1x, s)} - 1 + \frac{1}{M(\ell x, F_2y, s)} - 1\right) \end{aligned} \tag{3.1}$$

for every $s \gg \vartheta$ and $\beta, \gamma, \delta \in [0, \infty)$ with $\beta + 2\gamma + 2\delta < 1$. If $F_1(X) \cup F_2(X) \subset \ell(X)$ and $\ell(X)$ is a complete subspace of X, then F_1, F_2, and ℓ have a unique point of coincidence. Moreover, if (F_1, ℓ) and (F_2, ℓ) are weakly compatible. Then F_1, F_2, and ℓ have a unique common fixed point in X.

Proof Fix $x_0 \in X$ and use the condition $F_1(X) \cup F_2(X) \subset \ell(X)$. We define some iterative sequences in X such that

$$\ell x_{2i+1} = F_1x_{2i} \quad \text{and} \quad \ell x_{2i+2} = F_2x_{2i+1}, \quad \text{for all } i \geq 0.$$

Now, by (3.1) for $s \gg \vartheta$,

$$\begin{aligned}
&\frac{1}{M(\ell x_{2i+1}, \ell x_{2i+2}, s)} - 1 = \frac{1}{M(F_1 x_{2i}, F_2 x_{2i+1}, s)} - 1 \\
&\quad \leq \beta\left(\frac{1}{M(\ell x_{2i}, \ell x_{2i+1}, s)} - 1\right) + \gamma\left(\frac{1}{M(\ell x_{2i}, F_1 x_{2i}, s)} - 1 + \frac{1}{M(\ell x_{2i+1}, F_2 x_{2i+1}, s)} - 1\right) \\
&\qquad + \delta\left(\frac{1}{M(\ell x_{2i+1}, F_1 x_{2i}, s)} - 1 + \frac{1}{M(\ell x_{2i}, F_2 x_{2i+1}, s)} - 1\right) \\
&\quad \leq \beta\left(\frac{1}{M(\ell x_{2i}, \ell x_{2i+1}, s)} - 1\right) + \gamma\left(\frac{1}{M(\ell x_{2i}, \ell x_{2i+1}, s)} - 1 + \frac{1}{M(\ell x_{2i+1}, \ell x_{2i+2}, s)} - 1\right) \\
&\qquad + \delta\left(\frac{1}{M(\ell x_{2i}, \ell x_{2i+1}, s)} - 1 + \frac{1}{M(\ell x_{2i+1}, \ell x_{2i+2}, s)} - 1\right).
\end{aligned}$$

Then

$$\frac{1}{M(\ell x_{2i+1}, \ell x_{2i+2}, s)} - 1 \leq \alpha\left(\frac{1}{M(\ell x_{2i}, \ell x_{2i+1}, s)} - 1\right), \quad \text{for } s \gg \vartheta,$$

where $\alpha = \frac{\beta+\gamma+\delta}{1-(\gamma+\delta)} < 1$. Similarly,

$$\begin{aligned}
&\frac{1}{M(\ell x_{2i+2}, \ell x_{2i+3}, s)} - 1 \\
&\quad \leq \alpha\left(\frac{1}{M(\ell x_{2i+1}, \ell x_{2i+2}, s)} - 1\right) \leq \alpha^2\left(\frac{1}{M(\ell x_{2i}, \ell x_{2i+1}, s)} - 1\right) \leq \cdots \\
&\quad \leq \alpha^{2i+2}\left(\frac{1}{M(\ell x_0, \ell x_1, s)} - 1\right),
\end{aligned}$$

which shows that a sequence $(\ell x_i)_{i\geq 0}$ is fuzzy cone contractive. Hence,

$$\lim_{i\to\infty} M(\ell x_i, \ell x_{i+1}, s) = 1 \quad \text{for } s \gg \vartheta. \tag{3.2}$$

Since M is triangular, for all $j > i > i_0$,

$$\begin{aligned}
&\frac{1}{M(\ell x_i, \ell x_j, s)} - 1 \\
&\quad \leq \left(\frac{1}{M(\ell x_i, \ell x_{i+1}, s)} - 1\right) + \left(\frac{1}{M(\ell x_{i+1}, \ell x_{i+2}, s)} - 1\right) + \cdots + \left(\frac{1}{M(\ell x_{j-1}, \ell x_j, s)} - 1\right) \\
&\quad \leq \left(\alpha^i + \alpha^{i+1} + \cdots + \alpha^{j-1}\right)\left(\frac{1}{M(\ell x_1, \ell x_0, s)} - 1\right) \\
&\quad \leq \frac{\alpha^i}{1-\alpha}\left(\frac{1}{M(\ell x_1, \ell x_0, s)} - 1\right) \\
&\quad \to 0 \quad \text{as } i \to \infty,
\end{aligned}$$

which shows that a sequence (ℓx_i) is Cauchy sequence and $\ell(X)$ is a complete subspace of X. Hence $\exists u, v \in X$ such that $\ell x_i \to u = \ell v$ as $i \to \infty$, i.e.,

$$\lim_{i\to\infty} M(u, \ell x_i, s) = 1 \quad \text{for } s \gg \vartheta. \tag{3.3}$$

Since M is triangular, we have

$$\frac{1}{M(\ell v, F_1 v, s)} - 1 \le \left(\frac{1}{M(\ell v, \ell x_{2i+2}, s)} - 1\right) + \left(\frac{1}{M(\ell x_{2i+2}, F_1 v, s)} - 1\right), \quad \text{for } s \gg \vartheta. \tag{3.4}$$

Now by (3.1), (3.2), and (3.3), for $s \gg \vartheta$,

$$\begin{aligned}
\frac{1}{M(\ell x_{2i+2}, F_1 v, s)} - 1 &= \frac{1}{M(F_2 x_{2i+1}, F_1 v, s)} - 1 \\
&\le \beta\left(\frac{1}{M(\ell v, \ell x_{2i+1}, s)} - 1\right) + \gamma\left(\frac{1}{M(\ell v, F_1 v, s)} - 1 + \frac{1}{M(\ell x_{2i+1}, F_2 x_{2i+1}, s)} - 1\right) \\
&\quad + \delta\left(\frac{1}{M(\ell x_{2i+1}, F_1 v, s)} - 1 + \frac{1}{M(\ell v, F_2 x_{2i+1}, s)} - 1\right) \\
&= \beta\left(\frac{1}{M(\ell v, \ell x_{2i+1}, s)} - 1\right) + \gamma\left(\frac{1}{M(\ell v, F_1 v, s)} - 1 + \frac{1}{M(\ell x_{2i+1}, \ell x_{2i+2}, s)} - 1\right) \\
&\quad + \delta\left(\frac{1}{M(\ell x_{2i+1}, F_1 v, s)} - 1 + \frac{1}{M(\ell v, \ell x_{2i+2}, s)} - 1\right) \\
&\to (\gamma + \delta)\left(\frac{1}{M(u, F_1 v, s)} - 1\right) \quad \text{as } i \to \infty.
\end{aligned}$$

Then,

$$\limsup_{i\to\infty}\left(\frac{1}{M(\ell x_{2i+2}, F_1 v, s)} - 1\right) \le (\gamma + \delta)\left(\frac{1}{M(u, F_1 v, s)} - 1\right), \quad \text{for } s \gg \vartheta.$$

Thus, from (3.3) and (3.4), we have

$$\left(\frac{1}{M(u, F_1 v, s)} - 1\right) \le (\gamma + \delta)\left(\frac{1}{M(u, F_1 v, s)} - 1\right), \quad \text{for } s \gg \vartheta.$$

$\gamma + \delta < 1$, since $\beta + 2\gamma + 2\delta < 1$, then $M(\ell v, F_1 v, s) = M(u, F_1 v, s) = 1$, i.e., $u = \ell v = F_1 v$.

Similarly, we can prove that $u = \ell v = F_2 v$. It follows that u is a common coincidence point of the mappings ℓ, F_1, and F_2 in X such that $u = \ell v = F_1 v = F_2 v$.

Now we prove the uniqueness of the point of coincidence in X for the mappings F_1, F_2, and ℓ. Let u^* be the other point in X such that

$$u^* = \ell v^* = F_1 v^* = F_2 v^*$$

for some $v^* \in X$. Then, by using (3.1) for $s \gg \vartheta$,

$$\begin{aligned}
\frac{1}{M(u, u^*, s)} - 1 &= \frac{1}{M(F_1 v, F_2 v^*, s)} - 1 \\
&\le \beta\left(\frac{1}{M(\ell v, \ell v^*, s)} - 1\right) + \gamma\left(\frac{1}{M(\ell v, F_1 v, s)} - 1 + \frac{1}{M(\ell v^*, F_2 v^*, s)} - 1\right) \\
&\quad + \delta\left(\frac{1}{M(\ell v^*, F_1 v, s)} - 1 + \frac{1}{M(\ell v, F_2 v^*, s)} - 1\right) \\
&= \beta\left(\frac{1}{M(u, u^*, s)} - 1\right) + \gamma\left(\frac{1}{M(u, u, s)} - 1 + \frac{1}{M(u^*, u^*, s)} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&+\delta\left(\frac{1}{M(u^*,u,s)}-1+\frac{1}{M(u,u^*,s)}-1\right)\\
&=(\beta+2\delta)\left(\frac{1}{M(u^*,u,s)}-1\right),
\end{aligned}
$$

$\beta + 2\delta < 1$, since $\beta + 2\gamma + 2\delta < 1$. Thus we get that $M(u, u^*, t) = 1$, that is, $u = u^*$. By using the weak compatibility of (F_1, ℓ), (F_2, ℓ) and Proposition 2.11, we can get a unique common fixed point of F_1, F_2, and ℓ, that is, $\ell v = F_1 v = F_2 v = v$. □

By using the map $\ell = I_x$ and by taking into account that every self-mapping is weakly compatible with identity map, i.e., I_x, we can get the following corollary.

Corollary 3.2 *Let $(X, M, *)$ be a complete fuzzy cone metric space in which M is triangular and the mappings $F_1, F_2 : X \to X$ satisfy*

$$
\begin{aligned}
\frac{1}{M(F_1x,F_2y,s)}-1 \le{}& \beta\left(\frac{1}{M(x,y,s)}-1\right)+\gamma\left(\frac{1}{M(x,F_1x,s)}-1+\frac{1}{M(y,F_2y,s)}-1\right)\\
&+\delta\left(\frac{1}{M(y,F_1x,s)}-1+\frac{1}{M(x,F_2y,s)}-1\right)
\end{aligned}
\tag{3.5}
$$

for all $x, y \in X, s \gg \vartheta$, and $\beta, \gamma, \delta \in [0, \infty)$ with $\beta + 2\gamma + 2\delta < 1$. Then F_1 and F_2 have a unique common fixed point in X. Moreover, the fixed point of F_1 is to be a fixed point of F_2 and conversely.

Example 3.3 Let $X = [0, 1]$, $*$ be a continuous t-norm, and $M : X^2 \times (0, \infty) \to [0, 1]$ be written as

$$
M(x,y,s)=\frac{s}{s+|x-y|}
$$

$\forall x, y \in X$ and $s > 0$. Then easily one can verify that M is triangular and $(X, M, *)$ is a complete FCM-space. Now we can define the mappings $F_1, F_2, \ell : X \to X$ as

$$
F_1z=F_2z=\frac{z}{z+6} \quad \text{and} \quad \ell z=\frac{z}{3}
$$

for every $z \in X$. Then from (3.1) we have that

$$
\begin{aligned}
\frac{1}{M(F_1x,F_2y,s)}-1 &= \left|\frac{F_1x-F_2y}{s}\right| = \frac{1}{s}\left|\frac{x}{x+6}-\frac{y}{y+6}\right|\\
&= \frac{1}{s}\left|\frac{x(y+6)-y(x+6)}{(x+6)(y+6)}\right|\\
&\le \frac{1}{s}\left|\frac{6x-6y}{36}\right| = \frac{1}{2}\left(\frac{1}{M(\ell x,\ell y,s)}-1\right)\\
&\le \beta\left(\frac{1}{M(\ell x,\ell y,s)}-1\right)+\gamma\left(\frac{1}{M(\ell x,F_1x,s)}-1+\frac{1}{M(\ell y,F_2y,s)}-1\right)\\
&\quad+\delta\left(\frac{1}{M(\ell y,F_1x,s)}-1+\frac{1}{M(\ell x,F_2y,s)}-1\right).
\end{aligned}
$$

Hence all the conditions of Theorem 3.1 are satisfied with $\beta = 1/2$, $\gamma = 2/15$, and $\delta = 1/9$. Thus, F_1, F_2, and ℓ have a unique common fixed point in X, that is, 0.

4 Quasi-contraction results in *FCM*-spaces

Definition 4.1 Let $(X, M, *)$ be an FCM-space, and let ℓ, F_1 be two self-maps on X. Then F_1 is called a fuzzy cone quasi-contraction (resp; ℓ-quasi-contraction) if, for some $q_c \in [0,1)$, for all $x, y \in X$ and $s \gg \vartheta$, there exists

$$\mathcal{U} \in C(x,y,s) = \begin{Bmatrix} M(x,y,s), M(x,F_1x,s), M(x,F_1y,s), \\ M(y,F_1x,s), M(y,F_1y,s) \end{Bmatrix} \tag{4.1}$$

$$\big(\text{resp;}\mathcal{U} \in C(\ell;x,y,s)\big) = \begin{Bmatrix} M(\ell x,\ell y,s), M(\ell x,F_1x,s), M(\ell x,F_1y,s), \\ M(\ell y,F_1x,s), M(\ell y,F_1y,s) \end{Bmatrix} \tag{4.2}$$

such that

$$\frac{1}{M(F_1x,F_1y,s)} - 1 \le q_c\left(\frac{1}{\mathcal{U}} - 1\right). \tag{4.3}$$

Theorem 4.2 *Let $F_1, \ell : X \to X$ be two self-maps and M be triangular in a complete FCM-space $(X, M, *)$ such that $F_1(X) \subset \ell(X)$ and $\ell(X)$ is closed. If F_1 is an ℓ-quasi-contraction with constant $q_c \in [0,1)$, then ℓ and F_1 have a unique point of coincidence. Moreover, if a pair (ℓ, F_1) is occasionally weakly compatible, then F_1 and ℓ have a unique common fixed point in X.*

Proof Fix $x_0 \in X$ and use the condition $F_1(X) \subset \ell(X)$. We construct a sequence (y_i) in X such that

$$y_i = F_1x_i = \ell x_{i+1} \quad \text{for all } i \ge 0.$$

Now, we have to show that (y_i) is a Cauchy sequence. First, we prove that

$$\frac{1}{M(y_i,y_{i+1},s)} - 1 \le \frac{q_c}{1-q_c}\left(\frac{1}{M(y_{i-1},y_i,s)} - 1\right) \tag{4.4}$$

for all $i \ge 1$ and $s \gg \vartheta$. Indeed,

$$\frac{1}{M(y_i,y_{i+1},s)} - 1 = \frac{1}{M(F_1x_i,F_1x_{i+1},s)} - 1 \le q_c\left(\frac{1}{\mathcal{U}_i} - 1\right), \tag{4.5}$$

where

$$\begin{aligned} \mathcal{U}_i &\in \begin{Bmatrix} M(\ell x_i,\ell x_{i+1},s), M(\ell x_i,F_1x_i,s), M(\ell x_i,F_1x_{i+1},s), \\ M(\ell x_{i+1},F_1x_i,s), M(\ell x_{i+1},F_1x_{i+1},s) \end{Bmatrix} \\ &= \begin{Bmatrix} M(y_{i-1},y_i,s), M(y_{i-1},y_i,s), M(y_{i-1},y_{i+1},s), \\ M(y_i,y_i,s), M(y_i,y_{i+1},s) \end{Bmatrix} \\ &= \big\{M(y_{i-1},y_i,s), M(y_{i-1},y_{i+1},s), 1, M(y_i,y_{i+1},s)\big\}. \end{aligned} \tag{4.6}$$

Then we may have the following four cases:

(i) First,

$$\begin{aligned}\frac{1}{M(y_i, y_{i+1}, s)} - 1 &\leq q_c\left(\frac{1}{M(y_{i-1}, y_i, s)} - 1\right) \\ &\leq \frac{q_c}{1-q_c}\left(\frac{1}{M(y_{i-1}, y_i, s)} - 1\right), \quad \text{for } s \gg \vartheta.\end{aligned}$$

Thus (4.4) holds as $q_c < q_c/(1-q_c)$ since $q_c \in [0, 1)$.

(ii) Second, by using the M triangular property, we have

$$\begin{aligned}\frac{1}{M(y_i, y_{i+1}, s)} - 1 &\leq q_c\left(\frac{1}{M(y_{i-1}, y_{i+1}, s)} - 1\right) \\ &\leq q_c\left(\frac{1}{M(y_{i-1}, y_i, s)} - 1 + \frac{1}{M(y_i, y_{i+1}, s)} - 1\right) \\ &\leq \frac{q_c}{1-q_c}\left(\frac{1}{M(y_{i-1}, y_i, s)} - 1\right), \quad \text{for } s \gg \vartheta.\end{aligned}$$

It follows that (4.4) holds.

(iii) Third,

$$\frac{1}{M(y_i, y_{i+1}, s)} - 1 \leq q_c.0, \quad \text{which implies that } M(y_i, y_{i+1}, s) = 1 \text{ for } s \gg \vartheta.$$

Hence (4.4) holds.

(iv) Fourth,

$$\frac{1}{M(y_i, y_{i+1}, s)} - 1 \leq q_c\left(\frac{1}{M(y_i, y_{i+1}, s)} - 1\right),$$

which implies $M(y_i, y_{i+1}, s) = 1$ for $s \gg \vartheta$.

In this case, immediately (4.4) follows since $q_c \in [0, 1)$.

Now, we may assume that $\delta = \frac{q_c}{1-q_c} < 1$, then we have that

$$\frac{1}{M(y_i, y_{i+1}, s)} - 1 \leq \delta\left(\frac{1}{M(y_{i-1}, y_i, s)}\right), \quad \text{for } s \gg \vartheta.$$

In view of (4.4),

$$\frac{1}{M(y_i, y_{i+1}, s)} - 1 \leq \delta\left(\frac{1}{M(y_{i-1}, y_i, s)}\right) \leq \cdots \leq \delta^i\left(\frac{1}{M(y_0, y_1, s)}\right)$$

for all $i \geq 1$ and $s \gg \vartheta$, which shows that (y_i) is a fuzzy cone contractive sequence in X such that

$$\lim_{i\to\infty} M(y_i, y_{i+1}, s) = 1 \quad \text{for } s \gg \vartheta. \tag{4.7}$$

Since M is triangular, then for all $j > i \geq i_0$,

$$\frac{1}{M(y_i, y_j, s)} - 1$$

$$\begin{aligned}
&\leq \left(\frac{1}{M(y_i, y_{i+1}, s)} - 1\right) + \left(\frac{1}{M(y_{i+1}, y_{i+2}, s)} - 1\right) + \cdots + \left(\frac{1}{M(y_{j-1}, y_j, s)} - 1\right) \\
&\leq \left(\delta^i + \delta^{i+1} + \cdots + \delta^{j-1}\right)\left(\frac{1}{M(y_0, y_1, s)} - 1\right) \\
&\leq \frac{\delta^i}{1-\delta}\left(\frac{1}{M(y_0, y_1, s)} - 1\right) \\
&\to 0 \quad \text{as } i \to \infty,
\end{aligned}$$

which shows that (y_i) is a Cauchy sequence in X. Since $(X, M, *)$ is complete and $\ell(X)$ is closed, $\exists v \in X$ such that $y_i = F_1 x_i = \ell x_{i+1} \to \ell v$ as $i \to \infty$, i.e.,

$$\lim_{i\to\infty} M(y_i, \ell v, s) = 1 \quad \text{for } s \gg \vartheta. \tag{4.8}$$

Now we have to show that $\ell v = F_1 v$. By using the triangularity of M, we have

$$\frac{1}{M(\ell v, F_1 v, s)} - 1 \leq \left(\frac{1}{M(\ell v, y_i, s)} - 1\right) + \left(\frac{1}{M(y_i, F_1 v, s)} - 1\right), \quad \text{for } s \gg \vartheta. \tag{4.9}$$

By the definition of ℓ-quasi-contraction, we have that

$$\frac{1}{M(y_i, F_1 v, s)} - 1 = \frac{1}{M(F_1 x_i, F_1 v, s)} - 1 \leq q_c\left(\frac{1}{\mathcal{U}_i} - 1\right), \quad \text{for } s \gg \vartheta, \tag{4.10}$$

where

$$\begin{aligned}
\mathcal{U}_i &\in \left\{\begin{matrix} M(\ell x_i, \ell v, s), M(\ell x_i, F_1 x_i, s), M(\ell x_i, F_1 v, s), \\ M(\ell v, F_1 x_i, s), M(\ell v, F_1 v, s) \end{matrix}\right\} \\
&= \left\{\begin{matrix} M(\ell x_i, \ell v, s), M(\ell x_i, \ell x_{i+1}, s), M(\ell x_i, F_1 v, s), \\ M(\ell v, \ell x_{i+1}, s), M(\ell v, F_1 v, s) \end{matrix}\right\} \\
&\to \left\{1, 1, M(\ell v, F_1 v, s), 1, M(\ell v, F_1 v, s)\right\} \quad \text{as } i \to \infty.
\end{aligned}$$

This implies

$$\mathcal{U}_i \to \left\{1, M(\ell v, F_1 v, s)\right\} \quad \text{as } i \to \infty$$

for $s \gg \vartheta$. Then we have the following two cases:

Case i: If $\mathcal{U}_i \to 1$ as $i \to \infty$. Then from (4.8), (4.9), and (4.10), we get that $M(\ell v, F_1 v, s) = 1$ as $i \to \infty$ for $s \gg \vartheta$. That is, $\ell v = F_1 v = u$.

Case ii: If $\mathcal{U}_i \to M(\ell v, F_1 v, s)$ as $i \to \infty$. Then from (4.10) we have that

$$\limsup_{i\to\infty}\left(\frac{1}{M(y_i, F_1 v, s)} - 1\right) \leq q_c\left(\frac{1}{M(\ell v, F_1 v, s)} - 1\right), \quad \text{for } s \gg \vartheta.$$

Now, this together with (4.8) and (4.9) gives,

$$\frac{1}{M(\ell v, F_1 v, s)} - 1 \leq q_c\left(\frac{1}{M(\ell v, F_1 v, s)} - 1\right), \quad \text{for } s \gg \vartheta.$$

Since $q_c < 1$, therefore $M(\ell\nu, F_1\nu, s) = 1$, i.e., $\ell\nu = F_1\nu = u$. Thus from both cases we get that $\ell\nu = F_1\nu = u$. Hence, the same as in Theorem 3.1, ν is the coincidence point of (ℓ, F_1) and u is its coincidence point in X. The uniqueness of the coincidence point can be shown by the standard way. By using Lemma 2.13, one can readily obtain that, when (ℓ, F_1) is occasionally weakly compatible, then u is a unique common fixed point of ℓ and F_1 in X. □

Theorem 4.3 *Let ℓ be a self-map on X and M be triangular in a complete FCM-space $(X, M, *)$ such that ℓ^2 is continuous. Let the self-map $F_1 : X \to X$ that commutes with ℓ. Further, we assume that F_1 and ℓ satisfy*

$$F_1\ell(X) \subset \ell^2(X), \tag{4.11}$$

and let F_1 be an ℓ-quasi-contraction. Then F_1 and ℓ have a unique common fixed point in X.

Proof By condition (4.11), starting with fix $x_0 \in \ell(X)$, define a sequence (x_i) in X such that

$$y_i = F_1x_i = \ell x_{i+1} \quad \text{for } i \geq 0,$$

as in Theorem 4.2. Now

$$\ell y_{i+1} = \ell F_1x_{i+1} = F_1\ell x_{i+1} = F_1y_i = \nu_i \quad \text{for } i \geq 1.$$

The same as in Theorem 4.2, we can get that (ν_i) is a Cauchy sequence and convergent to some point $\nu \in X$ such that

$$\lim_{i\to\infty} M(\ell y_{i+1}, \nu, s) = 1 \quad \text{for } s \gg \vartheta.$$

Further, we have to show that $\ell^2\nu = F_1\ell\nu$. Since,

$$\lim_{i\to\infty} \ell y_i = \lim_{i\to\infty} \ell F_1x_i = \lim_{i\to\infty} F_1\ell x_i = \lim_{i\to\infty} F_1y_{i-1} = \lim_{i\to\infty} \nu_{i-1} = \nu, \tag{4.12}$$

by the continuity of ℓ^2, it follows that

$$\lim_{i\to\infty} \ell^4x_i = \lim_{i\to\infty} \ell^3F_1x_{i-1} = \ell^2\nu. \tag{4.13}$$

Now, by the triangular property of M, we have

$$\begin{aligned}\frac{1}{M(\ell^2\nu, F_1\ell\nu, s)} - 1 \leq{}& \left(\frac{1}{M(\ell^2\nu, \ell^3F_1x_i, s)} - 1\right) \\ &+ \left(\frac{1}{M(\ell^3F_1x_i, F_1\ell\nu, s)} - 1\right), \quad \text{for } s \gg \vartheta\end{aligned} \tag{4.14}$$

and

$$\frac{1}{M(\ell^3F_1x_i, F_1\ell\nu, s)} - 1 = \frac{1}{M(F_1\ell^3x_i, F_1\ell\nu, s)} - 1 \leq q_c\left(\frac{1}{\mathcal{U}_i} - 1\right), \tag{4.15}$$

where

$$\mathcal{U}_i \in \left\{ \begin{array}{l} M(\ell^4 x_i, \ell^2 v, s), M(\ell^4 x_i, F_1 \ell^3 x_i, s), M(\ell^2 v, F_1 \ell v, s), \\ M(\ell^4 x_i, F_1 \ell v, s), M(\ell^2 v, F_1 \ell^3 x_i, s) \end{array} \right\}. \tag{4.16}$$

Now, by using (4.13) for $s \gg \vartheta$, we can get the following:

$$\begin{aligned}
&M(\ell^4 x_i, \ell^2 v, s) \to M(\ell^2 v, \ell^2 v, s) = 1 \quad \text{as } i \to \infty, \\
&M(\ell^4 x_i, F_1 \ell^3 x_i, s) \to M(\ell^2 v, \ell^2 v, s) = 1 \quad \text{as } i \to \infty, \\
&M(\ell^4 x_i, F_1 \ell v, s) \to M(\ell^2 v, F_1 \ell v, s) \quad \text{as } i \to \infty, \\
&M(\ell^2 v, F_1 \ell^3 x_i, s) \to M(\ell^2 v, \ell^2 v, s) = 1 \quad \text{as } i \to \infty.
\end{aligned}$$

Equation (4.16) can be written as

$$\mathcal{U}_i \in \{1, 1, M(\ell^2 v, F_1 \ell v, s), M(\ell^2 v, F_1 \ell v, s), 1\} = \{1, M(\ell^2 v, F_1 \ell v, s)\}$$

as $i \to \infty$. Then we have the following two cases:

Case i: If $\mathcal{U}_i \to 1$ as $i \to \infty$, then from (4.13), (4.14), and (4.15), we can get $M(\ell^2 v, F_1 \ell v, s) = 1$, for $s \gg \vartheta$. This implies that $F_1 \ell v = \ell^2 v$.

Case ii: If $\mathcal{U}_i \to M(\ell^2 v, F_1 \ell v, s)$ as $i \to \infty$, for $s \gg \vartheta$. Then we have

$$\limsup_{i \to \infty} \left(\frac{1}{M(\ell^3 F_1 x_i, F_1 \ell v, s)} - 1 \right) \leq q_c \left(\frac{1}{M(\ell^2 v, F_1 \ell v, s)} - 1 \right), \quad \text{for } s \gg \vartheta.$$

This together with (4.13) and (4.14) leads to

$$\frac{1}{M(\ell^2 v, F_1 \ell v, s)} - 1 \leq q_c \left(\frac{1}{M(\ell^2 v, F_1 \ell v, s)} - 1 \right), \quad \text{for } s \gg \vartheta.$$

Since $0 \leq q_c < 1$, this implies that $M(\ell^2 v, F_1 \ell v, s) = 1$, that is, $F_1 \ell v = \ell^2 v$. Thus from both cases we get that $F_1 \ell v = \ell^2 v$. This implies that ℓv is the common fixed point of ℓ and F_1.

Now we prove the uniqueness. Assume that $\ell v = w$ such that $F_1 w = \ell w$, and let w^* be the other common fixed point of the mappings ℓ and F_1 such that $F_1 w^* = \ell w^*$. Then, by the standard way of ℓ-quasi-contraction, easily we can get that $w = w^*$. This completes the proof. □

5 Application

In this section, we present an integral type application, which is the new direction in *FCM*-spaces. For this purpose, we present the two Urysohn integral type equations, or shortly UITEs, to prove the existence result for common solution. Assume that $X = [0, 1]$, and let $\mathbf{E}$ be the real-valued functions on X. Then $\mathbf{E}$ is a vector space over $\mathbb{R}$ under the following operations:

$$(x + y)(l) = x(l) + y(l), \qquad (\beta x)(l) = \beta x(l)$$

for all $x, y \in \mathbf{E}$ and $\beta \in \mathbb{R}$, and

$$P = \{x \in \mathbf{E} | x(l) \geq 0, \forall l \in [0, 1]\}.$$

$*$ is a continuous s-norm and an FM-space $M : \mathbf{E} \times \mathbf{E} \times (0, \infty) \to [0, 1]$ can be expressed as

$$M(x, y, s) = \frac{s}{s + d(x, y)}, \quad \text{where } d(x, y) = \|x - y\|$$

for all $x, y \in \mathbf{E}$ and $s > 0$. Then easily we can show that M is triangular and $(\mathbf{E}, M, *)$ is a complete FCM-space.

Theorem 5.1 *The two UITEs are*

$$\begin{aligned} x(l) &= \int_0^1 K_1\big(l, v, x(v)\big)\,dv + g(l), \\ y(l) &= \int_0^1 K_2\big(l, v, y(v)\big)\,dv + g(l), \end{aligned} \tag{5.1}$$

where $l \in [0, 1]$ *and* $x, y, g \in \mathbf{E}$.

Assume that $K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ *are such that* $A_x, B_y \in \mathbf{E}$ *for every* $x, y \in \mathbf{E}$, *where*

$$\begin{aligned} A_x(l) &= \int_0^1 K_1\big(l, v, x(v)\big)\,dv + g(l), \\ B_y(l) &= \int_0^1 K_2\big(l, v, y(v)\big)\,dv + g(l), \end{aligned} \tag{5.2}$$

where $l \in [0, 1]$. *If there exists* $\lambda \in (0, 1)$ *such that, for all* $x, y \in X$,

$$\|A_x - B_y\| \leq \lambda N(x, y), \tag{5.3}$$

where

$$N(x, y) = \max\big\{\|x - y\|, \big(\|A_x - x\| + \|B_y - y\|\big), \big(\|A_x - y\| + \|B_y - x\|\big)\big\}.$$

Then the two UITEs (5.1) *have a unique common solution.*

Proof Define the mappings $F_1, F_2, \ell : \mathbf{E} \to \mathbf{E}$:

$$\ell(x) = x, \qquad F_1(x) = A_x \quad \text{and} \quad F_2(y) = B_y.$$

If

$$N(x, y) = \|x - y\|,$$

then

$$\big\|F_1(x) - F_2(y)\big\| \leq \delta \|x - y\|,$$

$\forall x, y \in \mathbf{E}$, by Theorem 3.1 with $\lambda = \beta$ and $\gamma = \delta = 0$ in Theorem 3.1. Then the two UITEs (5.1) have a unique common solution. If

$$N(x,y) = \|A_x - x\| + \|B_y - y\|,$$

then

$$\|F_1(x) - F_2(y)\| \le \lambda\big(\|F_1(x) - x\| + \|F_2(y) - y\|\big),$$

$\forall x, y \in \mathbf{E}$, by Theorem 3.1 with $\lambda = \gamma$ and $\beta = \delta = 0$. Then the two UITEs (5.1) have a unique common solution. Again, if

$$N(x,y) = \|A_x - y\| + \|B_y - x\|,$$

then

$$\|F_1(x) - F_2(y)\| \le \lambda\big(\|F_1(x) - y\| + \|F_2(y) - x\|\big),$$

$\forall x, y \in \mathbf{E}$, by Theorem 3.1 with $\lambda = \delta$ and $\beta = \gamma = 0$. Then from the two UITEs (5.1), we have a unique common solution. □

Now, we present a special type of example for UITEs.

Example 5.2 Let $X = [0,1]$ and the following integral equation be of the form

$$\begin{aligned} x(l) &= \int_0^1 \frac{1}{3(l + 1 + x(v))}\,dv + \frac{l}{3}, \\ y(l) &= \int_0^1 \frac{1}{3(l + 1 + y(v))}\,dv + \frac{l}{3}. \end{aligned} \tag{5.4}$$

The problem system of equations (5.4) is a special kind of problem system of equations (5.1), where $g(l) = \frac{l}{3}$ and $l \in [0,1]$, and

$$K_i\big(l, v, w_i(v)\big) = \frac{1}{3(l + 1 + w_i(v))}, \quad \text{where } i = 1, 2.$$

Then we have

$$\begin{aligned} \big\|K_1\big(l, v, x(v)\big) - K_2\big(l, v, y(v)\big)\big\| &= \left\| \frac{1}{3(l + 1 + x(v))} - \frac{1}{3(l + 1 + y(v))} \right\| \\ &= \frac{1}{3} \left\| \frac{x(v) - y(v)}{(l + 1 + x(v))(l + 1 + y(v))} \right\| \\ &\le \frac{1}{3} \big\|x(v) - y(v)\big\| \\ &= \frac{1}{3} N(x,y), \end{aligned}$$

where $N(x,y) = \|x(v) - y(v)\|$. Now, we have to show that $\|A_x(l) - B_y(l)\| \leq \lambda N(x,y)$, from the system of equations (5.2), we have

$$\begin{aligned}
\|A_x(l) - B_y(l)\| &= \left\| \int_0^1 K_1(l,v,x(v))\,dv - \int_0^1 K_2(l,v,y(v))\,dv \right\| \\
&= \int_0^1 \|K_1(l,v,x(v)) - K_2(l,v,y(v))\|\,dv \\
&\leq \int_0^1 \frac{1}{3}\|x(v) - y(v)\|\,dv \\
&= \int_0^1 \frac{1}{3}N(x,y)\,dv \\
&= \frac{1}{3}N(x,y)\int_0^1 dv \\
&= \frac{1}{3}N(x,y).
\end{aligned}$$

Hence, all the conditions of Theorem 5.1 with $\lambda = \frac{1}{3} < 1$ hold. The problem system of equations (5.4) has a unique common solution by using Theorem 5.1.

6 Conclusion

We defined weakly compatible self-mappings in fuzzy cone metric spaces and proved some coincidence point and common fixed point theorems under the fuzzy cone contraction condition without the assumption that fuzzy cone contractive sequences are Cauchy by using the "M triangular condition". This change, to use "M triangular condition" to weaken the "fuzzy cone contractive sequences are Cauchy", is expected to bring a wider range of applications of fixed point theorems in fuzzy cone metric spaces. We also gave the concept of quasi-contraction and proved some common fixed point theorems in fuzzy cone metric spaces. Moreover, we presented an application of the two Urysohn integral type equations for common solution to support our result. We also presented some illustrative examples to support our theoretical work.

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