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Existence of solutions for an ordinary second-order hybrid functional differential equation

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Abstract

In this paper, we study the existence of solutions for an initial value problem of an ordinary second-order hybrid functional differential equation (SHDE) using a fixed point theorem of Dhage. Example is given to illustrate the obtained result.

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Keywords: Second-order hybrid functional differential equation; Dhage fixed point theorem; Existence of solutions

1 Introduction

In this paper, we consider the initial value problems of the second-order hybrid functional differential equation (in short SHDE):

$$\begin{cases} \frac{d^2}{dt^2} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) = g(t, x(\varphi_3(t))) & t \in J = [0, T], \\ x(0) = h(0, x(0)) \quad \text{and} \quad x'(0) = \frac{dh}{dt} \Big|_{t=0}, \end{cases} \quad (1.1)$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, $h \in C(J \times \mathbb{R}, \mathbb{R})$, and $\varphi_i \in C(J)$ with $\varphi_i(0) = 0$, $i = 1, 2, 3$. By a solution of SHDE (1.1) we mean a function $x \in C(J, \mathbb{R})$ such that

- the function $t \rightarrow \frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))}$ is continuous for each $x \in C(J, \mathbb{R})$ and
- x satisfies the equations in (1.1).

The importance of the investigations of hybrid differential equations lies in the fact that they include several dynamic systems as special cases. The consideration of hybrid differential equations is implicit in the works of Krasnoselskii [1] and extensively treated in several papers on hybrid differential equations with different perturbations. See [2–9] and [10] and the references therein. This class of hybrid differential equations includes the perturbations of original differential equations in different ways.

Here we study the existence of solutions for the initial value problem of second-order hybrid differential equation (1.1). Some remarks and an example to illustrate our results are given.

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This paper is organized as follows: In Sect. 2, we recall some useful preliminaries. In Sect. 3, we prove an auxiliary theorem related to the linear variant of problem (1.1) and state sufficient conditions which guarantee the existence of solutions to problem (1.1). Also, conditions are added to our problem in order to obtain a new existence theorem, and an illustrative example is presented.

2 Preliminaries

In this section, we introduce some basic definitions and preliminary facts which we need in the sequel.

Definition 2.1 ([11]) An algebra X is a vector space endowed with an internal composition law noted by

$$(\cdot): X \times X \rightarrow X, \quad (x, y) \rightarrow x.y,$$

which is associative and bilinear. A normed algebra is an algebra endowed with a norm satisfying the following property:

For all $x, y \in X$, we have

$$\|x.y\| \leq \|x\| \cdot \|y\|.$$

A complete normed algebra is called a Banach algebra.

Definition 2.2 ([11]) Let X be a normed vector space. A mapping $T: X \rightarrow X$ is called D -Lipschitzian if there exists a continuous and nondecreasing function $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leq \phi(\|x - y\|)$$

for all $x, y \in X$, where $\phi(0) = 0$.

Sometimes, we call the function ϕ to be a D -function of the mapping T on X . Obviously, every Lipschitzian mapping is D -Lipschitzian. Further, if $\phi(r) < r$, for $r > 0$, then T is called a nonlinear contraction on X . An important fixed point theorem that has been commonly used in the theory of nonlinear integral equations is a generalization of the Banach contraction mapping principle proved in [11].

Recently Dhage in [12] has proven a fixed point theorem involving three operators in a Banach algebra by blending the Banach fixed point theorem with Schauder's fixed point principle.

Lemma 2.3 ([13]) Let S be a nonempty, closed convex, and bounded subset of a Banach algebra X , and let $A, C: X \rightarrow X$ and $B: S \rightarrow X$ be three operators such that:

- (a) A and C are Lipschitzian with Lipschitz constants δ and ρ , respectively;
- (b) B is compact and continuous;
- (c) $x = AxBy + Cx \Rightarrow x \in S$ for all $y \in S$;
- (d) $\delta M + \rho < r$ for $r > 0$ where $M = \|B(S)\|$.

Then the operator equation $AxBx + Cx = x$ has a solution in S .

3 Main results

In this section, we formulate our main result for SHDE (1.1) depending on the fixed point theorems due to Dhage [13].

Let $X = C(J, \mathbb{R})$ of the vector of all real-valued continuous functions on $J = [0, T]$. We equip the space X with the norm $\|x\| = \sup_{t \in J} |x(t)|$. Clearly, $C(J, \mathbb{R})$ is a complete normed algebra with respect to this supremum norm. Consider the following assumptions:

- (A₁) The functions $f : J \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ and $h : J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and there exist two functions $k, L \in C(J, \mathbb{R}_+)$, with norms $\|k\|$ and $\|L\|$ respectively, such that

$$|h(t, x) - h(t, y)| \leq k(t)|x - y|,$$

$$|f(t, x) - f(t, y)| \leq L(t)|x - y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

- (A₂) $g : J \times \mathbb{R} \rightarrow \mathbb{R}$. There exist a function $p \in C(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\Psi : [0, \infty) \rightarrow (0, \infty)$ such that

$$|g(t, x)| \leq p(t)\Psi(|x|), \quad \forall (t, x) \in J \times \mathbb{R}.$$

- (A₃) $\varphi_i : J \rightarrow J$ are continuous functions with $\varphi_i(0) = 0$, $i = 1, 2, 3$.

- (A₄) There exists a number $r > 0$ such that

$$r \geq \frac{H + G\|p\|\Psi(r)\frac{T^2}{2}}{1 - (\|L\|\|p\|\Psi(r)\frac{T^2}{2} + \|k\|)}, \quad (3.1)$$

where $G = \sup_{t \in J} |f(t, 0)|$, $H = \sup_{t \in J} |h(t, 0)|$, and

$$\|L\|\|p\|\Psi(r)\frac{T^2}{2} + \|k\| < 1. \quad (3.2)$$

Now, we shall prove the following lemma.

Lemma 3.1 Assume that hypotheses (A₁) – (A₄) hold. Then a function $x \in C(J, \mathbb{R})$ is a solution of SHDE (1.1) if, and only if, it satisfies the following quadratic integral equation:

$$x = h(t, x(\varphi_1(t))) + f(t, x(\varphi_2(t))) \int_0^t (t-s)g(s, x(\varphi_3(s))) ds. \quad (3.3)$$

Proof First, assume that x is a solution of SHDE (1.1), applying integration to both sides of (1.1) from 0 to t , we obtain

$$\frac{d}{dt} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) - \frac{d}{dt} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) \Big|_{t=0} = \int_0^t g(s, x(\varphi_3(s))) ds.$$

On the other hand (due to the fact that $f(0, x(0)) \neq 0$ and $\varphi_i(0) = 0$, $i = 1, 2, 3$), we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) \Big|_{t=0} \\ &= \frac{f(0, x(0))(x'(0) - \frac{dh}{dt}|_{t=0}) - (x(0) - h(0, x(0)))\frac{df}{dt}|_{t=0}}{f^2(0, x(0))} = 0. \end{aligned}$$

Since

$$x(0) = h(0, x(0)),$$

$$x'(0) = \left. \frac{dh}{dt} \right|_{t=0},$$

therefore, we get

$$\frac{d}{dt} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) = \int_0^t g(s, x(\varphi_3(s))) ds. \quad (3.4)$$

Again integrating both sides of (3.4) from 0 to t , we obtain

$$\left. \frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} - \frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right|_{t=0} = \int_0^t (t-s)g(s, x(\varphi_3(s))) ds, \quad (3.5)$$

and we have

$$\left. \frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right|_{t=0} = \frac{x(0) - h(0, x(0))}{f(0, x(0))} = 0.$$

Hence, Eq. (3.5) becomes

$$\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} = \int_0^t (t-s)g(s, x(\varphi_3(s))) ds,$$

i.e.,

$$x(t) = h(t, x(\varphi_1(t))) + f(t, x(\varphi_2(t))) \int_0^t (t-s)g(s, x(\varphi_3(s))) ds.$$

Thus, Eq. (3.3) holds.

Conversely, assume that x satisfies Eq. (3.3). Then dividing by $f(t, x(t))$ and making direct differentiation for both sides of Eq. (3.3), we obtain

$$\frac{d}{dt} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) = \int_0^t g(s, x(\varphi_3(s))) ds.$$

Then, again by direct differentiation, Eq. (1.1) is satisfied.

$$\frac{d^2}{dt^2} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) = g(t, x(\varphi_3(t))).$$

Again, substituting $t = 0$ in Eq. (3.3) (due to the fact that $f(0, x(0)) \neq 0$ and $\varphi_i(0) = 0$, $i = 1, 2, 3$) yields

$$\frac{x(0) - h(0, x(0))}{f(0, x(0))} \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

hence $x(0) = h(0, x(0))$, and

$$\left. \frac{d}{dt} \left(\frac{x(t) - h(t, x(\varphi_1(t)))}{f(t, x(\varphi_2(t)))} \right) \right|_{t=0} = 0,$$

$$\frac{f(t, x(t))(x'(t) - \frac{dh}{dt}) - (x(t) - h(t, x(t))) \frac{df}{dt}}{f^2(t, x(t))} \Big|_{t=0} = 0,$$

$$f(0, x(0)) \left(x'(0) - \frac{dh}{dt} \Big|_{t=0} \right) - (x(0) - h(0, x(0))) \frac{df}{dt} \Big|_{t=0} = 0.$$

Since we have proven $x(0) = h(0, x(0))$, this yields $x'(0) = \frac{dh}{dt} \Big|_{t=0}$. The proof is completed. \square

3.1 Existence of solution

Now, our target is to prove the following existence theorems.

Theorem 3.2 *Assume that hypotheses $(A_1) - (A_4)$ hold. Then SHDE (1.1) has at least one solution defined on J .*

Proof By Lemma 3.1, problem (1.1) is equivalent to the quadratic functional integral equation (3.3). Set $X = C(J, \mathbb{R})$ and define a subset S of X as

$$S := \{x \in X, \|x\| \leq r\},$$

where r satisfies inequality (3.1).

Clearly S is a closed, convex, and bounded subset of the Banach space X .

Corresponding to the functions f , g , and h , we introduce the three operators $A : X \rightarrow X$, $B : S \rightarrow X$, and $C : X \rightarrow X$ defined by

$$(Ax)(t) = f(t, x(\varphi_2(t))), \quad t \in J, \quad (3.6)$$

$$(Bx)(t) = \int_0^t (t-s)g(s, x(\varphi_3(t))) ds, \quad t \in J, \quad (3.7)$$

$$(Cx)(t) = h(t, x(\varphi_1(t))), \quad t \in J. \quad (3.8)$$

Then the integral equation (3.3) can be rewritten as follows:

$$x(t) = Ax(t) + Bx(t) + Cx(t), \quad t \in J. \quad (3.9)$$

We shall show that A , B , and C satisfy all the conditions of Lemma 2.3. This will be achieved in the following series of steps.

Step 1. To show that A and C are Lipschitzian on X , let $x, y \in X$. So

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(\varphi_2(t))) - f(t, y(\varphi_2(t)))| \\ &\leq L(t)|x(\varphi_2(t)) - y(\varphi_2(t))| \leq \|L\| \|x - y\|, \end{aligned}$$

which implies $\|Ax - Ay\| \leq \|L\| \|x - y\|$ for all $x, y \in X$. Therefore, A is a Lipschitzian on X with Lipschitz constant $\|L\|$.

Similarly, for any $x, y \in X$, we have

$$\begin{aligned} |Cx(t) - Cy(t)| &= |h(t, x(\varphi_1(t))) - h(t, y(\varphi_1(t)))| \\ &\leq k(t)|x(\varphi_1(t)) - y(\varphi_1(t))| \leq \|k\| \|x - y\|. \end{aligned}$$

Consequently,

$$\|Cx - Cy\| \leq \|k\| \|x - y\|.$$

This shows that C is a Lipschitz mapping on X with the Lipschitz constant $\|k\|$.

Step 2. To prove that B is a compact and continuous operator on S into X .

First we show that B is continuous on X . Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then, by the Lebesgue dominated convergence theorem, let us assume that $t \in J$, and since $\varphi_3(t)$ is a continuous function and $g(t, x(t))$ is continuous in x , then $g(t, x_n(\varphi_3(t)))$ converges to $g(t, x(\varphi_3(t)))$, (see assumption (A_2)). Applying the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n(t) &= \lim_{n \rightarrow \infty} \int_0^t (t-s)g(s, x_n(\varphi_3(s))) ds \\ &= \int_0^t (t-s)g(s, x(\varphi_3(s))) ds \\ &= Bx(t). \end{aligned}$$

Thus, $Bx_n \rightarrow Bx$ as $n \rightarrow \infty$ uniformly on \mathbb{R}_+ , and hence B is a continuous operator on S into S .

Now, we show that B is a compact operator on S . It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in X . To see this, let $x \in S$ be arbitrary. Then, by hypothesis (A_2) ,

$$\begin{aligned} |Bx(t)| &\leq \int_0^t (t-s)|g(s, x(\varphi_3(s)))| ds \\ &\leq \int_0^t (t-s)p(s)\Psi(|x|) ds \\ &\leq \|p\|\Psi(r) \int_0^t (t-s) ds \\ &\leq \|p\|\Psi(r) \frac{T^2}{2} = K \end{aligned}$$

for all $t \in J$. Taking supremum over t ,

$$\|Bx(t)\| \leq K$$

for all $x \in S$. This shows that B is uniformly bounded on S .

Now, we proceed to showing that $B(S)$ is also an equicontinuous set in X . Let $t_1, t_2 \in J$, and $x \in S$ (without loss of generality assume that $t_1 < t_2$), then we have

$$\begin{aligned} (Bx)(t_2) - (Bx)(t_1) &= \int_0^{t_2} (t_2-s)g(s, x(\varphi_3(s))) ds - \int_0^{t_1} (t_1-s)g(s, x(\varphi_3(s))) ds \\ &\leq \int_0^{t_1} (t_2-s)g(s, x(\varphi_3(s))) ds + \int_{t_1}^{t_2} (t_2-s)g(s, x(\varphi_3(s))) ds \end{aligned}$$

$$\begin{aligned}
& - \int_0^{t_1} (t_1 - s)g(s, x(\varphi_3(s))) \, ds \\
& \leq \int_0^{t_1} [(t_2 - s) - (t_1 - s)]g(s, x(\varphi_3(s))) \, ds + \int_{t_1}^{t_2} (t_2 - s)g(s, x(\varphi_3(s))) \, ds
\end{aligned}$$

and

$$\begin{aligned}
& |(Bx)(t_2) - (Bx)(t_1)| \\
& \leq \int_0^{t_1} (t_2 - t_1)|g(s, x(\varphi_3(s)))| \, ds + \int_{t_1}^{t_2} (t_2 - s)|g(s, x(\varphi_3(s)))| \, ds \\
& \leq \int_0^{t_1} (t_2 - t_1)p(t)\Psi(|x|) \, ds + \int_{t_1}^{t_2} (t_2 - s)p(t)\Psi(|x|) \, ds \\
& \leq \|p\|\Psi(r) \left[T(t_2 - t_1) + \int_{t_1}^{t_2} (t_2 - s) \, ds \right] \\
& \leq \|p\|\Psi(r) \left[T(t_2 - t_1) + \frac{(t_2 - t_1)^2}{2} \right],
\end{aligned}$$

i.e.,

$$|(Bx)(t_2) - (Bx)(t_1)| \leq \|p\|\Psi(r) \left[T(t_2 - t_1) + \frac{(t_2 - t_1)^2}{2} \right],$$

which is independent of $x \in S$. Hence, for $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta \implies |(Bx)(t_2) - (Bx)(t_1)| < \epsilon$$

for all $t_2, t_1 \in J$ and for all $x \in S$. This shows that $B(S)$ is an equicontinuous set in X . Now, the set $B(S)$ is a uniformly bounded and equicontinuous set in X , so it is compact by the Arzela–Ascoli theorem. As a result, B is a complete continuous operator on S .

Step 3. Hypothesis (c) of Lemma 2.3 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = AxBy + Cx$. Then we have

$$\begin{aligned}
|x(t)| & \leq |Ax(t)| + |By(t)| + |Cx(t)| \\
& \leq |f(t, x(\varphi_2(t)))| \int_0^t (t - s)|g(s, y(\varphi_3(s)))| \, ds + |h(t, x(\varphi_1(t)))| \\
& \leq (|f(t, x(\varphi_2(t))) - f(t, 0)| + |f(t, 0)|) \int_0^t (t - s)p(t)\Psi(|y|) \, ds \\
& \quad + (|h(t, x(\varphi_1(t))) - h(t, 0)| + |h(t, 0)|) \\
& \leq (\|L\||x(\varphi_2(t))| + G)\|p\|\Psi(r) \int_0^t (t - s) \, ds + \|k\||x(\varphi_3(t))| + H \\
& \leq (\|L\|r + G)\|p\|\Psi(r) \frac{T^2}{2} + \|k\|r + H.
\end{aligned}$$

Consequently,

$$|x(t)| \leq (\|L\|r + G)\|p\|\Psi(r) \frac{T^2}{2} + \|k\|r + H.$$

Taking supremum over t ,

$$\|x\| \leq (\|L\|r + G)\|p\|\Psi(r)\frac{T^2}{2} + \|k\|r + H.$$

Therefore, $x \in S$.

Step 4. Finally we show that $\delta M + \rho < 1$, that is, (d) of Lemma 2.3 holds.

Since

$$\begin{aligned} M &= \|B(S)\| \\ &= \sup_{x \in S} \left\{ \sup_{t \in J} |Bx(t)| \right\} \\ &\leq \|p\|\Psi(r)\frac{T^2}{2}, \end{aligned}$$

and by (A_4) , we have

$$\|L\|M + \|k\| < 1$$

with $\delta = \|L\|$ and $\rho = \|k\|$.

Thus all the conditions of Lemma 2.3 are satisfied, and hence the operator equation $x = Ax + Bx + Cx$ has a solution in S . In consequence, problem (1.1) has a solution on J . This completes the proof. \square

3.2 Remarks and examples

- If we replace conditions (A_2) and (A_4) with the following conditions:

(A'_2) $g : J \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Caratheodory condition, i.e., g is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0, T]$.

There exist two positive real functions $t \rightarrow a(t)$, $t \rightarrow b(t)$ such that

$$|g(t, x)| \leq a(t) + b(t)|x|, \quad \forall (t, x) \in J \times \mathbb{R};$$

(A'_4) There exists a number $r > 0$ such that

$$r \leq \frac{\|L\| \cdot \|a\| T^2 + G\|b\| T^2 + \|k\|}{2\|b\| \cdot \|L\| T^2}, \quad (3.10)$$

where $G = \sup_{t \in J} |f(t, 0)|$ and $(\|L\| \|a\| + G\|b\|) T^2 + \|k\| < 1$,

and *Step 3* in the proof can be replaced with the following. Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = Ax + By + Cx$. Then we have

$$\begin{aligned} |x(t)| &\leq |Ax(t)| + |By(t)| + |Cx(t)| \\ &\leq |f(t, x(\varphi_2(t)))| \int_0^t (t-s) |g(s, y(\varphi_3(s)))| ds + |h(t, x(\varphi_1(t)))| \\ &\leq (|f(t, x(\varphi_2(t))) - f(t, 0)| + |f(t, 0)|) \int_0^t (t-s) (a(s) + b(s)|x(\varphi_3(s))|) ds \\ &\quad + (|h(t, x(\varphi_1(t))) - h(t, 0)| + |h(t, 0)|) \end{aligned}$$

$$\begin{aligned} &\leq (\|L\| |x(\varphi_2(t))| + G)(\|a\| + \|b\|r) \int_0^t (t-s) ds + \|k\| |x(\varphi_3(t))| + H \\ &\leq (\|L\|r + G)(\|a\| + \|b\|r) \frac{T^2}{2} + \|k\|r + H. \end{aligned}$$

Consequently,

$$|x(t)| \leq (\|L\|r + G)(\|a\| + \|b\|r)T^2 + \|k\|r + H.$$

Taking supremum over t ,

$$\|x\| \leq (\|L\|r + G)(\|a\| + \|b\|r)T^2 + \|k\|r + H.$$

From this estimate we show that the operator A maps the set S into itself with

$$r = \frac{\|L\| \cdot \|a\| T^2 + G \|b\| T^2 + \|k\|}{2 \|b\| \cdot \|L\| T^2}.$$

- Our results can be obtained using the technique of measures of noncompactness in the Banach algebras (under assumptions $(A_1) - (A_4)$) and a fixed point theorem for the product of two operators verifying a Dhage type condition as follows in [2].

Example Consider the second-order functional differential equation

$$\begin{cases} \frac{d^2}{dt^2} \left(\frac{x(t) - \frac{\cos \pi t + 2t^2}{1+5t^2} |x(t)|}{\left(\frac{|x(t)|+1}{|x(t)|+2} \right) \frac{7-e^t}{2\sqrt{25-t^2}} + \frac{2-t}{10}} \right) = \frac{(t-1)^2+3}{35(13-t^2)} (7|x(t)| + 15), & t \in J = [0, 1], \\ x(0) = h(0, x(0)) \quad \text{and} \quad x'(0) = \frac{dh}{dt} \Big|_{t=0}, \end{cases} \quad (3.11)$$

where

$$f(t, x(t)) = \left(\frac{|x(t)| + 1}{|x(t)| + 2} \right) \frac{7 - e^t}{2\sqrt{25 - t^2}} + \frac{2 - t}{10},$$

$$|f(t, x(t)) - f(t, y(t))| \leq \left(\frac{7 - e^t}{2\sqrt{25 - t^2}} \right) |x - y|,$$

$$h(t, x(t)) = \frac{\cos \pi t + 2t^2}{1 + 5t^2} |x(t)|,$$

$$|h(t, x(t)) - h(t, y(t))| \leq \left(\frac{1 + 2t^2}{1 + 5t^2} \right) |x - y|,$$

and

$$g(t, x(t)) = \frac{(t-1)^2+3}{35(13-t^2)} (7|x(t)| + 15) = \left(\frac{(t-1)^2+3}{13-t^2} \right) \left(\frac{|x|}{5} + \frac{3}{7} \right).$$

Take $p(t) = \frac{(t-1)^2+3}{13-t^2}$ and $\Psi(x) = \frac{|x|}{5} + \frac{3}{7}$.

We can easily verify that $x(0) = h(0, x(0))$, $x'(0) = \frac{dh}{dt} \Big|_{t=0}$.

$\|k\| = 1/2$, $\|p\| = 4/13$, $\|L\| = 7/10$, and $G = 1$.

For condition $\|L\| \|p\| \Psi(r) \frac{T^2}{2} + k < 1$ is satisfied, r should be chosen $r < 21.07$.

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